POSITIONAL NUMERAL SYSTEMS OVER POLYADIC RINGS

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arXiv:2506.12930v1 [math.NT] 15 Jun 2025

ABSTRACT. We construct positional numeral systems that work natively over nonderived polyadic (m,n)-rings whose addition takes m arguments and multiplication takes n. In such rings, the length of an admissible additive word and a multiplicative tower are not arbitrary (as in the binary case), but "quantized". Our main contributions are the following. Existence: every commutative (m,n)-ring admits a base-p place-value expansion that respects the word length constraint in terms of numbers of operation compositions $\ell_{mult} = \ell_{add}(m-1)+1$. Lower bound: the minimum number of digits is greater than or equal to the arity of addition m. Representability gap: for $m,n \geqslant 3$ only a proper subset of ring elements possess finite expansions, characterized by congruence-class arity shape invariants $I^{(m)}$ and $J^{(n)}$. Mixed-base "polyadic clocks": allowing a different base at each position enlarges the design space quadratically in the digit count. Catalogues: explicit tables for the integer rings $\mathbb{Z}_{4,3}$ and $\mathbb{Z}_{6,5}$ illustrate how ordinary integers lift to distinct polyadic variables. These results lay the groundwork for faster arity-aware arithmetic, exotic coding schemes, and hardware that exploits operations beyond the binary pair.

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Date: *of start* May 25, 2025. *Date*: *of completion* June 13, 2025.
Total: 11 references.
2010 *Mathematics Subject Classification.* 11-03, 11A07, 11A67, 20N15.
Key words and phrases. positional number system, arity, polyadic structure, polyadic number, congruence class, numeral, polyadic ring, mixed-base, querelement.

1. Introduction

Classical positional numeral systems (for a review see, e.g. Conway and Guy [1996], Ifrah [2000]) answer the question: how to encode quantities efficiently? The linked question, which discusses how to operate on those encodings, has the narrow answer: the admissible operations are binary, that is addition and multiplication decompose into chains of pairwise interactions. Nowadays in group theory, abstract algebra, coding theory, ultra-metric analysis and physics n-ary (or polyadic) operations, those that swallow n inputs at once, crop up with growing frequency (for a review, see Duplij [2022] and refs therein), nevertheless their foundational connections to numeral systems have remained largely unexplored (with the exception of Duplij [2017]). This paper fills that gap and bridges two seemingly disparate domains: the positional numeral systems that represent integers Menninger and Broneer [1971], Fomin [1974] and polyadic rings that generalize ring theory to n-ary operations Leeson and Butson [1980].

The main novel and unexpected phenomena which were uncovered in this paper are the following:

Double quantization of word length. As a starting point is the rich body of work on polyadic algebraic structures, we observe that in an n-ary magma the fundamental operation μ_n combines n elements at a time, such that closeness and total associativity now "quantizes" the admissible word lengths to the arithmetic progression (opposed to the binary case, where word length is arbitrary). Adding a second m-operation ν_m yields an (m,n)-ring, provided the two interact through a generalized distributive law and obey "double quantization" of word length. The polyadic rings enjoy exotic features: multiple identities, two kinds of querelements replacing inverses (of addition and multiplication), and the possibility that zero or one may fail to exist at all.

Numeral polyadization. We introduce a novel systematic recipe that translates the classical place-value expansion into a special genuine composition of m-ary additions and n-ary multiplications which reduces to the standard positional numeral presentation in the binary $(2,2)$ case. The key feature of polyadic numeral presentation is the unforeseen fact that the number of summands and the number of multiplicative factors are no longer independent, but lock together through the arities, forcing numeral strings to respect a built-in "polyadic uncertainty principle".

Numeral representability is no longer guaranteed. Unlike the binary case ($m = n = 2$), not every element of an (m,n)-ring admits a numeral expansion. We arrive to the theorem claiming that for $m,n \geqslant 3$, the set of numbers with a finite positional expansion becomes a strict subset (characterized by the shape invariants $I^{(m)}$ and $J^{(n)}$) of the (m,n)-ring, unless $m = n = 2$.

Mixed-base and "polyadic clock" systems. Generalizing time-of-day notation, we allow each digit position to carry its own n-ary tower of bases, yielding a combinatorial explosion of admissible encodings and opening the door to cryptographic and coding-theoretic applications, as well as turn coin-change, currency, and time-keeping problems to higher arity.

In general, polyadic numeral systems promise new tools wherever data are naturally grouped in arity-specific chunks: multi-qubit gates in quantum circuits, multiary neural activations, and even higher-order interaction terms in field theory.

The structure of the paper is as follows. Section 2 recaps the essentials of polyadic one set algebraic structures with an emphasis on the quantization of admissible word lengths and the various exotic element types (querelements, polyadic powers and idempotents, etc.). Section 3 develops the theory of (m,n)-rings built from integer representatives of congruence classes, culminating in the arity-shape map that dictates which classes admit the polyadic ring structure. Section 4 our main contribution constructs positional numeral systems over the (m,n)-rings, proves the minimal-digit theorem, and works through explicit examples in the $(4,3)$ and $(6,5)$ settings. We close with a catalogue of open problems, ranging from polyadic floating-point arithmetic to potential links with higher-categories.

2. PRELIMINARIES

For self-consistency, we briefly recall polyadic structure notation and general properties of polyadic rings (for details, see DUPLIJ [2022]).

Let X be a non-empty set and $X^{\times n}$ be its Cartesian product consisting of elements $(x_1, \ldots, x_n)$ called polyads (or n-tuples) and denoted by $(\mathbf{x}) \in X^{\times n}$, and an n-tuple with equal elements is (x^n). On $X^{\times n}$ the polyadic (n-ary operation) is defined by $\boldsymbol{\mu}_n : X^{\times n} \longrightarrow X$ and is denoted as $\boldsymbol{\mu}_n[\mathbf{x}]$. A polyadic structure $\langle X \mid \boldsymbol{\mu}_{n_i} \rangle$ is the set X that is closed with respect to polyadic operations $\boldsymbol{\mu}_{n_i}$ (see, e.g., KUROSH [1963], COHN [1965]).

