C*-simplicity of discrete groups

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Abstract

A discrete group is called C*-simple if its reduced C*-algebra is simple. A dynamical characterization of C*-simplicity was recently obtained by Kalantar and Kennedy, namely that a discrete group is C*-simple if and only if it acts freely on its Furstenberg boundary. Shortly afterwards, a group-theoretic characterization was obtained as a consequence by Kennedy - C*-simplicity is equivalent to the group admitting no amenable recurrent subgroups. We investigate these characterizations, and use them to give proofs of various groups being C*-simple. In addition, we also briefly discuss the relationship between C*-simplicity and the unique trace property - the property that the reduced group C*-algebra admits a unique tracial state. It is shown that C*-simplicity implies the unique trace property, and we mention counterexamples to the converse.

Contents

1 Motivation and history

Let G be a finite group. A linear representation of G is an action of G on a complex vector space V by linear transformations. Recall that the group ring $\mathbb{C}[G]$ is an algebra with basis elements $\{\lambda_g \mid g \in G\}$, and product given by $\lambda_g \lambda_h = \lambda_{gh}$ (extend linearly). For any G, there is always a canonical representation on $V = \mathbb{C}[G]$, where $g \cdot \lambda_h = \lambda_{gh}$. This representation, and any representation of G in general, can be naturally viewed as being a $\mathbb{C}[G]$-module by extending the action of G linearly.

Now, $\mathbb{C}[G]$ (both the canonical representation and the algebra) carries much of the group-theoretic properties in the case of finite G. For example, $\mathbb{C}[G]$ as a representation breaks up into a direct sum of the irreducible representations of G, with multiplicity of each irreducible representation given by its dimension. As the algebra acting on this representation, $\mathbb{C}[G]$ breaks up as a direct sum of matrix algebras by the Artin-Wedderburn theorem, with sizes again given by the dimensions of each irreducible representation. (It is also worth noting that these spaces are finite dimensional and, as one learns in an introductory functional analysis course, finite-dimensional vector spaces are "complete" in every reasonable sense. Hence, there is nothing deep to say about the analytic structure of $\mathbb{C}[G]$ in this case).

Now assume G is an infinite discrete group. There is natural analytic representation theory associated to our group - instead of considering $\mathbb{C}[G]$, a vector space with algebraic basis given by the elements of G, one can consider $\ell^2(G)$, a Hilbert space with orthonormal basis given by the elements of G. Note that $\mathbb{C}[G]$ embeds densely into $\ell^2(G)$, and the action of G on $\mathbb{C}[G]$ extends to an action by unitaries on $\ell^2(G)$. The algebra $\mathbb{C}[G]$ embeds naturally into the algebra $\mathcal{B}(\ell^2(G))$ of bounded linear operators on $\ell^2(G)$, however it is no longer closed under the operator norm - closing it under this norm yields the reduced group C*-algebra $C_r^*(G)$. There is also the option to close it under the weak operator topology, yielding the group von Neumann algebra, which we will not consider here. Note that we are often, but not always, interested in countable groups. (Uncountable groups usually come equipped with some natural non-discrete topology, in which case the story becomes much different - see Section 3). On the other hand, it can be shown that any locally compact countable group is discrete.

Our main concern is that of knowing a fundamental property of the group C*-algebra, namely when it is simple (that is, it has no nontrivial, closed, two-sided ideals). The first result on this matter was given by Powers in 1975, when he showed that $\mathbb{F}_2$, the free group on two generators, is C*-simple. Since then, considerable efforts have been made in finding C*-simple groups. The following is a small sample of discrete groups which were proven to be C*-simple prior to the results discussed in this paper:

Class of groups	Author(s)	Date	Reference
$\mathbb{F}_2$	Powers	1975	[Pow75]
Large class of groups containing: - Fuchsian groups - F-groups - Free products of at least 2 cyclic groups (except $\mathbb{Z}_2 * \mathbb{Z}_2$)	Akemann	1981	[Ake81]
Powers groups	de la Harpe	1985	[Har85]
Weak Powers groups	Boca and Niţică	1988	[BN88]
Countable linear groups with no nontrivial amenable subgroups	Poznansky	2008	[Poz08]
Certain groups with non-vanishing ℓ^2-Betti numbers	Peterson and Thom	2011	[PT11]
Countable groups containing a non-degenerate hyperbolically embedded subgroup and no nontrivial finite normal subgroups	Dahmani, Guirardel, and Osin	2017	[DGO17]
Free Burnside groups $B(m,n)$ with $m \geq 2$ and n odd and sufficiently large	Olshanskii and Osin	2014	[OO14]

Since Powers' result in 1975 on $\mathbb{F}_2$ being C*-simple, the proof has been abstracted to various classes of groups satisfying some form of Power's property (see [Har85] for the definition of the original property), and this had been about the only method for determining the C*-simplicity of any given group.

Recently, a dynamical characterization of C*-simplicity was obtained - a group G is C*-simple if and only if it acts freely on its Furstenberg boundary $\partial_F G$, if and only if it acts topologically freely on some G-boundary [KK17]. We prove this characterization in Section 4. An intrinsic characterization of C*-simplicity was also obtained as a corollary, namely that a group is C*-simple if and only if it has no amenable recurrent subgroups (equivalently, no nontrivial amenable uniformly recurrent subgroups) [Ken15]. This is discussed in Section 5, along with various improvements. Both of these characterizations were used to give (easier) proofs of previously-known C*-simplicity results, as well as new results. We present some of these and other examples in Section 7, along with alternate proofs.

Up until recently, it was also an open question whether being C*-simple is equivalent to the reduced group C*-algebra admitting a unique tracial state - the canonical tracial state given by $\tau_\lambda(a) = \langle a\delta_e \mid \delta_e \rangle$. This conjecture is true if, instead, one considers the group von Neumann algebra. The dynamical characterization of C*-simplicity gives an easy proof that C*-simplicity implies the unique trace property, while Le Boudec showed that there were counterexamples to the converse. We discuss this topic in Section 6.

2 Preliminaries

The reader is assumed to be familiar with the basics of operator algebras, in particular C*-algebras. A good introduction to the topic is given in [Arv76] and [Dav96]. To make this paper more self-contained, and to establish terminology, we provide a brief recap of various concepts we will use. Note that, unless stated otherwise, all our groups are assumed to be discrete.

2.1 Group representations and group C*-algebras

Let G denote a discrete group. A unitary representation of G is a group homomorphism $\pi : G \to U(H)$, where H is a Hilbert space, and $U(H)$ denotes its group of unitary operators. We do not consider the zero representation to be a unitary representation. Such a representation is called irreducible if H admits no nontrivial closed invariant subspaces. Note that there is a natural extension to a *-representation $\pi : \mathbb{C}[G] \to \mathcal{B}(H)$ by linearity. It is also convenient sometimes to identify $\mathbb{C}[G]$ with $C_c(G)$, the continuous, compactly supported (hence, finitely supported) functions on G. Equipping $\mathbb{C}[G]$ with the 1-norm (where the norm of $f = \sum_{g\in G} \alpha_g g \in \mathbb{C}[G]$ is given by $\|f\|_1 = \sum_{g\in G} |\alpha_g|$), there is a further extension to a contractive *-representation of $\ell^1(G)$ by continuity. A representation H is called cyclic if the span of $\pi(G)\xi = \{\pi(g)\xi \mid g \in G\}$ is dense in H for some $\xi \in H$, with ξ being called a cyclic vector. In particular, it can easily be shown that a representation is irreducible if and only if every nonzero vector in H is cyclic.

By $\lambda : G \to \mathcal{B}(\ell^2(G))$, we denote the left-regular representation. Here, $\ell^2(G)$ is the Hilbert space of square-summable functions on G (that is, functions $f : G \to \mathbb{C}$ satisfying $\sum_{g\in G} |f(g)|^2 < \infty$), together with inner product given by

$$\langle f_1 \mid f_2 \rangle = \sum_{g\in G} f_1(g)\overline{f_2(g)} \quad \text{for } f_1, f_2 \in \ell^2(G).$$

The action of G is left-translation: for any $f \in \ell^2(G)$, $g \in G$, and $x \in G$, we have $(g \cdot f)(x) = f(g^{-1}x)$. This is, in a sense, the canonical unitary representation of G.

Note that, for consistency, we will always write $g \cdot f$ instead of $\lambda(g) f$, as left-translation of functions also shows up in contexts other than the left-regular representation λ on $\ell^2(G)$.

Definition 2.1. The reduced group C*-algebra of G is defined as $C^*_r(G) \coloneqq C^*(\lambda(G))$, i.e. it is C*-algebra generated by the image of G under the left-regular representation.

Of interest to us is to consider when the above C*-algebra is simple, that is, it has no nontrivial ideals (here, ideals are always assumed to be closed and two-sided). Groups having this property are said to be C*-simple. In general, given any unitary representation π, one may define the C*-algebra $C^*_\pi(G)$ analogously.

Finally, we also construct the (universal) C*-algebra of G, denoted $C^*(G)$. Note that there is a restriction on the cardinality of any cyclic representation - if ξ is a cyclic vector in H, then the set $\{\sum_{\text{finite}} \alpha_g g \cdot \xi \mid g \in G, \alpha_g \in \mathbb{Q}[i]\}$ with cardinality at most $\aleph_0 |G|$ is dense in the span of $\pi(G)\xi$, and hence in H. As any point in H is a sequence of elements in this set, then the cardinality of H is bounded by $(\aleph_0 |G|)^{\aleph_0}$. Consequently, it makes sense to define π_U to be the direct sum of all up-to-unitary-equivalence cyclic representations of G. Again, we let $C^*(G) = C^*(\pi_U(G))$. Equivalently, for $f \in \mathbb{C}[G]$, one can define

$$\|f\|_U \coloneqq \sup\{\|\pi(f)\| \mid \pi \text{ cyclic representation of } G\}$$

and let $C^*(G)$ be the completion of $\mathbb{C}[G]$ with respect to this norm. It is not hard to show that $C^*(G)$ is indeed universal. Given any unitary representation $\pi : G \to U(H)$, it can be broken up into a direct sum of cyclic representations. To see this, note that:

- Every unitary representation H admits a cyclic subrepresentation. Just pick any nonzero $\xi \in H$, and consider $\overline{\operatorname{span}\pi(G)\xi}$, the cyclic subrepresentation generated by ξ
- If $K \subseteq H$ is a subrepresentation, then so is $K^\perp$. Indeed, this follows from the fact that, if $\eta \in K^\perp$, then for any $\xi \in K$, we have $\langle \xi \mid g\eta \rangle = \langle g^{-1}\xi \mid \eta \rangle = 0$.

A standard Zorn's lemma argument then applies on the set of families of cyclic subrepresentations of H that are pairwise orthogonal. Now, the identity map id : $\mathbb{C}[G] \subseteq C^*(G) \to \mathbb{C}[G] \subseteq C^*_\pi(G)$ is contractive, as $\|f\|_\pi \le \|f\|_U$ for any $f \in \mathbb{C}[G]$ almost by construction, and hence extends to a *-homomorphism $\varphi : C^*(G) \to C^*_\pi(G)$. It is a basic result of C*-algebras that the image under any *-homomorphism is always closed, following from the fact that the quotient by an ideal is also a C*-algebra, and an injective *-homomorphism is an isometry. Thus, φ is surjective, and $C^*(G)/\ker\varphi \cong C^*_\pi(G)$. We denote $\ker\varphi$ by $C^*\ker\pi$. Note that the above argument also shows that the unitary representations of G are in bijection with the (nondegenerate) *-representations of $C^*(G)$.

We also wish to explore the tracial states of $C^*_r(G)$. For this, we recall some definitions:

Definition 2.2. Let A be a unital C*-algebra.

1. A *state* on A is a positive, unital, bounded linear functional on A. The set of all states on A, known as the *state space* of A, is denoted by $S(A)$. Unless stated otherwise, this space will be equipped with the weak* topology.
2. A *tracial state* is a state τ that also satisfies $\tau(ab) = \tau(ba)$.

We say that G has the *unique trace property* if $C^*_r(G)$ has a unique tracial state. Note that there always exists a canonical tracial state given by $\tau_\lambda(a) = \langle a\delta_e \mid \delta_e \rangle$.

2.2 Weak containment of group representations

Recall that a representation $\pi : G \to U(H)$ is said to be contained in a representation $\rho : G \to U(K)$ if π is unitarily equivalent to a subrepresentation of ρ. For example, for finite groups, any irreducible representation is contained in the left-regular representation.

Another form of containment of representations is the following. We say that π is *weakly contained* in ρ, denoted $\pi \prec \rho$, if for any $\xi \in H$, and any finite $F \subseteq G$, there are $\eta_1, \ldots, \eta_n \in K$ such that the diagonal matrix coefficient $\langle \pi(\cdot)\xi \,|\, \xi \rangle$ is approximated uniformly on F by $\sum_{i=1}^{n} \langle \rho(\cdot)\eta_i \,|\, \eta_i \rangle$. That is, for any $\varepsilon > 0$, there are $\eta_1, \ldots, \eta_n \in K$ such that

$$\left| \langle \pi(g)\xi \,|\, \xi \rangle - \sum_{i=1}^{n} \langle \rho(g)\eta_i \,|\, \eta_i \rangle \right| < \varepsilon \quad \forall g \in F.$$

Also, π and ρ are called *weakly equivalent* if both $\pi \prec \rho$ and $\rho \prec \pi$, denoted by $\pi \sim \rho$. An elaborate discussion on weak containment can be found in [BHV08, Appendix F].

Remark 2.3. Call a diagonal matrix coefficient $\varphi = \langle \pi(\cdot)\xi \,|\, \xi \rangle$ *normalized* if we have $\varphi(e) = \|\xi\|^2 = 1$.

1. By scaling, it clearly suffices to check that normalized diagonal matrix coefficients can be approximated as above.

2. If $\varphi = \langle \pi(\cdot)\xi \,|\, \xi \rangle$ is a normalized diagonal matrix coefficient, then it can be approximated as above if and only if it can be approximated by a convex combination of normalized diagonal matrix coefficients.

   Indeed, assume $F \subseteq G$ is finite, $\varepsilon > 0$, and φ is $\frac{\varepsilon}{2}$-approximated by $\sum_{i=1}^{n} \psi_i$ as above, where $\psi_i = \langle \rho(\cdot)\eta_i \,|\, \eta_i \rangle$ (we may assume η_i is never zero by removing the corresponding ψ_i from this summation). Without loss of generality, we may also assume $e \in F$, in which case

   $$\left| 1 - \sum_{i=1}^{n} \psi_i(e) \right| = \left| \varphi(e) - \sum_{i=1}^{n} \psi_i(e) \right| < \frac{\varepsilon}{2}.$$

Consequently, for any $g \in F$, we have the following inequality:

$$\begin{aligned}
&\left|\varphi(g) - \sum_{i=1}^{n}\left(\frac{\psi_i(e)}{\psi_1(e)+\cdots+\psi_n(e)}\right)\frac{\psi_i(g)}{\psi_i(e)}\right| \\
&\leq \left|\varphi(g) - \sum_{i=1}^{n}\psi_i(g)\right| + \left|\sum_{i=1}^{n}\left(1 - \frac{1}{\psi_1(e)+\cdots+\psi_n(e)}\right)\psi_i(g)\right| \\
&< \frac{\varepsilon}{2} + \left|1 - \frac{1}{\psi_1(e)+\cdots+\psi_n(e)}\right|\sum_{i=1}^{n}|\psi_i(g)| \\
&\leq \frac{\varepsilon}{2} + \frac{|\psi_1(e)+\cdots+\psi_n(e)-1|}{\psi_1(e)+\cdots+\psi_n(e)}\left(\psi_1(e)+\cdots+\psi_n(e)\right) \\
&< \varepsilon.
\end{aligned}$$

As the sum

$$\sum_{i=1}^{n}\left(\frac{\psi_i(e)}{\psi_1(e)+\cdots+\psi_n(e)}\right)\frac{\psi_i(g)}{\psi_i(e)}$$

is a convex combination of normalized diagonal matrix coefficients, we are done.

Conversely, if φ can be approximated by some convex combination $\sum_{i=1}^{n}\alpha_i\psi_i$ of normalized diagonal matrix coefficients as above, where $\psi_i = \langle\, \rho(\cdot)\,\eta_i \,|\, \eta_i \,\rangle$, then we can just write $\alpha_i\psi_i = \langle\, \rho(\cdot)\,\sqrt{\alpha_i}\eta_i \,|\, \sqrt{\alpha_i}\eta_i \,\rangle$.

3. Similar to the previous point, the map φ can be approximated by convex combinations of normalized diagonal matrix coefficients if and only if it can be approximated by averages of such functions, i.e. $\frac{1}{n}\sum_{i=1}^{n}\psi_i$, where $\psi_i = \langle\, \rho(\cdot)\,\eta_i \,|\, \eta_i \,\rangle$ is normalized.

   First, assume $F \subseteq G$ is finite, $\varepsilon > 0$, and φ can be $\frac{\varepsilon}{3}$-approximated as above by some convex combination of normalized diagonal matrix coefficients $\sum_{i=1}^{n}\alpha_i\psi_i$. We may choose rational numbers $\frac{a_i}{b} \leq \alpha_i$ (common denominator b) so that $\left|1 - \frac{a_i}{b}\right| < \frac{\varepsilon}{3n}$. Now consider the sum

$$\sum_{i=1}^{n}\frac{a_i}{b}\psi_i + \left(1 - \sum_{i=1}^{n}\frac{a_i}{b}\right)\psi_1.$$

   Expanding each fraction of the form $\frac{m}{b}$ as $\frac{1}{b}+\cdots+\frac{1}{b}$, this sum indeed becomes an average of normalized diagonal matrix coefficients. Further, for any $g \in G$, we have

$$\begin{aligned}
&\left|\varphi(g) - \left(\sum_{i=1}^{n} \frac{a_i}{b}\psi_i(g) + \left(1 - \sum_{i=1}^{n}\frac{a_i}{b}\right)\psi_1(g)\right)\right| \\
&\leq \left|\varphi(g) - \sum_{i=1}^{n}\alpha_i\psi_i(g)\right| + \sum_{i=1}^{n}\left|\alpha_i - \frac{a_i}{b}\right| |\psi_i(g)| + \left|1 - \sum_{i=1}^{n}\frac{a_i}{b}\right| |\psi_1(g)| \\
&< \frac{\varepsilon}{3} + \sum_{i=1}^{n}\frac{\varepsilon}{3n}\cdot 1 + \frac{\varepsilon}{3}\cdot 1 \\
&= \varepsilon.
\end{aligned}$$

This shows one direction of our claim. The reverse direction is obvious.

Weak containment shows up in various characterizations of amenability and C*-simplicity. Of particular importance to us will be the following equivalence, given in [BHV08, Theorem F.4.4] for arbitrary topological groups.

Theorem 2.4. *Let $\pi : G \to U(H)$ and $\rho : G \to U(K)$ be unitary representations of G. The following are equivalent:*

1. $\pi \prec \rho$.
2. $C^* \ker \rho \subseteq C^* \ker \pi$.
3. *There is a *-homomorphism $\phi : C^*_\rho(G) \to C^*_\pi(G)$ mapping $\rho(g)$ to $\pi(g)$ for every $g \in G$.*
4. $\|\pi(f)\| \leq \|\rho(f)\|$ *for all* $f \in \mathbb{C}[G]$.

Proof. (2) $\implies$ (3): There is a canonical surjective *-homomorphism from $C^*_\rho(G)$ to $C^*_\pi(G)$, given by

$$C^*_\rho(G) \cong C^*(G)/C^* \ker \rho \to C^*(G)/C^* \ker \pi \cong C^*_\pi(G).$$

Following through with the *-homomorphism, we map $\rho(g)$ to $\pi(g)$ for every $g \in G$.

(3) $\implies$ (4): Extending linearly, our *-homomorphism $\phi : C^*_\rho(G) \to C^*_\pi(G)$ maps $\rho(f)$ to $\pi(f)$ for all $f \in \mathbb{C}[G]$. Then use the fact that *-homomorphisms between C*-algebras are automatically contractions.

(4) $\implies$ (2): Again, consider the canonical map given by

$$C^*(G)/C^* \ker \rho \cong C^*_\rho(G) \xrightarrow{\phi} C^*_\pi(G) \cong C^*(G)/C^* \ker \pi.$$

Extending linearly from ϕ, this map sends $f + C^* \ker \rho$ (where $f \in \mathbb{C}[G]$) to $f + C^* \ker \pi$. In order for this map to be linear, zero must map to zero. In other words, for any $f \in \mathbb{C}[G]$, if $f \in C^* \ker \rho$, i.e. $f + C^* \ker \rho = 0$, then $f + C^* \ker \pi = 0$, i.e. $f \in C^* \ker \pi$. Hence, $C^* \ker \rho \subseteq C^* \ker \pi$.

(1) $\implies$ (4): First, using the fact that $\|\pi(f)\|^2 = \|\pi(f^*f)\|$ and $\|\rho(f)\|^2 = \|\rho(f^*f)\|$, it suffices to show that $\|\pi(f)\| \le \|\rho(f)\|$ holds for the positive elements of $\mathbb{C}[G]$. Let $f = \sum_{g\in G} \alpha_g g$ be such an element. Our aim is to show that any value $\langle \pi(f)\xi \mid \xi \rangle$, where $\xi \in H$ is of norm 1, can be approximated arbitrarily well by convex combinations of the form $\sum_{i=1}^n \beta_i \langle \rho(f)\eta_i \mid \eta_i \rangle$, where $\eta_i \in K$ are all of norm 1. To this end, let $F = \{g \in G \mid \alpha_g \neq 0\}$. We know that, as $\pi \prec \rho$, the normalized diagonal matrix coefficient $\langle \pi(\cdot)\xi \mid \xi \rangle$ can be uniformly approximated on F by convex combinations of the form $\sum_{i=1}^n \beta_i \langle \rho(\cdot)\eta_i \mid \eta_i \rangle$. Fix an $\varepsilon > 0$, and pick such a sum as above so that

$$\left| \langle \pi(g)\xi \mid \xi \rangle - \sum_{i=1}^n \beta_i \langle \rho(g)\eta_i \mid \eta_i \rangle \right| < \varepsilon \quad \forall g \in F.$$

This gives us that:

$$\begin{aligned}
&\left| \langle \pi(f)\xi \mid \xi \rangle - \sum_{i=1}^n \beta_i \langle \rho(f)\eta_i \mid \eta_i \rangle \right| \\
&= \left| \sum_{g\in G} \alpha_g \langle \pi(g)\xi \mid \xi \rangle - \sum_{i=1}^n \beta_i \sum_{g\in G} \alpha_g \langle \rho(g)\eta_i \mid \eta_i \rangle \right| \\
&= \left| \sum_{g\in G} \alpha_g \left(\langle \pi(g)\xi \mid \xi \rangle - \sum_{i=1}^n \beta_i \langle \rho(g)\eta_i \mid \eta_i \rangle \right) \right| \\
&\le \sum_{g\in G} |\alpha_g| \cdot \varepsilon,
\end{aligned}$$

which can be made arbitrarily small. Now assume $\|\rho(f)\| < \|\pi(f)\|$, so that $\|\rho(f)\| \le \|\pi(f)\| - \varepsilon$ for some $\varepsilon > 0$. For any convex combination of the form $\sum_{i=1}^n \beta_i \langle \rho(f)\eta_i \mid \eta_i \rangle$ as above, it is the case that

$$\sum_{i=1}^n \beta_i \langle \rho(f)\eta_i \mid \eta_i \rangle \le \sum_{i=1}^n \beta_i \|\rho(f)\| \le \|\rho(f)\| \le \|\pi(f)\| - \varepsilon.$$

However, as $\pi(f)$ is positive, then $\langle \pi(f)\xi \mid \xi \rangle$ ($\|\xi\| = 1$) can be made arbitrarily close to $\|\pi(f)\|$, which is a contradiction.

