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GENERALIZED INTERACTION IN MULTIGRAVITY

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We consider a general approach to describing the interaction in multigravity models in a D-dimensional space–time. We present various possibilities for generalizing the invariant volume. We derive the most general form of the interaction potential, which becomes a Pauli–Fierz-type model in the bigravity case. Analyzing this model in detail in the (3+1)-expansion formalism and also requiring the absence of ghosts leads to this bigravity model being completely equivalent to the Pauli–Fierz model. We thus in a concrete example show that introducing an interaction between metrics is equivalent to introducing the graviton mass.

Keywords: multigravity, bigravity, massive gravity, invariant volume, interaction potential, Pauli–Fierz model

1. Introduction

Multigravity together with conformal gravity [1] and scalar theories [2] is one possible extension of general relativity [3], [4]. In early papers, a particular case of multigravity was called the f–g theory or strong gravity [5]–[7]. This construction was later successfully applied in quantum gravity and brane theory [8]–[10], in theories with discrete dimensions [11], [12], in renormalization theory [13], and in massive gravity [14] and was used to explain such experimental facts as dark energy and dark matter [15]–[17] and the accelerated expansion of the Universe [18], [19]. Considering nonlinear formulations of multigravity is therefore important (this was done for bigravity in [20]).

On the other hand, progress in the theory of massive gravity was achieved in [21], where the Pauli–Fierz mass term was extended in a linearized gravity and it was shown that such a model is free of ghost modes [22]. The theory was further extended to the case of an arbitrary additional metric [23]. The main properties of such theories were considered in [24], [25], and the absence of ghost terms in nonlinear models was proved in [26]. In theories of gravity with nonzero mass, we encounter a singularity because of which a theory does not tend to general relativity as the graviton mass tends to zero [27], [28]. The Vainshtein mechanism [29] allows avoiding such an inhomogeneity in the parameter space [23], [30]; moreover, such an inhomogeneity can be eliminated in the case of a nonflat background metric [31], [32].

Here, we describe the general approach for describing interactions in multigravity models in a D-dimensional space–time ($D > 3$). In Sec. 2, we study various possibilities of generalizing the invariant volume of interaction $d\Omega_{\text{int}}^{(N)}$, which is restricted by the conditions that the invariant volume $d\Omega_{\text{int}}^{(N)}$ must be a scalar that passes to the standard volume $\sqrt{g}\,d^D x$ in the limit in which all metrics coincide. The function $d\Omega_{\text{int}}^{(N)}$ must also be monotonic and uniform in all the metrics. In Sec. 3, we derive the most general form of the interaction potential and show that in the simplest case of two metrics (bigravity), it is given by a Pauli–Fierz-type model. A detailed analysis of this model in the formalism of (3+1)-expansion under the condition that ghosts are absent leads to this bigravity model in the weak-field limit being completely

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equivalent to the Pauli–Fierz model. In fact, this means that introducing an interaction between the tensor fields $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(2)}$ is equivalent to introducing a graviton mass. In the appendix, we present a new method for calculating $\sqrt{g}$ for small excitations, which can be used with any background metric. In the case of a flat Minkowski background space–time, we obtain the standard expression.

2. Multigravity and the generalization of the invariant volume of interaction

We consider the union of N different universes each of which is described by its metric $\mathsf{g}_{\mu\nu}^{(i)}$, where $i = 1, \ldots, N$. We use the signature $(+, \overbrace{-, \ldots, -}^{D-1})$ in the D-dimensional space–time. We write the action for the ith universe in the form

$$S_{G(i)} = \int d\Omega^{(i)} [L_{\text{gr}}^{(i)}(\mathsf{g}^{(i)}) + L_{\text{mat}}(\mathsf{g}^{(i)}, \Phi^{(i)})], \tag{1}$$

where $d\Omega^{(i)} = d^4x \sqrt{g^{(i)}}$ is the invariant volume, $g^{(i)} = |\det(\mathsf{g}_{\mu\nu}^{(i)})|$ is the scalar density with weight two, $\mathsf{g}_{\mu\nu}^{(i)}$ is the metric tensor of the ith universe, $L_{\text{gr}}^{(i)}(\mathsf{g}^{(i)})$ is the Lagrangian describing the gravitational field, and the Lagrangian $L_{\text{mat}}^{(i)}(\mathsf{g}^{(i)}, \Phi^{(i)})$ describes the coupling between the gravity and matter fields $\Phi^{(i)}$. The integral in (1) is taken over the total manifold of N universes.

Assuming "weakly coupled worlds" [20] and the "no-go" theorem [33], we can write the general action for N massless gravitons as a sum of purely gravitational actions of form (1):

$$S_0 = \sum_{i=1}^{N} S_{G(i)}.$$

Assuming that "weakly coupled worlds" mutually interact only through gravitational fields, we can write the complete multigravity action in the form

$$S_{\text{full}} = \sum_{i}^{N} S_{G(i)} + S_{\text{int}},$$

where the last term, S_{int}, describes the interaction between the universes. Choosing this term is crucial when describing multigravity models [34].