The basic one-operation polyadic structure is called a (n-ary) magma $\mathcal{M} = \langle X \mid \boldsymbol{\mu}_n \rangle$. The (totally) commutative n-ary magma $\mathcal{M} = \langle X \mid \boldsymbol{\mu}_n \mid comm \rangle$ is defined by $\boldsymbol{\mu}_n = \boldsymbol{\mu}_n \circ \sigma$, where σ is an arbitrary n-permutation. Additional requirements lead to various structures called group-like ones. For instance, polyadic associative magma is an n-ary semigroup $\mathcal{S}_n = \langle X \mid \boldsymbol{\mu}_n \mid assoc \rangle$. The polyadic associativity can be defined as the invariance DUPLIJ [2018]

$$\boldsymbol{\mu}_n[\mathbf{x}, \boldsymbol{\mu}_n[\mathbf{y}], \mathbf{z}] = invariant, \tag{2.1}$$

where the internal multiplication can be on any of $n-1$ places (giving $n-1$ relations) and the polyads $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are of needed size, the total number of elements in (2.1) is fixed $2n-1$ (cf. with the binary associativity $(x \cdot y) \cdot z = x \cdot (y \cdot z)$). This allows us to omit internal bracket in compositions and define the iterated multiplication

$$\boldsymbol{\mu}_n^{\circ \ell_\mu}[\mathbf{x}] = \overbrace{\boldsymbol{\mu}_n[\boldsymbol{\mu}_n[\ldots \boldsymbol{\mu}_n[\mathbf{x}]]]}^{\ell_\mu}, \quad \mathbf{x} \in X^{\ell_\mu(n-1)+1}, \tag{2.2}$$

where ℓ_μ is the "number" of operations (now multiplications). From (2.2) it follows the main and crucial peculiarity of polyadic structures which differs them from ordinary binary ones ($n = 2$): the length of a word $w_\mu(n)$ inside composition of an n-ary multiplications is not arbitrary, but it is "quantized", obeying the following admissible values

$$w_\mu^{admiss}(n) = \ell_\mu(n-1) + 1, \tag{2.3}$$

which means that we can multiply $w_\mu^{admiss}(n)$ elements only.

Besides, one can treat the l.h.s. of (2.3) as another operation (the iterated multiplication) having higher arity

$$\bar{\boldsymbol{\mu}}_{\ell_\mu(n-1)+1} = \boldsymbol{\mu}_n^{\circ \ell_\mu}[\mathbf{x}]. \tag{2.4}$$

This allows us to divide all polyadic operations into two classes: iterated from lower arity operations and noniterated, or derived and nonderived. Such division can also depend on the underlying set.

For simplest example, the ternary operation $\boldsymbol{\mu}_3[x, y, z] = xyz$ (product of three integers from $\mathbb{Z}$) is derived for positive integers $X = \mathbb{Z}^+$ (because $\boldsymbol{\mu}_3[x, y, z] = \boldsymbol{\mu}_2[\boldsymbol{\mu}_2[x, y], z] = (x \cdot y) \cdot z$), but $\boldsymbol{\mu}_3$ is nonderived for negative integers $X = \mathbb{Z}^-$, because the product of two negative integers is positive, and so the binary operation $\boldsymbol{\mu}_2$ is not closed, i.e. the iteration does not exist. The ternary associativity is obvious in both cases, and therefore the polyadic structures $\mathcal{S}_3^+ = \langle \mathbb{Z}^+ \mid \boldsymbol{\mu}_3 \mid assoc \rangle$ and $\mathcal{S}_3^- = \langle \mathbb{Z}^- \mid \boldsymbol{\mu}_3 \mid assoc \rangle$ are ternary semigroups, nevertheless only the second one $\mathcal{S}_3^-$ is nonderived. Below we will consider barely the nonderived polyadic operations and the corresponding polyadic structures. Recall definitions of important elements in polyadic structures which will be used later on.

For each element $x \in X$ we can construct its ℓ_μ-polyadic power by

$$x^{\langle \ell_\mu \rangle} = \boldsymbol{\mu}_n^{\circ \ell_\mu}\left[x^{\ell_\mu(n-1)+1}\right], \tag{2.5}$$

and in the binary case $n = 2$ the polyadic power differs by 1 from the ordinary power $x^{\langle \ell_\mu \rangle} = x^{\ell_\mu + 1}$.

A polyadic idempotent x_{id} (if exists) is

$$x_{id}^{\langle \ell_\mu \rangle} = x_{id}, \quad x_{id} \in X. \tag{2.6}$$

A polyadic zero is defined unambiguously by the $n-1$ relations

$$\boldsymbol{\mu}_n [z, \mathbf{x}] = z, \quad \mathbf{x} \in X^{n-1}, \tag{2.7}$$

where z can be on any of $n-1$ places. A polyadic nilpotent element x_{nil} is defined by one relation

$$x_{nil}^{\langle \ell_\mu \rangle} = z, \quad x_{nil} \in X. \tag{2.8}$$

The neutral $(n-1)$-polyad $\mathbf{e}$ (in general non-unique) is given by

$$\boldsymbol{\mu}_n [x, \mathbf{e}] = x, \quad \mathbf{e} \in X^{n-1}. \tag{2.9}$$

If all elements in the neutral polyad are equal $\mathbf{e} = e^{n-1}$, then

$$\boldsymbol{\mu}_n \left[x, e^{n-1}\right] = x, \tag{2.10}$$

and e can be on any of $n-1$ places and is called an identity of $\langle X \mid \boldsymbol{\mu}_n \rangle$.

If the identity e exists, we can define polyadic ℓ_μ-reflection by

$$x_{refl}^{\langle \ell_\mu \rangle} = e, \quad x_{refl} \in X. \tag{2.11}$$

As follows from (2.7) with $\mathbf{x} = z^{n-1}$ and (2.10) with $x = e$, polyadic zero z and identity e are idempotents satisfying (2.6). There are exotic polyadic structures without idempotents, zero, identity at all or with several identities.

For instance, the above ternary semigroup of negative integers $\mathcal{S}_3^-$ contains two neutral sequences $\mathbf{e}_+ = (+1, +1)$, $\mathbf{e}_- = (-1, -1)$ and two ternary identities $e_+ = +1$ and $e_- = -1$.

The invertibility in the polyadic case ($n \geqslant 3$) is not connected with identity (2.10), but is guided by the polyadic analogue of inverse, the querelement $\bar{x} = \bar{x}(x)$ defined by the $n-1$ relations DÖRNTE [1929]

$$\boldsymbol{\mu}_n \left[\bar{x}, x^{n-1}\right] = x, \quad x \in X, \tag{2.12}$$

which should be satisfied with $\bar{x}$ be on all of $n-1$ places, and such x is called polyadically invertible. The polyadic solvability is defined as the unique solution for x_0 with given $\mathbf{u}, y$ of the $n-1$ equations

$$\boldsymbol{\mu}_n [x_0, \mathbf{u}] = y, \quad x_0, y \in X, \quad \mathbf{u} \in X^{n-1} \tag{2.13}$$

with x_0 be on any of all $n-1$ places. If each element of an n-ary semigroup $\mathcal{S}_n$ is polyadically invertible, or equivalently, (2.13) is solvable on any place, then $\mathcal{S}_n$ becomes an n-ary (polyadic) group $\mathcal{G}_n = \langle X \mid \boldsymbol{\mu}_n \mid assoc \mid solv \rangle$. The existence of identity is not necessary for polyadic groups.

For example, the imaginary fractions ix/y with $x, y \in \mathbb{Z}^{odd}$ ($i^2 = -1$, operations are in $\mathbb{C}$) form a nonderived ternary group $\mathcal{G}_3^{odd}$ without identity, each element is polyadically invertible and has its unique querelement, that is $\overline{(ix/y)} = -iy/x$.

3. POLYADIC RINGS

A set with two polyadic operations belongs to the so-called ring-like polyadic structures (see, e.g., LEESON AND BUTSON [1980], DUPLIJ [2022] and refs. therein).

