Non-linear Wave Equations – Week 1

Gustav Holzegel

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1. In lectures we solved the initial value problem in $3+1$ dimensions. A similar derivation works in all *odd* spatial dimensions leading to the strong Huygens' principle. For even dimensions one can use **Hadamard's method of descent**, which we discuss here for $n = 2$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth. Our task is to obtain a representation formula for the solution to the Cauchy problem in $2+1$ dimensions:

$$\begin{cases} \Box_{2+1}\phi = 0 \\ \phi(0, x_1, x_2) = f(x_1, x_2) \\ \partial_t\phi(0, x_1, x_2) = g(x_1, x_2). \end{cases} \tag{1}$$

   (a) Show that by solving the initial value problem in $3+1$ dimensions

$$\begin{cases} \Box_{3+1}\phi = 0 \\ \phi(0, x_1, x_2, x_3) = f(x_1, x_2) \\ \partial_t\phi(0, x_1, x_2, x_3) = g(x_1, x_2) \end{cases} \tag{2}$$

   we can obtain a solution to (1).

   (b) Write down the representation formula for (2) and convert the integrals over the spheres $|x-y| = t$ centred at $(x_1, x_2, 0)$ to integrals over the disks $(x_1 - y_1)^2 + (x_2 - y_2)^2 \leq t^2$ to deduce the representation formula

$$\phi(t, x_1, x_2) = \frac{1}{2\pi}\int_{r\leq t}\frac{g(y_1, y_2)}{\sqrt{t^2 - r^2}}dy_1dy_2 + \frac{1}{2\pi}\partial_t\int_{r\leq t}\frac{f(y_1, y_2)}{\sqrt{t^2 - r^2}}dy_1dy_2\,, \tag{3}$$

   where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Interpret the formula geometrically.

   (c) Using (3) prove a pointwise decay estimate for solutions to (1) analogous to the one proven in lectures.

2. Focussing of discontinuities in dimension $n \geq 2$ (F. John, p.132). We discuss here the phenomenon that leads to the loss of C^k regularity in evolution for the wave equation seen in lectures.

   (a) Show that for $n = 3$ the general solution of the wave equation with spherical symmetry around the origin has the form

$$\phi(t, r) = \frac{F(r+t) + G(r-t)}{r} \quad , \quad r = |x| \tag{4}$$

   (b) Show that the solution with initial data of the form

$$\phi = 0 \quad , \quad \phi_t = g(r)$$

   with g an even function of r, is given by

$$\phi(t, r) = \frac{1}{2r}\int_{r-t}^{r+t}\rho g(\rho)d\rho\,. \tag{5}$$

For

$$g(r) = \begin{cases} 1 & \text{if } 0 \le r \le a \\ 0 & \text{if } r > a \end{cases} \tag{6}$$

obtain the solution from (5) in the different regions bounded by the cones $r = a \pm t$. Conclude that ϕ is discontinuous at $t = a$, $r = 0$.

(c) Can you use the above ideas to construct a solution $f \in C^2$, $g \in C^1$ such that the solution ϕ ceases to be C^2 at some time $t > 0$?

3. F. John (p.133). For $n = 3$ let $\phi(t, x) \in C^2$ and $\Box\phi = 0$ for $x \in \mathbb{R}^3$, $t \ge 0$. Assume moreover that

$$\Phi(t) = \sum_{|\alpha| \le 2} \int |D^\alpha \phi(t, x)| dx < \infty \quad \text{for } t = 0.$$

(a) Show that there exists a constant K independent of ϕ such that

$$|\phi(t, x)| \le \frac{K}{t} \Phi(0) \quad \text{for } t > 0.$$

Hint: Write the integrand in the representation formula as

$$\sum_i \left[t^{-1}(tg(y) + f(y))(y_i - x_i) + t f_{y_i}(y) \right] \xi_i \quad \text{where } \xi_i = \frac{y_i - x_i}{t}.$$

Note that ξ is the unit-normal vector to the spheres and convert the integral into one over the solid sphere $|y - x| < t$ using Stokes' theorem.

HINT: For small t you might want to also use that the assumption implies $f \in L^3(\mathbb{R}^n)$.

(b) Show that

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = 0$$

implies that ϕ vanishes identically. Interpretation?

Hint: Apply (a) to the function $\psi(t, x, T) = \phi(T - t, x)$ for large T.

4. Littman's Theorem for $n = 3$. Consider for $n = 3$ the initial value problem

$$\Box\phi = 0 \quad , \quad \phi(0, x) = 0 \quad , \quad \phi_t(0, x) = g(x)$$

Prove that for $n = 3$, $p \ne 2$ and $t \ne 0$

$$\sup_{g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}} \frac{\|\phi_t(t)\|_{L^p(\mathbb{R}^n)}}{\|\phi_t(0)\|_{L^p(\mathbb{R}^n)}} = \infty$$

Hint: Use Problem 2. What happens for $p = 2$? Interpret the result.

5. Show that $\Box\phi = (\partial_t \phi)^2$ has a blow up solution. (Hint: consider initial data independent of x.) Now, can you construct a blow up solution with compactly supported initial data? (Hint: use finite speed of propagation.) Finally, given $\epsilon > 0$, can you moreover construct a blow up solution with compactly supported initial data (f, g) such that the initial energy satisfies $\int_{t=0} d^n x \left(|\nabla f|^2 + g^2 \right) \le \epsilon$? (Hint: use scaling.)

6. Prove Theorem 1.1.2 in the lecture notes. Hint: Write down the equation for $\psi = e^\phi - 1$.

Analysis Review Problems

1. Prove the theorem on the Fourier transform in the lecture notes.

2. Define the function space $H^s(\mathbb{R}^n)$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^s(\mathbb{R}^n)} := \||(1+|\xi|^2)^{\frac{s}{2}}\hat{f}\|_{L^2(\mathbb{R}^n)}.$$

   Show that when s is a non-negative integer, there exists a constant C depending only on s such that

$$\frac{1}{C}\sum_{|\alpha|\leq s}\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C\sum_{|\alpha|\leq s}\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

3. (Sobolev embedding theorem, simplest case) Prove that there exists a constant $C = C(n,s) > 0$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$ we have

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$