(3) $\implies$ (1): Let $\langle \pi(\cdot)\xi \mid \xi \rangle$ be a normalized ($\|\xi\| = 1$) diagonal matrix coefficient with respect to π. This extends to a state on $C^*_\pi(G)$, and by our assumption, it canonically also becomes a state on $C^*_\rho(G)$ - call it ψ, for convenience. We wish to show that the weak*-closed convex hull of states on $C^*_\rho(G)$ of the form $\langle \rho(\cdot)\eta \mid \eta \rangle$ ($\|\eta\| = 1$) is the whole state space $S\left(C^*_\rho(G)\right)$.

Assume otherwise, so that there is some state ψ that does not lie in the weak*-closed convex hull. Recall that, given a locally convex topological vector space X, we have that the dual of (X^*, w^*) is X, viewing each element $x \in X$ as the evaluator-functional $\widehat{x}$ given by $\widehat{x}(f) = f(x)$. Hence, by the Hahn-Banach separation theorem, there is some $a \in C_r^*(G)$ and $c \in \mathbb{R}$ such that $\operatorname{Re}\psi'(a) \geq c$ for all ψ' in the closed convex hull, but $\operatorname{Re}\psi(a) < c$. In particular, $\operatorname{Re}\langle a\eta \,|\, \eta \rangle \geq c$ for all $\eta \in K$ with $\|\eta\| = 1$. Using the fact that for any state ψ', we have $\operatorname{Re}\psi'(a) = \psi'(\operatorname{Re} a)$, then replacing a with $\operatorname{Re} a$, we may assume:

- a is self-adjoint.
- $\langle a\eta \,|\, \eta \rangle \geq c$ for all ψ' in the closed convex hull, while $\psi(a) < c$.

Now, replacing a with $a + c1$, we may assume that $c = 0$. But then $\langle a\eta \,|\, \eta \rangle \geq 0$ for all $\eta \in K$, and so $a \geq 0$. As ψ is a state, this forces $\psi(a) \geq 0$, a contradiction. Hence, the state $\psi = \langle \pi(\cdot)\eta \,|\, \eta \rangle$ on $C_\rho^*(G)$ must indeed be a weak*-limit of convex combinations of states of the form $\langle \rho(\cdot)\eta \,|\, \eta \rangle$ $(\|\eta\| = 1)$. Using the fact that $\rho(G) \subseteq C_\rho^*(G)$, the claim that $\pi \prec \rho$ now immediately follows. ■

This next proposition substantially cuts down on the diagonal matrix coefficients we need to show can be approximated, and will come in useful later. The proof can be found in [BHV08, Lemma F.1.3].

Proposition 2.5. *Assume $\pi : G \to U(H)$ and $\rho : G \to U(K)$ are unitary representations of G, and $V \subseteq H$ is such that:*

1. *The set $\pi(G)V = \{\pi(g)\xi \mid g \in G, \xi \in V\}$ has dense linear span in H.*
2. *Every diagonal matrix coefficient $\langle \pi(\cdot)\xi \,|\, \xi \rangle$, where $\xi \in V$, can be approximated uniformly on finite subsets by sums of diagonal matrix coefficients of ρ.*

Then π is weakly contained in ρ.

2.3 Induced representations

Assume H is a subgroup of G, and let (K, σ) be a representation of H. The induced representation $\operatorname{Ind}_H^G \sigma$ is, in some sense, the natural extension of σ to a representation of G that contains K as an H-subrepresentation. It can be constructed as follows.

Let T be a transversal of the left-coset space G/H - that is, a choice of one representative from each coset in G/H (hence, each $g \in G$ decomposes uniquely as $g = rh$, where $r \in T$ and $h \in H$). Let $K' = \oplus_{r \in T} K_r$ be a Hilbert space direct sum of copies of K indexed by T. Given any $r \in T$, denote any element of the r-th copy of K_r by (r, ξ), where $\xi \in K$. Now given any $g \in G$, we define the action of g on any $(r, \xi) \in K_r$ as follows: assume $gr = r'h$ for some $r' \in T$ and $h \in H$. We let $g \cdot (r, \xi) = (r', h \cdot \xi)$. It is not hard to check this action of g extends linearly and continuously to a unitary operator on K', giving us an extension $\sigma' : G \to U(K')$.

Definition 2.6. With the above construction, we define the induced representation $\operatorname{Ind}_H^G \sigma$ of (K, σ) to be (K', σ').

We see that K embeds into K' as K_e as an H-subrepresentation (restricting our new representation back down to H), and that $K' \cong \oplus_{r \in T} rK$ as a Hilbert space. It can also be shown that $\operatorname{Ind}_H^G \sigma$ is independent of the transversal we chose. See, for example, [BHV08, Appendix E] (done for general topological groups). The following fact about induced representations will be important for us:

Proposition 2.7 (Continuity of induction)**.** *Let H be a subgroup of G, and $\sigma : H \to U(K_1)$, $\tau : H \to U(K_2)$ representations of H with $\sigma \prec \tau$. Then $\operatorname{Ind}_H^G \sigma \prec \operatorname{Ind}_H^G \tau$.*

Proof. Denote $\operatorname{Ind}_H^G \sigma$ by π and $\operatorname{Ind}_H^G \tau$ by ρ. For convenience, we may choose the same transversal T of G/H for both of our representations, and assume $e \in T$. Then the set $V = \{(e, \xi) \mid \xi \in K_1\}$ is such that $\pi(G) V$ has dense linear span in K_1. Assume $F \subseteq G$ is finite, and $\varepsilon > 0$. Note that, for any $\xi \in K_1$ and $g \in G$, it is the case that

$$\langle \pi(g)(e, \xi) \,|\, (e, \xi) \rangle = \begin{cases} \langle \sigma(g) \xi \,|\, \xi \rangle & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}.$$

Thus, if $\langle \sigma(\cdot) \xi \,|\, \xi \rangle$ is ε-uniformly-approximated by $\sum_{i=1}^n \langle \tau(\cdot) \eta_i \,|\, \eta_i \rangle$ ($\eta_i \in K_2$) on $F \cap H$, we have that $\langle \pi(\cdot)(e, \xi) \,|\, (e, \xi) \rangle$ is ε-uniformly-approximated by the sum $\sum_{i=1}^n \langle \rho(\cdot)(e, \eta_i) \,|\, (e, \eta_i) \rangle$ on F. By Proposition 2.5, $\pi \prec \rho$. ■

2.4 Dynamics

Here, we present the basics of dynamical systems.

Definition 2.8. Assume X is a compact Hausdorff space. A G-action on X is an action $G \curvearrowright X$, where G acts by homeomorphisms. The space X is known as a G-space, or a dynamical system over G.

Here, all G-spaces are assumed to be compact and Hausdorff. Since G-actions are just actions with continuity built in, it is not hard to find a suitable candidate for what morphisms between two G-spaces should be.

Definition 2.9. Assume X and Y are two compact G-spaces, and $f : X \to Y$ is some map. We call f a G-map if it is continuous and G-equivariant (that is, $f(g \cdot x) = g \cdot f(x)$). For convenience, we will also define *G-image* to mean the image under a G-map, and *G-isomorphism* to mean a G-equivariant homeomorphism.

Of course, we also have an analogue of irreducible representations for dynamical systems.

Definition 2.10. Assume X is a G-space, and $Y \subseteq X$ is non-empty, closed, and G-invariant subset. We say that Y is a subsystem. If the only subsystem of X is itself, then we say X is minimal.

Proposition 2.11. *A G-space X is minimal if and only if the G-orbit of any point $x \in X$, i.e. $G \cdot x = \{g \cdot x \mid g \in G\}$, is dense in X.*

Proof. It is easy to check that $\overline{G \cdot x}$ is always a subsystem of X. Hence, if X is minimal, then we always have $\overline{G \cdot x} = X$. Conversely, assume $\overline{G \cdot x}$ is always dense. Let $Y \subseteq X$ be any subsystem, and pick any $y \in Y$. Then $X = \overline{G \cdot y} \subseteq Y \subseteq X$. ■

By the Riesz-Markov-Kakutani representation theorem, the dual space $C(X)^*$ can be identified with the algebra of complex Radon measures $M(X)$. For this reason, given any $\mu \in M(X)$ and $f \in C(X)$, we will often write $\mu(f)$ to mean $\int_X f \, d\mu$. It is relatively straightforward to check that $\mathcal{P}(X)$, the set of all probability Radon measures on X, is weak*-closed, and it is clearly also contained in the closed unit ball of $M(X)$. Consequently, by the Banach-Alaoglu theorem, it is weak*-compact. Note that, just as we have left-translation of functions on G, there is a natural action of G on $\mathcal{P}(X)$, given by $(g \cdot \mu)(E) = \mu(g^{-1} \cdot E)$ for any $g \in G$ and Borel $E \subseteq X$. With this, we have the following definitions:

Definition 2.12. A G-space X is called proximal if, given any $x, y \in X$, there is a net $(g_\lambda) \subseteq G$ such that $\lim_\lambda g_\lambda x = \lim_\lambda g_\lambda y$. A G-space X is called strongly proximal if $\mathcal{P}(X)$ is proximal.

An immediate observation is the following:

Proposition 2.13. *The map from X to $\mathcal{P}(X)$, mapping each singleton x to the Dirac mass δ_x, is a G-isomorphism from X onto its image.*

Proof. Almost by definition of our action on $\mathcal{P}(X)$, it is easy to check that $g\delta_x = \delta_{gx}$. Continuity of this map follows from the definition of w*-convergence: if $x_\lambda \to x$, then $\int_X f d\delta_{x_\lambda} = f(x_\lambda) \to f(x) = \int_X f \, d\delta_x$ for all $f \in C(X)$. Finally, continuous bijections from a compact space to a Hausdorff space are automatically homeomorphisms. ■

From now on, we will identify X with the subset of Dirac masses in $P(X)$. From this embedding, we see that, as the name suggests, being strongly proximal is stronger than being proximal.

Definition 2.14. The action of G (on any set) is called *free* if the set of fixed points of any nonidentity element $g \in G$, i.e. $X_g = \{x \in X \mid gx = x\}$, is empty. If X is a G-space, then the action of G on X is called *topologically free* if the set of fixed points X_g of any given nonidentity element $g \in G$ has empty interior.

Finally, the following is an easy lemma, but it will be used several times throughout this paper:

Lemma 2.15. *Assume X is a minimal G-space, and $U \subseteq X$ is a nonempty open subset of X. Then X is covered by finitely many translates of U, i.e. $X = g_1 U \cup \cdots \cup g_n U$ for some $g_1, \ldots, g_n \in G$.*

Proof. Fix any $x \in U$, and let $y \in X$ be arbitrary. As $x \in \overline{Gy}$, then $Gy \cap U \neq \emptyset$. Consequently, $g_y y \in U$ for some $g_y \in G$, or in other words, $y \in g_y^{-1}U$. This shows $\left\{g_y^{-1}U\right\}_{y\in X}$ is an open cover of X. Compactness allows us to reduce this to a finite subcover. ■

2.5 Crossed products

Similar to Section 2.4, we now consider the action of a group G acting on a C*-algebra. For our purposes, we will always work with unital C*-algebras.

Definition 2.16. A G-C*-algebra A is a unital C*-algebra A, equipped with an action $G \curvearrowright A$, where G acts by automorphisms (cf. Definition 2.8).

Similar to how we defined the group C*-algebras of G, our aim now is to embed both A and G into a larger C*-algebra in which the action of any $g \in G$ on A is inner, i.e. so that for all $g \in G$, there is a unitary u_g such that we have $g \cdot a = u_g a u_g^*$ for all $a \in A$. For this, we take inspiration from how we construct the semidirect product of two groups.

Consider the space $A \otimes \mathbb{C}[G]$, which consists of finite sums of elements of the form $a \otimes g$, where $a \in A$ and $g \in G$. Now define a product on our space given by $(a \otimes g)(b \otimes h) = a(g \cdot b) \otimes gh$ (extend linearly), and an involution given by $(a \otimes g)^* = \left(g^{-1} \cdot a^*\right) \otimes g^{-1}$ (extend linearly). Embedding $a \in A$ as $a \otimes e$, and $g \in G$ as $u_g = 1 \otimes g$, we have that $A \otimes \mathbb{C}[G]$ consists of sums of the form $\sum_{g\in G} a_g u_g$ (only finitely many nonzero a_g), $u_g a u_g^* = g \cdot a$, and $u_g^* = u_g^{-1} = u_{g^{-1}}$.

This determines the algebraic structure for the space we want. There is only the matter of completing it under a C*-norm (the analytic structure). Similar to group C*-algebras, we will use representations to accomplish this, except here we consider simultaneous representations of A and G satisfying a quite restrictive property. Namely, a *covariant representation* (u, π, H) is a triple consisting of a unitary representation (u, H) of G and *-representation (π, H) of A such that

$$u(g)\,\pi(a)\,u(g)^* = \pi(g \cdot a)\,.$$

This gives rise to a *-representation $\rho : A \otimes \mathbb{C}[G] \to \mathcal{B}(H)$ by $\rho(a \otimes g) = \pi(a)\,u(g)$ (extend linearly), and we define the crossed product corresponding to (u, π, H) as $C^*(\rho(A \otimes \mathbb{C}[G])) = \overline{\rho(A \otimes \mathbb{C}[G])}^{\|\cdot\|}$. In particular, there is again always a "canonical" crossed product, given as follows.

Let A be a G-C*-algebra. We can assume A is represented faithfully as $A \subseteq \mathcal{B}(H)$ for some Hilbert space H. Consider the Hilbert space $H \otimes \ell^2(G)$. We have a natural analogue of the left-regular representation $\lambda : G \to \mathcal{B}\left(H \otimes \ell^2(G)\right)$ determined by $g \cdot (\xi \otimes f) = \xi \otimes (g \cdot f)$, and a *-representation $\pi : A \to \mathcal{B}\left(H \otimes \ell^2(G)\right)$ determined by $a \cdot (\xi \otimes \delta_g) = \left(\left(g^{-1} \cdot a\right)\xi\right) \otimes \delta_g$.

Definition 2.17. The *reduced crossed product* $A \rtimes_r G$ is the crossed product corresponding to the covariant representation $\left(\lambda, \pi, H \otimes \ell^2(G)\right)$, as defined above.

It is worth noting that, instead of using $H \otimes \ell^2(G)$, one could use the isomorphic space $\ell^2(G, H)$ of H-valued, square-summable functions on G. Here, the inner product is given by

$$\langle \varphi \mid \psi \rangle = \sum_{x \in G} \langle \varphi(x) \mid \psi(x) \rangle \qquad \text{for } \varphi, \psi \in \ell^2(G, H).$$

Then G would act on $f \in \ell^2(G, H)$ by left-translation, and $a \in A$ would act by $(a \cdot f)(g) = (g^{-1} \cdot a) f(g)$ for all $g \in G$.

Proposition 2.18. *The construction given in Definition 2.17 is well-defined, i.e. it does not depend on the choice of the faithful *-representation $A \hookrightarrow \mathcal{B}(H)$.*

A proof of the above proposition is given in [BO08, Proposition 4.1.5].

Proposition 2.19. *Both A and $C_r^*(G)$ embed isometrically into $A \rtimes_r G$ in the canonical way (with $a \in A$ mapping to $a \otimes e$, and $b \in \mathbb{C}[G] \subseteq C_r^*(G)$ mapping to $1 \otimes b$).*

Proof. By the density of $\mathbb{C}[G]$ in $C_r^*(G)$, to show $C_r^*(G)$ embeds into $A \rtimes_r G$, it suffices to show that the norm of any element in $\mathbb{C}[G]$ is the same in $C_r^*(G)$ as it is in $A \rtimes_r G$. To this end, note that $b \in \mathbb{C}[G]$ acts on $H \otimes \ell^2(G)$ as $I \otimes \lambda(b)$, where $\lambda : \mathbb{C}[G] \to \mathcal{B}(\ell^2(G))$ is the left-regular representation, from which our claim immediately follows.

Now assume $a \in A$, and let $f = \sum_{g \in G} \xi_g \otimes \delta_g \in H \otimes \ell^2(G)$. Then

$$(a \otimes e) \cdot f = \sum_{g \in G} \left(\left(g^{-1} \cdot a \right) \xi_g \right) \otimes \delta_g,$$

which shows

$$\begin{aligned} \|(a \otimes e) \cdot f\|^2 &= \sum_{g \in G} \left\| \left(g^{-1} \cdot a \right) \xi_g \right\|^2 \leq \sum_{g \in G} \left\| g^{-1} \cdot a \right\|^2 \|\xi_g\|^2 = \|a\|^2 \sum_{g \in G} \|\xi_g\|^2 \\ &= \|a\|^2 \|f\|^2 . \end{aligned}$$

Hence, $\|a \otimes e\|^2 \leq \|a\|^2$. The fact that equality is attained follows from

$$\|(a \otimes e) \cdot (\xi \otimes \delta_e)\| = \|a\xi \otimes \delta_e\| = \|a\xi\|,$$

and taking sup over all $\|\xi\| = 1$. ■

Note that there is also a natural action of G on $A \rtimes_r G$ by automorphisms, with $g \in G$ acting by conjugation by u_g.

Now we quickly recall the notion of a conditional expectation. This definition, and the following theorem as well, can be found in [BO08, pp. 12–13].

Definition 2.20. Given C*-algebras $B \subseteq A$, we say that a linear map $E : A \to B$ is a *conditional expectation* if:

1. $E(b) = b$ for all $b \in B$, that is, it is a *projection* from A onto B.
2. E is contractive.
3. E is completely positive.
4. $E(bab') = bE(a)b'$ for all $a \in A$ and $b, b' \in B$.

The following theorem, which we will not prove here, is quite useful in checking if a map is indeed a conditional expectation:

Theorem 2.21 (Tomiyama)**.** *Let $E : A \to B$ be a projection from a C*-algebra A to a C*-algebra $B \subseteq A$. The following are equivalent:*

1. *E is a conditional expectation.*
2. *E is contractive and completely positive.*
3. *E is contractive.*

Any crossed product $A \rtimes_r G$ always has a canonical conditional expectation onto A, determined by $E\left(\sum_{g\in G} a_g\lambda_g\right) = a_e$ for $\sum_{g\in G} a_g\lambda_g \in A\otimes\mathbb{C}[G]$. Indeed, let $\xi \in H$ be arbitrary of norm 1, and consider $\xi \otimes \delta_e \in H \otimes \ell^2(G)$. We have

$$\begin{aligned}\left\|\sum_{g\in G} a_g\lambda_g\right\| &= \sup_{\substack{f\in H\otimes\ell^2(G)\\ \|f\|=1}} \left\|\left(\sum_{g\in G} a_g\lambda_g\right) f\right\| \geq \left\|\left(\sum_{g\in G} a_g\lambda_g\right)(\xi\otimes\delta_e)\right\| \\ &\geq \left\|\sum_{g\in G}\left(\left(g^{-1}a_g\right)\xi\right)\otimes\delta_g\right\| \geq \|a_e\xi\otimes\delta_e\| \geq \|a_e\xi\|\,.\end{aligned}$$

Taking supremum over all such ξ gives us that $\|a_e\| \leq \left\|\sum_{g\in G} a_g\lambda_g\right\|$. Consequently, this map extends to a contractive linear map $E : A \rtimes_r G \to A$. The fact that this map is a projection onto A is clear. Consequently, applying Tomiyama's theorem (Theorem 2.21), we conclude that E is a conditional expectation.

It can also be shown that this conditional expectation is faithful, i.e. nonzero on positive elements. For this, we direct the reader to [BC15] for a discussion on the Fourier series of a crossed product. In short, every element $x \in A \rtimes_r G$ has a Fourier series given by $\widehat{x} = \sum_{g\in G} x_g\lambda_g$ (formal series), where $x_g = E\left(x\lambda_g^*\right)$. The map $x \mapsto \widehat{x}$ is injective, and $E(x^*x) = \sum_{g\in G} g^{-1}\cdot\left(x_g^*x_g\right)$ (convergent in operator norm). Recall that every positive element in $A \rtimes_r G$ (or any C*-algebra in general) is of the form x^*x, and if this element is nonzero, x must be nonzero as well (so x_g is nonzero for some $g \in G$). As every element $g^{-1}\cdot\left(x_g^*x_g\right)$ is positive, the sum $\sum_{g\in G} g^{-1}\cdot\left(x_g^*x_g\right) = E(x^*x)$ is nonzero, as at least one of the summands is nonzero.

2.6 Operator systems and completely positive maps

Now we recall the definitions of operator spaces/operator systems, and the appropriate morphisms between them (completely bounded/completely positive maps). A good discussion on this topic can be found in [Pau03].

Definition 2.22. An *operator space* M is a subspace of a C*-algebra A. An *operator system* S is a self-adjoint, unital subspace of a unital C*-algebra A.

A couple of remarks are to be made. First, any operator system is an operator space, and given any operator space $M \subseteq A$, the smallest operator system containing it is $S = M + M^* + \mathbb{C}1$. Now, viewing $A \subseteq \mathcal{B}(H)$, we may consider the C*-algebra $M_n(A)$ of $n \times n$ matrices with elements in A, acting on H^n in the canonical way. This defines a norm structure on $M_n(M)$. Further, any linear map $\phi : M \to B$ from an operator system M to a C*-algebra B has a natural extension to a map $\phi_n : M_n(M) \to M_n(B)$, given by

$$\phi_n\left(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} \phi(a_{11}) & \dots & \phi(a_{1n}) \\ \vdots & & \vdots \\ \phi(a_{n1}) & \dots & \phi(a_{nn}) \end{bmatrix}.$$

It is often convenient to use the identification $M_n(A) \cong A \otimes M_n(\mathbb{C})$, in which case our map ϕ_n is determined by $\phi_n(a \otimes X) = \phi_n(a) \otimes X$, i.e. $\phi_n \cong \phi \otimes \mathrm{id}$. Using these maps, we define the following class of morphisms.

Definition 2.23. Assume M is an operator space, and $\phi : M \to B$ is a linear map from M to some C*-algebra B. We say ϕ is *completely bounded* if $\sup_{n\in\mathbb{N}} \|\phi_n\| < \infty$, and let $\|\phi\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\phi_n\|$. Further, we say ϕ is *completely contractive* if $\|\phi\|_{\mathrm{cb}} \leq 1$.

Now assume S is an operator system, and $\phi : S \to B$ is a linear map. We say ϕ is *n-positive* if the map ϕ_n is positive, i.e. maps positive elements to positive elements. We say ϕ is *completely positive* if it is n-positive for all $n \in \mathbb{N}$.

These (completely) positive and (completely) bounded maps have some particularly nice properties. A summary of the properties we will use is given below, all of which can be found in [Pau03, Chapters 2, 3, 4, 7]. Here, A denotes a unital C*-algebra, while S denotes an operator subsystem of A. We also let B denote another (not necessarily unital) C*-algebra. Further, X denotes a compact Hausdorff space, so that $C(X)$ denotes an arbitrary commutative unital C*-algebra. Finally, of course, H denotes a Hilbert space.

1. Assume $\phi : S \to B$ is positive. Then ϕ is automatically bounded, and $\|\phi\| \leq 2\|\phi(1)\|$.

2. Assume $\phi : S \to C(X)$ is a positive linear map. Then ϕ is completely positive.

3. (Arveson's extension theorem) Assume $\phi : S \to \mathcal{B}(H)$ is completely positive. Then ϕ extends to a completely positive map $\widetilde{\phi} : A \to \mathcal{B}(H)$.