A polyadic or (m,n)-ring $\mathcal{R}_{m,n} = \langle X \mid \boldsymbol{\nu}_m, \boldsymbol{\mu}_n \rangle$ is a nonempty set X with m-ary addition $\boldsymbol{\nu}_m : X^m \to X$ and n-ary multiplication $\boldsymbol{\mu}_n : X^n \to X$, such that additively $\langle X \mid \boldsymbol{\nu}_m \mid assoc \mid comm \mid solv \rangle$ is an m-ary commutative group, and multiplicatively $\langle X \mid \boldsymbol{\mu}_n \mid assoc \rangle$ is a n-ary semigroup, while the operations

$\boldsymbol{\nu}_m$ and $\boldsymbol{\mu}_n$ are not arbitrary, but connected by the following distributivity property having n relations

$$\boldsymbol{\mu}_n\left[\boldsymbol{\nu}_m\left[x_1,\ldots x_m\right],y_2,y_3,\ldots y_n\right]$$
$$=\boldsymbol{\nu}_m\left[\boldsymbol{\mu}_n\left[x_1,y_2,y_3,\ldots y_n\right],\boldsymbol{\mu}_n\left[x_2,y_2,y_3,\ldots y_n\right],\ldots\boldsymbol{\mu}_n\left[x_m,y_2,y_3,\ldots y_n\right]\right] \quad (3.1)$$
$$\boldsymbol{\mu}_n\left[y_1,\boldsymbol{\nu}_m\left[x_1,\ldots x_m\right],y_3,\ldots y_n\right]$$
$$=\boldsymbol{\nu}_m\left[\boldsymbol{\mu}_n\left[y_1,x_1,y_3,\ldots y_n\right],\boldsymbol{\mu}_n\left[y_1,x_2,y_3,\ldots y_n\right],\ldots\boldsymbol{\mu}_n\left[y_1,x_m,y_3,\ldots y_n\right]\right] \quad (3.2)$$
$$\vdots$$
$$\boldsymbol{\mu}_n\left[y_1,y_2,\ldots y_{n-1},\boldsymbol{\nu}_m\left[x_1,\ldots x_m\right]\right]$$
$$=\boldsymbol{\nu}_m\left[\boldsymbol{\mu}_n\left[y_1,y_2,\ldots y_{n-1},x_1\right],\boldsymbol{\mu}_n\left[y_1,y_2,\ldots y_{n-1},x_2\right],\ldots\boldsymbol{\mu}_n\left[y_1,y_2,\ldots y_{n-1},x_m\right]\right], \quad (3.3)$$

where $x_i,y_j\in X$, $i=1,\ldots,m$, $j=1,\ldots,n$.

If not all relations of distributivity (3.1)–(3.3), as well as associativity (2.1) are not satisfied, they are called partial (as opposed to the total ones), which gives enormous different versions of polyadic rings.

For instance, the exotic $(3,2)$-ring over rationals $\mathscr{R}_{3,2}=\langle\mathbb{R}\mid\boldsymbol{\nu}_3,\boldsymbol{\mu}_2\rangle$ with the derived ternary addition $\boldsymbol{\nu}_3\left[x,y,z\right]=x\cdot y\cdot z$ and the binary multiplication $\boldsymbol{\mu}_2\left[x,y\right]=x^y$ is noncommutative, because $x^y\neq y^x$, nonassociative, since $\left(x^y\right)^z\neq x^{\left(y^z\right)}$, and partial (left) distributive, because only the first relation (3.1) holds $\left(xyz\right)^t=x^ty^tz^t$, but $t^{xyz}\neq t^xt^yt^z$, $x,y,z,t\in\mathbb{R}$.

The simplest ternary ring is the commutative, totally associative and totally distributive $(2,3)$-ring of negative integer numbers $\mathscr{R}_{2,3}^{(-)}=\langle\mathbb{Z}^-\mid\boldsymbol{\nu}_2,\boldsymbol{\mu}_3\rangle$ with the derived ternary addition $\boldsymbol{\nu}_2\left[x,y\right]=x+y$ and the nonderived ternary multiplication $\boldsymbol{\mu}_2\left[x,y\right]=x\cdot y\cdot z$ (the binary product $x\cdot y$ gives result out of the underlying set $\mathbb{Z}^-$ that is in positive integers $\mathbb{Z}^+$).

More general polyadic number rings which are formed by representatives of congruence (residue) classes were introduced in DUPLIJ [2017, 2019]. Recall that a congruence class of a non-negative integer $a\in\mathbb{Z}^+$, modulo natural $b\in\mathbb{N}$, is defined by

$$\left[\left[a\right]\right]_b=\left\{x_k^{[a,b]}\mid k\in\mathbb{Z}\right\}, \quad (3.4)$$

where

$$x_k^{[a,b]}=a+bk, \quad (3.5)$$
$$0\leqslant a\leqslant b-1, \quad (3.6)$$

is a representative of the class being infinite. The set of representatives $X^{[a,b]}\equiv\left\{x_k^{[a,b]}\right\}$ (as elements of the congruence class $\left[\left[a\right]\right]_b$) was never considered before in details, because it does not form any binary structure, while the "internal" operations are not simultaneously closed $x_{k_1}^{[a,b]}+x_{k_2}^{[a,b]}\notin X^{[a,b]}$, $x_{k_1}^{[a,b]}\cdot x_{k_2}^{[a,b]}\notin X^{[a,b]}$ for arbitrary a, b (obviously, $X^{[01]}\in\mathbb{Z}$ are ordinary binary integers). This was a reason, why only operations between classes (the binary class addition $+'$ and binary class multiplication $\cdot'$) were considered, we call them "external" operations DUPLIJ [2017].

Nevertheless, for special values of the congruence class parameters a and b polyadic (nonbinary) operations on $X^{[a,b]}$ can be defined, and they are closed DUPLIJ [2017]. Indeed, the closeness of m additions and n multiplications in $\mathbb{Z}$

$$x_{k_1}^{[a,b]}+\ldots+x_{k_m}^{[a,b]}\in X^{[a,b]}\subset\mathbb{Z}, \quad (3.7)$$
$$x_{k_1}^{[a,b]}\cdot\ldots\cdot x_{k_n}^{[a,b]}\in X^{[a,b]}\subset\mathbb{Z},\qquad k_i,m,n\in\mathbb{N}, \quad (3.8)$$

hold valid, if the following "quantization" conditions

$$ma' \equiv a' \,(\mathrm{mod}\, b') \Longleftrightarrow \frac{(m-1)\,a'}{b'} = I^{(m)}\,(a', b') \equiv I = \text{integer}, \tag{3.9}$$

$$a'^{n} \equiv a' \,(\mathrm{mod}\, b') \Longleftrightarrow \frac{a'^{n} - a'}{b'} = J^{(n)}\,(a', b') = J = \text{integer}, \tag{3.10}$$

are satisfied with the special values of parameters as solutions $\{a'\} \subset \{a\}$, $\{b'\} \subset \{b\}$, $X^{[a',b']} \subset X^{[a,b]}$. The corresponding mapping is called the arity shape

$$\Psi^{[a'b']}_{m,n} : (a', b') \longrightarrow (m, n)\,, \tag{3.11}$$

and it is presented for lowest values in TABLE 1 of DUPLIJ [2019], however $\Psi^{[a'b']}_{m,n}$ cannot be presented by a formula. The arity shape mapping (3.11) is injective and non-surjective (empty cells in TABLE 1): for the congruence classes $[[2]]_4$, $[[2]]_8$, $[[3]]_9$, $[[4]]_8$, $[[6]]_8$ and $[[6]]_9$ there are no solutions of the "quantization" conditions (3.9)–(3.10), while, e.g., $m = 5$, $n = 6$ corresponds to different congruence classes $[[2]]_5$, $[[3]]_5$, $[[2]]_{10}$, and $[[8]]_{10}$.

