Getting mathematics students to focus less on memorization and more on deep understanding

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GS 902 research paper

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1 Introduction

Before we begin, it is important to fully establish the question that we will be studying. Loosely speaking, there are two different mentalities one can when learning mathematics and solving mathematical problems. The first is *imitative*: viewing mathematics as facts to be memorized, and rigid algorithms to be applied to solve problems. The second is perhaps a bit harder to fully describe. It is the mentality of having a deeper understanding on why something is true, why something is defined the way it is, and why a given algorithm works. It is the realization that mathematics is not a collection of separate topics, and that one should form connections between old and new concepts. Most importantly, it is being able to combine what was previously learned to solve new types of problems never encountered before, and conjecture your own results. Because of this last description in particular, we will refer to this reasoning as *creative*. This is elaborated on in more detail in Section 2, where we give an overview of Lithner's framework.

It is recognized in [Hiebert, 2003] that "students learn what they have an opportunity to learn". This is elaborated on by noting that it is more complicated than just receiving information (for example, through lectures). Rather, if students are to effectively learn what you want them to learn, one must:

1. Take into account the student's existing background knowledge on the subject.

2. Provide them with relevant activities that they will engage in and will give the students an opportunity to develop the knowlege and skills that we want.

An good concrete instance of this second point at work can be observed in [Matić, 2015], where first-year engineering students were assigned several non-routine tasks in their calculus course, and eventually became more creative in their reasoning at the end of the course.

Given this, the next few sections address some of the main ways with which students studying mathematics are typically provided opportunities to learn. Section 3 discusses the design of assignments and assignment questions, and this is in my opinion the most important source of such an opportunity. Section 4 discusses textbooks, both from the perspective of the instructor using an already written textbook, and from the perspective of the author (who may be the instructor writing their own set of lecture notes). Section 5 briefly talks about threshold concepts in general, and structuring a course around them.

It is also worth briefly mentioning the intended education level of students that this is meant to apply to. Undergraduate students are a major focus, seeing as they both have a higher level of maturity when it comes to learning, and the undergraduate level is also when the material to be learned both starts significantly increasing in difficulty. However, I suspect that much of what will be discussed can apply to lower levels, such as secondary school students. Indeed, it may even be worthwhile if such students adopt a better learning mentality as early on as possible.

2 Creative and imitative reasoning and their consequences

This section aims to quickly give an overview of Lithner's framework [Lithner, 2008], which in short serves to characterize the various forms of reasoning one can use in terms of level of depth and understanding. We will use this as a rigorous basis with respect to which the rest of the discussion in this project will proceed.

We start with the more basic of the two: *imitative reasoning*. Lithner splits this off into two forms, the first of which he calls *memorized reasoning* (MR), which as the name suggests involves simply recalling an answer and writing it down. Perhaps more interesting is the second form: *algorithmic reasoning* (AR). This is characterized by choosing an algorithm to solve a problem, and simply following the procedure set out, with this in particular being trivial for the reasoner (for example, only having to perform some numerical computation as specified by the algorithm).

It is worth focusing on algorithmic reasoning a bit more. The reason for the particular choice of algorithm can vary. The type of task at hand can either be familiar to the reasoner and a suitable algorithm is recalled (*familiar AR*). Alternatively, the task may be unfamiliar and an algorithm is chosen and verified using only surface-level considerations (*delimited AR*), or external guidance is sought and implemented, either from some text or some person, but without any form of verificative argumentation (*guided AR*). However, in all cases above, the pattern is that the choice of algorithm is not based on any deep understanding of the task at hand on the part of the reader. Likewise, the execution is sparingly verified to make sure that either the computation, logic, or final answer actually make sense, and that the choice of algorithm is correct.

In contrast, we have *creative reasoning* (CR). According to Lithner, this is any reasoning that fulfills all of the following criteria (which, in my opinion, should always be viewed as tying into one another, as opposed to being entirely separate).

