General Relativity and Black Holes – Week 10

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This example sheet (mostly) covers some basic Sobolev inequalities used in lectures.

1 Exercises

1. Let $(S^2, \gamma = d\theta^2 + \sin^2\theta d\phi^2)$ denote the round sphere of radius 1.

   (a) Show that the vectorfields

   $$\Omega_1 = \partial_\phi \quad , \quad \Omega_2 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi \quad , \quad \Omega_3 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$$

   generate the Lie algebra of rotations (i.e. $[\Omega_i, \Omega_j] = \epsilon_{ijk}\Omega_k$). Do the vectorfields extend regularly to the poles? Show also the identity $\mathring{\Delta} = \sum_{i=1}^{3} \Omega_i\Omega_i$, where $\mathring{\Delta} = \gamma^{AB}\mathring{\nabla}_A\mathring{\nabla}_B$ denotes the covariant Laplacian on the sphere and $\mathring{\nabla}$ is the covariant derivative of γ.

   (b) For $\psi : S^2 \to \mathbb{R}$ smooth show the estimate

   $$|\mathring{\nabla}\psi|^2_\gamma := \gamma^{AB}\partial_A\psi\partial_B\psi \leq C\sum_{i=1}^{3}(\Omega_i\psi)^2 \, .$$

   (c) Suppose ψ satisfies the Poisson equation

   $$\mathring{\Delta}\psi = F$$

   on S^2.[1] Prove the estimate

   $$\int_{S^2} \sin\theta d\theta d\phi \left(|\mathring{\nabla}\mathring{\nabla}\psi|^2_\gamma + |\mathring{\nabla}\psi|^2_\gamma\right) \leq \int_{S^2} \sin\theta d\theta d\phi |F|^2 \, . \tag{1}$$

   Conclude that $\int_{S^2} \sin\theta d\theta d\phi \left(|\mathring{\nabla}\mathring{\nabla}\psi|^2_\gamma + |\mathring{\nabla}\psi|^2_\gamma\right) \leq \sum_{i=1}^{3}\int_{S^2} \sin\theta d\theta d\phi |\Omega_i\Omega_i\psi|^2$.

   HINT: Start from $|\mathring{\Delta}\psi|^2 = |F|^2$ integrated over S^2 and commute covariant derivatives on the left. Unlike in the Euclidean case, the commutators will now produce a term involving the (positive) curvature of the sphere.

2. Let $(S^2, \gamma = d\theta^2 + \sin^2\theta d\phi^2)$ denote the round sphere of radius 1. For $\psi : S^2 \to \mathbb{R}$ smooth show the Sobolev embedding estimate

   $$\sup|\psi|_{S^2} \leq \|\psi\|_{H^2(S^2)} := \sqrt{\int_{S^2} \sin\theta d\theta d\phi \left(|\mathring{\nabla}\mathring{\nabla}\psi|^2_\gamma + |\mathring{\nabla}\psi|^2_\gamma + |\psi|^2\right)}$$

   HINT: One option is to use two charts on the sphere, apply partition of unity and reduce the estimate to the embedding estimate you proved on $\mathbb{R}^n$.

3. Prove the Commutation Lemma from Lectures: If ψ is a smooth solution of the equation $\Box_g\psi = 0$ and X is a vectorfield. Then

   $$\Box_g(X\psi) = 2^{(X)}\pi^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi + \left(2\nabla^{\alpha(X)}\pi_{\alpha\mu} - \nabla_\mu(tr^{(X)}\pi)\right)\nabla^\mu\psi \, . \tag{2}$$

   What happens for the (scaling) vectorfield $S = t\partial_t + \sum_{i=1}^{3} x^i\partial_i$ in Minkowski space $g = \eta$?

[1] Note that this requires $\int_{S^2} F d\gamma = 0$ to hold (why)?

2 Problems and Discussion

1. *(Sobolev Inequality on a Bounded Set, Fritz John, Partial Differential Equations p. 170).* Given a point $x \in \mathbb{R}^n$ and a (relatively) open subset σ of the unit sphere in $\mathbb{R}^n$, we define a cone with vertex x to be the set of points $x + t\xi$, where $t \in [0, \infty)$ and $\xi \in \sigma$. The *solid angle* ω of σ, and correspondingly of the cone, is the volume of σ with respect to the measure on S^{n-1}. We define a *conical sector* Γ with vertex x and radius h to be the intersection of a cone with a ball with center x and radius h; i.e.

$$\Gamma = \{y | y = x + t\xi : t \in [0, h], \xi \in \sigma\}. \tag{3}$$

An open set $\Omega \subset \mathbb{R}^n$ has the *cone property* if there exist positive constants h, ω such that, for every $x \in \Omega$, there exists a conical sector Γ_x with radius h and solid angle ω.

Show that if a subset $\Omega \subset \mathbb{R}^n$ has the cone property, there exists a constant C such that for any $g \in C^s(\Omega)$, where $s = \lfloor \frac{n}{2} \rfloor + 1$ (i.e. the smallest integer greater than $n/2$),

$$|g(x)| \leq C_\Omega \|g\|_{H^s}. \tag{4}$$

   (a) Take a generic function $\phi \in C^s(\mathbb{R})$ such that $\phi(t) = 0$ for $t \geq h$ and integrate by parts s times to get
   $$\phi(0) = C \int_0^h t^{s-1} \phi^{(s)}(t)\, dt,$$
   where C is a constant depending on s, h.

   (b) Use the Cauchy-Schwarz inequality to show that
   $$\phi^2(0) \leq C \int_0^h t^{n-1} (\phi^{(s)}(t))^2\, dt. \tag{5}$$

   (c) Let $\zeta \in C^\infty(\mathbb{R})$ be a cutoff function such that $\zeta(0) = 1$ and $\zeta(t) = 0$ for $t \geq h$. Define
   $$\phi_\xi(t) = \zeta(t) g(x + t\xi). \tag{6}$$
   Give a bound for $\phi_\xi(t)$ using derivatives of g and ζ.

   (d) Write
   $$\phi^2(x) = \frac{1}{\omega} \int_\sigma \phi_\xi^2(0)\, d\xi \tag{7}$$
   and use the results of the previous parts to prove our result.

2. Discuss how the above results are used to obtain Proposition 5.6-Corollary 5.10 in the lecture notes.