The closeness (3.7)–(3.8) and the "quantization" conditions (3.9)–(3.10) allow us to define two abstract polyadic operations on the set of the abstract elements $\mathsf{X}^{[a',b']} = \left\{\mathsf{x}^{[a',b']}_k\right\} \equiv \{\mathsf{x}_k\}$ reflecting the representatives of the fixed congruence class $[[a']]_{b'}$: m-ary addition and n-ary multiplication

$$\boldsymbol{\nu}_m \left[\mathsf{x}_{k_1}, \ldots, \mathsf{x}_{k_m}\right] = \mathsf{x}_{k_1} + \ldots + \mathsf{x}_{k_m} = \mathsf{x}_{k_0}, \tag{3.12}$$

$$\boldsymbol{\mu}_n \left[\mathsf{x}_{r_1}, \ldots, \mathsf{x}_{r_n}\right] = \mathsf{x}_{r_1} \cdot \ldots \cdot \mathsf{x}_{r_n} = \mathsf{x}_{r_0}, \tag{3.13}$$

where

$$k_0 = k_1 + \ldots + k_m + I^{(m)}\,(a', b')\,, \tag{3.14}$$

$$r_0 = s\,(r_i, a', b') + J^{(n)}\,(a', b')\,, \tag{3.15}$$

where the invariants $I^{(m)}\,(a', b')$ and $J^{(n)}\,(a', b')$ are in (3.9)–(3.10), and the integer $s\,(r_i, a', b')$ is a special polynomial of r_i, a', b', which follows from the presentation (3.5). For instance, in the ternary case $n = 3$ we have $s\,(r_1, r_2, r_3, a', b') = a'^2 r_1 + a'^2 r_2 + a'^2 r_3 + a'b'r_1r_2 + a'b'r_1r_3 + a'b'r_2r_3 + b'^2 r_1r_2r_3$.

Because the abstract polyadic operations $\boldsymbol{\nu}_m$ and $\boldsymbol{\mu}_n$ are closed, commutative, totally associative and totally distributive (two latter ones follow from the binary associativity and distributivity), we can define the abstract commutative polyadic (m, n)-ring of integers

$$\mathbf{Z}^{[a',b']}_{m,n} = \left\langle \mathsf{X}^{[a',b']} \mid \boldsymbol{\nu}_m, \boldsymbol{\mu}_n \mid comm \right\rangle. \tag{3.16}$$

It follows from (2.3), that we have "double quantization": in the (m, n)-ring we can add

$$w^{admiss}_{\nu}\,(m) = \ell_{\nu}\,(n-1) + 1 \tag{3.17}$$

elements (ℓ_{ν} is quantity of composed m-ary additions) and multiply

$$w^{admiss}_{\mu}\,(n) = \ell_{\mu}\,(n-1) + 1 \tag{3.18}$$

elements, where ℓ_{μ} is quantity of composed n-ary multiplications, to be in the same underlying set $\mathsf{X}^{[a',b']}$. It is obvious, that $(2, 2)$-ring $\mathbf{Z}^{[0,1]}_{2,2} = \mathbb{Z}$ is the binary ring of ordinary integers, and $w^{admiss}_{\nu}\,(m) = \ell_{\nu}$, $w^{admiss}_{\mu}\,(n) = \ell_{\mu}$ are any natural numbers $\mathbb{N}$ (without any "quantization").

For example, in the polyadic $(8, 7)$-ring $\mathbf{Z}_{8,7} = \mathbf{Z}^{[5,7]}_{8,7}$ we can add only $7\ell_{\nu} + 1 = 8, 15, 22 \ldots$ elements x_i and multiply $6\ell_{\mu} + 1 = 7, 13, 19 \ldots$ ones.

The elements of the (m,n)-ring $\mathbf{Z}_{m,n} = \mathbf{Z}_{m,n}^{[a',b']}$ (3.16) are abstract variables x_k obeying the m-ary addition (3.12) and n-ary multiplication (3.13) which inherit the "internal" operations (3.7)–(3.8) in the congruence class $[[a']]_{b'}$. Therefore, the elements of (m,n)-ring $\mathbf{Z}_{m,n}$ should carry the arities as additional indices to distinguish elements corresponding to the same representative (as decimal number) of the initial congruence classes.

For instance, consider two infinite zeroless and unitless nonderived abstract polyadic rings

$$\mathbf{Z}_{4,3} = \{\ldots -13, -10, -7_{4,3}, -4, -1, 2, 5, 8_{4,3}, 11, 14, 17\ldots\}, \tag{3.19}$$

$$\mathbf{Z}_{6,5} = \{\ldots -22, -17, -12, -7_{6,5}, -2, 3, 8_{6,5}, 13, 18, 23, 28\ldots\}, \tag{3.20}$$

which are constructed from the congruence classes $[[2]_3]$ and $[[3]]_5$, respectively. The intersection of the generating classes for $|k| \leqslant 5$ consists of two integer numbers $U_{[3,5]}^{[2,3]} = [[2]_3] \cap [[3]]_5 = \{-7, 8\}$ (in general, $U_{[3,5]}^{[2,3]}$ this is an infinite set). We treat the elements of the rings (3.19)–(3.20) not as ordinary decimal numbers, but as abstract variables (or abstract symbols, as, e.g., the letters A, B, C in the hexadecimal numeral system) $\mathsf{x}_k^{[2,3]}$ and $\mathsf{x}_k^{[3,5]}$ obeying (3.12)–(3.15) and carrying additional arity lower indices (which are written manifestly in the needed cases only). In this way, we conclude that the elements corresponding to the same numbers in the class intersection $U_{[3,5]}^{[2,3]}$ are different $(-7_{4,3}) \neq (-7_{4,3})$, $8_{4,3} \neq 8_{6,5}$, in the sense that they obey different operations and their arities in distinct polyadic rings $\mathbf{Z}_{4,3}$ and $\mathbf{Z}_{6,5}$. Indeed, e.g., $8_{4,3} = \mathsf{x}_2^{[2,3]} \in \mathbf{Z}_{4,3}$ and $8_{6,5} = \mathsf{x}_1^{[3,5]} \in \mathbf{Z}_{6,5}$, and their first polyadic powers (2.5) are different $(8_{4,3})^{\langle 1\rangle} = \boldsymbol{\mu}_3\left[(8_{4,3})^3\right] = 512_{4,3} = \mathsf{x}_{102}^{[2,3]}$, while $(8_{6,5})^{\langle 1\rangle} = \boldsymbol{\mu}_5\left[(8_{6,5})^5\right] = 32768_{6,5} = \mathsf{x}_{6553}^{[3,5]}$.