1. It should be novel, either by synthesizing new arguments altogether or recreating old ones.

2. The choice of strategy, along with its execution and final results, should have plausible arguments supporting them.

3. Any reasoning is based on intrinsic mathematical properties of the task at hand.

There are a few consequences to using algorithmic reasoning over creative reasoning. The first is that mistakes in either the choice or execution of algorithm arise more easily and are much more difficult to spot and correct. The second is that too much reliance on algorithmic reasoning, and in general rote learning without a deeper understanding of the material, leads to an inability to adapt and solve more complex problems (which I personally view as a required skill in many professions). These are observed at all levels of education, for example in [Hiebert, 2003], which deals with the K–12 level, and in an earlier empirical study of Lithner [Lithner, 2000] with first-year undergraduate students.

This is not to say that rote learning does not have any merits. For example, consider [van Merriënboer and Sweller, 2005]. Cognitive load theory assumes that humans have a limited working memory and cannot process more than a few pieces of completely novel information. Conversely, information can be organized and committed to long-term memory, and eventually retrieval and application can be come automated (algorithmic), which in turn can free up working memory and allow it to focus on other tasks. Consider, for example, being able to automatically simplify $a^b \cdot a^c$ as part of a larger computation without needing to explicitly think about any of the details of exponentiation.

However, I do not view any of this to be incompatible with the previously mentioned goals of creative reasoning. Firstly, over-reliance on imitative reasoning still has the same problems mentioned earlier. Secondly, cognitive load theory simply mentions that for effective learning to occur, schemata must be constructed and committed to long-term memory to be used later in effective problem solving. This is done through repeated practice, which I wholly agree is important in learning mathematics. It does not, however, mention *what* those schemata necessarily must be, and in my opinion, they can and should involve as many connections between topics and as much inherent understanding in the material as possible.

I would also like to personally speculate on a few more benefits to creative reasoning. The first is that the mathematics that one learns becomes increasingly more complex over time, and builds on what was previously learned. Choosing to memorize and have a shallow understanding of everything is simply less efficient in the long run as a result, due to the increased difficulty of recalling previous material, and also the difficulty of being able to anchor the new material in it. Second, knowing the theory behind something both reinforces the concept and serves as a fallback in case some of the details are forgotten. For example, consider purely memorizing the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for the derivative of a function f at a point x. This can be easy to mess up if one is clueless to what is going on. But having at least has the slightly hand-wavy intuition that this is just the slope of the tangent line, computed by approximating using points getting closer and closer to x, this formula is suddenly much easier to remember or re-derive (no pun intended) if necessary.

3 Optimizing design of assignment questions

Perhaps this may sound like a more traditional viewpoint, but the (correct) use of homework assignments is, in my opinion, the most effective way to teach mathematics. I wish to elaborate on this. Mathematics, especially higher level mathematics, is full of subtle technicalities and unintuitive (to the student first encountering the material) concepts. It takes time to fully digest what was learned. More importantly, it takes practice, and working hands-on with the material in many different contexts, to fully form the cognitive connections that go beyond simple memorization of facts. As such, having an environment in which one can take as much time as they need to complete meaningful and well-thought-out problems that give students the experience and opportunities they need is one of the best ways to go about this.

Motivating students to take homework assignments seriously is a separate topic on its own, and one which will not be discussed much here. However, I will again give my own personal opinion on this. One of the best motivations to complete homework assignments is having them be worth a certain percentage of your final grade. It is worth noting that we are still viewing assignments as mostly formative, even if they have a grade attached to them. To mitigate the impact of cheating (especially among lower-year university courses), it is important to make sure that the weight of the assignments is not set too high. This is not to say that students should not be allowed to collaborate. It can still be an effective form of learning, especially if students are encouraged to write out their final solutions individually and make sure they truly understand them. Moreover, collaboration as a skill is again an important one to have in many professions.

With the above out of the way, let us focus on what this section aims to discuss. We will give several different types of assignment questions that one can ask that go beyond the typical algorithmic computation one usually encounters, and instead forsters the desired skills that were mentioned earlier.