4. (Stinespring's dilation theorem) Assume $\phi : A \to \mathcal{B}(H)$ is a completely positive map. There is a unital *-representation $\pi : A \to \mathcal{B}(K)$ onto some Hilbert space K, together with a bounded linear operator $V : H \to K$ with $\|V\|^2 = \|\phi\| = \|\phi(1)\|$, such that $\phi(a) = V^* \pi(a) V$. We call (π, V, K) a *Stinespring representation* or *Stinespring dilation* of ϕ.

2.7 Amenability

Several characterizations of amenability exist. Here, we provide what one usually takes as the definition.

Definition 2.24. G is called amenable if there exists a positive unital functional $M \in \ell^\infty(G)^*$ (i.e. a state) such that $M(g \cdot \varphi) = M(\varphi)$. Such a map is known as a left-invariant mean.

Example 2.25. If G is finite, then the state $M \in \ell^\infty(G)^*$ defined by $M(\varphi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g)$ for $\varphi \in \ell^\infty(G)$ is a left-invariant mean. Hence, finite groups are amenable.

It is well-known that amenability of G is equivalent to *Reiter's condition*, given below:

Proposition 2.26. *The following are equivalent:*

1. *G is amenable.*

2. *(Reiter's condition) Given any finite $F \subseteq G$ and any $\varepsilon > 0$, there is some $f \in \ell^1(G)$ with $\|f\|_1 = 1$ and $f \geq 0$ such that $\|g \cdot f - f\|_1 < \varepsilon$ for all $g \in F$.*

We may also relate amenability to weak containment as follows:

Proposition 2.27. *The following are equivalent:*

1. *G is amenable.*

2. *The trivial representation 1_G is weakly contained in the left-regular representation λ_G.*

Before we prove this, we first show that weak containment of the trivial representation is equivalent to admitting *almost invariant vectors.*

Proposition 2.28. *Given any unitary representation $\pi : G \to U(H)$, the trivial representation 1_G is weakly contained in π if and only if for any finite subset $F \subseteq G$ and any $\varepsilon > 0$, there is a vector $\xi \in H$ of norm 1 with $\|\pi(g)\xi - \xi\| < \varepsilon$ for all $g \in F$.*

Proof. Note that any normalized diagonal matrix coefficient of the trivial representation is always the constant function 1. First, assume π admits almost invariant vectors. Let $F \subseteq G$ be finite and $\varepsilon > 0$, and let $\xi \in H$ be of norm 1 with $\|\pi(g)\xi - \xi\| < \varepsilon$ for all $g \in F$. Consequently, for all $g \in F$, we have

$$\begin{aligned} |1 - \langle \pi(g)\xi \mid \xi \rangle| &= |\langle \xi \mid \xi \rangle - \langle \pi(g)\xi \mid \xi \rangle| = |\langle \xi - \pi(g)\xi \mid \xi \rangle| \\ &\leq \|\xi - \pi(g)\xi\| \, \|\xi\| < \varepsilon, \end{aligned}$$

which shows $1_G \prec \pi$.

Conversely, start with the assumption that $1_G \prec \pi$. Assume that π does NOT admit any almost invariant vector - there must exist a finite $F \subseteq G$ and $\varepsilon > 0$ such that for any $\xi \in H$ of norm 1, there is always a $g \in F$ with $\|\pi(g)\xi - \xi\| \geq \varepsilon$. Using the fact that

$$\|\pi(g)\xi - \xi\|^2 = \langle \pi(g)\xi - \xi \mid \pi(g)\xi - \xi \rangle = 2\left(1 - \operatorname{Re}\langle \pi(g)\xi \mid \xi \rangle\right),$$

it must be the case that $1 - \operatorname{Re}\langle \pi(g)\xi \mid \xi \rangle \geq \frac{\varepsilon^2}{2}$ for the appropriate $g \in F$. Note that for all $g \in F$, we have $1 - \operatorname{Re}\langle \pi(g)\xi \mid \xi \rangle \geq 0$. Hence, summing over $g \in F$, we obtain the following inequality:

$$|F| - \sum_{g \in F} \operatorname{Re}\left(\langle \pi(g)\xi \mid \xi \rangle\right) \geq \frac{\varepsilon^2}{2}.$$

In general, for any nonzero $\xi \in H$ (with no assumption on the norm), we can obtain the following inequality by replacing ξ with $\frac{1}{\|\xi\|}\xi$ in the previous inequality:

$$|F| \, \|\xi\|^2 - \sum_{g \in F} \operatorname{Re}\left(\langle \pi(g)\xi \mid \xi \rangle\right) \geq \frac{\varepsilon^2}{2} \|\xi\|^2.$$

(Note that this inequality also trivially holds for $\xi = 0$).

By the weak containment $1_G \prec \pi$, we may choose $\xi_1, \ldots, \xi_n \in H$ of norm 1 with $|1 - \sum_{i=1}^n \alpha_i \langle \pi(g)\xi_i \mid \xi_i \rangle| < \frac{\varepsilon^2}{2|F|}$ for all $g \in F$ (with the sum being a convex combination). Substituting ξ with $\sqrt{\alpha_i}\xi_i$ in the previous inequality, and summing over all i, we have that

$$\begin{aligned} & |F| - \sum_{i=1}^n \sum_{g \in F} \operatorname{Re}\left(\langle \pi(g)\sqrt{\alpha_i}\xi_i \mid \sqrt{\alpha_i}\xi_i \rangle\right) \geq \frac{\varepsilon^2}{2} \\ \iff & \sum_{g \in F} \left(1 - \sum_{i=1}^n \alpha_i \operatorname{Re}\left(\langle \pi(g)\xi_i \mid \xi_i \rangle\right)\right) \geq \frac{\varepsilon^2}{2}. \end{aligned}$$

Hence,

$$1 - \sum_{i=1}^n \operatorname{Re}\left(\langle \pi(g)\xi_i \mid \xi_i \rangle\right) \geq \frac{\varepsilon^2}{2|F|}$$

for some $g \in F$. Thus, it is the case that

$$\left|1 - \sum_{i=1}^{n} \alpha_i \langle \pi(g) \xi_i \mid \xi_i \rangle\right| \geq 1 - \sum_{i=1}^{n} \operatorname{Re}(\langle \pi(g) \xi_i \mid \xi_i \rangle) \geq \frac{\varepsilon^2}{2|F|},$$

which contradicts how we chose the ξ_i. ∎

Using this, together with Reiter's condition, we can easily prove Proposition 2.27, with the basic idea being that there is a natural way to convert $\ell^2(G)$ functions into positive $\ell^1(G)$ functions, and vice versa.

Proof of Proposition 2.27. First, assume we have the weak containment $1_G \prec \lambda_G$. Then $\ell^2(G)$ contains almost invariant vectors. Let $F \subseteq G$ be finite, and let $\varepsilon > 0$. There is some $f \in \ell^2(G)$ with $\|f\|_2 = 1$ and $\|g \cdot f - f\|_2 < \frac{\varepsilon}{2}$ for all $g \in F$. Now consider $|f|^2 \in \ell^1(G)$ ($|f|^2 \geq 0$ and $\left\||f|^2\right\|_1 = \|f\|_2 = 1$). We have

$$\begin{aligned}\left\|g \cdot |f|^2 - |f|^2\right\|_1 &= \left\|(g \cdot |f|)^2 - |f|^2\right\|_1 = \|(g \cdot |f| - |f|)(g \cdot |f| + |f|)\|_1 \\ &\leq \|g \cdot |f| - |f|\|_2 \, \|g \cdot |f| + |f|\|_2 \leq \frac{\varepsilon}{2} \cdot 2 = \varepsilon.\end{aligned}$$

Hence, Reiter's condition holds, and so G is amenable.

Conversely, assume G is amenable, and let $F \subseteq G$ be finite, $\varepsilon > 0$, and $f \in \ell^1(G)$ satisfying Reiter's condition. We will show $\sqrt{f} \in \ell^2(G)$ is an almost invariant vector. Note that for any real numbers $a, b \geq 0$, we have $|a - b| \leq a + b$, and so $|a - b|^2 \leq |a - b|(a + b) = |a^2 - b^2|$. Hence,

$$\begin{aligned}\left\|g \cdot \sqrt{f} - \sqrt{f}\right\|_2^2 &= \sum_{x \in G} \left|\sqrt{f(g^{-1}x)} - \sqrt{f(x)}\right|^2 \leq \sum_{x \in G} \left|\left(\sqrt{f(g^{-1}x)}\right)^2 - \left(\sqrt{f(x)}\right)^2\right| \\ &= \|g \cdot f - f\|_1 < \varepsilon.\end{aligned}$$

∎

There is a dynamical characterization of amenability that will come in handy often (see Definition 2.8 for the definition of a G-space). For convenience, we recall the Stone-Čech compactification here, as we will use it several times throughout this paper. The proof of the existence of the Stone-Čech compactification can be found in [Run05, Theorem 4.2.4], for example.

Theorem 2.29 (Stone-Čech compactification)**.** *Let X be a completely regular topological space. There is a compact Hausdorff topological space βX, the Stone-Čech compactification of X, together with a continuous map $\iota : X \to \beta X$ which homeomorphically maps X onto a dense subset, satisfying the following universal property: given any compact Hausdorff space K, and any continuous map $f : X \to K$, there is a unique continuous extension $\widetilde{f} : \beta X \to K$ such that $\widetilde{f} \circ \iota = f$. Further, βX is unique up to homeomorphism.*

Remark 2.30. It is a standard result in topology/functional analysis that the canonical action of G on itself by left-multiplication extends to an action on βG that makes it a G-space, and that $\ell^\infty(G)$ and $C(\beta G)$ are isomorphic as G-C*-algebras.

Proposition 2.31. *The following are equivalent:*

1. *G is amenable.*

2. *For any G-space X, there is a G-invariant probability measure $\mu \in P(X)$.*

Proof. Here, it is most convenient to identify $P(X)$ with the state space of $C(X)$. First, assume G is amenable, and let $M \in \ell^\infty(G)^*$ be a left-invariant mean. Fix some $x_0 \in X$, and define $\varphi : C(X) \to \ell^\infty(G)$ by $\varphi(f)(g) = f(gx_0)$ for all $f \in C(X)$ and $g \in G$. It is easy to check that φ is a G-equivariant, positive, unital, linear map. Hence, $M \circ \varphi : C(X) \to \mathbb{C}$ is a G-invariant state on $C(X)$.

Conversely, assume any G-space X admits a G-invariant probability measure. Consider $X = \beta G$, the Stone-Čech compactification of G. Then there is a G-invariant $\mu \in P(\beta G) \cong S(C(\beta G)) \cong S(\ell^\infty(G))$, i.e. a left-invariant mean on $\ell^\infty(G)$. ■

Many of the examples of C*-simplicity will deal with whether our group has any nontrivial amenable normal subgroups (here, nontrivial just means not equal to $\{e\}$). As such, we introduce the following concept, originally given in [Day57, Section 4, Lemma 1]:

Proposition 2.32. *Any group G has a largest amenable normal subgroup (in the sense that it contains all other amenable normal subgroups).*

Proof. First, we show there is a maximal amenable normal subgroup, for which we use Zorn's lemma. Consider the set of all amenable normal subgroups of G, partially ordered by inclusion. Let $(H_\lambda)_{\lambda\in\Lambda}$ be an ascending chain of amenable normal subgroups in G, and let $H \coloneqq \bigcup_{\lambda\in\Lambda} H_\lambda$. It is clear that H is still a normal subgroup. To show it is amenable, pick a left-invariant mean $M_\lambda : \ell^\infty(H_\lambda) \to \mathbb{C}$ for each H_λ. Given any $f \in \ell^\infty(H)$, there is no reason to expect that $\lim_\lambda M_\lambda(f|_{H_\lambda})$ exists. That doesn't stop us, however, from making it exist. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$, and consider the product space $\prod_{f\in\ell^\infty(H)} \|f\|_\infty \overline{\mathbb{D}}$ (compact by Tychonoff's theorem). Viewing $(M_\lambda(f|_{H_\lambda}))_{f\in\ell^\infty(H)}$ as a net in this space, then we admit a convergent subnet indexed by, say, λ_i for $i \in I$. Note that the union of the elements in our subnet is still H. Now defining $M : \ell^\infty(H) \to \mathbb{C}$ by $M(f) \coloneqq \lim_i M_{\lambda_i}\left(f|_{H_{\lambda_i}}\right)$, it is easy to see that this is a left-invariant mean on $\ell^\infty(H)$. Hence, H is always an upper bound to our chain, so G admits a maximal amenable normal subgroup (call it H).

Assume K is any other amenable normal subgroup. Then we know KH is a normal subgroup of G (easy to check), and $KH/H \cong K/(K \cap H)$ (the second isomorphism theorem). As quotients of amenable groups are amenable, then KH/H is amenable. But H is also amenable, so KH is amenable. As $KH \supseteq H$, this implies $KH = H$, which is only possible if $K \subseteq H$. ■

Definition 2.33. Given a group G, its largest amenable normal subgroup is called the *amenable radical*, and is denoted by $R_a(G)$.

Clearly, a group is amenable if and only if $R_a(G) = G$. We say that G has *trivial amenable radical* if $R_a(G) = \{e\}$, i.e. G has no nontrivial amenable normal subgroups.

3 A word on non-discrete groups

Most of the above preliminaries can be generalized to non-discrete groups, but not the main results in this paper. We summarize the biggest roadblock here.

Assume G is an arbitrary locally compact group. There is a nonzero Radon measure m on G that is left-translation invariant, known as a (left) Haar measure. By left-translation invariant, we mean that for any Borel subset $E \subseteq G$, and any $g \in G$, we have $m(gE) = m(E)$. Such a measure is unique up to scaling by a positive real number, and hence we usually refer to it as *the* Haar measure. More details can be found in [Fol15].

Quite naturally, the spaces $\ell^p(G)$ are replaced with $L^p(G)$, where the measure used is the Haar measure. Perhaps not as natural is the following. The left-regular representation of G on $L^2(G)$ (or any unitary representation, for that matter) induces a representation of $L^1(G)$: for any $F \in L^1(G)$ and $f \in L^2(G)$, define

$$F \cdot f = \int_G F(g)\, g \cdot f \, dm(g)$$

(where the integral above is the Bochner integral). With this, we define the reduced group C*-algebra of G to be the closure of the image of $L^1(G)$ (or $C_c(G)$) under the left-regular representation. An immediate observation is that G may no longer embed into $C_r^*(G)$. From this, one might expect a vastly different flavor between the discrete and non-discrete cases, and rightly so. Indeed, it can be shown that $C_r^*(G)$ is unital if and only if G is discrete! [Dav96, Chapter VII] gives a discussion of group C*-algebras for non-discrete groups.

Remark 3.1. Let G be discrete. The Haar measure on G is just the counting measure (and this measure is both left and right-translation invariant). Our spaces $L^p(G)$ just become $\ell^p(G)$. Using the canonical correspondence between $C_c(G)$ and $\mathbb{C}[G]$, we see that our general definition for the group C*-algebra coincides with the old one.

4 Dynamical characterization of C*-simplicity

In this section, we present some background on what is known as the Furstenberg boundary of a group G, which was originally introduced by Furstenberg in the study of Lie groups, beginning with [Fur63]. We then show how it can be used to give a characterization of C*-simplicity.

4.1 Boundary actions and the Furstenberg boundary

This section builds on the preliminaries, particularly what is discussed in Section 2.4. Again, we always assume G is a discrete group, and G-spaces are compact and Hausdorff. Much of the material here can be found in [Fur73], which presents a good overview of boundary theory of groups. To start, we begin with the definition of a boundary.

Definition 4.1. A G-boundary is a minimal, strongly proximal G-space.

It is worth noting that this is not the definition given by Furstenberg in [Fur73].

Proposition 4.2. *Let X be a G-space. The following are equivalent:*

1. *The space X is a G-boundary.*
2. *The space X is a minimal G-space where for any $\nu \in P(X)$, we have that $\overline{G\nu}^{\mathrm{w}^*}$ always contains some Dirac mass (the definition given in [Fur73]). By minimality of X, $\overline{G\nu}^{\mathrm{w}^*}$ must in fact contain all Dirac masses.*

Proof. Start with the second assumption. Assume $x, y \in X$, and consider $\nu = \frac{1}{2}(\delta_x + \delta_y)$. By assumption, there is a net $(g_\lambda) \subseteq G$ with $g_\lambda \nu \xrightarrow{\mathrm{w}^*} \delta_z$ for some $z \in X$. Dropping to a subnet, we may also assume that the nets $(g_\lambda x)$ and $(g_\lambda y)$ are also convergent to some x' and y' in X, respectively. Then $g_\lambda \nu = \frac{1}{2}(\delta_{g_\lambda x} + \delta_{g_\lambda y}) \xrightarrow{\mathrm{w}^*} \frac{1}{2}(\delta_{x'} + \delta_{y'})$. This shows that $\frac{1}{2}(\delta_{x'} + \delta_{y'}) = \delta_z$, which is only possible if $x' = y' = z$. Thus, X is proximal.

By definition, the action on X is proximal if and only if for any $(x, y) \in X \times X$, we have that $\overline{G(x, y)}$ contains an element of the diagonal $\{(z, z) \mid z \in X\}$. This is what we will show is true for $P(X)$. Let $(\mu, \nu) \in P(X) \times P(X)$. By assumption, there is some net $(g_\lambda) \subseteq G$ with $g_\lambda \mu \to \delta_x$, for some $x \in X$. Dropping to a subnet as appropriate, we also have that $g_\lambda \nu$ is convergent to some $\nu' \in P(X)$. Hence, $(\delta_x, \nu') \in \overline{G(\mu, \nu)}$. As the Dirac masses are a weak*-closed subset of $P(X)$, then applying an analogous argument, we have that $(\delta_{x'}, \delta_y)$ lies in $\overline{G(\mu, \nu)}$, for some $x', y \in X$. By proximality of X, some (δ_z, δ_z) lies in $\overline{G(\mu, \nu)}$. Thus, the action on X is in fact strongly proximal.

Conversely, start with Definition 2.12, and let $\nu \in P(X)$. Then choosing any $\delta_z \in P(X)$, there is a net $(g_\lambda) \subseteq G$ with $g_\lambda \nu$ and $g_\lambda \delta_z = \delta_{g_\lambda z}$ having the same weak*-limits. As the set of Dirac masses is weak*-closed, the limit $g_\lambda \nu$ must be a Dirac mass. ∎

In some sense, boundaries measure how far a group strays from being amenable. As we will show later, any boundary of an amenable group is always trivial, i.e. always consists of a singleton.

The following result on the structure of boundaries will come in useful later:

Proposition 4.3. *Assume X is a nontrivial G-boundary, i.e. it consists of more than just one element. Then X has no isolated points.*

Proof. Assume otherwise, so that $x \in X$ is an isolated point, and let $U = \{x\}$ be the singleton-neighbourhood of x. Lemma 2.15 tells us that X is finite. But then the normalized counting measure $\mu \in P(X)$ is a G-fixed point, as G acts by bijections on X. This is only possible if X is a singleton, by strong proximality. ■

Corollary 4.4. *Assume X is a nontrivial G-boundary. Then X is uncountable.*

Proof. Given any $x \in X$, the set $\{x\}$ is a nowhere-dense, closed set. But $X = \bigcup_{x \in X} \{x\}$, so the Baire category theorem (which applies to compact Hausdorff spaces) tells us that X cannot be countable. ■

We now show the existence and uniqueness of a universal boundary of G, known as the Furstenberg boundary. First, the following results will come in useful:

Lemma 4.5. *Every compact G-space Y contains a minimal subsystem.*

Proof. Consider the set of subsystems of Y (call it $\mathcal{S}$). Assume $(Z_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{S}$ is a descending chain. Inductively, this chain has the finite intersection property, and so by compactness, the intersection $\bigcap_{\lambda \in \Lambda} Z_\lambda$ is nonempty. It is also clear that this intersection is closed and G-invariant, making it a lower bound to our chain. By the power of Zorn's lemma, $\mathcal{S}$ contains a minimal element. ■

Proposition 4.6. *Assume Y is any compact G-space, X is some G-boundary, and $\varphi : Y \to P(X)$ is some G-map. Then Y contains X in its range. Further, if Y is minimal, then $\varphi(Y) = X$, and φ is unique (assuming such a map exists).*

Proof. First, note that if we can prove the results on the case of Y being minimal, then our first claim follows immediately by restricting down to a minimal subsystem. Assume Y is minimal. G-equivariance implies that $\varphi(Y)$ is G-invariant. It is also closed, as it is the continuous image of the compact set Y. As X is a boundary, then $\varphi(Y)$ must contain at least one, hence all, Dirac masses, i.e. $X \subseteq \varphi(Y)$. Further, $\varphi^{-1}(X)$ can similarly be checked to be a subsystem of Y (it is closed, as X is weak*-closed in $P(X)$, and φ is continuous). Minimality of Y implies that $\varphi^{-1}(X) = Y$, and so $\varphi(Y) = X$.

Now assume that $\varphi_1 : Y \to P(X)$ and $\varphi_2 : Y \to P(X)$ are two such maps (again, with Y minimal). As their ranges are X, then the map $\Phi(y) \coloneqq \frac{1}{2}\left(\delta_{\varphi_1(y)} + \delta_{\varphi_2(y)}\right)$ is well-defined, and is easily checked to be a G-map. As its range is X, i.e. $\frac{1}{2}\left(\delta_{\varphi_1(y)} + \delta_{\varphi_2(y)}\right)$ is always a Dirac mass, this forces $\varphi_1(y) = \varphi_2(y)$ for all y. ■

Corollary 4.7. *The only G-map from a G-boundary X to $P(X)$ is the canonical embedding $x \mapsto \delta_x$.*

Corollary 4.8. *The only G-map from a G-boundary X to itself is the identity map.*

Corollary 4.9. *Any G-map from some compact G-space Y to a G-boundary X is surjective. If Y is minimal, then this map is unique (if it exists).*

(Corollary 4.8 and Corollary 4.9 are immediately obtained by composing with the inclusion map $\iota_X : X \to P(X)$). The following technical results will also come in handy in the proof of the next theorem:

Lemma 4.10. *Assume $X_1, \ldots, X_n$ are strongly proximal G-spaces, and $\mu_i \in P(X_i)$. Then there is a common net (g_λ) such that $g_\lambda \mu_i$ converges to a Dirac mass for all i.*

Proof. It suffices to show that, in the compact G-space $P(X_1) \times \cdots \times P(X_n)$, we have that $\overline{G(\mu_1, \ldots, \mu_n)}$ contains an n-tuple of Dirac masses. First, choose a net $(g_\lambda) \subseteq G$ such that $g_\lambda \mu_1 \xrightarrow{\mathrm{w}^*} \delta_{x_1}$, for some $x_1 \in X_1$. Then, dropping to a subnet as appropriate, we have that $g_\lambda(\mu_1, \mu_2, \ldots, \mu_n) \to (\delta_{x_1}, \mu_2', \ldots, \mu_n')$ for some potentially different measures $\mu_i' \in P(X_i)$ $(i \geq 2)$. Repeating the same argument again with μ_2', we have that $\left(\delta_{x_1'}, \delta_{x_2}, \mu_3'', \ldots, \mu_n''\right)$ also lies in $\overline{G(\mu_1, \ldots, \mu_n)}$. Note that the Dirac masses are always weak*-closed, which is why we still have a (potentially different) Dirac mass in the first coordinate. Inductively, we see that our claim is true. ■

We also recall the notion of the *push-forward* of a measure:

Definition 4.11. Let X and Y be measure spaces, $f : X \to Y$ some measurable function, and μ some measure on X. The push-forward measure $f_*(\mu)$ on Y is given by $f_*(\mu)(E) \coloneqq \mu(f^{-1}(E))$.