4. POLYADIZATION OF POSITIONAL NUMERAL SYSTEMS

Let us remind the standard positional numeral system, as the presentation of a number by a special sequence of its digits (for numerous extended versions and history, see, e.g., IFRAH [2000]).

In the manifest form, the presentation of a number over the (binary) ring of non-negative integers $\mathbb{Z}^+ = 0, 1, 2\ldots$ is defined by the base (radix) $p \in \mathbb{Z}^+$ and the digits $y(i) \in \mathbb{Z}^+$ as follows (place-value notation)

$$N^{(\ell)}(p) = \sum_{i=0}^{\ell-1} y(i)\, p^i = y(\ell-1)\, p^{\ell-1} + y(\ell-2)\, p^{\ell-2} + \ldots + y(1)\, p + y(0)$$
$$\Longrightarrow \left(y(\ell-1)\, y(\ell-2) \ldots y(1)\, y(0)\right)_p^{(\ell)}, \tag{4.1}$$
$$0 \leqslant y(i) \leqslant p-1, \tag{4.2}$$

where the natural $\ell = l(p) \in \mathbb{N}$ is the quantity of digits (function of the base p), being simultaneously the amount of terms in the l.h.s., $\ell - 1$ (in the binary case only) coincides with the quantity of additions and the total amount of multiplications in the first term, and usually $p^0 = 1$ (to have the same summation formula with $i = 0$). Commonly, the leading zeroes in the r.h.s. of (4.1) are omitted. The change of the base $p \mapsto p'$ leads to another presentations of the same number, usually with the different quantity of digits $\ell \mapsto \ell' = l(p')$

$$N^{(\ell')}(p') = \sum_{i=0}^{\ell'-1} y'(i)\, p'^i \Longrightarrow \left(y'(\ell-1)\, y'(\ell-2) \ldots y'(1)\, y'(0)\right)_{p'}^{(\ell')}, \quad 0 \leqslant y'(i) \leqslant p'-1. \tag{4.3}$$

In the binary case, using ℓ digits one can describe $E(p) = p^\ell$ numbers: $0, \ldots p^\ell - 1 \in \mathbb{Z}^+$. The efficiency of a numeral system is the ability to represent as many numbers as possible using the smallest total number of symbols s. In this case, the number of digits becomes s/p, and the quantity of the described numbers (the efficiency function) is

$$E(p) = p^{\frac{s}{p}}. \tag{4.4}$$

The function $E(p)$ reaches its maximum (in p), when $p = e \approx 2.718...$ (the Euler's number), and therefore the most efficient numeral system (with integer p) is $p_{\max} = 3$.

For further details about positional numeral systems (over the binary ring $\mathbb{Z}^+$), see, e.g., IFRAH [2000], and refs. therein.

Now we generalize the above construction to the polyadic integer numbers DUPLIJ [2017, 2019], that is we build an analog of positional numeral system over polyadic (m,n)-rings $\mathbf{Z}_{m,n}$ considered in the previous section. The polyadic analog of binary numeral systems have complicated and nontrivial structure, because the underlying (m,n)-rings obey unusual peculiarities, for instance, some of them can not have zero or/and unity, absence of the natural ordering, etc. (see, for details, DUPLIJ [2022]).

First, we express the binary positional numeral system (presentation (4.1)) in the polyadic notation from SECTION 2. Indeed, let the binary ring of integers is $\mathbb{Z} = \langle \nu, \mu \mid comm \mid assoc \rangle$, where ν and μ are the ordinary binary addition $\nu[x,y] = x + y$ and multiplication $\mu[x,y] = y \cdot y$, $x, y \in \mathbb{Z}$. In this notation, each term in (4.1) can be written as the composition of $i+1$ multiplications (see (2.2)) $\mu^{\circ(i+1)}\left[y(i), \overbrace{p, \ldots, p}^{i}\right] \equiv \mu^{\circ(i+1)}\left[y(i), p^i\right]$, the first term becomes $\mu^{\circ(\ell_\mu - 1)}\left[y(\ell_\mu - 1), p^{\ell_\mu - 1}\right]$, and the sum as the composition of ℓ_ν additions, such that the place-value presentation takes the form of operation compositions

$$N^{(\ell_\nu,\ell_\mu)}(p) = \nu^{\circ\ell_\nu}\left[\overbrace{\mu^{\circ(\ell_\mu-1)}\left[y(\ell_\mu-1), p^{\ell_\mu-1}\right], \mu^{\circ(\ell_\mu-1)}\left[y(\ell_\mu-2), p^{\ell_\mu-2}\right], \ldots, \mu[y(1), p], y(0)}^{\ell_\nu+1}\right]$$

$$\Longrightarrow \left(\overbrace{y(\ell_\mu-1)\, y(\ell_\mu-2) \ldots y(1)\, y(0)}^{\ell_\nu+1}\right)_p^{(\ell_v,\ell_\mu)}, \quad 0 \leqslant y(i) \leqslant p-1 \in \mathbb{Z}^+, \tag{4.5}$$

where the important additional consistency condition

$$\ell_\mu = \ell_\nu + 1 = \ell, \tag{4.6}$$

which follows from the construction itself. Note that in the composition form we do not write multiplier for the last degit $y(0)$ at all for consistency with the higher arity generalizations (not all multiplicative parts of polyadic rings contain unity, see SECTION 3, however we need the last digit in any case).

For example, in the case $p = 7$ and $\ell = \ell_\mu = \ell_\nu + 1 = 3$ we have the ordinary binary positional numeral presentation in the polyadic notation

$$(165)_7^{(2,2)} = \nu^{\circ 2}\left[\mu^{\circ 2}[1,7,7], \mu[6,7], 5\right] = 1 \cdot 7 \cdot 7 + 6 \cdot 7 + 5 = (96)_{10}^{(1,1)}. \tag{4.7}$$

Now the polyadization of the binary place-value presentation in the composition form (4.5) can be done in the straightforward way. We assume that all ordinary numbers become polyadic numbers, that is the initial number ring should be exchange to the polyadic (m,n)-ring: $\mathbb{Z} = \langle \nu, \mu \rangle \longrightarrow \mathbf{Z}_{m,n} = \langle \boldsymbol{\nu}_m, \boldsymbol{\mu}_n \rangle$. Thus, modifying (4.5) consistently by $\nu \to \boldsymbol{\nu}_m$ and $\mu \to \boldsymbol{\mu}_n$, taking into account the admissible length of words (3.17)-(3.18), we propose the direct polyadic generalization of the standard place-value presentation