In the general case of N-gravity in D dimensions, the action S_{int} is

$$S_{\text{int}} = \int d^Dx\, W(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)}),$$

where d^Dx and $W(\mathsf{g}^{(1)}, \ldots, \mathsf{g}^{(N)})$ are scalar densities of opposite weights. By analogy with the standard invariant volume $d\Omega = d^4x \sqrt{g}$ in general relativity [3], [4], we represent the expression $d^Dx\, W(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)})$ as

$$d^Dx\, f(\sqrt{g_1}, \ldots, \sqrt{g_N})V(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)}).$$

In this expression, $V(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)}) \equiv V(\mathsf{g}^{(i)})$ is the scalar interaction potential, and $f(\sqrt{g_1}, \ldots, \sqrt{g_N})$ is a smooth positive function of N positive real arguments and has the weight $+1$. We introduce the invariant volume of interaction

$$d\Omega_{\text{int}}^{(N)} = d^Dx\, f(\sqrt{g_1}, \ldots, \sqrt{g_N}), \tag{2}$$

which must be a scalar. Moreover, in the limit of coinciding arguments [34]

$$\mathsf{g}_{\mu\nu}^{(i)} = \cdots = \mathsf{g}_{\mu\nu}^{(N)} \equiv \mathsf{g}_{\mu\nu},$$

the invariant volume of interaction must transform into the standard invariant volume, $d\Omega_{\text{int}}^{(N)} \to d\Omega$. To satisfy all these requirements, the function $f(\sqrt{g_1}, \ldots, \sqrt{g_N})$ must have the following properties:

1. idempotency in the limit of coinciding arguments, $f(\sqrt{g},\ldots,\sqrt{g}\,)=\sqrt{g}$,

2. monotonicity,

3. homogeneity under a rescaling of all arguments, $f(t\sqrt{g_1},\ldots,t\sqrt{g_N}\,)=t^{\alpha}f(\sqrt{g_1},\ldots,\sqrt{g_N}\,)$ (idempotency implies that $\alpha=1$), and

4. total symmetricity in all arguments.

The homogeneity and symmetricity conditions for $f(\sqrt{g_1},\ldots,\sqrt{g_N}\,)$ imply that the invariant volume of interaction can be represented in the form [34]

$$d\Omega^{(N)}_{\text{int}}=d^Dx\,f(\sqrt{g_1},\ldots,\sqrt{g_N}\,)=d^Dx\,\sqrt[2N]{g_1\ldots g_N}\,f(y^{(N)}_1,\ldots,y^{(N)}_N),$$

where

$$\begin{aligned}
y^{(N)}_1 &= \sqrt[2N]{g_1^{N-1}g_2^{-1}\ldots g_N^{-1}},\\
y^{(N)}_2 &= \sqrt[2N]{g_1^{-1}g_2^{N-1}\ldots g_N^{-1}},\\
&\vdots\\
y^{(N)}_N &= \sqrt[2N]{g_1^{-1}g_2^{-1}\ldots g_{N-1}^{-1}g_N^{N-1}}.
\end{aligned}$$

The variables $y^{(N)}_i$ obviously satisfy the identity

$$y^{(N)}_1y^{(N)}_2\cdots y^{(N)}_N=1,$$

the function f is therefore in fact a function of $N-1$ arguments, and we can write the invariant volume of interaction in the form

$$d\Omega^{(N)}_{\text{int}}=d^4x\,f(\sqrt{g_1},\ldots,\sqrt{g_N}\,)=d^4x\,\sqrt[2N]{g_1\cdots g_N}\,\hat{f}(y^{(N)}_1,\ldots,y^{(N)}_{N-1}),$$

where

$$\hat{f}(y^{(N)}_1,\ldots,y^{(N)}_{N-1})\stackrel{\text{def}}{=}f\left(y^{(N)}_1,\ldots,y^{(N)}_{N-1},\frac{1}{y^{(N)}_1y^{(N)}_2\cdots y^{(N)}_{N-1}}\right).$$

We note that $y^{(N)}_i=1$ and $f(1,\ldots,1)=1$ in the limit of coinciding metrics.

We choose the invariant volume of interaction as a sum of three means: the arithmetic mean, the geometric mean, and the harmonic mean taken with arbitrary real coefficients α, β, and γ. Then

$$d\Omega^{(N)}_{\text{int}}=d^Dx\,\sqrt[2N]{g_1\cdots g_N}\frac{1}{\alpha+\beta+\gamma}\left[\frac{\alpha}{N}\sum_{i=1}^{N}y^{(N)}_i+\beta+\gamma\frac{N}{\sum_{i=1}^{N}1/y^{(N)}_i}\right],\tag{3}$$

where $\alpha+\beta+\gamma\neq 0$. For simplicity, we restrict ourself to this natural expression (3) for the invariant volume of interaction in multigravity. We note that a particular case of (3) with $\alpha=\gamma=0$ and $\beta=1$ for bigravity (for $N=2$) was considered in [20].