If X and Y are compact Hausdorff spaces, f is continuous, and μ is a Radon measure on X, then it can be shown that $f_*(\mu)$ is a Radon measure on Y. In fact, $f_* : M(X) \to M(Y)$ is linear, and maps probability measures to probability measures. Further, it is also easy to check that if X and Y are G-spaces, and f is G-equivariant, then f_* is also G-equivariant.

Consider two compact Hausdorff spaces X and Y, and a measure $m \in P(X \times Y)$. Letting $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the canonical projections, a canonical set of measures on X and Y corresponding to m are $\pi_X^*(m) \in P(X)$ and $\pi_Y^*(m) \in P(Y)$, respectively. In general, m is not equal to $\pi_X^*(m) \times \pi_Y^*(m)$, or any product measure. This is true, however, if one of the measures $\pi_X^*(m)$ or $\pi_Y^*(m)$ is a Dirac mass.

Lemma 4.12. *Let X and Y be compact Hausdorff spaces, let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the canonical projections, and let $m \in P(X \times Y)$ be such that $\pi_X^*(m)$ is a Dirac mass δ_x for some $x \in X$. Then $m = \delta_x \times \nu$ for some $\nu \in P(Y)$.*

Proof. First, we claim that m is supported on $\{x\} \times Y$. Assume otherwise, so that $(x', y') \in \operatorname{supp} m$ with $x' \neq x$. Pick some $f \geq 0$ in $C(X)$ such that $f(x') = 1$, but $f(x) = 0$. Then we have both $(\pi_X^*(m))(f) = \delta_x(f) = f(x) = 0$, and

$(\pi_X^* (m))(f) = m(f \otimes 1_Y) > 0$, a contradiction. Thus, m is indeed supported on $\{x\} \times Y$.

Now let $\nu = \pi_Y^* m$. We claim that $m = \delta_x \times \nu$. To show this, let $f_1 \in C(X)$ and $f_2 \in C(Y)$ be arbitrary, and consider $f_1 \otimes f_2 \in C(X \times Y)$. As $f_1 \otimes f_2$ and $f_1(x) 1_X \otimes f_2$ agree on $\operatorname{supp} m \subseteq \{x\} \times Y$, we have

$$\begin{aligned} m(f_1 \otimes f_2) &= m(f_1(x) 1_X \otimes f_2) = f_1(x) m(1_X \otimes f_2) = f_1(x) \nu(f_2) \\ &= (\delta_x \times \nu)(f_1 \otimes f_2). \end{aligned}$$

Extending linearly, and using the fact that $\{\sum_{\text{finite}} f_1 \otimes f_2 \mid f_1 \in C(X), f_2 \in C(Y)\}$ is dense in $C(X \times Y)$ by the Stone-Weierstrass theorem, we get that $m = \delta_x \times \nu$. ∎

Corollary 4.13. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of compact Hausdorff spaces, let $X := \prod_{\alpha \in A} X_\alpha$ denote their product, and for each $\alpha \in A$, let $\pi_\alpha : X \to X_\alpha$ denote the canonical projection. If $m \in P(X)$ is such that $\pi_\alpha^*(m) = \delta_{x_\alpha}$ for all $\alpha \in F$, where $F \subseteq A$ is finite, then $m = \prod_{\alpha \in A} \delta_{x_\alpha} \times \nu$ for some $\nu \in P\left(\prod_{\alpha \in A \setminus F} X_\alpha\right)$.*

Proof. Our claim is clearly true for $F = \emptyset$, if we interpret $\prod_{\alpha \in A} \delta_{x_\alpha} \times \nu$ as ν. Assume F is nonempty. Given any nonempty $I \subseteq A$, let $X_I := \prod_{\alpha \in I} X_\alpha$, and define $\pi_I : X \to X_I$ to be the canonical projection, for convenience. Further, given any nonempty $J \subseteq I$, let $\pi_{I,J} : X_I \to X_J$ be the canonical projection. We see that the following diagram commutes:

$$\begin{tikzcd} P(X) \arrow[r, "\pi_I"] \arrow[dr, "\pi_J"] & P(X_I) \arrow[d, "\pi_{I,J}"] \\ & P(X_J) \end{tikzcd}$$

Indeed, if $E \subseteq X_J$ is a Borel subset, then

$$\begin{aligned} \left(\pi_{I,J}^* (\pi_I^* (m))\right)(E) &= (\pi_I^*(m))\left(E \times \prod_{\alpha \in I \setminus J} X_\alpha\right) = m\left(E \times \prod_{\alpha \in I \setminus J} X_\alpha \times \prod_{\alpha \in A \setminus I} X_\alpha\right) \\ &= m\left(E \times \prod_{\alpha \in A \setminus J} X_\alpha\right) = (\pi_J^*(m))(E). \end{aligned}$$

Let $F = \{\alpha_1, \dots, \alpha_n\}$. We will inductively show that $m = \prod_{i=1}^k \delta_{x_{\alpha_i}} \times \nu_{k+1}$ (for some $\nu_{k+1} \in P\left(X_{A \setminus \{\alpha_1, \dots, \alpha_k\}}\right)$) for $k = 1, \dots, n$. Our base case $k = 1$ follows from Lemma 4.12. Now assume our claim holds true for some $k < n$. By our above commutative diagram, we have that $\pi^*_{A \setminus \{\alpha_1, \dots, \alpha_k\}, \{\alpha_{k+1}\}}(\nu_{k+1}) = \delta_{x_{\alpha_{k+1}}}$ (as $\pi^*_{A \setminus \{\alpha_1, \dots, \alpha_k\}}(m) = \nu_{k+1}$ and $\pi_{\alpha_{k+1}}(m) = \delta_{x_{\alpha_{k+1}}}$). Applying Lemma 4.12 again, we have that ν_{k+1} itself decomposes as $\delta_{x_{\alpha_{k+1}}} \times \nu_{k+2}$ for some $\nu_{k+2} \in P\left(X_{A \setminus \{\alpha_1, \dots, \alpha_{k+1}\}}\right)$. By induction, our claim follows. ∎

Theorem 4.14. *There exists a boundary of G, denoted $\partial_F G$ and known as the Furstenberg boundary, which is universal in the following sense: any G-boundary X is the image of $\partial_F G$ under some G-map. Further, the Furstenberg boundary is unique up to G-isomorphism.*

Proof. First, we show that there is a limit on the cardinality of any boundary of G. Let X be a boundary, and fix some $x \in X$. The inclusion map $\iota : Gx \to X$ extends to a continuous map $\widetilde{\iota} : \beta(Gx) \to X$, where $\beta(Gx)$ is the Stone-Čech compactification of Gx. By continuity, $\widetilde{\iota}(\beta(Gx))$ is a compact, hence closed, superset of Gx, which forces $\widetilde{\iota}(\beta(Gx)) = X$ by density of Gx. In particular, $|X| \leq |\beta(Gx)|$. The bound on $|Gx|$ (namely, $|Gx| \leq |G|$) implies the existence of a bound on $|\beta(Gx)|$, and consequently a bound on $|X|$. Thus, it makes sense to index all G-boundaries up to G-isomorphism - call this set $\{X_\alpha\}_{\alpha \in A}$. Note that this set is nonempty, as a singleton with the trivial action is always a boundary.

Let $X = \prod_{\alpha \in A} X_\alpha$, equipped with the product topology. This is compact (Tychonoff's theorem) and Hausdorff, and it is a G-space under pointwise action of G. Let $\partial_F G$ be a minimal subsystem of X. We only need to show this space is strongly proximal. First, we will show that X is strongly proximal.

To this end, let $\mu \in P(X)$. Given any subset $I \subseteq A$, let $X_I \coloneqq \prod_{\alpha \in I} X_\alpha$ (in particular, $X_{\{\alpha_0\}} = X_{\alpha_0}$ for any $\alpha_0 \in A$), and let $\mu_I \in P(X_I)$ be the push-forward $\pi_I^*(\mu)$, where $\pi_I : X \to X_I$ is the canonical projection. For convenience, for any $\alpha_0 \in A$, we will also let $\mu_{\alpha_0} \coloneqq \mu_{\{\alpha_0\}}$. Similarly, any $f \in C(X_I)$ extends canonically to $f \otimes 1_{A \setminus I} \in C(X)$. We can and will view $C(X_I) \subseteq C(X)$ this way. We know $\overline{G\mu_\alpha}^{\text{w*}}$ always contains the Dirac masses on X_α. Hence, given any finite subset $F \subseteq A$, there is some common net (g_λ) such that $g_\lambda \mu_\alpha \xrightarrow{\text{w*}} \delta_{x_\alpha^F}$ (for some $x_\alpha^F \in X_\alpha$) for all $\alpha \in F$ (see Lemma 4.10). Dropping to a subnet, we may also assume that $g_\lambda \mu \xrightarrow{\text{w*}} m_F$ for some $m_F \in P(X)$. Consequently, as the map $\mu \mapsto \mu_\alpha$ is just the push-forwards of the canonical projection $\pi_\alpha : X \to X_\alpha$ (a G-map), we have that

$$(m_F)_\alpha = \lim_\lambda (g_\lambda \mu)_\alpha = \lim_\lambda (g_\lambda \mu_\alpha) = \delta_{x_\alpha^F},$$

and so $m_F = \prod_{\alpha \in F} \delta_{x_\alpha^F} \times \nu_{A \setminus F}$ for some $\nu_{A \setminus F} \in P\left(X_{A \setminus F}\right)$ by Corollary 4.13. Now consider the net (m_F), indexed by the finite subsets of A and ordered by inclusion. Once again by compactness, this net admits a subnet (m_{F_i}) weak*-convergent to some $m \in P(X)$ (so $m \in \overline{G\mu}^{\text{w*}}$). Note that, given any fixed $\alpha \in A$, eventually $\alpha \in F_i$, and so for any $f_\alpha \in C(X_\alpha)$, it is the case that

$$m(f_\alpha) = \lim_i m_{F_i}(f_\alpha) = \lim_i f_\alpha\left(x_\alpha^{F_i}\right)$$

(so this last limit exists). This forces $\left(x_\alpha^{F_i}\right)$ to be convergent to some $x_\alpha \in X_\alpha$ (otherwise, if it would admit two cluster points $x_\alpha^{(1)}, x_\alpha^{(2)} \in X_\alpha$, then Urysohn's lemma guarantees we could pick an $f_\alpha \in C(X_\alpha)$ so that $f_\alpha\left(x_\alpha^{(1)}\right) = 1$ and $f_\alpha\left(x_\alpha^{(2)}\right) = 0$, and so we would reach a contradiction). Now given $F \subseteq A$ finite, and $f_\alpha \in C(X_\alpha)$ (where $\alpha \in F$), consider the product $\prod_{\alpha \in F} f_\alpha \in C(X)$. As, eventually, $F \subseteq F_i$, we have the following:

$$m\left(\prod_{\alpha \in F} f_\alpha\right) = \lim_i m_{F_i}\left(\prod_{\alpha \in F} f_\alpha\right) = \lim_i \prod_{\alpha \in F} f_\alpha\left(x_\alpha^{F_i}\right) = \prod_{\alpha \in F} f_\alpha(x_\alpha).$$

Extending linearly, and noting that $\left\{\sum_{\text{finite}} \prod_{\text{finite}} f_\alpha \mid f_\alpha \in C(X_\alpha)\right\}$ is dense in $C(X)$ by the Stone-Weierstrass theorem, we have $m = \delta_{(x_\alpha)}$. Thus, X is strongly proximal.

Now we show $\partial_F G$ must be strongly proximal. We will show that, in general, for any strongly proximal G-space X, any subsystem $Y \subseteq X$ must also be strongly proximal. Assume $\mu \in P(Y)$, and consider the extension $\widetilde{\mu} \in P(X)$ given by $\widetilde{\mu}(E) = \mu(E \cap Y)$ (in other words, given $f \in C(X)$, we have $\widetilde{\mu}(f) = \mu(f|_Y)$). By strong proximality of X, there is a net $(g_\lambda) \subseteq G$ such that $g_\lambda \widetilde{\mu} \xrightarrow{\text{w}^*} \delta_x$, for some $x \in X$. Now, given any $f \in C(Y)$, there is always at least one extension $\widetilde{f} \in C(X)$ (Tietze's extension theorem). Hence,

$$\lim_\lambda (g_\lambda \mu)(f) = \lim_\lambda \mu\left(g_\lambda^{-1} f\right) = \lim_\lambda \widetilde{\mu}\left(g_\lambda^{-1} \widetilde{f}\right) = \lim_\lambda (g_\lambda \widetilde{\mu})\left(\widetilde{f}\right) = \delta_x\left(\widetilde{f}\right) = \widetilde{f}(x).$$

Of course, this implies $x \in Y$, as otherwise, Urysohn's lemma guarantees the existence of an $\widetilde{f} \in C(X)$ with $f := \widetilde{f}|_Y = 0$ and $\widetilde{f}(x) = 1$, giving us a contradiction (we would have $(g_\lambda \mu)(f) = 0$ for all λ, but the limit $\widetilde{f}(x)$ would be 1). Thus, $\overline{G\mu}^{\text{w}^*}$ always contains a Dirac mass, showing Y is strongly proximal. As a consequence, $\partial_F G$ is strongly proximal.

The fact that each boundary X_{α_0} is a G-image of $\partial_F G$ is an immediate consequence of Corollary 4.9 applied to $\pi_{\alpha_0}|_{\partial_F G} : \partial_F G \to X_{\alpha_0}$, where $\pi_{\alpha_0} : \prod_{\alpha \in A} X_\alpha \to X_{\alpha_0}$ is the canonical projection.

Now we prove uniqueness. Assume X_1 and X_2 are two universal boundaries. Then by assumption, there exist surjective G-maps $\varphi_1 : X_1 \to X_2$ and $\varphi_2 : X_2 \to X_1$. Applying Corollary 4.8 to $\varphi_2 \circ \varphi_1$, we get that φ_1 and φ_2 are isomorphisms of G-spaces. ■

The converse to the above claim, i.e. the claim that the image of $\partial_F G$ under a G-map is a G-boundary, is also true.

Proposition 4.15. *The image of any G-boundary X under a G-map is still a G-boundary.*

Proof. Let X be a G-boundary, and let $p : X \to Y$ be a surjective G-map onto a G-space Y. First, given any $y \in Y$, we have that $y = p(x)$ for some $x \in X$. Hence, $Gy = Gp(x) = p(Gx)$ is dense in Y, as continuous surjections map dense sets to dense sets. This shows that Y is minimal.

Now consider the push-forward map $p_* : P(X) \to P(Y)$. We see that $p_*(\delta_x) = \delta_{p(x)}$, so all Dirac masses of Y are contained in $p_*(P(X))$, by surjectivity of p. As the convex hull of the Dirac masses is weak*-dense in $P(Y)$, and the image of p_* is closed and convex, it must be the case that p_* is surjective. Hence, given any $\nu \in P(Y)$, there is a $\mu \in P(X)$ with $p_*(\mu) = \nu$. Choose a net $(g_\lambda) \subseteq G$ such that $g_\lambda \mu \xrightarrow{\text{w}^*} \delta_x$ for some $x \in X$. We have that

$$g_\lambda \nu = g_\lambda p_*(\mu) = p_*(g_\lambda \mu) \to p_*(\delta_x) = \delta_{p_*(x)}.$$

Thus, Y is strongly proximal, as $\overline{G\nu}^{\text{w}^*}$ always contains a Dirac mass. ■

We now discuss an important source of boundaries. Recall that, given a real vector space V and a convex subset $K \subseteq V$, a map $f : K \to K$ is called *affine* if it respects convex combinations. That is, for any convex combination $\sum_{i=1}^n \lambda_i v_i$ (where $v_i \in K$, $\lambda_i \geq 0$, and $\sum_{i=1}^n \lambda_i = 1$), we have $f\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i f(v_i)$. Now assume G acts by affine homeomorphisms on a compact convex subset K of a locally convex topological vector space V. We say that K is *irreducible* if the only G-invariant closed convex subset is K itself. The following proof is taken from [Gla76, Chapter III, Theorem 2.3].

Theorem 4.16. *Assume G acts by affine homeomorphisms on a compact convex subset K of a locally convex topological vector space, and that K is irreducible. Then the closure of the set of the extreme points of K, $\overline{\text{ex}}(K)$, is the unique G-boundary contained in K.*

Proof. First, we show that $\overline{\text{ex}}(K)$ is the unique minimal subsystem of K. It is easy to check that $\text{ex}(K)$ is a G-invariant subset. Indeed, if x is an extreme point of K and $g \in G$, assume $gx = \alpha y + (1-\alpha) z$ for $y, z \in K$ and $\alpha \in [0,1]$. Then $x = \alpha g^{-1} y + (1-\alpha) g^{-1} z$, showing $g^{-1} y = g^{-1} z$, i.e. $y = z$. Consequently, $\overline{\text{ex}}(K)$ is a closed, G-invariant subset. Now let $X \subseteq K$ be a minimal subset, and let $x \in X$ be arbitrary. We know that the closed convex G-invariant subset generated by x is all of K, i.e. $\overline{\text{conv}}(Gx) = K$. By Milman's converse to the Krein-Milman theorem, $\text{ex}(K) \subseteq \overline{Gx} = X$, so $\overline{\text{ex}}(K) \subseteq X$. In summary, $\overline{\text{ex}}(K)$ is a subsystem, and it is contained in every minimal subsystem. Consequently, $\overline{\text{ex}}(K)$ must be the unique minimal subsystem.

Now we show that K is strongly proximal. For this, we recall some basic Choquet theory - see [Phe01] for a good overview. For any measure $\mu \in P(K)$, there is a unique point $x_\mu \in K$ such that

$$f(x_\mu) = \int_K f \, d\mu$$

for any continuous, affine, real-valued function $f : K \to \mathbb{R}$ (with x_μ known as the *barycenter* of μ). Denoting the map $\mu \mapsto x_\mu$ by $\beta : P(K) \to K$, the *barycenter map*, we have that β is a surjective, affine G-map. Using this, assume $\mu \in$

$P(K)$. Then $\overline{\operatorname{conv}}\left(\overline{G\mu}\right)$ is a closed, convex, G-invariant subset of $P(K)$, and hence $\beta\left(\overline{\operatorname{conv}}\left(\overline{G\mu}\right)\right)$ is such a subset of K. Irreducibility forces $\beta\left(\overline{\operatorname{conv}}\left(\overline{G\mu}\right)\right) = K$. It is easy to check that $\beta\left(\overline{\operatorname{conv}}\left(\overline{G\mu}\right)\right) = \overline{\operatorname{conv}}\left(\beta\left(\overline{G\mu}\right)\right)$, and so Milman's converse to the Krein-Milman theorem tells us that $\overline{\operatorname{ex}}(K) \subseteq \beta\left(\overline{G\mu}\right)$. Now pick any $x \in \operatorname{ex}(K)$. There must exist some $\nu \in \overline{G\mu}$ such that $\beta(\nu) = x$.

We will show that $\nu = \delta_x$. Assume otherwise, and note that $\operatorname{supp}\nu$ must consist of at least two points. Using the fact that the dual of a locally convex topological vector space separates points, we could write ν as some nontrivial convex combination $\alpha_1\nu_1 + \alpha_2\nu_2$ of probability measures, where ν_1 is supported on a compact convex subset of K not containing x. As the barycenter of any measure lies in the closed convex hull of its support, then $\alpha_1\beta(\nu_1) + \alpha_2\beta(\nu_2)$ is a nontrivial convex combination that results in x, a contradiction to x being an extreme point of K. Thus, $\nu = \delta_x$, and so K is strongly proximal. As any subsystem of a strongly proximal G-space is strongly proximal (a proof is contained in the proof of Theorem 4.14), then we are done. ■

Corollary 4.17. *Assume G acts by affine homeomorphisms on a compact convex subset K of a locally convex topological vector space (without any assumption on the irreducibility of K). Then K contains a G-boundary.*

Proof. The proof of Lemma 4.5 can easily be modified to show that every such space K contains a compact convex irreducible subsystem $K' \subseteq K$. Then just apply Theorem 4.16 to K'. ■

Corollary 4.18. *Assume G acts by affine homeomorphisms on a compact convex subset K of a locally convex topological vector space, and K contains a G-boundary X such that $\overline{\operatorname{conv}}(X) = K$. Then K is irreducible.*

Proof. First, note that by Milman's converse to the Krein-Milman theorem, we have $\overline{\operatorname{ex}}(K) \subseteq X$. By minimality, $X = \overline{\operatorname{ex}}(K)$. Viewing $P(\overline{\operatorname{ex}}(K)) \subseteq P(K)$, it is a straightforward application of the Krein-Milman theorem that $\beta(P(\overline{\operatorname{ex}}(K)))$ is all of K, where $\beta : P(K) \to K$ is the barycenter map (discussed in the proof of Theorem 4.16).

Now let $x \in K$ and $y \in \overline{\operatorname{ex}}(K)$ be arbitrary. By the above remark, there is some $\mu \in P(\overline{\operatorname{ex}}(K))$ with with barycenter x. As $\overline{\operatorname{ex}}(K)$ is a G-boundary, then there is a net $(g_\lambda) \subseteq G$ with $g_\lambda\mu \to \delta_y$. Hence, we have that

$$\lim_\lambda g_\lambda x = \lim_\lambda g_\lambda \beta(\mu) = \lim_\lambda \beta(g_\lambda\mu) = \beta(\delta_y) = y.$$

This shows $\overline{\operatorname{ex}}(K) \subseteq \overline{Gx}$, and so $\overline{\operatorname{conv}}\left(\overline{Gx}\right) = K$ by the Krein-Milman theorem. As x was arbitrary, K is irreducible. ■

Note that, for any compact G-space X, G acts by affine homeomorphisms on the convex space $P(X)$. Indeed, this action is just the restriction of the action of G on $M(X)$ by isometric isomorphisms. Consequently, $P(X)$ contains a G-boundary, and so there exists a G-map $b : \partial_F G \to P(X)$. Such a map is known as a *boundary map.*

We also recall the following disconnectedness property for topological spaces:

Definition 4.19. A topological space X is called *extremally disconnected* if the closure of any open set is open.