(4.1) of a polyadic number $\boldsymbol{N}\in\mathbf{Z}_{m,n}$ by the polyadic digits $\boldsymbol{y}(i)\in\mathbf{Z}_{m,n}$ and the polyadic base $\boldsymbol{p}\in\mathbf{Z}_{m,n}$

$$\boldsymbol{N}_{m,n}^{(\ell_\nu,\ell_\mu)}(\boldsymbol{p})=\boldsymbol{\nu}_m^{\circ\ell_\nu}\Bigg[\overbrace{\boldsymbol{\mu}_n^{\circ(\ell_\mu-1)}\left[\boldsymbol{y}(\ell_\mu-1),\boldsymbol{p}^{\ell_\mu(n-1)}\right],\boldsymbol{\mu}_n^{\circ(\ell_\mu-1)}\left[\boldsymbol{y}(\ell_\mu-2),\boldsymbol{p}^{\ell_\mu(n-1)-1}\right],\ldots,\boldsymbol{\mu}_n\left[\boldsymbol{y}(1),\boldsymbol{p}^{n-1}\right],\boldsymbol{y}(0)}^{\ell_\nu(m-1)+1}\Bigg]$$
$$\Longrightarrow\left(\overbrace{\boldsymbol{y}(\ell_\mu-1)\,\boldsymbol{y}(\ell_\mu-2)\ldots\boldsymbol{y}(1)\,\boldsymbol{y}(0)}^{\ell_\nu(m-1)+1}\right)_{m,n;\boldsymbol{p}}^{(\ell_v,\ell_\mu)}, \tag{4.8}$$

where the arity indices m,n, when they are obvious, can be omitted for conciseness. The polyadic analog of the consistency condition (4.6) now is

$$\ell_\mu=\ell_\nu(m-1)+1, \tag{4.9}$$

and therefore the minimal quantity of multiplications in (4.8) is $\ell_\mu\geqslant m$, which coincides with minimal amount of digits in the place-value presentation over (m,n)-ring.

Theorem 4.1. *In the polyadic numeral system over (m,n)-ring the minimal number of digits is more or equal than the arity of addition m.*

The m-ary addition of polyadic numbers $\boldsymbol{N}$ in the place-value presentation (4.8) can be done using the total polyadic distributivity (3.1)–(3.3) in the general form by adding digits (as in the binary case, prime sign is not a derivative, but numerates variables)

$$\boldsymbol{\nu}_m\left[\boldsymbol{N}_{m,n}^{\prime(\ell_\nu,\ell_\mu)}(\boldsymbol{p}),\boldsymbol{N}_{m,n}^{\prime\prime(\ell_\nu,\ell_\mu)}(\boldsymbol{p}),\ldots,\boldsymbol{N}_{m,n}^{\prime\prime\prime(\ell_\nu,\ell_\mu)}(\boldsymbol{p})\right]=\boldsymbol{N}_{m,n}^{(\ell_\nu,\ell_\mu)}(\boldsymbol{p}), \tag{4.10}$$
$$\boldsymbol{\nu}_m\left[\boldsymbol{y}'(i),\boldsymbol{y}''(i),\ldots,\boldsymbol{y}'''(i)\right]=\boldsymbol{y}(i),\quad i=0,\ldots,\ell_\mu. \tag{4.11}$$

The n-ary multiplication of polyadic numbers $\boldsymbol{N}$ is more complicated and should be made in each concrete case manifestly.

In general, the direct polyadization formula (4.8) for place-value presentation can be considered for any commutative polyadic (m,n)-ring.

Here we will study some examples for polyadic rings of integer numbers from the previous SECTION 3. Let us consider the abstract polyadic number ring $\mathbf{Z}_{4,3}^{[2,3]}$ (3.19) generated by the congruence class $[[2]]_3$. The polyadic integer numbers $\mathsf{x}_k^{[2,3]}$ are "symmetric" with respect to $\mathsf{x}_{k=0}^{[2,3]}=a=2$ (playing the role of zero in the binary case $a=0$, $b=1$ and $x_k=k$), therefore instead of (3.6), we use

$$0\leqslant k_{y(i)}\leqslant k_p-1, \tag{4.12}$$

where $\boldsymbol{y}(i)=\mathsf{x}_{k_{y(i)}}$, $\boldsymbol{p}=\mathsf{x}_{k_p}$, and $\mathsf{x}_k=\mathsf{x}_k^{[2,3]}\in\mathsf{X}^{[2,3]}$. In this way, for the simplest polyadic base $\boldsymbol{p}=\mathsf{x}_2=8$, $\ell_\nu=1$ and $\ell_\mu=4$ (see (4.9)) we have the 4-digit ($\boldsymbol{y}(i)=2,5$, $i=0,1,2,3$) polyadic numeral presentation for a number from $\mathbf{Z}_{4,3}^{[2,3]}$ as

$$\boldsymbol{N}_{4,3}^{(1,3)}(8)=\boldsymbol{\nu}_4\left[\boldsymbol{\mu}_3^{\circ3}\left[\boldsymbol{y}(3),8,8,8,8,8,8\right],\boldsymbol{\mu}_3^{\circ2}\left[\boldsymbol{y}(2),8,8,8,8\right],\boldsymbol{\mu}_3\left[\boldsymbol{y}(1),8,8\right],\boldsymbol{y}(0)\right]$$
$$\Longrightarrow\left(\boldsymbol{y}(3)\,\boldsymbol{y}(2)\,\boldsymbol{y}(1)\,\boldsymbol{y}(0)\right)_{4,3;8}^{(1,3)}\equiv\left(\boldsymbol{y}(3)\,\boldsymbol{y}(2)\,\boldsymbol{y}(1)\,\boldsymbol{y}(0)\right)_8. \tag{4.13}$$

The polyadic numerals (polyadic numbers of the (m,n)-ring $\mathbf{Z}_{4,3}^{[2,3]}$ that can be presented in base-8 place-value form by 4 digits) are

$$\begin{gathered}(2,2,2,2)_8 = 532610 = \mathsf{x}_{177536}, (2,2,2,5)_8 = 532613 = \mathsf{x}_{177537}, (2,2,5,2)_8 = 532802 = \mathsf{x}_{177600},\\ (2,2,5,5)_8 = 532805 = \mathsf{x}_{177601}, (2,5,2,2)_8 = 544898 = \mathsf{x}_{181632}, (2,5,2,5)_8 = 544901 = \mathsf{x}_{181633},\\ (2,5,5,2)_8 = 545090 = \mathsf{x}_{181696}, (2,5,5,5)_8 = 545093 = \mathsf{x}_{181697}, (5,2,2,2)_8 = 1319042 = \mathsf{x}_{439680},\\ (5,2,2,5)_8 = 1319045 = \mathsf{x}_{439681}, (5,2,5,2)_8 = 1319234 = \mathsf{x}_{439744}, (5,2,5,5)_8 = 1319237 = \mathsf{x}_{439745},\\ (5,5,2,2)_8 = 1331330 = \mathsf{x}_{443776}, (5,5,2,5)_8 = 1331333 = \mathsf{x}_{443777}, (5,5,5,2)_8 = 1331522 = \mathsf{x}_{443840},\\ (5,5,5,5)_8 = 1331525 = \mathsf{x}_{443841}, \qquad (4.14)\end{gathered}$$

which correspond to the ordinary base-2 (or binary) numerals with two digits $y(i) = 0, 1$.

