General Relativity and Black Holes – Week 1

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1 Exercises

1. Prove that a smooth manifold (as defined in lectures) has a countable atlas.

2. Let V be an n-dimensional real vector space. Prove that there exists a unique manifold structure (up to diffeomorphism) on V such that every linear isomorphism into $\mathbb{R}^n$ is a coordinate chart.

3. Prove that the tangent space T_pM of an n-dimensional manifold M (as defined in lectures) is an n-dimensional vectorspace. Show also that tangent vectors satisfy the Leibniz product rule, i.e. if $X_p \in T_pM$ and $f, g \in C_p^\infty(M)$, then

$$X_p(fg) = f(p)X_p(g) + X_p(f)g(p)\,,$$

   where the product of fg is defined by $fg(p) = f(p)g(p)$.

4. Let V be a vectorfield field on an n-dimensional manifold M and p a point such that $V(p) \neq 0$. Then there is a coordinate system $x^1, x^2, \ldots, x^n$ at p such that $V = \frac{\partial}{\partial x^1}$ on this coordinate neighbourhood.

5. Let X, Y, Z be smooth vector fields on a manifold M. Additionally, let a and b be real numbers, and let f and g be smooth functions on M. Prove that the commutator (also called the Lie bracket) satisfies the following:

   (a) $[X, Y] = -[Y, X]$ (*anticommutativity*)

   (b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ (*linearity*)

   (c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*)

   (d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$

2 Problems and Discussion

We say that an open cover $\mathcal{U}$ of a manifold M is *locally finite* if every point $p \in M$ has a neighborhood which intersects (non-trivially) finitely many members of $\mathcal{U}$. Additionally, we say that an open cover $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every member of $\mathcal{V}$ is contained within a member of $\mathcal{U}$. Given an open cover $\mathcal{U}$ of a manifold M, we define a *partition of unity* subordinate to $\mathcal{U}$ to be a countable collection of smooth maps $f_i : M \to [0, 1]$ such that

- There exists a locally finite refinement $\mathcal{V}$ of $\mathcal{U}$ such that supp $f_i \subset V_i$ for some $V_i \in \mathcal{V}$.
- $\sum_i f_i(p) = 1$ for all $p \in M$.

We will generally take $\mathcal{U}$ to consist of the domains of coordinate charts. Note that the first condition implies that, for $p \in \mathcal{M}$, there exists a neighborhood of p in which f_i is identically zero for all but finitely many i.

1. (*Existence for partitions of unity*) Let $\mathcal{U}$ be an open covering of a smooth manifold M. Show that there exists a locally finite open cover $\{V_i\}$ such that the closure $\overline{V}_i$ of each V_i is compact and contained in an element of $\mathcal{U}$. Consequently, there exists a partition of unity subordinate to $\mathcal{U}$

   (a) Show that the set of basis elements V_β which have compact closure, and which are subsets of both an element of $\mathcal{U}$ and the domain of a coordinate chart, form a countable open cover.

   (b) Construct a countable sequence of compact sets $K_1, K_2, \ldots$ such that $K_i \subset \text{int } K_{i+1}$, $\cup K_i = M$, and each K_i is the union of finitely many $\overline{V}_\beta$.

   (c) Use the construction of $\{K_i\}$ to construct a locally finite open cover.

   (d) Use this cover to construct a partition of unity.

2. (*Existence, uniqueness, and continuous dependence on initial data for ODEs*) Recall the following:

   Let $x : \mathbb{R} \to \mathbb{R}^n$ satisfy the initial value problem

$$x'(t) = f(t, x), \qquad x(t_0) = x_0 \tag{1}$$

   If f is continuous in t and locally Lipschitz continuous in x, then there exist $\delta_1, \delta_2 > 0$ such that, for x_1 satisfying $|x_1 - x_0| < \delta_1$, the IVP

$$x'(t) = f(t, x), \qquad x(t_0) = x_1 \tag{2}$$

   has a unique solution on $(t_0 - \delta_2, t_0 + \delta_2)$. For $t_1 \in (t_0 - \delta_2, t_0 + \delta_2)$, then $x(t_1)$ is a continuous function of x_1.