Proposition 4.20. *The Furstenberg boundary is extremally disconnected.*

Proof. Assume $U \subseteq \partial_F G$ is open, and let $Y = \overline{U} \times \{0\} \cup U^{\complement} \times \{1\} \subseteq \partial_F G \times \{0,1\}$. Also, let $\pi : Y \to \partial_F G$ be the canonical projection. Fix $x_0 \in \partial_F G$, and let $\phi : G \to Y$ be given by

$$\phi(g) = \begin{cases} (gx_0, 0) & \text{if } gx_0 \in U \\ (gx_0, 1) & \text{if } gx_0 \notin U \end{cases}.$$

As G is discrete, then $\phi : G \to Y$ is continuous, and so there is an extension to βG, the Stone-Čech compactification of G. We will also denote this map by ϕ, i.e. $\phi : \beta G \to Y$. Further, recall that βG is naturally a G-space. By our earlier remark on boundary maps, there also exists some G-map $b : \partial_F G \to P(\beta G)$. Now consider the following composition of maps:

$$\begin{array}{ccccccc} \partial_F G & \xrightarrow{b} & P(\beta G) & \xrightarrow{\phi_*} & P(Y) & \xrightarrow{\pi_*} & P(\partial_F G) \\ x & \mapsto & b_x & \mapsto & \mu_x & \mapsto & \nu_x \end{array}$$

As all of the above maps are continuous, so is their composition. Further, $b : \partial_F G \to P(\beta G)$ is a G-map. While ϕ_* and π_* are not G-maps (we did not even define a G-action on Y), we claim that their composition $\pi_* \circ \phi_* = (\pi \circ \phi)_*$ is a G-map. To this end, note that $(\pi \circ \phi)(g) = gx_0$ for all $g \in G$, so $(\pi \circ \phi)|_G : G \to \partial_F G$ is clearly G-equivariant. Now given any $g \in G$ and $w \in \beta G$, we may choose a net $(w_\lambda) \subseteq G$ with $w_\lambda \to w$, and so

$$(\pi \circ \phi)(gw) = \lim_\lambda (\pi \circ \phi)(gw_\lambda) = \lim_\lambda g(\pi \circ \phi)(w_\lambda) = g(\pi \circ \phi)(w).$$

In other words, $\pi \circ \phi : \beta G \to \partial_F G$ is G-equivariant, and hence so is the pushforward map $(\pi \circ \phi)_* : P(\beta G) \to P(\partial_F G)$.

Thus, the overall composition $\pi_* \circ \phi_* \circ b : \partial_F G \to P(\partial_F G)$ is a G-map. By Corollary 4.7, we have $\nu_x = \delta_x$, and so μ_x is supported on $\pi^{-1}(x) = \{x\} \times \{0,1\}$. Consequently, $\mu_x\left(U^{\complement} \times \{1\}\right) = 0$ if $x \in U$, and $\mu_x\left(U^{\complement} \times \{1\}\right) = 1$ if $x \notin \overline{U}$ (note that $(x,0)$ would not be an element of Y if $x \notin \overline{U}$). However, we also note that the map $x \mapsto \mu_x\left(U^{\complement} \times \{1\}\right)$ is continuous (use the fact that the indicator function $1_{U^{\complement} \times \{1\}}$ lies in $C(Y)$, and the fact that if $(x_\lambda) \subseteq \partial_F G$ is a net with $x_\lambda \to x \in \partial_F G$, then $\mu_{x_\lambda} \xrightarrow{w^*} \mu_x$). Continuity forces this map to vanish on $\overline{U}$, making it the indicator function on $\overline{U}^{\complement}$. This can only be continuous if $\overline{U}$ is clopen. ■

4.2 Application to C*-simplicity

Our aim is to prove the following theorem:

Theorem 4.21. *The following are equivalent:*

1. *G is C*-simple.*
2. *The action of G on its Furstenberg boundary $\partial_F G$ is free.*
3. *There exists a G-boundary for which the action is topologically free.*

We require some machinery ahead of time, before we are able to prove this result.

Theorem 4.22 (Frolík's theorem, special case)**.** *Let X be any extremally disconnected Hausdorff topological space, and let $h : X \to X$ be a homeomorphism of X. Then the set of fixed points of h, $F = \{x \in X \mid h(x) = x\}$, is clopen.*

Proof. Proven in [Pit17, Proposition 2.11] in a slightly more general context. Call an open set $U \subseteq X$ h-simple if $h(U) \cap U = \emptyset$. Considering the set of h-simple subsets of X, partially ordered by inclusion, there exists a maximal element by Zorn's lemma (the upper bound to any chain is the union of all elements in that chain). Denote this maximal element by U. We claim that $\overline{U}$ is also h-simple. Indeed, it is easy to convince yourself that if A and B are disjoint open sets, then A and $\overline{B}$ are also disjoint. Hence, $h\left(\overline{U}\right) \cap U \subseteq \overline{h(U)} \cap U = \emptyset$. But $\overline{U}$ is open (hence, so is $h\left(\overline{U}\right)$), and so $h\left(\overline{U}\right) \cap \overline{U} = \emptyset$, proving our claim. By maximality of U, we have that $\overline{U} = U$, and so U is clopen.

As U is h-simple, then it cannot admit any h-fixed points. Consequently, neither can $h(U)$ and $h^{-1}(U)$, and thus $M := h^{-1}(U) \cup U \cup h(U)$ can't either. In other words, $F \subseteq M^{\complement}$. We wish to show we have equality here. Assume otherwise, so that there exists some $x \in M^{\complement}$ that is not a fixed point of h. We may choose a neighbourhood H of x with $h(H) \cap H = \emptyset$. (Why? Again, otherwise, for every neighbourhood H of x, we could choose some $x_H \in H$ with $h(x_H) \in H$. Then the nets (x_H) and $(h(x_H))$, indexed by the neighbourhoods of x ordered under reverse inclusion, would both converge to x. But $h(x_H) \to h(x)$, and so $h(x) = x$, a contradiction.) As M is closed, and $x \notin M$, then we may shrink H so that $H \cap M = \emptyset$ as well. Thus, $H \cap U = \emptyset$ (so $H \cup U$ is a proper superset of U), $H \cap h(U) = \emptyset$, and $H \cap h^{-1}(U) = \emptyset$. Using this,

$$\begin{aligned} h(H \cup U) \cap (H \cup U) &= (h(H) \cup h(U)) \cap (H \cup U) \\ &= (h(H) \cap H) \cup (h(H) \cap U) \cup (h(U) \cap H) \cup (h(U) \cap U) \\ &= \emptyset. \end{aligned}$$

This contradicts maximality of U, and so $F = M^{\complement}$. As M is clopen, then so is F. ■

Proof of Theorem 4.21, (2) $\iff$ (3). It is clear that (2) $\implies$ (3), as the empty set always has empty interior.

To show (3) $\implies$ (2), assume X is a G-boundary on which the action of G is topologically free, and let $\varphi : \partial_F G \to X$ be a surjective G-map. Assume G does NOT act topologically freely on $\partial_F G$, so that the set of fixed points $(\partial_F G)_g$ for some nonidentity $g \in G$ is has nonempty interior. Let $U \subseteq (\partial_F G)_g$ be nonempty and open in $\partial_F G$. By Lemma 2.15, there are $g_1, \ldots, g_n \in G$ such that $\partial_F G = g_1 U \cup \cdots \cup g_n U$. Consequently,

$$X = \varphi(\partial_F G) = g_1 \varphi(U) \cup \cdots \cup g_n \varphi(U).$$

This forces $\varphi(U)$ to have nonempty interior. But $\varphi(U) \subseteq \varphi\left((\partial_F G)_g\right) \subseteq X_g$ (this last inclusion follows from G-equivariance of φ), so X_g has nonempty interior. This contradicts G acting topologically freely on X, and so the action of G on $\partial_F G$ must be topologically free. More can easily be said, as Theorem 4.22 tells us that $(\partial_F G)_g$ is open. Hence, it can only have nonempty interior if it is empty, showing the action of G on $\partial_F G$ is in fact free. ∎

Lemma 4.23. *G is C*-simple if and only if, for any unitary representation π with $\pi \prec \lambda$, we have $\pi \sim \lambda$.*

Proof. First, assume G is C*-simple, making any *-homomorphisms from $C_r^*(G)$ to any other C*-algebra either zero or injective. Let π be a unitary representation of G with $\pi \prec \lambda$. By Theorem 2.4, there is a (surjective, unital) *-homomorphism $\phi : C_r^*(G) \to C_\pi^*(G)$ mapping $\lambda(g)$ to $\pi(g)$ for all $g \in G$. As $C_r^*(G)$ is simple, then ϕ must be injective, and consequently a *-isomorphism. Thus, $\phi^{-1} : C_\pi^*(G) \to C_r^*(G)$ is a *-homomorphism mapping $\pi(g)$ to $\lambda(g)$ for all $g \in G$, and so $\lambda \prec \pi$ as well.

Now assume that G is NOT C*-simple, so that $C_r^*(G)$ admits a nontrivial ideal I. Consider the following composition of maps:

$$C^*(G) \twoheadrightarrow C^*(G) / C^* \ker \lambda \cong C_r^*(G) \twoheadrightarrow C_r^*(G) / I.$$

We see that $C^*(G)$ must admit a proper ideal J with $C^* \ker \lambda \subsetneqq J$. By the GNS construction, $C^*(G)/J$ embeds into $B(H)$ for some cyclic Hilbert space H. Hence, the unitary representation $\pi : G \to U(H)$ resulting from

$$C^*(G) \twoheadrightarrow C^*(G)/J \hookrightarrow B(H)$$

satisfies $C^* \ker \pi = J \supsetneqq C^* \ker \lambda$. Again by Theorem 2.4, we get that $\pi \prec \lambda$, but $\lambda \not\prec \pi$. ∎

Recall that for discrete groups, given any subgroup $H \leq G$, the quasi-regular representation $\lambda_{G/H}$ of G on $\ell^2(G/H)$ is given by $(g \cdot f)(xH) = f(g^{-1}xH)$.

Proposition 4.24. *Assume H is an amenable subgroup of G. Then $\lambda_{G/H} \prec \lambda_G$.*

Proof. We know that the trivial representation 1_H is weakly contained in the left-regular representation λ_H, by amenability of H. By construction of the induced representation (see Section 2.3), it is almost immediate that $\operatorname{Ind}_H^G 1_H \cong \lambda_{G/H}$ and $\operatorname{Ind}_H^G \lambda_H \cong \lambda_G$. Continuity of induction gives us the result we want. ■

Lemma 4.25. *Given any subgroup $H \leq G$, there is a positive, unital, H-equivariant, isometric linear embedding of $\ell^\infty(H)$ into $\ell^\infty(G)$.*

Proof. Let T be a transversal of the right-coset space $H\backslash G$ (a choice of one representative from each coset). Then any $g \in G$ decomposes uniquely as $g = hr$, for some $h \in H$ and $r \in T$. Now given $f \in \ell^\infty(H)$, define an extension $\iota(f)$ to $\ell^\infty(G)$ by $(\iota(f))(hr) := f(h)$. It is easy to see that this extension is unital, linear, isometric and positive. Finally, given any $k \in H$, and any $hr \in G$, we have

$$(k \cdot \iota(f))(hr) = \iota(f)\left(k^{-1}hr\right) = f\left(k^{-1}h\right) = (k \cdot f)(h) = \iota(k \cdot f)(hr),$$

making the embedding H-invariant as well. ■

Proposition 4.26. *There is a duality of G-maps given as follows:*

1. *Assume X and Y are G-spaces. There is a bijective correspondence between G-equivariant, unital, positive, linear maps $\phi : C(X) \to C(Y)$ and G-maps $\widetilde{\phi} : Y \to P(X)$, given by $\widetilde{\phi}(y)(f) = \phi(f)(y)$ (for $y \in Y$ and $f \in C(X)$).*
2. *More generally, assume A is a G-C*-algebra, and Y is a G-space. There is a bijective correspondence between G-equivariant, unital, positive, linear maps $\phi : A \to C(Y)$ and G-maps $\widetilde{\phi} : Y \to S(A)$, given by $\widetilde{\phi}(y)(a) = \phi(a)(y)$ (for $y \in Y$ and $a \in A$).*
3. *Assume A is a G-C*-algebra, and X is a G-space. Any G-equivariant, unital, positive, linear map $\phi : C(X) \to A$ induces an affine G-map $\widetilde{\phi} : S(A) \to P(X)$, given by $\widetilde{\phi}(\psi)(f) = \psi(\phi(f))$ (for $\psi \in S(A)$ and $f \in C(X)$). The converse here may not necessarily be true though.*

Proof. Here, it is most convenient to always view $P(X)$ as the state space of $C(X)$.

1. Any G-equivariant, unital, positive, linear map $\phi : C(X) \to C(Y)$ can be viewed as a map $\phi' : C(X) \times Y \to \mathbb{C}$ such that:
   (a) For any $f \in C(X)$ and $y \in Y$, $\phi'(g \cdot f, y) = \phi'(f, g^{-1}y)$ (ϕ is G-equivariant).
   (b) For any $y \in Y$, $\phi'(1, y) = 1$ (ϕ is unital).
   (c) For any $f \geq 0$ in $C(X)$ and $y \in Y$, $\phi'(f, y) \geq 0$ (ϕ is positive).
   (d) The map $\phi' : C(X) \times Y \to \mathbb{C}$ is linear in the first variable (ϕ is linear).

(e) For any $f \in C(X)$, the map $\phi'(f,\cdot) : Y \to \mathbb{C}$ is continuous ($\phi(f)$ is continuous).

Similarly, any G-map $\widetilde{\phi} : Y \to P(X)$ can be viewed as a map $\widetilde{\phi}' : Y \times C(X) \to \mathbb{C}$ such that:

(a) For any $y \in Y$ and $f \in C(X)$, $\widetilde{\phi}'(gy, f) = \widetilde{\phi}'(y, g^{-1} \cdot f)$ ($\widetilde{\phi}$ is G-equivariant).

(b) For any $y \in Y$, $\widetilde{\phi}'(y, 1) = 1$ ($\widetilde{\phi}(y)$ is unital).

(c) For any $y \in Y$ and $f \geq 0$ in $C(X)$, $\widetilde{\phi}'(y, f) \geq 0$ ($\widetilde{\phi}(y)$ is positive).

(d) The map $\widetilde{\phi}' : Y \times C(X) \to \mathbb{C}$ is linear in the second variable (for any $y \in Y$, $\widetilde{\phi}(y)$ is linear).

(e) For any $f \in C(X)$, the map $\widetilde{\phi}'(\cdot, f) : Y \to \mathbb{C}$ ($\widetilde{\phi}$ is continuous).

From here, it is clear that the correspondence $\phi \leftrightarrow \widetilde{\phi}$ is bijective.

2. The proof is the same as the previous case.

3. Let $\phi : C(X) \to A$ be such a map. The corresponding dual map $\widetilde{\phi} : S(A) \to P(X)$ is just the restriction of the adjoint map $\phi^* : A^* \to C(X)^* = M(X)$ to $S(A)$, making $\widetilde{\phi}$ affine. Note that if $\psi \in S(A)$, then the following are true:

- If $f \geq 0$ in $C(X)$, then $\phi^*(\psi)(f) = \psi(\phi(f)) \geq 0$, as $\phi(f) \geq 0$.
- We have $\phi^*(\psi)(1) = \psi(\phi(1)) = \psi(1) = 1$.

Thus, $\phi^*(S(A)) \subseteq S(C(X)) = P(X)$, and so $\widetilde{\phi} : S(A) \to P(X)$ is indeed well-defined. Further, assume $(\psi_\lambda) \subseteq S(A)$ is a net with $\psi_\lambda \xrightarrow{\text{w}^*} \psi \in S(A)$. Then for any $f \in C(X)$, we have

$$\widetilde{\phi}(\psi_\lambda)(f) = \psi_\lambda(\phi(f)) \to \psi(\phi(f)) = \widetilde{\phi}(\psi)(f).$$

In other words, $\widetilde{\phi}(\psi_\lambda) \xrightarrow{\text{w}^*} \widetilde{\phi}(\psi)$, showing $\widetilde{\phi}$ is weak*-weak* continuous. Finally, if $g \in G$, $\psi \in S(A)$, and $f \in C(X)$, then

$$\begin{aligned}\widetilde{\phi}(g \cdot \psi)(f) &= (g \cdot \psi)(\phi(f)) = \psi\left(g^{-1} \cdot (\phi(f))\right) = \psi\left(\phi\left(g^{-1} \cdot f\right)\right) \\ &= \widetilde{\phi}(\psi)\left(g^{-1} \cdot f\right) = \left(g \cdot \left(\widetilde{\phi}(\psi)\right)\right)(f).\end{aligned}$$

This shows $\widetilde{\phi}$ is G-equivariant.

■

Proposition 4.27. *Given any $x \in \partial_F G$, the stabilizer $G_x = \{g \in G \mid gx = x\}$ is amenable.*

Proof. Consider βG, the Stone-Čech compactification of G, and recall that $\ell^{\infty}(G) \cong C(\beta G)$ as G-C*-algebras. There exists a boundary map $b : \partial_F G \to P(\beta G)$, and consequently by Proposition 4.26 a G-equivariant, unital, positive, linear dual map $r : C(\beta G) \to C(\partial_F G)$, i.e. $r : \ell^{\infty}(G) \to C(\partial_F G)$. Letting $e_x : C(\partial_F G) \to \mathbb{C}$ denote the evaluation map at x, we have that $e_x \circ r : \ell^{\infty}(G) \to \mathbb{C}$ is a state on $\ell^{\infty}(G)$. We can also check that it is G_x-invariant. Given any $f \in \ell^{\infty}(G)$ and $g \in G_x$, we have

$$e_x \circ r(g \cdot f) = r(g \cdot f)(x) = (g \cdot r(f))(x) = r(f)\left(g^{-1} \cdot x\right) = r(f)(x) = e_x \circ r(f).$$

Viewing $\ell^{\infty}(G_x) \subseteq \ell^{\infty}(G)$ using Lemma 4.25, we have that $(e_x \circ r)|_{\ell^{\infty}(G_x)}$ is a left-invariant mean on $\ell^{\infty}(G_x)$. ■

Proposition 4.28. *Assume $G \neq \{e\}$, and X is a G-boundary which G does not act topologically freely on, i.e. there is a nontrivial $s \in G$ such that the set of fixed points has nonempty interior. Then given any $x \in X$, $\lambda_G \not\prec \lambda_{G/G_x}$.*

First, we prove the following lemma:

Lemma 4.29. *Assume $G \neq \{e\}$, and let X be any G-boundary. Given any nonempty open subset $U \subseteq X$ and $\varepsilon > 0$, there is a finite subset $F \subseteq G \setminus \{e\}$ such that for any $\mu \in P(X)$, there is some $t \in F$ with $\mu(tU) > 1 - \varepsilon$.*

Proof. First, note that this lemma is easy enough if X is a singleton, as we may just let $F = \{g\}$ for any non-identity element $g \in G$. Assume that X consists of more than just a single element. Fix any $x \in U$, and let $\mu \in P(X)$. We wish to show there is always some $t_\mu \in G \setminus \{e\}$ such that

$$\left(\delta_x - t_\mu^{-1}\mu\right)(U) = 1 - \mu(t_\mu U) < \varepsilon.$$

- Assume $\mu = \delta_x$. We wish to show that there is always a nontrivial $t_\mu \in G$ with $x \in t_\mu U$. By Proposition 4.3, we may choose some $y \in U$ distinct from x. Minimality tells us that Gx is dense in X, and so there is a net $(g_\lambda) \subseteq G$ with $g_\lambda x \to y$. As $y \neq x$, we may assume no g_λ is the identity element. Since U is also a neighbourhood of y, then there is some λ_0 with $g_{\lambda_0} x \in U$, i.e. $x \in g_{\lambda_0}^{-1} U$, and so we may choose $t_\mu = g_{\lambda_0}^{-1}$.

- Assume $\mu \neq \delta_x$. Recall that strong proximality is equivalent to X being minimal, and always having $\overline{G\mu}^{\mathrm{w}^*}$ contain at least one, hence all, Dirac masses. In particular, there is a net $(g_\lambda) \subseteq G$ with $g_\lambda \mu \xrightarrow{\mathrm{w}^*} \delta_x$. Again, we may assume that no g_λ is the identity element. It suffices to show that $g_\lambda \mu(U) \to 1$, as we could just let $t_\mu = g_{\lambda_0}^{-1}$, where λ_0 is such that $1 - g_{\lambda_0}\mu(U) < \varepsilon$. To this end, Urysohn's lemma allows us to pick $f \in C(X)$ such that $f(x) = 1$, and $0 \leq f \leq 1_U$. Knowing this, we have

$$1 \geq g_\lambda \mu(U) \geq \int_X f \, dg_\lambda \mu \to \int_X f \, d\delta_x = f(x) = 1,$$

which proves our claim.

In either case, we have $\mu(t_\mu U) > 1-\varepsilon$, and so we may pick $f \in C(X)$, $0 \leq f \leq 1_{t_\mu U}$, such that $\mu(f) > 1-\varepsilon$. By continuity of the evaluation map $\nu \mapsto \nu(f)$ (for $\nu \in P(X)$), there is a weak*-open neighbourhood V_μ of μ such that $\nu(f) > 1-\varepsilon$ for all $\nu \in V_\mu$. But $\nu(t_\lambda U) \geq \nu(f)$, and so $\nu(t_\lambda U) > 1-\varepsilon$. Of course, $\{V_\mu\}_{\mu \in P(X)}$ is an open cover of $P(X)$, and so by weak*-compactness, there is a finite subcover $V_{\mu_1}, \ldots, V_{\mu_n}$. Hence, letting $F = t_{\mu_1}, \ldots, t_{\mu_n}$, we are done. ∎

Proof of Proposition 4.28. Let $s \neq e$ be such that the set of fixed points X_s has nonempty interior. We will argue by contradiction - assume that we do have the weak containment $\lambda_G \prec \lambda_{G/G_x}$. Consider the normalized matrix coefficient $\langle \lambda_G(\cdot)\,\delta_e \,|\, \delta_e \rangle$. By Remark 2.3, it suffices to show that this function can be approximated uniformly on any finite $F \subseteq G$ by averages of the form $\frac{1}{m}\sum_{j=1}^{m} \left\langle \lambda_{G/G_x}(\cdot)\,\xi_j \,\middle|\, \xi_j \right\rangle$ of normalized diagonal matrix coefficients. Let U be the interior of X_s, and let F be as in Lemma 4.29. Let $F' = \left\{tst^{-1} \;\middle|\; t \in F\right\}$, let $\varepsilon = \frac{1}{3}$, and let ξ_j be as above, approximating $\langle \lambda_G(\cdot)\,\delta_e \,|\, \delta_e \rangle$ on F'.