The more complicated case is the 7-digit ($\boldsymbol{y}(i) = 2, 5, 8, 11$, $i = 0, \ldots, 7$) polyadic numeral presentation for numbers from $\mathbf{Z}_{4,3}^{[2,3]}$ with the polyadic base (see (3.19)) $\boldsymbol{p} = \mathsf{x}_4 = 14$, composition of two m-ary additions $\ell_\nu = 2$ and seven n-ary multiplications $\ell_\mu = 7$ (see (4.9))

$$\begin{gathered}\boldsymbol{N}_{4,3}^{(1,3)}(14) = \boldsymbol{\nu}_4^{\circ 2}\left[\boldsymbol{\mu}_3^{\circ 6}\left[\boldsymbol{y}(6), 14^{12}\right], \boldsymbol{\mu}_3^{\circ 5}\left[\boldsymbol{y}(5), 14^{10}\right], \boldsymbol{\mu}_3^{\circ 4}\left[\boldsymbol{y}(4), 14^{8}\right], \boldsymbol{\mu}_3^{\circ 3}\left[\boldsymbol{y}(3), 14^{6}\right],\right.\\ \left.\boldsymbol{\mu}_3^{\circ 2}\left[\boldsymbol{y}(2), 14, 14, 14, 14\right], \boldsymbol{\mu}_3\left[\boldsymbol{y}(1), 14, 14\right], \boldsymbol{y}(0)\right]\\ \Longrightarrow (\boldsymbol{y}(6)\,\boldsymbol{y}(5)\,\boldsymbol{y}(4)\,\boldsymbol{y}(3)\,\boldsymbol{y}(2)\,\boldsymbol{y}(1)\,\boldsymbol{y}(0))_{4,3;14}^{(1,3)} \equiv (\boldsymbol{y}(6)\,\boldsymbol{y}(5)\,\boldsymbol{y}(4)\,\boldsymbol{y}(3)\,\boldsymbol{y}(2)\,\boldsymbol{y}(1)\,\boldsymbol{y}(0))_{14}. \qquad (4.15)\end{gathered}$$

The first several numerals in (4.15) are

$$\begin{aligned}(2,2,2,2,2,2,2)_{14} &= 113\,969\,300\,774\,954,\\ (2,2,2,2,2,2,11)_{14} &= 113\,969\,300\,774\,963,\\ (2,2,2,2,2,5,2)_{14} &= 113\,969\,300\,775\,542,\\ (2,2,2,2,2,8,2)_{14} &= 113\,969\,300\,776\,130,\\ (2,2,2,2,2,11,11)_{14} &= 113\,969\,300\,776\,727,\\ (2,5,2,2,2,2,2)_{14} &= 114\,837\,064\,739\,882,\\ (2,11,2,2,2,2,2)_{14} &= 116\,572\,592\,669\,738,\\ (11,2,2,2,2,2,2)_{14} &= 624\,214\,512\,152\,618,\\ (11,11,11,11,11,11,11)_{14} &= 626\,831\,154\,262\,247. \qquad (4.16)\end{aligned}$$

It follows from the examples (4.14) and (4.16), that not all polyadic integer numbers $\boldsymbol{N}$ can be represented by the polyadic numeral formula in the place-value form (4.8). We call such numbers numerally representable $\{\boldsymbol{N}_{reps}\} \subseteq \{\boldsymbol{N}\}$.

Theorem 4.2. *The set of polyadic numbers which are representable in the numeral form is a subset of the set of all numbers, while the equality is reached in the binary case only* $\boldsymbol{N}_{reps} = \boldsymbol{N} \Longleftrightarrow \boldsymbol{N}_{reps}, \boldsymbol{N} \in \mathbf{Z}_{2,2}^{[0,1]} \equiv \mathbb{Z}$. *In the polyadic nonbinary case* $m \geqslant 3 \vee n \geqslant 3$, *the set of representable numbers is a proper (strict) subset of all numbers* $\{\boldsymbol{N}_{reps}\} \subset \{\boldsymbol{N}\}$.

Now we introduce the polyadic analog of the mixed-base (radix) positional numeral system. In the polyadic notation, the mixed-base version of the binary (4.5) can be written as the sum as the composition of ℓ_ν additions having different bases $p(i)$ for each digit $y(i)$, and we assume that for the last digit $y(0)$ the basis multiplier is absent (see note after (4.6)). In such assumption the connection between the quantity of digits N_y and the quantity of bases N_p is

$$N_p = N_y - 1. \tag{4.17}$$

Then the place-value presentation takes the form of operation compositions

$$N^{(\ell_\nu,\ell_\mu)}(p) = \nu^{\circ\ell_\nu}\Bigg[\overbrace{\mu^{\circ(\ell_\mu-1)}\left[y(\ell_\mu-1), p(\ell_\mu-1)\right], \mu^{\circ(\ell_\mu-1)}\left[y(\ell_\mu-2), p(\ell_\mu-2)\right], \ldots, \mu\left[y(1), p(1)\right], y(0)}^{\ell_\nu+1}\Bigg]$$

$$\Longrightarrow \left(\overbrace{y(\ell_\mu-1)\,y(\ell_\mu-2)\ldots y(1)\,y(0)}^{\ell_\nu+1}\right)^{(\ell_v,\ell_\mu)}_{p(\ell_\mu-1),p(\ell_\mu-2)\ldots p(1)}, \quad 0 \leqslant y(i) \leqslant p(i) - 1 \in \mathbb{Z}^+. \tag{4.18}$$

The simplest binary example of the mixed-base numeral system is computation of currency in different banknote numbers $y(i)$ of the denomination $p(i)$. Sometimes the bases are composed and connected between them taking the “recurrent” form

$$p(i+1) = p(i)\,p(i-1)\ldots p(1). \tag{4.19}$$

A binary example of mixed-base recurrent numeral system is the time calculation in seconds (“clock”/“timer”) using number of days $y(days)$, hours $y(hours)$, minutes $y(\mathit{min})$, seconds $y(\mathit{seconds})$, and the lengths $p(days) = 24$, $p(hours) = 60$, $p(\mathit{minutes}) = 60$. E.g. the polyadic/composition form (4.18) of “2 days, 7 hours, 35 minutes, 48 seconds” (having $m = n = 2$, $\ell_\nu = 2$, $\ell_\mu = 4$), is (in seconds)

$$\nu^{\circ2}\left[\mu^{\circ3}[2,24,60,60], \mu^{\circ2}[7,60,60], \mu[35,60], 48\right] = 2\cdot24\cdot60\cdot60 + 7\cdot60\cdot60 + 35\cdot60 + 48 = 200\,148. \tag{4.20}$$