Many good types of questions and examples are presented in [Cardetti and LeMay, 2019]. This paper focuses on mathematical argumentation at the undergraduate level, viewing it as highly effective for having students develop higher-level mathematical thinking. With this in mind, they present five different categories of tasks that allow students to develop their argumentation skills in tandem with focusing on developing a deep conceptual knowledge. Each category is summarized and briefly discussed below:

1. *Making sense of procedures.* In a way, this is perhaps the most straightforward way to combat the problem of over-reliance on imitative reasoning on the part of the students. Design questions that ask the students to answer *why* they chose a specific approach to solve the a given problem at hand. Or instead, get them to solve it through multiple approaches and compare the approaches. Ask them whether an approach is feasible in different contexts.

2. *Analyzing misconceptions.* Some interesting takes are provided here. One option is to create problems where after solving it normally, students are asked where someone else solving the problem might make a mistake, why it might arise, and how it might affect the rest of the problem. Alternatively, give them an incorrect solution ahead of time, and get them to find the mistake and elaborate on it. Taking this further, it would be interesting if it would be feasible to structure an entire assignment like this: give a student an assignment that is already completed, and have them "grade" it on their own. Continuing with some personal experience of mine, counterexamples give another good class of problems. For example, tell the student a common misconception

(ex: $\lim_i(\lim_j x_{i,j}) = \lim_j(\lim_i x_{i,j})$), and have them come up with a counterexample to prove that this is false. The reasoning required to come up with them is often extremely novel and creative, and involves a lot of reflection on precise definitions, intuition, and previous examples. As such, they are often some of the hardest problems in mathematics (especially at the research level!), but are also an excellent learning tool.

3. *Tying concepts together.* I view this as quite similar to category 5, which involves having multiple representations on the same material. As such, I won't comment much on it here. But briefly, it emphasizes that forming connections between the different concepts that one learns (for example, between the various algebraic and geometric notions and interpretations that one learns) allows students to think from multiple perspectives, and promotes more advanced mathematical thinking. The example questions given involves having the students experiment with drawing functions with negative derivative on its domain (ideally, should have specified on an open interval, as non-connected domains may not quite behave as expected), (hopefully) having the students observe that all such functions necessarily are decreasing, and finally having them rigorously prove this observation using the mean value theorem.

4. *Connections to prior knowledge.* Similar to the previous and next categories, the mathematics that one learns almost always builds upon what was previously learned, and it is important to not neglect this fact. A good example of this (from personal experience, but I believe this is the case in most universities) is the typical progression of a pure mathematics student, taking some calculus courses in first and second year, and afterwards taking some real analysis courses in third year. In a sense, much of what is learned in the real analysis courses simply generalizes the calculus content (for example, the notions limits, continuity, etc...), and as a result many of the concepts and even proofs presented will be quite similar to what one has seen before. The example

questions presented in this paper start off by having the student recall the desired prior knowledge, and then eventually guiding them to proving something entirely new on their own with this. In the context of real analysis, a good question in this same spirit could be to remind the students the result from calculus that given two continuous functions $f, g : (a, b) \to \mathbb{R}$, we have that $f + g$, αf ($\alpha \in \mathbb{R}$), and fg are all still continuous. The question could then encourage the students to look back at the proofs they learned in these early courses, and have the students prove for themselves that the same is true when replacing (a, b) with an arbitrary metric space X. This would also give them the practice they need with identifying what is the same and what is different in this new context.

5. *Connections between representations.* Having multiple perspectives on the same material is always advantageous, and this is something I wholeheartedly agree with. One reason is that not every perspective and representation is equally intuitive when applied to different contexts. A basic example question is given in the aforementioned paper with the function "$y = \ln x$", where the question starts out by giving this function (algebraic), along with a graph and a table of some values. It then proceeds to ask various questions with an emphasis on asking the student to describe the various strengths and weaknesses of each representation, such as in the context of finding the domain of the function, or discussing differentiability.