Recall that there is a canonical correspondence between G/G_x and Gx, given by $gG_x \leftrightarrow gx$. Under this identification, we define the following probability measures on X:

$$\mu_j \coloneqq \sum_{y \in Gx} |\xi_j(y)|^2\,\delta_y \quad \text{and} \quad \mu \coloneqq \frac{1}{m}\sum_{j=1}^{m} \mu_j.$$

(As $\sum_{y\in Gx} |\xi_j(y)|^2 = 1$, this first sum is indeed convergent, and it converges to a probability measure). Lemma 4.29 gives us that there is a $t \in F$ with $\mu\left(tU^{\complement}\right) < \varepsilon$. Now let $v_j \coloneqq t^{-1}\xi_j$. For convenience, given any $\xi, \eta \in \ell^2(G/G_x)$ and any subset $A \subseteq X$, we define

$$\langle \xi \,|\, \eta \rangle_A \coloneqq \sum_{y \in A\cap Gx} \xi(y)\,\overline{\eta(y)} \quad \text{and} \quad \|\xi\|_A \coloneqq \langle \xi \,|\, \xi \rangle_A^{1/2}.$$

Note that U, hence $U^{\complement}$ as well, is s-invariant. Thus,

$$\begin{aligned}
|1 - \langle sv_j \,|\, v_j \rangle| &= \left|1 - \langle sv_j \,|\, v_j \rangle_U - \langle sv_j \,|\, v_j \rangle_{U^{\complement}}\right| \\
&\leq \left|1 - \langle sv_j \,|\, v_j \rangle_U\right| + \left|\langle sv_j \,|\, v_j \rangle_{U^{\complement}}\right| \\
&\leq \left|1 - \langle v_j \,|\, v_j \rangle_U\right| + \|v_j\|_{U^{\complement}}^2 \\
&= 2\,\|v_j\|_{U^{\complement}}^2 \\
&= 2 \sum_{y \in U^{\complement}\cap Gx} |\xi_j(ty)|^2 \\
&= 2\mu_j\left(tU^{\complement}\right).
\end{aligned}$$

Using the fact that $\langle sv_j \mid v_j \rangle = \langle tst^{-1}\xi_j \mid \xi_j \rangle$, then taking an average over $j = 1, \ldots, m$, we get

$$\frac{1}{m}\sum_{j=1}^{m}\left|1 - \left\langle tst^{-1}\xi_j \,\middle|\, \xi_j \right\rangle\right| \leq 2\frac{1}{m}\sum_{j=1}^{m}\mu_j\left(tU^{\complement}\right) = 2\mu\left(tU^{\complement}\right) < 2\varepsilon = \frac{2}{3}.$$

However, as $e \notin F$, then $e \notin F'$, and so we always have $\langle g\delta_e \mid \delta_e \rangle = 0$ for all $g \in F'$. Consequently, by our weak containment approximation, we have

$$\left|\frac{1}{m}\sum_{j=1}^{m}\left\langle tst^{-1}\xi_j \,\middle|\, \xi_j \right\rangle\right| < \varepsilon = \frac{1}{3}.$$

But letting $\alpha_j \coloneqq \langle tst^{-1}\xi_j \mid \xi_j \rangle$, we see that

$$\frac{2}{3} > \frac{1}{m}\sum_{j=1}^{m}|1 - \alpha_j| \geq 1 - \frac{1}{m}\sum_{j=1}^{m}|\alpha_j| \geq 1 - \left|\frac{1}{m}\sum_{j=1}^{m}\alpha_j\right| > 1 - \frac{1}{3} = \frac{2}{3},$$

which is nonsense. ∎

Proof of Theorem 4.21, (1) $\implies$ (2). Note that Theorem 4.21 is obvious for $G = \{e\}$, as the trivial group is always C*-simple, and any boundary always consists of a singleton. Hence, we now focus our attention to when G is not the trivial group. Assume that G does NOT act freely on its Furstenberg boundary (consequently, there is a nontrivial $g \in G$ admitting a set of fixed points of nonempty interior). Then given any $x \in X$, we have $\lambda_G \not\prec \lambda_{G/G_x}$ by Proposition 4.28. But G_x is amenable by Proposition 4.27, and so $\lambda_{G/G_x} \prec \lambda_G$ by Proposition 4.24. Hence, G cannot be C*-simple by Lemma 4.23. ∎

The following are some easy lemmas:

Lemma 4.30. *Let A be a unital C*-algebra. Then A is simple if and only if every unital *-representation $\pi : A \to \mathcal{B}(H)$, where H is a nonzero Hilbert space.*

Proof. Assume A is simple. If $\ker \pi = A$, then $\pi = 0$. In particular, π is nonunital, contradicting our assumption. Hence, $\ker \pi = \{0\}$, making π injective. Now assume A is NOT simple, in particular admitting some nontrivial ideal I. By the GNS construction, there is a faithful, unital *-representation of A/I into $\mathcal{B}(H)$ for some H. Then composing with the canonical projection onto A/I, i.e.

$$A \twoheadrightarrow A/I \hookrightarrow \mathcal{B}(H),$$

we obtain a non-injective, unital *-representation of A onto $\mathcal{B}(H)$. ∎

Recall that a *-representation $\pi : A \to \mathcal{B}(H)$ of a C*-algebra is called *faithful* if it is nonzero on positive, nonzero elements.

Lemma 4.31. *Let A be a C^*-algebra. A $*$-representation $\pi : A \to \mathcal{B}(H)$ is faithful if and only if it is injective.*

Proof. It is clear that being injective implies being faithful. Conversely, assume π is faithful, and let $a \in A$ be arbitrary with $\pi(a) = 0$. We have $\pi(a^*a) = \pi(a)^* \pi(a) = 0$. But a^*a is positive, so this implies $a^*a = 0$, and so $a = 0$ (using the fact that $\|a\|^2 = \|a^*a\|$). ■

The following results are consequences of the duality of G-maps given in Proposition 4.26.

Proposition 4.32. *Assume $\phi : C(\partial_F G) \to C(\partial_F G)$ is a G-equivariant unital positive linear map. Then ϕ is the identity map.*

Proof. By Proposition 4.26, there is a dual G-map $\widetilde{\phi} : \partial_F G \to P(\partial_F G)$ given by $\widetilde{\phi}(x)(f) = \phi(f)(x)$ for any $x \in \partial_F G$ and $f \in C(\partial_F G)$. But this map is the canonical embedding by Corollary 4.7, i.e. $\widetilde{\phi}(x) = \delta_x$. Consequently, $\phi(f)(x) = \widetilde{\phi}(x)(f) = \delta_x(f) = f(x)$, i.e. $\phi(f) = f$. ■

Proposition 4.33. *Let A be a G-C^*-algebra. Every G-equivariant unital positive linear map $\phi : C(\partial_F G) \to A$ is an isometric embedding.*

Proof. First recall that, because the domain is commutative, positivity guarantees ϕ is automatically a contraction, without any prior assumptions on continuity. Now consider the dual map $\widetilde{\phi} : S(A) \to P(\partial_F G)$ given by $\widetilde{\phi}(\psi)(f) = \psi(\phi(f))$ (see Proposition 4.26). Proposition 4.6 says that this map contains $\partial_F G$ in its range. Consequently, for any $x \in \partial_F G$, there is some $\psi_x \in S(A)$ such that $\widetilde{\phi}(\psi_x) = \delta_x$, i.e. $\psi_x(\phi(f)) = \widetilde{\phi}(\psi)(f) = \delta_x(f) = f(x)$ for all $f \in C(X)$. Hence,

$$\|f\|_\infty = \sup_{x \in \partial_F G} |f(x)| = \sup_{x \in \partial_F G} |\psi_x(\phi(f))| \leq \sup_{\psi \in S(A)} |\psi(\phi(f))| \leq \|\phi(f)\|,$$

finishing the proof that ϕ is an isometry. ■

Finally, we wish to prove injectivity of $C(\partial_F G)$. For this, we the following "projectivity" result on $\partial_F G$.

Proposition 4.34. *The Furstenberg boundary is "projective" with respect to affine G-spaces. That is, assume K and K' are compact convex G-spaces on which G acts by affine homeomorphisms, $p : K' \to K$ is a surjective affine G-map, and $a : \partial_F G \to K$ is any G-map. Then there is a lifting to a G-map $c : \partial_F G \to K'$ such that $p \circ c = a$, i.e. the following diagram commutes:*

$$\begin{tikzcd} & K' \arrow[d, two heads, "p"] \\ \partial_F G \arrow[ur, dashed, "c"] \arrow[r, "a"] & K \end{tikzcd}$$

Proof. By Proposition 4.15, $a(\partial_F G)$ is a G-boundary. Let $C = \overline{\mathrm{conv}} a(\partial_F G) \subseteq K$. This is a compact, convex, G-invariant subset of K, and it is easy to see this forces $p^{-1}(C)$ to be such a subset of K' as well. Thus, $p^{-1}(C)$ contains a G-boundary X, by Corollary 4.17. Further, C is irreducible by Corollary 4.18, so $a(\partial_F G)$ is in fact the *unique* boundary in C by Theorem 4.16. Applying Proposition 4.15 again, this forces $p(X) = a(\partial_F G)$. Now, the universality of $\partial_F G$ gives us the existence of some G-map $c : \partial_F G \to X \subseteq K'$. But $\mathrm{im}(p \circ c) = a(\partial_F G)$, so Corollary 4.9 tells us that $p \circ c = a$. ∎

Proposition 4.35. *In the category of unital G-C*-algebras, with G-equivariant unital (completely) positive maps as morphisms, $C(\partial_F G)$ is injective, in the following sense: given any two unital G-C*-algebras $A \subseteq B$, and any G-equivariant unital positive map $\phi : A \to C(\partial_F G)$, there is an extension to a G-equivariant unital positive map $\psi : B \to C(\partial_F G)$.*

This same result holds in the category of G-operator systems, i.e. operator systems where G acts by unital complete order isomorphisms. Here, by a complete order isomorphism $\phi : S \to S$ from an operator system S to itself, we mean a linear map ϕ that is invertible with both ϕ and ϕ^{-1} completely positive.

Proof. First, recall that a G-equivariant unital positive map $\phi : A \to C(\partial_F G)$ corresponds to a dual G-map $\widetilde{\phi} : \partial_F G \to S(A)$ (and vice versa), using by $\widetilde{\phi}(x)(a) = \phi(a)(x)$ (see Proposition 4.26). Further, note that the restriction map $p : S(B) \to S(A)$ is surjective, as any state on A is completely positive, and hence extends to a state on B by Arveson's extension theorem applied to $\mathcal{B}(\mathbb{C}) \cong \mathbb{C}$. Thus, applying Proposition 4.34, there is a lifting $\widetilde{\psi} : \partial_F G \to S(B)$ such that $p \circ \widetilde{\psi} = \widetilde{\phi}$. Then the map $\psi : B \to C(\partial_F G)$ corresponding to $\widetilde{\psi} : \partial_F G \to S(B)$ is an extension of A. Indeed, if $a \in A$ and $x \in \partial_F G$, then $\widetilde{\psi}(x)|_A = \widetilde{\phi}(x)$, and so

$$\psi(a)(x) = \widetilde{\psi}(x)(a) = \widetilde{\phi}(x)(a) = \phi(a)(x).$$

The exact same proof applies in the case of G-operator systems. ∎

Definition 4.36. Given a contractive, completely positive map $\phi : A \to B$, where A is a unital C*-algebra, we define the *multiplicative domain* of ϕ as follows:

$$D_\phi = \{a \in A \mid \phi(a^*a) = \phi(a)^* \phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}.$$

It can be shown that D_ϕ is a C*-subalgebra of A. For our purposes, we only need the following property [BO08, Proposition 1.5.7 (2)] (which shows the name *multiplicative domain* is indeed warranted):

Proposition 4.37. *Assume $\phi : A \to B$ is a contractive, completely positive map, where A is a unital C*-algebra, and $a \in D_\phi$, $b \in A$. Then $\phi(ba) = \phi(b)\phi(a)$ and $\phi(ab) = \phi(a)\phi(b)$.*

Proof. By the GNS construction, we may view $B \subseteq \mathcal{B}(H)$ for some Hilbert space H, and consequently $\phi : A \to \mathcal{B}(H)$. Let (π, V, K) be a Stinespring dilation of ϕ. Assume $\phi(a^*a) = \phi(a)^* \phi(a)$ for some $a \in A$, and let $b \in A$ be arbitrary. We have

$$\begin{aligned}
&\phi(a^*a) = \phi(a)^* \phi(a) \\
\iff &\phi(a^*a) - \phi(a)^* \phi(a) = 0 \\
\iff &V^*\pi(a)^* \pi(a) V - V^*\pi(a)^* VV^*\pi(a) V = 0 \\
\iff &V^*\pi(a)^* (I - VV^*) \pi(a) V = 0 \\
\iff &\left((I - VV^*)^{1/2} \pi(a) V\right)^* \left((I - VV^*)^{1/2} \pi(a) V\right) = 0 \\
\iff &(I - VV^*)^{1/2} \pi(a) V^* = 0.
\end{aligned}$$

(Note that $\|V\|^2 = \|\phi\| \leq 1$, so $I - VV^*$ is positive, and so $(I - VV^*)^{1/2}$ indeed exists). Using this, we get

$$\begin{aligned}
\phi(ba) - \phi(b)\phi(a) &= V^*\pi(b)\pi(a) V - V^*\pi(b) VV^*\pi(a) V \\
&= V^*\pi(b)(I - VV^*)\pi(a) V \\
&= V^*\pi(b)(I - VV^*)^{1/2} (I - VV^*)^{1/2} \pi(a) V \\
&= 0.
\end{aligned}$$

The proof of our other claim is analogous. ∎

Proof of Theorem 4.21, (2) $\implies$ (1). Assume G acts freely on its Furstenberg boundary. To show G is C*-simple, Lemma 4.30 tells us that it suffices to show every unital *-representation $\pi : C_r^*(G) \to \mathcal{B}(H)$ (where $H \neq \{0\}$) is injective. It is easy to see that a *-homomorphism $\pi : A \to B$ between C*-algebras is always completely positive, as it is in particular positive (follows from $\pi(a^*a) = \pi(a)^* \pi(a)$), and the extension $\pi_n : M_n(A) \to M_n(B)$ is always a *-homomorphism. Hence, viewing $C_r^*(G) \subseteq C(\partial_F G) \rtimes_r G$, Arveson's extension theorem tells us that there is a unital, completely positive extension $\phi : C(\partial_F G) \rtimes_r G \to \mathcal{B}(H)$. As ϕ is an extension of π, we clearly have for any $a \in C_r^*(G)$ that $\phi(a^*a) = \phi(a)^* \phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a)^*$. Hence, $C_r^*(G)$ lies in the multiplicative domain D_ϕ of ϕ. Applying Proposition 4.37, we see that for any $a \in C(\partial_F G) \rtimes_r G$ and $g \in G$, we have

$$\phi(g \cdot a) = \phi\left(\lambda_g a \lambda_g^*\right) = \pi(\lambda_g)\phi(a)\pi(\lambda_g)^* = g \cdot \phi(a).$$

(Here, G acts on $C(\partial_F G) \rtimes_r G$ by conjugation by the unitary λ_g, and on $\mathcal{B}(H)$ by conjugation by the unitary $\pi(\lambda_g)$). In other words, ϕ is G-equivariant.

Consider $\phi|_{C(\partial_F G)} : C(\partial_F G) \to \mathcal{B}(H)$. By Proposition 4.33, this map is an isometric embedding onto $\phi(C(\partial_F G))$. Hence, we may consider the inverse map $\left(\phi|_{C(\partial_F G)}\right)^{-1} : \phi(C(\partial_F G)) \to C(\partial_F G)$. As $\phi(C(\partial_F G))$ is at least a G-operator system, by Proposition 4.35, this map extends to a G-equivariant unital positive map $\tau : \mathcal{B}(H) \to C(\partial_F G)$. Hence, we obtain a G-equivariant unital positive map $\psi := \tau \circ \phi : C(\partial_F G) \rtimes_r G \to C(\partial_F G)$.

Now we wish to show ψ is the canonical conditional expectation from $C(\partial_F G) \rtimes_r G$ to $C(\partial_F G)$. To this end, note that $\psi|_{C(\partial_F G)} : C(\partial_F G) \to C(\partial_F G)$ is the identity map by Proposition 4.32. Consequently, $C(\partial_F G) \subseteq D_\psi$, and so letting $s \in G$ be a nonidentity element and $f \in C(\partial_F G)$, we may apply Proposition 4.37 to get

$$\psi(\lambda_s) f = \psi(\lambda_s)\psi(f) = \psi(\lambda_s f) = \psi((s \cdot f)\lambda_s) = \psi(s \cdot f)\psi(\lambda_s) = (s \cdot f)\psi(\lambda_s).$$

As s is a nonidentity element, and the action $G \curvearrowright \partial_F G$ is free, then for any $x \in \partial_F G$, we have $s^{-1}x \neq x$. Thus, we may choose $f \in C(\partial_F G)$ with $f(x) = 1$, and $f(s^{-1}x) = 0$, by Urysohn's lemma. Evaluating the above expression at x gives us that $\psi(\lambda_s)(x) = 0$, and so $\psi(\lambda_s) = 0$. This shows that for any $s \neq e$ and $f \in C(\partial_F G)$, we have $\psi(f\lambda_s) = \psi(f)\psi(\lambda_s) = 0$, while $\psi(f\lambda_e) = \psi(f)\psi(\lambda_e) = f \cdot 1 = f$. By linearity, as ψ and the canonical expectation $E : C(\partial_F G) \rtimes_r G \to C(\partial_F G)$ agree on the dense subset $C(\partial_F G) \otimes \mathbb{C}[G]$ (algebraic tensor), they must be equal. But E, and hence $\psi = \tau \circ \phi$, is faithful, showing ϕ is faithful as well (otherwise, if ϕ vanished on some nonzero positive element, so would ψ). This implies $\pi = \phi|_{C_r^*(G)}$ is faithful, and hence injective, finishing the proof. ■

5 Intrinsic characterization of C*-simplicity

Here, we provide an intrinsic characterization of C*-simplicity, i.e. one which depends only on the internal structure of our group. Note that we view amenability as an intrinsic property. See, for example, the Følner characterization of amenability. Much of the hard work was already done in Section 4, and the main results in this section follow without too much difficulty.

To avoid confusion with the space of probability measures $P(G)$, let 2^G denote the power set of G, i.e. the set of all subsets. Recall that there is a natural correspondence between 2^G and $\{0,1\}^G$, where the subset $A \subseteq G$ maps to $(x_g)_{g \in G}$, with

$$x_g = \begin{cases} 1 & \text{if } g \in S \\ 0 & \text{if } g \notin S \end{cases}.$$

Equipped with the product topology, this becomes a compact (Hausdorff) space by Tychonoff's theorem. A convenient way to visualize convergence in this space is as follows: let $(A_\lambda) \subseteq 2^G$ be a net. Then $A_\lambda \to A \in 2^G$ if and only if both of the following are true:

1. For every $g \in A$, we have that g eventually lies in A_λ.
2. For every $g \notin A$, we have that g eventually never lies in A_λ.

Definition 5.1. $S(G) \subseteq 2^G$ denotes the space of subgroups of G, equipped with the relative topology induced from 2^G, known as the *Chabauty topology*.

From now on, unless stated otherwise, $S(G)$ will always be equipped with the Chabauty topology.

Proposition 5.2. *$S(G)$ is compact under the Chabauty topology.*

Proof. This is equivalent to showing it is a closed subset of 2^G. To this end, assume $(H_\lambda) \subseteq S(G)$ is a net in $S(G)$ convergent to $H \in 2^G$. Let $g, h \in H$. Then eventually, g lies in H_λ, and the same can be said for h, showing the product gh eventually lies in H_λ, and thus $gh \in H$. Similarly, as g eventually lies in H_λ, so does g^{-1}, showing $g^{-1} \in H$. Hence, $H \in S(G)$. ■

It is easy to check that the action of any $g \in G$ on $S(G)$ by conjugation, i.e. $g \cdot H = gHg^{-1}$ for $H \in S(G)$, is continuous. In other words, $S(G)$ is a G-space under conjugation.

Finally, we let $S_a(G) \subseteq S(G)$ denote the space of amenable subgroups. It is clear that this space is G-invariant, as conjugate subgroups are isomorphic. We can also check it is compact, making it a G-space under conjugation as well.

Proposition 5.3. *$S_a(G)$ is compact under the Chabauty topology.*

Proof. It suffices to show this is a closed subset of $S(G)$, which we know is compact. Assume $(H_\lambda) \subseteq S_a(G)$ is a net with $H_\lambda \to H \in S(G)$. Let $M_\lambda \in \ell^\infty(H_\lambda)^*$ be a left-invariant mean on H_λ. Recall that there is a positive, unital, $H_\lambda \cap H$-equivariant, isometric linear embedding $\iota_\lambda : \ell^\infty(H_\lambda \cap H) \to \ell^\infty(H_\lambda)$ (Lemma 4.25).

Now, letting $\mathbb{D}$ denote the unit disk in $\mathbb{C}$, consider the space $\prod_{f \in \ell^\infty(H)} \|f\|_\infty \overline{\mathbb{D}}$, and the net $\left(\alpha_f^{(\lambda)}\right)$ given by $\alpha_f^{(\lambda)} = M_\lambda\left(\iota_\lambda\left(f|_{H \cap H_\lambda}\right)\right)$. As our space is compact by Tychonoff's theorem, our net admits a convergent subnet. For convenience, we will just reindex our original net (H_λ) so that $\left(\alpha_f^{(\lambda)}\right)$ is convergent.

Using this, define $M : \ell^\infty(H) \to \mathbb{C}$ by $M(f) \coloneqq \lim_\lambda M_\lambda\left(\iota_\lambda\left(f|_{H \cap H_\lambda}\right)\right)$. Clearly, M is linear, unital, positive. H-equivariance follows from the fact that eventually, any $h \in H$ lies in $H \cap H_\lambda$, and so

$$\begin{aligned}
M(h \cdot f) &= \lim_\lambda M_\lambda\left(\iota_\lambda\left((h \cdot f)|_{H \cap H_\lambda}\right)\right) \\
&= \lim_\lambda M_\lambda\left(\iota_\lambda\left(h \cdot \left(f|_{H \cap H_\lambda}\right)\right)\right) \\
&= \lim_\lambda M_\lambda\left(h \cdot \iota_\lambda\left(f|_{H \cap H_\lambda}\right)\right) \\
&= \lim_\lambda M_\lambda\left(\iota_\lambda\left(f|_{H \cap H_\lambda}\right)\right) \\
&= M(f).
\end{aligned}$$

■

Finally, we give some definitions:

Definition 5.4. A *uniformly recurrent subgroup* of G is a minimal subsystem $X \subseteq S(G)$. X is said to be *amenable* if $X \subseteq S_a(G)$, and *nontrivial* if $X \neq \{\{e\}\}$.

Definition 5.5. A *recurrent subgroup* of G is a subgroup $H \leq G$ such that the following is true: there is a finite subset $F \subseteq G \setminus \{e\}$ such that for all $g \in G$, we have $F \cap gHg^{-1} \neq \emptyset$.

The definition of recurrence given above is the most "group theoretic", as does not use topology. There are other characterizations, however, which are sometimes more convenient (in particular, for the proof of our main theorem). Given a subgroup $H \leq G$, denote the set of its conjugates by $\mathrm{Conj}(H) = \{gHg^{-1} \mid g \in G\} \subseteq S(G)$.

Proposition 5.6. *Let H be a subgroup of G. The following are equivalent:*

1. *H is recurrent.*
2. *There exists a finite subset $F \subseteq G \setminus \{e\}$ such that for all $K \in \overline{\mathrm{Conj}}(H)$, we have $F \cap K \neq \emptyset$.*
3. *$\{e\} \notin \overline{\mathrm{Conj}}(H)$.*
4. *Given any net $(g_\lambda) \subseteq G$, there exists a subnet (g_{λ_i}) such that $\bigcap_i g_{\lambda_i} H g_{\lambda_i}^{-1} \neq \{e\}$.*

Proof. (2) $\implies$ (1): This implication is clear, as $\mathrm{Conj}(H) \subseteq \overline{\mathrm{Conj}}(H)$.

(1) $\implies$ (4): Assume $(g_\lambda)_{\lambda \in \Lambda} \subseteq G$ is a net. Given $g \in F$, define $\Lambda_g := \left\{\lambda \in \Lambda \mid g \in g_\lambda H g_\lambda^{-1}\right\}$. By finiteness of F, and the fact that $\Lambda = \bigcup_{g \in F} \Lambda_g$, at least one of $(g_\lambda)_{\lambda \in \Lambda_g}$ is a subnet of $(g_\lambda)_{\lambda \in \Lambda} \subseteq G$. By construction, $g \in \bigcap_{\lambda \in \Lambda_g} g_\lambda H g_\lambda^{-1}$, so this intersection cannot be $\{e\}$.

(4) $\implies$ (3): Assume otherwise, i.e. $\{e\} \in \overline{\mathrm{Conj}}(H)$, so that there is some net $(g_\lambda) \subseteq G$ with $g_\lambda H g_\lambda^{-1} \to \{e\}$. There exists a subnet (g_{λ_i}) such that $\bigcap_i g_{\lambda_i} H g_{\lambda_i}^{-1} \neq \{e\}$. But clearly, this subnet cannot converge to $\{e\}$.