In general, the polyadic analog of the mixed-base (radix) positional numeral system can be defined by the formula similar to (4.8), but the base in each term in the m-ary sum will depend of its place $\boldsymbol{p} \longrightarrow \boldsymbol{p}(i)$, that is

$$\begin{gathered}\boldsymbol{N}^{(\ell_\nu,\ell_\mu)}_{m,n}(\boldsymbol{p}(1), \ldots \boldsymbol{p}(\ell_\mu-1)) = \boldsymbol{\nu}_m^{\circ\ell_\nu}\Big[\\ \boldsymbol{\mu}_n^{\circ(\ell_\mu-1)}\left[\boldsymbol{y}(\ell_\mu-1), \boldsymbol{p}(\ell_\mu-1)^{\ell_\mu(n-1)}\right], \boldsymbol{\mu}_n^{\circ(\ell_\mu-2)}\left[\boldsymbol{y}(\ell_\mu-2), \boldsymbol{p}(\ell_\mu-2)^{\ell_\mu(n-1)-1}\right], \ldots \\ \ldots, \boldsymbol{\mu}_n\left[\boldsymbol{y}(1), \boldsymbol{p}(1)^{n-1}\right], \boldsymbol{y}(0)\Big] \Longrightarrow \left(\overbrace{\boldsymbol{y}(\ell_\mu-1)\,\boldsymbol{y}(\ell_\mu-2)\ldots \boldsymbol{y}(1)\,\boldsymbol{y}(0)}^{\ell_\nu(m-1)+1}\right)^{(\ell_v,\ell_\mu)}_{m,n;\boldsymbol{p}(1),\ldots\boldsymbol{p}(\ell_\mu-1)}, \\ \boldsymbol{p}(1), \ldots \boldsymbol{p}(\ell_\mu-1) \in \mathbf{Z}_{m,n}.\end{gathered} \tag{4.21}$$

Here the connection between the quantity of digits N_y and the quantity of bases N_p is the same as in the binary case (4.17). Nevertheless, we can extend each ith term using in it different bases (which now have two indices $\boldsymbol{p}_j(i) \in \mathbf{Z}_{m,n}$) as

$$\boldsymbol{\mu}_n^{\circ(i-1)}\left[\boldsymbol{y}(i-1), \boldsymbol{p}_1(i-1), \boldsymbol{p}_2(i-1), \ldots, \boldsymbol{p}_{i(n-1)}(i-1)\right]. \tag{4.22}$$

Summing up in (4.21), instead of (4.17), for the quantity of polyadic bases $\boldsymbol{p}_j(i)$ through the number of digits N_y we will have

$$N_p = \frac{1}{2}N_y(N_y-1)(n-1). \tag{4.23}$$

For example, in the case $\ell_\nu = 1$ and $\ell_\mu = 4$ (see (4.9)) we have the 4-digit ($\boldsymbol{y}(i)$, $i = 0, 1, 2, 3$) polyadic mixed-base numeral presentation for a number from the $(4,3)$-ring $\mathbf{Z}_{4,3}^{[2,3]}$ as

$$\boldsymbol{N}_{4,3}^{(1,3)}\left(\boldsymbol{p}_j(i)\right) = \boldsymbol{\nu}_4\left[\boldsymbol{\mu}_3^{\circ 3}\left[\boldsymbol{y}(3), \boldsymbol{p}_1(3), \boldsymbol{p}_2(3), \boldsymbol{p}_3(3), \boldsymbol{p}_4(3), \boldsymbol{p}_5(3), \boldsymbol{p}_6(3)\right],\right.$$
$$\left.\boldsymbol{\mu}_3^{\circ 2}\left[\boldsymbol{y}(2), \boldsymbol{p}_1(2)\boldsymbol{p}_2(2)\boldsymbol{p}_3(2)\boldsymbol{p}_4(2)\right], \boldsymbol{\mu}_3\left[\boldsymbol{y}(1), \boldsymbol{p}_1(1), \boldsymbol{p}_2(1)\right], \boldsymbol{y}(0)\right]$$
$$\Longrightarrow \left(\boldsymbol{y}(3)\,\boldsymbol{y}(2)\,\boldsymbol{y}(1)\,\boldsymbol{y}(0)\right)_{4,3;\boldsymbol{p}_j(i)}^{(1,3)} \equiv \left(\boldsymbol{y}(3)\,\boldsymbol{y}(2)\,\boldsymbol{y}(1)\,\boldsymbol{y}(0)\right)_{\boldsymbol{p}_j(i)}. \tag{4.24}$$

Thus, the mixed-base polyadic 4-digit numerals over $(4,3)$-ring with $\ell_\nu = 1$ and $\ell_\mu = 4$ are described by 12 different polyadic bases $\boldsymbol{p}_j(i) \in \mathbf{Z}_{4,3}^{[2,3]}$. Comparing with the "binary clock" (4.18)–(4.20), we can call the presented construction (4.21)–(4.24) as the "polyadic clock".

5. Conclusions

In this paper we showed that by transplanting the entire positional-numeral paradigm into the realm of (m,n)-rings we uncovered a landscape where digit strings, carries, and even the very concept of "base" behave in ways impossible in the ordinary arithmetic. Let us distil the main messages.

1. Polyadic rings admit a native positional calculus presented here. Replacing plus by an m-ary sum and product by an n-ary multiplication forces a "double quantization" of word length.

2. Minimal-digit and representability theorems. We proved that in the positional expansion every numeral must respect the constraint $\ell_{mult} = \ell_{add}(m-1) + 1$, and so the shortest admissible numeral contains more positions (digits) than the additive arity $\ell_{mult} \geqslant m$ (Theorem 4.1), and that for $m, n \geqslant 3$ not every element of a polyadic ring possesses a finite positional expansion (Theorem 4.2). The gap is governed by the arity-shape invariants $I^{(m)}$ and $J^{(n)}$ originating in the polyadic features of congruence-class geometry.

3. Mixed bases are generalized naturally yielding a "polyadic clock". Allowing each digit to sit atop its own n-ary tower of bases replaces the linear count $N_p = N_y - 1$ by the quadratic formula

$$N_p = \frac{1}{2}N_y(N_y - 1)(n-1),$$

exploding the design space of numeral systems. Our exotic polyadic clock over $(4,3)$-ring, specified by twelve independent bases, hints at rich cryptographic and coding-theoretic applications.

4. Concrete catalogues. The worked-out examples in the $(4,3)$ and $(6,5)$ integer rings illustrate how ordinary integers lift to abstract polyadic variables that share a decimal "shadow" yet obey distinct algebraic laws. These catalogues provide ready test-beds for algorithmic experimentation.

Future directions. The present framework raises more questions than it settles:

Algorithms and complexity. What is the cost of addition, "carry" propagation, and comparison in a polyadic positional machine? Can we find arities for which arithmetic accelerates relative to binary RAM models?

Floating-point and analysis. A polyadic analogue of IEEE-754 would require rounding rules compatible with m-ary adders and n-ary multipliers. How do error bounds scale with the arity pair (m,n)?

Hardware realization. Ternary logic became tangible with modern CMOS; could polyadic ALUs (arithmetic logic units) unlock energy or area advantages?

Category-theoretic reformulation. Viewing (m,n)-rings as $\mathbb{N}^2$-graded monoidal categories may clarify the functorial behavior of numeral expansions.

Quantum and cryptographic angles. The inherent nonassociativity of many polyadic rings could serve as a source of hard problems, while the enlarged digit alphabet suggests fresh qudit encodings.

Acknowledgments. The author would like to express his deep thankfulness to Vladimir Tkach for productive discussions and to Qiang Guo and Raimund Vogl for valuable support.

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