It is also worthwhile to investigate the kinds of assignment questions that are *currently* being used in various courses, to see what is currently being done and where improvements can be made. Of course, this is a broad question that depends heavily on several factors, such as the course topic, year, the types of students taking it (ex: pure math vs. business students), the quality/status of the university, and even the particular instructor teaching the course. As such, a comprehensive study of this would be quite difficult, and we just present a summary of [Mac an Bhaird et al., 2017], which offers its own very brief literature at the start, and afterwards presents its own findings on three calculus courses at various degrees of specialization across two Irish universities, from the perspective of Lithner's framework discussed earlier.

In terms of both their literature review, they conclude that the majority of questions throughout various calculus courses in the UK, the US, and Sweden, both on assignments and exams, could be solved using imitative reasoning alone. Percentages varied, but it was often roughly on the order of 70-85%. Their own study showed a similar trend in a course for business students and another for science students (non-specialists), especially among the summative assessments used. The exception to this trend was a course for pure mathematics students, in which more than half of the practice questions and the graded assignment questions required creative reasoning of various depth, and the final exam still had a nontrivial portion of such questions. This is, in a sense, to be expected (know your audience, and one would expect pure mathematics students to have the deepest intended learning outcomes). However, they then go on to say that even the students in the non-specialist courses can still benefit from higher-order mathematical thinking, and therefore the low portion of creative reasoning tasks in their respective questions is not necessarily ideal. I personally agree. This is largely due to what was discussed in Section 2, much of which is not restricted to mathematics, but applies instead to many fields and professions.

4 Optimizing design and use of textbooks

Here, we investigate how students use the textbook they are assigned in a given course, and from this information, suggest improvements that can be made in the assignment or even structure of a textbook. Even with the latter, the information to be discussed is perhaps more useful to an instructor than it may seem at first glance, as it is not uncommon to have the instructor write his own set of lecture notes altogether to serve as a textbook.

Our discussion will primarily be based upon [Weinberg et al., 2012], which itself states that there is currently not much literature on this matter at the undergraduate level for mathematics students. Their literature review, however, suggests that students largely use textbooks through imitative reasoning, with a common strategy simply being using it to solve exercises by finding superficial similarities between the problem they are trying to solve and previous exercises/content in the textbook. Moreover, they also give other sources in the literature stating that this general problem may not be exclusive to mathematics students, but occurs in other disciplines as well.

The results obtained by the authors in this paper roughly coincide with what was previously mentioned. It was found that the majority of students do not use the chapter introductions/summaries when reading the textbooks (less than 30% would do this for each). Interestingly, only 63.3% would even read the chapter text. Conversely, the most popular component of textbooks was the examples scattered throughout, with 89.4% of students using them. Students were also surveyed to see what aspects of textbooks they value highly, and it was found that students valued having lots of examples as a tool to understand the material (77.5%) more than they valued the textbook explaining the big ideas in the course (66.4%) and explaining the underlying concepts of problems (68%). Moreover, the most valued aspect was highlighting important definitions and equations (80.3%).

From this, the authors infer that students are "looking for algorithms and shortcuts", especially in the context of solving homework problems or studying for exams (which were the most popular use cases for the examples). In terms of Lithner's framework, this is again largely imitative reasoning as described earlier. One of the downsides to this style of textbook usage is that the chapter introduction/text/summary are often what are used to build context, motivation, and (very importantly) intuition for the material that is to be learned.

In terms of remedying the situation, the authors suggest that instructors can clearly communicate to the students how they are expected to be using the textbook. In addition, another effective solution is to simply assign more problems that require creative reasoning and a deeper understanding on the part of the students. Just extracting meaning from previous examples becomes less plausible.