(3) $\implies$ (2): Assume otherwise, so that no such finite subset exists. Given any finite $F \subseteq G \setminus \{e\}$, we may pick some $K_F \in \overline{\mathrm{Conj}}(H)$ with $F \cap K_F = \emptyset$. Hence, given $g \neq e$, for any finite $F \supseteq \{g\}$, we have $g \notin K_F$. Thus, the net (K_F), indexed by finite $F \subseteq G \setminus \{e\}$ and directed under inclusion, must converge to $\{e\}$, showing $\{e\} \in \overline{\overline{\mathrm{Conj}}}(H) = \overline{\mathrm{Conj}}(H)$. ■

Our main theorem is the following:

Theorem 5.7. *The following are equivalent:*

1. *G is C^*-simple.*
2. *G admits no non-trivial amenable uniformly recurrent subgroups.*
3. *G admits no amenable recurrent subgroups.*

Proof of Theorem 5.7, (1) $\iff$ (2). To show (2) $\implies$ (1), assume G is NOT C*-simple, and let $X := \{G_x \mid x \in \partial_F G\}$. Clearly, X is nonempty. As the action $G \curvearrowright \partial_F G$ is not free by Theorem 4.21, then $G_x \supsetneq \{e\}$ for some $x \in \partial_F G$, showing X is nontrivial. Proposition 4.27 gives us that X is amenable as well. Finally, we wish to show X is a uniformly recurrent subgroup. Consider the following map:

$$\begin{aligned} \varphi :\ \partial_F G &\to S_a(G) \\ x \quad &\mapsto G_x \end{aligned}$$

This map is clearly G-equivariant, as

$$h \in G_{gx} \iff hgx = gx \iff g^{-1}hgx = x \iff g^{-1}hg \in G_x \iff h \in gG_xg^{-1}.$$

Further, this map is continuous: given any fixed $g_0 \in G$, consider the sub-basic open sets $A_{g_0} := \{H \in S_a(G) \mid g_0 \in H\}$ and $B_{g_0} := \{H \in S_a(G) \mid g_0 \notin H\}$ (note that $B_{g_0} = A_{g_0}^{\complement}$). We have that $\varphi^{-1}(A_{g_0}) = (\partial_F G)_{g_0}$, the set of fixed points of g_0. As $\partial_F G$ is extremally disconnected (Proposition 4.20), then Theorem 4.22 tells us that $(\partial_F G)_{g_0}$ is a clopen set. Consequently, so is $\varphi^{-1}(B_{g_0}) = \varphi^{-1}(A_{g_0})^{\complement}$. But sets of the form A_{g_0} and B_{g_0} ($g_0 \in G$) generate the topology on $S_A(G)$, proving φ is continuous. As $X = \varphi(\partial_F G)$, then X is a G-boundary, in particular a uniformly recurrent subgroup.

Conversely, to show (1) $\implies$ (2), assume G admits a nontrivial amenable uniformly recurrent subgroup X. Fix $H \in X$. Viewing $\partial_F G$ as an H-space, as H is amenable, we have that there is an H-fixed probability measure $\mu \in P(\partial_F G)$ by Proposition 2.31. Further, as $\partial_F G$ is a G-boundary, then $\overline{G\mu}^{\mathrm{w}^*}$ contains a Dirac mass. Thus, there is a net $(g_\lambda) \subseteq G$ with $g_\lambda \mu \xrightarrow{\mathrm{w}^*} \delta_x$ for some $x \in \partial_F G$. Our aim is to show that x is fixed by some nonidentity element of G.

Consider the net $\left(g_\lambda H g_\lambda^{-1}\right) \subseteq S_a(G)$. By compactness of $S_a(G)$, this net admits a convergent subnet, and so, dropping to this subnet, we may assume $\left(g_\lambda H g_\lambda^{-1}\right)$ converges to some $K \in S_a(G)$. Note that X is closed, so $K \in X$, and minimality of X guarantees that $K \neq \{e\}$ (otherwise, if $K = \{e\}$, then the conjugates of K would all be $\{e\}$, and so $\{gKg^{-1}\}_{g \in G}$ could never be dense in X, which is nontrivial). Pick a nonidentity element $g \in K$. Chopping off the start of the net $\left(g_\lambda H g_\lambda^{-1}\right)$ as appropriate, we may assume $g \in g_\lambda H g_\lambda^{-1}$ for all λ. Hence, given any λ, we have $g = g_\lambda h_\lambda g_\lambda^{-1}$ for some $h_\lambda \in H$, showing $gg_\lambda\mu = g_\lambda h_\lambda \mu = g_\lambda \mu$. Taking the limit, we get $g\delta_x = \delta_x$, i.e. $\delta_{gx} = \delta_x$, which is only possible if $gx = x$. As g was a nonidentity element, then the action $G \curvearrowright \partial_F G$ is not free, so G is not C*-simple by Theorem 4.21.

Now, (2) is easy to rephrase into (3). To show (2) $\implies$ (3), assume that there exists some amenable recurrent subgroup $H \leq G$. Then $\{e\} \notin \overline{\mathrm{Conj}}(H)$ by Proposition 5.6. Further, there is always some minimal subsystem $X \subseteq \overline{\mathrm{Conj}}(H)$ (see Lemma 4.5) - this is a nontrivial amenable uniformly recurrent subgroup.

Conversely, to show (3) $\implies$ (2), assume there is some nontrivial amenable uniformly recurrent subgroup $X \subseteq S_a(G)$. Pick any $H \in X$. Then $\overline{\operatorname{Conj}}(H) \subseteq X$ (in fact, we have equality by minimality), and $\{e\} \notin X$, so $\{e\} \notin \overline{\operatorname{Conj}}(H)$. Thus, H is an amenable recurrent subgroup. ■

Corollary 5.8. *Assume G is C*-simple. Then it has trivial amenable radical.*

Proof. This follows from the fact that any normal subgroup $N \neq \{e\}$ is clearly recurrent. ■

Corollary 5.9. *Assume G is C*-simple. Then it is icc. That is, the conjugacy class of every nonidentity element is infinite.*

Proof. Assume otherwise, so that there is some nonidentity $x \in G$ with finite conjugacy class. Then the conjugates of $\langle x \rangle$ are given by $\langle gxg^{-1} \rangle$ for $g \in G$ (finitely many). In particular, $\langle x \rangle$ is an amenable recurrent subgroup, contradicting G being C*-simple. ■

We now discuss a strengthening of this theorem that was remarked, but not elaborated much on, in [Ken15]. First, we introduce another definition:

Definition 5.10. A subgroup $H \leq G$ is called *normalish* if for every $t_1, \ldots, t_n \in G$, we have that $\bigcap_{i=1}^{n} t_i H t_i^{-1}$ is infinite.

The name is misleading, as finite normal subgroups are never normalish, but infinite normal subgroups always are. Such subgroups were used to give a sufficient, but not necessary, condition for C*-simplicity in [BKKO17, Theorem 6.2], namely the following:

Theorem 5.11. *If G has no finite normal subgroups other than $\{e\}$ and no amenable normalish subgroups, then G is C*-simple.*

As it turns out, we may combine these two characterizations to reduce the number of subgroups we need to check.

Theorem 5.12. *G is C*-simple if and only if it has no finite normal subgroups other than $\{e\}$, and no amenable recurrent normalish subgroups.*

Proof. Clearly, the forwards direction is given by Theorem 5.7 (note that finite normal subgroups other than $\{e\}$ are automatically amenable and recurrent).

Conversely, assume G is NOT C*-simple. If G admits a finite normal subgroup other than $\{e\}$, we are done. Assume no such subgroup exists. The proof of Theorem 5.7 actually gives us that $\{G_x \mid x \in \partial_F G\}$ is a nontrivial, amenable, uniformly recurrent subgroup of G. In other words, every G_x ($x \in \partial_F G$) is an amenable recurrent subgroup.

Fix any $x \in \partial_F G$. We claim that for any $t_1, \ldots, t_n \in G$, the intersection $\bigcap_{i=1}^{n} t_i G_x t_i^{-1}$ is not $\{e\}$. Pick any $s \neq e$ such that the set of fixed points $(\partial_F G)_s$

is nonempty (and open by Theorem 4.22). Consider the action of G on $(\partial_F G)^n$ - repeatedly applying proximality and dropping to a subnet as appropriate, we have that $\overline{G(t_1 x, \ldots, t_n x)}$ contains some element $(y, \ldots, y)$ of the diagonal. Hence, applying minimality, there is some simultaneous $r \in G$ such that $rt_i x \in X_s$ for all i. Using the fact that $t_i G_x t_i^{-1} = G_{t_i x}$, it is easy to see that $r^{-1} sr \in \bigcap_{i=1}^n t_i G_x t_i^{-1}$ (and $r^{-1} sr \neq e$, as $s \neq e$). Consequently, if this intersection is ever finite for some $t_1, \ldots, t_n \in G$, then $\bigcap_{t \in G} tG_x t^{-1}$ can be made an intersection of finitely many terms. Thus, the intersection $\bigcap_{t \in G} tG_x t^{-1}$ is not $\{e\}$, is finite, and is a normal subgroup, contradicting our earlier assumption that no such subgroups exist. This shows G_x is normalish. ■

6 Application to the unique trace property

It was an open question whether the reduced group C*-algebra of a discrete group is simple if and only if it has a unique tracial state, namely the canonical tracial state given by $\tau_\lambda(a) = \langle a\delta_e \mid \delta_e \rangle$. This is known to be true for the group von Neumann algebra. Here, we show one direction of this claim is true, and mention counterexamples obtained by Le Boudec for the other direction.

Conjecture 6.1 (false)**.** The following are equivalent:

1. G is C*-simple.
2. G has the unique trace property.
3. The amenable radical of G is trivial.

We've already shown that (1) $\implies$ (3) (Corollary 5.8). Now we wish to show the equivalence of (2) and (3).

Proposition 6.2. *The kernel of the action $G \curvearrowright \partial_F G$ is equal to $R_a(G)$.*

Proof. For convenience, denote the kernel by K. It is clear that K is normal. It is also amenable using the fact that $K = \bigcap_{x \in \partial_F G} G_x$, and that each point-stabilizer G_x is amenable (Proposition 4.27). Thus, $K \subseteq R_a(G)$.

Conversely, we know by amenability of $R_a(G)$ that $R_a(G)$ admits a fixed measure $\mu \in P(\partial_F G)$ (Proposition 2.31). Further, given any $g \in G$ and $r \in R_a(G)$, we have

$$rg\mu = g\left(g^{-1} rg\right)\mu = g\mu,$$

so $R_a(G)$ fixes $G\mu$, and hence $\overline{G\mu}^{\mathrm{w}^*}$. But $\partial_F G$ is a G-boundary, and so it contains all of the Dirac masses δ_x. Using the fact that $g\delta_x = \delta_{gx}$ for any $g \in G$, we have that $R_a(G)$ fixes every $x \in \partial_F G$, and so $R_a(G) \subseteq K$. ■

Proposition 6.3. *Assume τ is a tracial state on $C_r^*(G)$. Then given any $s \in G$ with $s \notin R_a(G)$, we have $\tau(\lambda_s) = 0$.*

Proof. We endow $\mathbb{C}$ with the trivial action of G, and $C_r^*(G)$ with conjugation by the unitaries λ_g. Under these actions, every tracial state τ on $C_r^*(G)$ is G-equivariant, as

$$\tau(g \cdot a) = \tau\left(\lambda_g a \lambda_g^*\right) = \tau\left(\lambda_g^* \lambda_g a\right) = \tau(a) = g \cdot \tau(a).$$

Further, under the scalar embedding $\mathbb{C} \hookrightarrow C(\partial_F G)$, we may view $\tau : C_r^*(G) \to C(\partial_F G)$. By Proposition 4.35, this map extends to a G-equivariant unital positive (hence, completely positive) map $\psi : C(\partial_F G) \rtimes_r G \to C(\partial_F G)$. Again, just like in the proof of Theorem 4.21, (2) $\implies$ (1), it can be shown that $\psi(\lambda_s) f = (s \cdot f) \psi(\lambda_s)$ for any $s \in G$ and $f \in C(\partial_F G)$. If $s \notin R_a(G)$, then by Proposition 6.2, it does not act trivially on $\partial_F G$. Hence, there is some $x \in \partial_F G$ such that $x \neq s^{-1}x$, and so by Urysohn's lemma, there is some $f \in C(\partial_F G)$ with $f(x) = 1$ and $f(s^{-1}x) = 0$. Thus, $\psi(\lambda_s)(x) = 0$. But $\psi(\lambda_s) = \tau(\lambda_s)$ is a constant function, so $\psi(\lambda_s) = 0$. ∎

Lemma 6.4. *Assume H is a subgroup of G. Then $C_r^*(H)$ embeds canonically into $C_r^*(G)$, and there is a canonical conditional expectation $E_H : C_r^*(G) \to C_r^*(H)$ where $E_H(\lambda_g) = 0$ for $g \notin H$.*

Proof. Let $H \backslash G$ denote the right-coset space. Define the canonical embedding $\iota_H : \mathbb{C}[H] \subseteq C_r^*(H) \to C_r^*(G)$ by mapping $\sum_{g \in H} \alpha_g \lambda_g$ to itself. Now viewing every $\ell^2(Hr)$ $(Hr \in H \backslash G)$ as a subset of $\ell^2(G)$ by making the rest of the coordinates zero (not to be confused with the embedding given in Lemma 4.25), any $f \in \ell^2(G)$ decomposes as $f = \sum_{Hr \in H \backslash G} f_{Hr}$, where $f_{Hr} \in \ell^2(Hr)$. Consequently, given any $a \in \mathbb{C}[H]$, we have

$$\|\iota_H(a) f\|^2 = \sum_{Hr \in H \backslash G} \|\iota_H(a) f_{Hr}\|^2 \leq \|a\|^2_{C_r^*(H)} \sum_{Hr \in H \backslash G} \|f_{Hr}\|^2 = \|a\|^2_{C_r^*(H)} \|f\|^2.$$

This shows that ι_H is contractive. In fact, ι_H is isometric, which follows from the fact that, for all $f \in \ell^2(H) \subseteq \ell^2(G)$, we have $\|\iota_H(a) f\| = \|af\|$. Thus, ι_H extends to an isometry on $C_r^*(H)$, which we will also denote ι_H. It is clear that ι_H is also a *-homomorphism.

Now, we may define a *-homomorphism $E_H : \mathbb{C}[G] \subseteq C_r^*(G) \to C_r^*(H)$, determined by

$$E_H(\lambda_g) = \begin{cases} \lambda_g & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}.$$

Let $f \in \ell^2(H) \subseteq \ell^2(G)$, and $a = \sum_{g \in G} \alpha_g \lambda_g \in \mathbb{C}[G]$. We have

$$\|af\| = \left\| \sum_{g \in H} \alpha_g \lambda_g f + \sum_{g \in H^\complement} \alpha_g \lambda_g f \right\| \geq \left\| \sum_{g \in H} \alpha_g \lambda_g f \right\| = \|E_H(a) f\|.$$

Consequently,

$$\|E_H(a)\| = \sup_{\substack{f \in \ell^2(H) \\ \|f\| = 1}} \|E_H(a) f\| \leq \sup_{\substack{f \in \ell^2(H) \\ \|f\| = 1}} \|af\| \leq \sup_{\substack{f \in \ell^2(G) \\ \|f\| = 1}} \|af\| = \|a\|,$$

which shows E_H extends to a contractive linear map on $C_r^*(G)$, which we will also denote E_H. The fact that it is a projection is clear. Applying Tomiyama's theorem (Theorem 2.21), E_H is a conditional expectation onto $C_r^*(G)$. ∎

Proof of Conjecture 6.1, (2) $\iff$ (3). To show (3) $\implies$ (2), assume G has trivial amenable radical. Then given any tracial state τ, we have that τ and τ_λ both vanish on λ_s for $s \neq e$ using Proposition 6.3. Further, they both evaluate to 1 on λ_e, the identity of $C_r^*(G)$. As $\operatorname{span}\{\lambda_g \mid g \in G\}$ is dense in $C_r^*(G)$, then $\tau = \tau_\lambda$.

Conversely, to show (2) $\implies$ (3), assume $R_a(G) \neq \{e\}$, and denote $N := R_a(G)$ for convenience. Recall that amenability is equivalent to having the trivial representation 1_G weakly contained in the left-regular representation λ (Proposition 2.27). Consequently, by Theorem 2.4, we obtain a *-homomorphism $\tau_N : C_r^*(N) \to \mathbb{C}$ that maps every group element λ_n $(n \in N)$ to 1. It is clear that this is in fact a tracial state on $C_r^*(N)$. Letting $\tau = \tau_N \circ E_N$, where $E_N : C_r^*(G) \to C_r^*(N)$ is the canonical conditional expectation, we obtain a new state. It is again tracial, which can be seen from the fact that, for any $g, h \in G$, we have

$$\tau(\lambda_g \lambda_h) = \tau(\lambda_{gh}) = \begin{cases} 1 & \text{if } gh \in N \\ 0 & \text{if } gh \notin N \end{cases} = \begin{cases} 1 & \text{if } hg \in N \\ 0 & \text{if } hg \notin N \end{cases} = \tau(\lambda_{hg}) = \tau(\lambda_h \lambda_g).$$

(As N is normal, then $gh \in N \iff g^{-1}ghg \in N \iff hg \in N$). It is also not equal to τ_λ, as $\tau(\lambda_n) = 1$ for any $n \in N$, and $N \supsetneq \{e\}$. ∎

The only missing piece of the puzzle is (2) $\implies$ (1). This would seem unintuitive at first glance, as using Proposition 6.2, the only time G would not act freely on its Furstenberg boundary is when there is some $g \neq e$ in G acting as the identity map. Counterexamples to this claim were originally given by Le Boudec in [LB17], namely various subgroups of automorphism groups on trees.

7 Examples

Here, we use the results we have obtained to deduce whether various classes of groups are C*-simple or not. For some examples below, we give multiple proofs for better insight.

7.1 Amenable groups

First, we show that there is nothing deep about boundaries for amenable groups.

Proposition 7.1. *Assume G is amenable. Any G-boundary X must be a singleton.*

Proof. Let $\mu \in P(X)$ be a G-invariant measure. We know there exists a net $(g_\lambda) \subseteq G$ such that $g_\lambda \mu$ converges weak* to some Dirac mass. However, by G-invariance,

$g_\lambda \mu = \mu$ for all λ, and so μ itself must be a Dirac mass, say, $\mu = \delta_x$ for some $x \in X$. But then for any $g \in G$, we have

$$\delta_{gx} = g\delta_x = g\mu = \mu = \delta_x,$$

showing $gx = x$. By minimality, this is only possible if $X = \{x\}$. ■

Theorem 7.2. *Assume G is amenable. If G is nontrivial, i.e. $G \neq \{e\}$, then G is not C*-simple. If $G = \{e\}$, then G is C*-simple.*

Dynamical approach. With the above fact, it is clear that any amenable group G does NOT act (topologically) freely on any G-boundary, unless G is the trivial group. Hence, G is never C*-simple by Theorem 4.21, except in the trivial case (in which case, $C_r^*(\{e\}) \cong \mathbb{C}$). ■

Uniformly recurrent subgroup approach. Assume $G \neq \{e\}$. Then $\{G\}$ is a nontrivial, amenable, uniformly recurrent subgroup, showing G is not C*-simple by Theorem 5.7. If $G = \{e\}$, then $\{\{e\}\}$ (the trivial case) is the only uniformly recurrent subgroup, showing $\{e\}$ is C*-simple. ■

Recurrent subgroup approach. Assume $G \neq \{e\}$. Then G itself is an amenable recurrent subgroup, again showing G is not C*-simple by Theorem 5.7. If $G = \{e\}$, then no subgroups are recurrent, so $\{e\}$ is C*-simple. ■

7.2 Free groups

Seeing as the canonical example of a non-amenable group is $\mathbb{F}_2$, the free group on two generators, it is worth using it as an example.

Theorem 7.3. *The free group $\mathbb{F}_2$ is C*-simple.*

Dynamical approach. First, we construct a boundary for $\mathbb{F}_2$, the details of which can be found [Fur73] (originally done for the free group on r generators).

Let $\mathbb{F}_2 = \langle a, b\rangle$, and let $Y = \left\{a, a^{-1}, b, b^{-1}\right\}^{\mathbb{N}}$ (i.e. sequences in the semigroup-generators of $\mathbb{F}_2$), equipped with the product topology (compact by Tychonoff's theorem). Now let X be the sequences in Y such that no two adjacent terms are inverses of each other. In other words, X is the set of formal countable reduced products of our semigroup-generators. This can also be identified with the infinite paths in the Cayley graph of $\mathbb{F}_2$ which start at the identity. It is easy to see that X is a closed subset of Y, hence compact. There is a canonical action of $\mathbb{F}_2$ on X as follows: for any generator γ, let

$$\gamma\left(w_1, w_2, \dots\right) = \begin{cases} \left(\gamma, w_1, w_2, \dots\right) & \text{if } w_1 \neq \gamma^{-1} \\ \left(w_2, w_3, w_4, \dots\right) & \text{if } w_1 = \gamma^{-1} \end{cases}.$$

In other words, for general $g \in \mathbb{F}_2$, the action on any sequence is the following: write g as a product of semigroup-generators, concatenate the result to the start of our sequence, and reduce. It is not hard to see that the action of any $g \in \mathbb{F}_2$ is continuous. We can also check that X is a boundary for $\mathbb{F}_2$.

To show X is minimal, let $x \in X$ be arbitrary. For convenience, let $A := (a, a, a, \ldots)$ and $A^{-1} := (a^{-1}, a^{-1}, a^{-1}, \ldots)$, and define B and B^{-1} similarly. Note that $\lim_{n\to\infty} a^n x$ always exists, and is equal to either A or A^{-1} (with the latter happening only when $x = A^{-1}$). Regardless of what this limit is (call it z), we have that $\lim_{n\to\infty} b^n z = B$, and so $B \in \overline{Gx}$. Continuing with a similar argument, we see that all of A, A^{-1}, B, and B^{-1} lie in $\overline{Gx}$. Now let $y = (w_1, w_2, w_3, \ldots) \in X$ be arbitrary. Letting $y^{(k)} = w_1 \ldots w_k (w_k, w_k, w_k, \ldots)$, we see that $y^{(k)} \in \overline{Gx}$, as $\overline{Gx}$ is G-invariant. Further, the first k coordinates of $y^{(k)}$ match up with the first k coordinates of y, and so $y = \lim_{k\to\infty} y^{(k)} \in \overline{Gx}$. In other words, we always have $\overline{Gx} = X$, and so X is minimal.

To show X is strongly proximal, again note that for any $x \in X$, the sequence $(a^n x)$ either converges to A or A^{-1}. Consequently, given any $f \in C(X)$, we have that $(a^{-n} \cdot f)$ converges pointwise to $f(A)\, 1_{\{A^{-1}\}^{\complement}} + f(A^{-1})\, 1_{\{A^{-1}\}}$. Let $\mu \in P(X)$ be arbitrary. Applying the Lebesgue dominated convergence theorem (all functions here are bounded by $\|f\|_\infty \cdot 1$), we get

$$\int_X f\, d(a^n \cdot \mu) = \int_X a^{-n} \cdot f\, d\mu \to f(A)\left(1 - \mu\left(\left\{A^{-1}\right\}\right)\right) + f\left(A^{-1}\right)\mu\left(\left\{A^{-1}\right\}\right).$$

Hence, $a^n \cdot \mu \xrightarrow{\mathrm{w}^*} (1 - \mu(\{A^{-1}\}))\, \delta_A + \mu(\{A^{-1}\})\, \delta_{A^{-1}}$, so this latter measure (call it ν) lies in $\overline{G\mu}^{\mathrm{w}^*}$. Using this, and the fact that $B^n \nu \xrightarrow{\mathrm{w}^*} \delta_B$, we see that $\overline{G\mu}^{\mathrm{w}^*}$ always contains a Dirac mass, and so we are done.

