Non-linear Wave Equations – Week 10

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1. Complete the proof of Proposition 7.4.6 of the lecture notes, i.e. prove that the formula for the commuted null form holds also for null-forms of type $Q_{\mu\nu}$.

2. (Integrated local energy decay estimate: Part I. This problem is a bit more advanced conceptually but actually only requires basic multivariable calculus.) Let ϕ be a smooth solution of $\Box\phi = 0$ in dimension $3+1$ arising from data (f, g) of compact support at $t = 0$.

   (a) Prove that

   $$\Big| \int_0^T \int_{\mathbb{R}^3} \frac{-(\partial_t\phi)^2 + (\partial_r\phi)^2}{r} dx dt \Big| \leq C \left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \right) \tag{1}$$

   holds for a constant C independent of T.
   HINT: Integrate by parts the expression $\int_0^T \int_{\mathbb{R}^3} \Box\phi(\partial_r\phi) dx dt$.

   (b) Use the previous part to establish the estimate

   $$\left| \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla\!\!\!/\phi|^2}{r} dx dt - \int_0^T \int_{\mathbb{R}^3} \frac{1}{2r} \Box(\phi^2) dx dt \right| \leq C \left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \right) . \tag{2}$$

   (c) Show that

   $$\int_0^\infty \int_{\mathbb{R}^3} \frac{|\nabla\!\!\!/\phi|^2}{r} dx dt + \int_0^\infty |\phi|^2(0,t) dt \leq C \left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \right) . \tag{3}$$

   Interpret the result.
   HINT: You may want to (prove and) use the Hardy inequality $\int_{\mathbb{R}^3} \frac{\phi^2}{r^2} dx \leq C \int_{\mathbb{R}^3} (\partial_r\phi)^2 dx$.

3. Consider in dimension $1+3$ the coupled system

   $$\begin{cases} \Box\phi = 0 \\ \Box\psi = (\partial_t\phi)^2. \end{cases} \tag{4}$$

   Show that there exist smooth and compactly supported initial data for the above system such that the following bound does **not** hold for any constant $C > 0$:

   $$\sup_x |\partial_t\psi|(t, x) \leq \frac{C}{1+t} . \tag{5}$$

Analysis Review Problems

1. (This will be useful in the proof of the Strichartz estimate that we will do in 1-2 weeks from now.) Recall the statement of the following inequalities: Hölder, Minkowski, Young and Hardy-Littlewood-Sobolev.