I wish to suggest a few other improvements that can be made. Again, as mentioned before, it is not such a rare occurrence for instructors to write their own set of lecture notes in lieu of a traditional textbook. This naturally gives freedom and flexibility in design and structure. As such, it may be worthwhile to repeat/emphasize content (whether it is material, context, or intuition) from the main body of the notes in the examples. For instance, let's take an introductory course on integration. Near the start of the course, integration is rigorously defined via limits of Riemann sums. The first few examples could include some graphs visualizing the function $f : [a, b] \to \mathbb{R}$ to be integrated, along with some other graphs also showing the rectangles (corresponding to some finite partition(s) of $[a, b]$) that approximate the area under f. Various approximations can be computed for a few partitions getting finer and finer, and hinting that they converge to an (ideally very nice) fixed value. A rigorous formula can then be established for certain partitions, and via taking limits, the actual area can finally be obtained. This natural progression goes through all of the intended cognitive processes—recalling the intention of Riemann integration (computing the area under graphs), having an (albeit slightly handwavy) intuition as to how we go about this (approximating the are with rectangles, with thinner rectangles giving better approximations), and finally making this rigorous through limits.

The only potential caveat I see with this style of examples is the question of how clear to make the essential vs the non-essential components of this example. Much of this, as mentioned earlier, was for reinforcing the concepts at hand. However, on a homework assignment or exam, for a question of the form "Compute the area underneath $f(x) = x^2$ on $[0, 1]$ using the definition of Riemann integration", a rigorous and complete solution would only consist of the very last bits with computing the Riemann sums for some general partitions (ex: fixed width $\frac{1}{n}$) and taking a limit. None of the aforementioned graphs and example partition computations would need to show up. However, too clearly labelling parts of the solution as "non-essential" may risk the students skipping those parts altogether. Hence, this may need some careful planning.

5 Emphasis on threshold concepts

In line with avoiding students' tendancy to memorize without understanding and reason imitatively, it is suggested in [Breen and O'Shea, 2016] that emphasis on certain *threshold concepts* in courses can be quite beneficial. These are concepts that satisfy five key properties:

1. They are *transformative*, in the sense that finally understanding them will result in a perspective shift on the topic at hand.
2. They are *irreversible*, in the sense that they are very unlikely to be forgotten.
3. They are *integrative*, connecting together different aspects of a subject.
4. Similarly, they are also *bounded*, by lying on the boundary between disciplinary areas. (Speaking as a mathematical analyst, this is a terrible choice of adjective, suggesting the exact opposite of what is intended. *Bridging* would have been better).
5. They are *troublesome*, often being conceptually difficult and taking a while to truly digest and understand.

In summary, these are concepts that are central to the topic at hand and future topics, but often make no intuitive sense at first and may take a while until they finally "click". I agree with the existence of such concepts from personal experience, and a few of the same topics that I struggled with as an undergraduate are given as examples in the above paper.

For instance, the ε-δ definition of a limit is often difficult for first-year students, usually being the first piece of rigorously defined mathematics that extends beyond a hand-wavy mental image. Understanding how to parse the definition involves understanding how to parse formal logic, including the $\forall$ and $\exists$ quantifiers (and the fact that order is important for these!) Using the definition is essentially their first introduction to proofs as well, rather than just computation where formulas and algorithms can be applied.

It is suggested that too much focus on algorithms and reproduction of large amounts of knowledge leads to shallow and imitative reasoning in students. Instead, courses should be structured to give enough attention to threshold concepts, given their central role in the subject and future topics to be learned. This can be accomplished through revisiting these concepts several times in a course, importantly from multiple perspectives (emphasizing forming a deeper understanding). In addition, they also state the importance of listening to students for feedback on what they are struggling with, reacting accordingly, and also letting students know that they are not alone in their difficulties, and that these concepts are genuinely not easy and take a little time. It is also important to keep in mind, as the instructor, that these threshold concepts might seem trivial to us now, forgetting the struggles we went through during undergrad. As such, careful attention should be paid to identify these concepts and to not gloss over them.

6 Conclusion

In this project, we investigated various methods of getting students to develop a more creative and less imitative approach to doing mathematics. This was largely based on the notion that students learn from the opportunities to learn that we give them. As such, we explored various assignment questions, textbook usage and structure, and emphasis on threshold concepts, that encourage or require the students to take a more creative approach to reasoning and form a deeper understanding of the material they are learning. Perhaps worth noting, much of what was discussed should not be too difficult to implement in existing courses. It does not involve changing the general structure of a course (such as in the case of flipped classrooms), but rather just involves changing the content in various places instead.

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