We claim that the action of $\mathbb{F}_2$ on this boundary is topologically free. Indeed, let $g = v_1 \ldots v_n$ be a nontrivial element of $\mathbb{F}_2$, where each v_i is a semigroup-generator of $\mathbb{F}_2$, and no two adjacent terms are inverses of each other. Assume $x = (w_1, w_2, \ldots)$ is a fixed point of g. Note that we have

$$gx = (v_1, \ldots, v_k, w_{n-k+1}, w_{n-k+2}, \ldots)$$

for some $0 \le k \le n$. We conclude that

$$\begin{cases} v_n = w_1^{-1} \\ \ldots \\ v_{k+1} = w_{n-k}^{-1} \end{cases} \qquad \begin{cases} v_1 = w_1 \\ \ldots \\ v_k = w_k \end{cases} \qquad \begin{cases} w_{k+1} = w_{n-k+1} \\ w_{k+2} = w_{n-k+2} \\ \ldots \end{cases}.$$

We consider cases:

- Assume $k < n - k$. Then

$$\begin{aligned} x &= (w_1, \ldots, w_k, w_{k+1}, \ldots, w_{n-k}, w_{k+1}, \ldots, w_{n-k}, \ldots) \\ &= \left(v_1, \ldots, v_k, v_{n-k}^{-1}, \ldots, v_{k+1}^{-1}, v_{n-k}^{-1}, \ldots, v_{k+1}^{-1}, \ldots\right). \end{aligned}$$

- Assume $k = n-k$. Then $v_{k+1} = w_{n-k}^{-1} = w_k^{-1} = v_k^{-1}$, which contradicts minimality of the expression $v_1 \dots v_n$.
- Assume $k > n-k$. Then

$$\begin{aligned} x &= (w_1, \dots, w_{n-k}, w_{n-k+1}, \dots, w_k, w_{n-k+1}, \dots, w_k, \dots) \\ &= (v_1, \dots, v_k, v_{n-k+1}, \dots, v_k, v_{n-k+1}, \dots, v_k, \dots) . \end{aligned}$$

In particular, the value of x is entirely dependent on the value of k, and so g admits only finitely many fixed points. However, as X is a nontrivial boundary, it has no isolated points (Proposition 4.3). Indeed, this is also easy to check manually - if $x = (w_1, w_2, \dots) \in X$, then the sequence $\left(x^{(k)}\right)_{k=1}^{\infty} \subseteq X$, where $x^{(k)}$ is the same as x except for having a different k-th term, converges to x. Consequently, the set of fixed points of g must have empty interior, and so $\mathbb{F}_2$ acts topologically freely on this boundary. By Theorem 4.21, $\mathbb{F}_2$ is C*-simple. Note that this same argument works for $\mathbb{F}_r$ for any $r \geq 2$, even if r is an infinite cardinal. ■

Recurrent subgroup approach. Recall that the Nielsen-Schreier theorem says that every subgroup of a free group is free. Hence, the only amenable subgroups of $\mathbb{F}_2$ are $\{e\}$ and $\langle x \rangle$, for $x \neq e$. As $\{e\}$ is never recurrent, it suffices to show that $\langle x \rangle$ is never recurrent for $x \neq e$. Assume such a subgroup is recurrent, and let $F \subseteq G \setminus \{e\}$ be a finite set that always intersects any conjugate of $\langle x \rangle$. Let $\mathbb{F}_2 = \langle a, b \rangle$, and without loss of generality, assume x contains some b or b^{-1} term in its reduced word. Then letting g be a sufficiently large power of a, the reduced word length of gxg^{-1} can be made arbitrarily large. Hence, the minimum reduced word length in $\langle gxg^{-1} \rangle \setminus \{e\}$ can be made arbitrarily large (the reduced word length of $gx^n g^{-1} = \left(gxg^{-1}\right)^n$, $n \neq 0$, is always at least that of gxg^{-1}). But there is a maximum reduced word length in F, which is a contradiction. Thus, no amenable recurrent subgroups exist in $\mathbb{F}_2$, and so it is C*-simple by Theorem 5.7. Again, this argument is easily adapted to work for any $\mathbb{F}_r$, where r is any (potentially infinite) cardinal with $r \geq 2$. ■

There is also another argument, but it is very similar to the above argument based on recurrent subgroups. In summary, one could show that any nontrivial cyclic subgroup of $\mathbb{F}_2$ is never normal, and then use the fact that $\mathbb{F}_2$ falls into the class of groups discussed in Section 7.3. Note that this argument only extends to countable free groups, i.e. $\mathbb{F}_r$ for $r \in \mathbb{N}$, $r \geq 2$, and $\mathbb{F}_{\aleph_0}$.

7.3 Groups with countably many amenable subgroups

First, we require the following lemma on amenable subgroups with few conjugates. As this lemma is not the main focus of this example, many of the technicalities in this proof will themselves be given without proof.

Lemma 7.4. *Assume the subgroup $H \leq G$ is amenable and only has finitely many distinct conjugates gHg^{-1} ($g \in G$). Then the normal subgroup generated by H is also amenable.*

Proof. First, we show this is true in the case of H having finite order. In this case, H is contained in the FC-center of G (the set of all elements of G with finite conjugacy class). As the FC-center of G is an amenable normal subgroup, then H is amenable.

Now assume $|H|$ is arbitrary. We know by the orbit-stabilizer theorem that $N_G(H)$, the normalizer of H in G, is of finite index. Hence, $N_G(H)$ contains a finite-index subgroup N_0 that is normal in G. For $g \in G$, let $N_g := N_0 \cap gHg^{-1}$. As this is a subgroup of $gHg^{-1} \cong H$, we have that N_g is always amenable. Further, N_g is normal in N_0, as N_0 is normal in G, and given any $g \in G$ and $n \in N_0$, we have:

$$ngHg^{-1}n^{-1} = g\left(g^{-1}ng\right)H\left(g^{-1}ng\right)^{-1}g^{-1} = gHg^{-1}.$$

Now consider $N := \langle N_g \rangle_{g \in G}$. As a subgroup generated by finitely many amenable normal subgroups is amenable, then N is an amenable subgroup of N_0 (as H only has finitely many conjugates, then the set $\{N_g \mid g \in G\}$ only has finitely many elements). It is also easy to see that N is normal in G, as the set $\{N_g \mid g \in G\}$ is closed under conjugation.

Let $\langle H \rangle_{\text{normal}}$ denote the normal subgroup generated by H. As N is amenable, then to show $\langle H \rangle_{\text{normal}}$ is amenable, it suffices to show that

$$\langle H \rangle_{\text{normal}}/N = \langle HN/N \rangle_{\text{normal}} \leq G/N$$

is amenable. But HN/N is finite, as

$$[HN : N] = [H : N \cap H] \leq [H : N_e] = [H : N_0 \cap H] = [HN_0 : N_0] \leq [G : N_0] < \infty.$$

Applying the fact that this lemma holds for finite subgroups, we are done. ■

Theorem 7.5. *Assume G has only countably many amenable subgroups. Then G is C*-simple if and only if it has trivial amenable radical.*

Uniformly recurrent subgroup approach. First, we know that being C*-simple implies having trivial amenable radical (Corollary 5.8).

To show the converse, assume G is not C*-simple. By Theorem 5.7, G admits a nontrivial, amenable, uniformly recurrent subgroup Y. As $S_a(G)$ is countable, then so is Y. But then the Baire category theorem for compact spaces implies that Y must admit an isolated point, call it K_0 (otherwise, $Y = \bigcup_{K \in Y} \{K\}$ would be a countable union of nowhere-dense sets). By minimality, Y must be finite (Lemma 2.15). Lemma 7.4 then tells us that the normal subgroup generated by the elements of Y is amenable. This is not the trivial subgroup, as Y is nontrivial. Hence, the amenable radical of G is not trivial. ■

7.4 Tarski monster groups

Recall that, given a fixed prime number p, a Tarski monster group is an infinite group G with the property that every nontrivial subgroup is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (cyclic of order p). Such groups were shown to exist for $p > 10^{75}$ and to be non-amenable [Ols80b], providing the first counterexample to the von Neumann conjecture - which stated that a group is non-amenable if and only if it contains a copy of $\mathbb{F}_2$ as a subgroup.

Proposition 7.6. *Tarski monster groups are simple.*

Proof. Assume otherwise, so that there is a non-simple Tarski monster group G. Let N be a nontrivial normal subgroup, and consider the canonical projection $\pi : G \to G/N$. By the fourth (lattice) isomorphism theorem for groups, there is a bijective correspondence between subgroups of G containing N, and subgroups of G/N. But there are no nontrivial subgroups between N and G, and so G/N has no nontrivial subgroups. Thus, $G/N \cong \mathbb{Z}/q\mathbb{Z}$ for some prime q. As $|N| = p$ and $[G : N] = q$, we have that $|G| = pq < \infty$, a contradiction. ■

Theorem 7.7. *Tarski monster groups are C*-simple.*

Countably many amenable subgroups approach. First, note that any Tarski monster group must be countable. Otherwise, the subgroup generated by two elements not lying in the same cyclic subgroup would form a nontrivial subgroup not isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Consequently, there are only countably many amenable subgroups - the subgroups generated by any singleton (note that G itself is non-amenable). Further, as Tarski monster groups are simple and non-amenable, then the amenable radical is trivial. Thus, Tarski monster groups fall into the class of groups given in Section 7.3, and so they are C*-simple. ■

Alternatively, there is a very clean, manual proof of this fact in terms of only recurrent subgroups.

Recurrent subgroup approach. Clearly, $\{e\}$ is never recurrent, and G is not amenable, so it suffices to consider nontrivial subgroups $H \leq G$. It is an easy exercise in group theory to show that, given any group, two cyclic subgroups of order p are always either equal, or have trivial intersection. Hence, to show H is never recurrent, it suffices to show it always has infinitely many conjugates. Assume otherwise, so that it has only finitely many conjugates. The orbit-stabilizer theorem tells us that the stabilizer of H, i.e. $N_G(H)$, is infinite, and hence must be all of G. Thus, H is a nontrivial normal subgroup, which contradicts G being simple. Hence, G has no amenable recurrent subgroups, so applying Theorem 5.7, we get that G is C*-simple. ■

Of course, Tarski monster groups also satisfy the stronger version of the above intrinsic characterization:

Stronger intrinsic approach. As G is simple, and the only amenable subgroups are finite cyclic, then by Theorem 5.12, we are done. ■

7.5 Torsion-free Tarski monster groups

Similar to Section 7.4, we consider the following variant: infinite simple groups for which every nontrivial subgroup is isomorphic to $\mathbb{Z}$. It is easy to see that such groups are necessarily torsion-free. This is what we will refer to as torsion-free Tarski monster groups, and denote by G for this example. Such groups were shown to exist - see [Ols80a] or [Ols91, Theorem 28.3] - and to be non-amenable. The proof of C*-simplicity given in [BKKO17] is to show that such groups fall into the category given in Section 7.3. Note that the requirement that torsion-free Tarski monster groups be simple is not mentioned in [BKKO17], but this requirement is necessary (indeed, without it, $\mathbb{Z}$ falls into this class of groups).

Proposition 7.8. *Torsion-free Tarski monster groups are countable.*

Proof. Our aim is to show that G is generated by two elements. First, note that G cannot be abelian, as it is both infinite and simple. We will show that G is Noetherian, i.e. any ascending chain of subgroups $H_1 \leq H_2 \leq \dots$ eventually stabilizes. Assume otherwise, and let $\langle x_1 \rangle \lneqq \langle x_2 \rangle \lneqq \dots$ be an infinite, strictly ascending chain of subgroups (clearly, G itself can never be part of such a chain). Then $\bigcup_{n=1}^{\infty} \langle x_n \rangle$ is an infinite abelian subgroup, but it is not cyclic. This is a contradiction, as this subgroup must be G, but G is non-abelian. Consequently, there is a maximal cyclic subgroup $\langle x \rangle \leq G$. Choosing $y \notin \langle x \rangle$, we see that $\langle x, y \rangle \supsetneqq \langle x \rangle$ is a noncyclic subgroup of G, so $\langle x, y \rangle = G$. ■

Theorem 7.9. *Torsion-free Tarski monster groups are C*-simple.*

Countably many amenable subgroups approach. As G is countable, then there are only countably many cyclic subgroups, i.e. the subgroups $\langle g \rangle$ given for every $g \in G$ (the amenable subgroups). As G is simple and non-amenable, then it has trivial amenable radical, and so G falls into the class of groups given in Section 7.3. ■

7.6 Free Burnside groups

The free Burnside groups were originally constructed to answer the Burnside problem: if G is finitely generated with the property that there is some fixed $n \in N$ such that $g^n = e$ for all $g \in G$, must G be finite? The free Burnside group $B(m, n)$ is the "largest" group with m generators and exponent n satisfying the above property. More specifically, it can be defined as follows: let $\mathbb{F}_m$ be the free group on m generators, and let N be the (normal) subgroup of $\mathbb{F}_m$ generated by $\{x^n \mid x \in \mathbb{F}_m\}$. Note that this generating set is invariant under conjugation, so the subgroup it generates is automatically normal. Then $B(m, n) = \mathbb{F}_m / N$. As it turns out, this group can be finite or infinite, depending on m and n. Again, [BKKO17] shows $B(m, n)$ is

C*-simple for $m \geq 2$ and n odd and sufficiently large by showing it falls into the class of groups in Section 7.3, but the proof uses $B(m, n)$ being countable.

We will do slightly better. Note that $B(I, n)$ can be analogously defined for any arbitrary set I, allowing the set of generators to have an arbitrary cardinality. For notation, we will always let $\mathbb{F}_I$ be generated by $\{x_i\}_{i \in I}$, and denote $A_I :=$ $\{x^n \mid x \in \mathbb{F}_I\}$, $N_I := \langle A_I \rangle$ (automatically normal), and $B(I, n) := \mathbb{F}_I / N_I$. We will also let $\{x_i\}_{i \in I}$ denote the set of generators for $B(I, n)$. Our aim is to show the same result on C*-simplicity holds if we remove the restriction that m be a natural number. It is worth mentioning that the C*-simplicity of these free Burnside groups, including the infinite-rank ones, was originally proven by Olshanskii and Osin in [OO14].

Remark 7.10. The free Burnside group $B(I, n)$ is universal among groups G with generators indexed by I, and satisfying $x^n = e$ for all $x \in G$, i.e. such groups G are a homomorphic image of $B(I, n)$. In fact, if $\{y_i\}_{i \in I}$ is a set of generators for G, then

$$\begin{aligned} \varphi : \ B(I, n) &\to G \\ x_i \quad &\mapsto y_i \end{aligned}$$

defines a surjective group homomorphism.

The following proposition is intuitive, but its validity may not be immediately obvious at first.

Proposition 7.11. *Let $J \subseteq I$. Then $B(J, n)$ embeds canonically into $B(I, n)$ by mapping the generators of $B(J, n)$ to their corresponding generators in $B(I, n)$.*

Proof. Consider the subgroup $H_J \leq B(I, n)$ generated by $\{x_j \mid j \in J\}$. By the universal property of the free Burnside groups, there is a surjective group homomorphism $\varphi_1 : B(J, n) \to H_J$ sending x_j to x_j. Also by the universal property, there is a surjective group homomorphism $\varphi_2 : B(I, n) \to B(J, n)$, mapping x_j to x_j for $j \in J$, and x_i to e for $i \in I \setminus J$. Now, the following composition of maps must overall yield the identity map:

$$B(J, n) \xrightarrow{\varphi_1} H_J \leq B(I, n) \xrightarrow{\varphi_2} B(J, n).$$

This forces φ_1 to be injective. ■

This proposition lets us extend facts we know about $B(m, n)$ for finite m to arbitrary $B(I, n)$. The following facts were mentioned/used in [BKKO17]. Assume $m \geq 2$ and n is odd and sufficiently large.

1. The group $B(m, n)$ is non-amenable.

2. Any non-cyclic subgroup of $H \leq B(m,n)$ contains a copy of $B(2,n)$, which is non-amenable, and so H is non-amenable.

3. Any cyclic subgroup $\langle x \rangle \leq B(m,n)$, where $x \neq e$, is not normal. This was not explicitly mentioned, but it was needed in order to claim that the amenable radical of $B(m,n)$ is trivial.

Theorem 7.12. *Assume I is any (potentially infinite) set with $|I| \geq 2$, and n is odd and sufficiently large. The free Burnside group $B(I,n)$ is C*-simple.*

Stronger intrinsic approach. We wish to extend the above results to $B(I,n)$, bootstrapping off of the finitely-generated case.

Let $H \leq B(I,n)$ be a non-cyclic subgroup. Pick any $x \in H$ of maximum order (this exists, as the order of any element is bounded by n). As H is not cyclic, then $\langle x \rangle \subsetneqq H$. Now choose any $y \in H \setminus \langle x \rangle$. If $\langle x, y \rangle$ were cyclic, say, $\langle x, y \rangle = \langle z \rangle$, then $\langle x \rangle \subsetneqq \langle z \rangle$, and so $|z| > |x|$, contradicting maximality of $|x|$. Thus, $\langle x, y \rangle$ is not cyclic. Writing $x = x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$ and $y = x_{j_1}^{\beta_1} \dots x_{j_l}^{\beta_l}$ as some product of generators, and letting $F = \{i_1, \dots, i_k, j_1, \dots, j_l\}$ (finite, and not hard to see $|F| \geq 2$ by construction of x and y), we see that $\langle x, y \rangle \leq B(F,n)$. Hence, $\langle x, y \rangle$, and thus H, contains a copy of the non-amenable group $B(2,n)$. This shows H is non-amenable.

Now let $x \in B(I,n)$ be a nonidentity element, and consider the cyclic subgroup $\langle x \rangle \leq B(I,n)$. As x can be written as some product of generators $x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}$, then letting $F = \{i_1, \dots, i_k\}$ (finite), we have $x \in B(F,n) \subseteq B(I,n)$. If F is a singleton, we can always enlarge it so that $|F| \geq 2$. Consequently, as $\langle x \rangle$ is not normal in $B(F,n)$, then it is certainly not normal in $B(I,n)$.

As the only amenable subgroups of $B(I,n)$ are finite cyclic, and the only such subgroup that is normal is $\{e\}$, then by Theorem 5.12, we are done. ■

References

[Ake81] Charles A. Akemann. "Operator algebras associated with Fuchsian groups". In: *Houston Journal of Mathematics* 7.3 (1981), pp. 295–301.

[Arv76] William Arveson. *An Invitation to C*-Algebras.* Vol. 39. Graduate Texts in Mathematics. New York: Springer-Verlag, 1976.

[BC15] Erik Bédos and Roberto Conti. "Fourier series and twisted C*-crossed products". In: *Journal of Fourier Analysis and Applications* 21.1 (2015), pp. 32–75.

[BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's Property (T).* New Mathematical Monographs. Cambridge University Press, 2008.

[BKKO17] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. "C*-simplicity and the unique trace property for discrete groups". In: *Publications mathématiques de l'IHÉS* 126.1 (2017), pp. 35–71.

[BN88] Florin Boca and Viorel Niţică. "Combinatorial properties of groups and simple C*-algebras with a unique trace". In: *Journal of Operator Theory* 20.1 (1988), pp. 183–196.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C*-Algebras and Finite-Dimensional Approximations*. Vol. 88. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2008.

[Dav96] Kenneth R. Davidson. *C*-Algebras by Example*. Vol. 6. Fields Institute Monographs. Providence, Rhode Island: American Mathematical Society, 1996.

[Day57] Mahlon M. Day. "Amenable semigroups". In: *Illinois Journal of Mathematics* 1.4 (1957), pp. 509–544.

[DGO17] François Dahmani, Vincent Guirardel, and Denis Osin. *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*. Vol. 245. Memoirs of the American Mathematical Society. Providence, Rhode Island: American Mathematical Society, 2017.

[Fol15] Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Second. Vol. 29. Textbooks in Mathematics. Boca Raton London New York: CRC Press/Taylor & Francis, 2015.

[Fur63] Harry Furstenberg. "A Poisson formula for semi-simple Lie groups". In: *Annals of Mathematics* 77.2 (1963), pp. 335–386.

[Fur73] Harry Furstenberg. "Boundary theory and stochastic processes on homogeneous spaces". In: *Harmonic Analysis on Homogeneous Spaces*. Ed. by Calvin C. Moore. Vol. 26. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, Providence, Rhode Island, 1973, pp. 193–229.

[Gla76] Shmuel Glasner. *Proximal Flows*. Vol. 517. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 1976.

[Har85] Pierre de la Harpe. "Reduced C*-algebras of discrete groups which are simple with a unique trace". In: *Operator Algebras and their Connections with Topology and Ergodic Theory. Proceedings of the OATE Conference held in Buşteni, Romania, August 29 - September 9, 1983*. Ed. by Huzihiro Araki, Calvin C. Moore, Serban-Valentin Stratila, and Dan-Virgil Voiculescu. Vol. 1132. Lecture Notes in Mathematics. Springer-Verlag, Berlin Heidelberg, 1985, pp. 230–253.

[Ken15] Matthew Kennedy. "An intrinsic characterization of C*-simplicity". In: *ArXiv e-prints* (2015). arXiv: 1509.01870 [math.OA].

[KK17] Mehrdad Kalantar and Matthew Kennedy. "Boundaries of reduced C*-algebras of discrete groups". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2017.727 (2017), pp. 247–267.

[LB17] Adrien Le Boudec. "C*-simplicity and the amenable radical". In: *Inventiones mathematicae* 209.1 (2017), pp. 159–174.

[Ols80a] Alexander Y. Olshanskii. "An infinite simple Noetherian group without torsion". In: *Izvestiya: Mathematics* 15.3 (1980), pp. 531–588.

[Ols80b] Alexander Y. Olshanskii. "On the problem of the existence of an invariant mean on a group". In: *Russian Mathematical Surveys* 35.4 (1980), pp. 180–181.

[Ols91] Alexander Y. Olshanskii. *Geometry of Defining Relations in Groups*. Vol. 70. Mathematics and its Applications. Netherlands: Springer, 1991.

[OO14] Alexander Y. Olshanskii and Denis V. Osin. "C*-simple groups without free subgroups". In: *Groups, Geometry, and Dynamics* 8.3 (2014), pp. 933–983.

[Pau03] Vern Paulsen. *Completely Bounded Maps and Operator Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2003.

[Phe01] Robert R. Phelps. *Lectures on Choquet's Theorem*. Second. Vol. 1757. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 2001.

[Pit17] David R. Pitts. "Structure for Regular Inclusions. I". In: *Journal of Operator Theory* 78.2 (2017), pp. 357–416.

[Pow75] Robert T. Powers. "Simplicity of the C*-algebra associated with the free group on two generators". In: *Duke Mathematical Journal* 42.1 (1975), pp. 151–156.

[Poz08] Tal Poznansky. "Characterization of linear groups whose reduced C*-algebras are simple". In: *ArXiv e-prints* (2008). arXiv: 0812.2486 [math.GR].

[PT11] Jesse Peterson and Andreas Thom. "Group cocycles and the ring of affiliated operators". In: *Inventiones mathematicae* 185.3 (2011), pp. 561–592.

[Run05] Volker Runde. *A Taste of Topology*. Universitext. New York: Springer-Verlag, 2005.