3. The generalized interaction potential

We consider the general form of the multigravity interaction described by a scalar potential $V(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)})$ determined as a function of N metrics $\mathsf{g}^{(i)}_{\mu\nu}$ in a D-dimensional space–time. The symmetry group of N universes is the direct product of groups of diffeomorphisms [20],

$$G_{\text{full}} = \text{Diff}(\varepsilon^{(i)}_{\mu}) \times \text{Diff}(\varepsilon^{(2)}_{\mu}) \times \cdots \times \text{Diff}(\varepsilon^{(N)}_{\mu}),$$

where a diffeomorphism $\text{Diff}(\varepsilon^{(i)}_{\mu})$ acts on the metric $\mathsf{g}^{(i)}_{\mu\nu}$ along the vector $\varepsilon^{(i)}_{\mu}(x)$. In accordance with the known theorem [33], we can reduce the group G_{full} to the diagonal subgroup when all vectors coincide, $\varepsilon^{(i)}_{\mu}(x) = \varepsilon_{\mu}(x)$. The infinitesimal transformations of each metric $\mathsf{g}^{(i)}_{\mu\nu}$ are then governed by the Lie derivative,

$$\delta\mathsf{g}^{(i)}_{\mu\nu} = \mathcal{L}_{\varepsilon}\mathsf{g}^{(i)}_{\mu\nu} = \varepsilon^{\rho}\,\partial_{\rho}\mathsf{g}^{(i)}_{\mu\nu} + \mathsf{g}^{(i)}_{\mu\rho}\,\partial_{\nu}\varepsilon^{\rho} + \mathsf{g}^{(i)}_{\rho\nu}\,\partial_{\mu}\varepsilon^{\rho}.$$

The scalar interaction potential must obviously be expressed via scalar functions of the metrics $\mathsf{g}^{(i)}_{\mu\nu}$. We can naturally choose these scalar functions as invariants of a tensor with one covariant and one contravariant index constructed from the metrics, $\mathsf{H}^{\mu}{}_{\nu} = \mathsf{H}^{\mu}{}_{\nu}(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)})$. Eigenvalues of the matrix $\widehat{\mathsf{H}}$ corresponding to the tensor $\mathsf{H}^{\mu}{}_{\nu}$ are then invariant under the action of general coordinate transformations $x^{\mu} \mapsto \tilde{x}^{\mu}$ because

$$\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\mathsf{H}^{\mu}{}_{\nu}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}} = \widetilde{\mathsf{H}}^{\alpha}{}_{\beta}.$$

We parameterize the matrix $\widehat{\mathsf{H}}(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)})$ using the following observation.

In most physically relevant models [4], the metric is diagonal,

$$\mathsf{g}^{(i)}_{\mu\nu} = \text{diag}(\lambda^{(i)}_{0}, \lambda^{(i)}_{1}, \ldots, \lambda^{(i)}_{D-1}), \tag{4}$$

where $\lambda^{(i)}_{a}$ are eigenvalues of the ith metric. Hence, we can describe the structure of the matrix $\widehat{\mathsf{H}}(\mathsf{g}^{(i)}, \ldots, \mathsf{g}^{(N)})$ analogously to that of the invariant volume of interaction constructed in Sec. 2. Namely, we construct N matrices $\mathsf{H}^{(i)\mu}_{\nu}$ as the product of diagonal matrices

$$\mathsf{H}^{(i)\mu}_{\nu} = \mathsf{g}^{(i)\mu\alpha_1}\mathsf{g}^{(i)}_{\alpha_1\rho_1}\mathsf{g}^{(i)\rho_1\beta_1}\mathsf{g}^{(2)}_{\beta_1\rho_2}\cdots\mathsf{g}^{(i)\rho_{j-1}\alpha_j}\mathsf{g}^{(j)}_{\alpha_j\rho_j}\mathsf{g}^{(i)\rho_j\beta_j}\mathsf{g}^{(j+1)}_{\beta_j\rho_{j+1}}\cdots$$
$$\cdots\mathsf{g}^{(i)\rho_{N-2}\alpha_{N-1}}\mathsf{g}^{(N-1)}_{\alpha_{N-1}\rho_{N-1}}\mathsf{g}^{(i)\rho_{N-1}\beta_{N-1}}\mathsf{g}^{(N)}_{\beta_{N-1}\nu}.$$

The thus constructed matrices $\widehat{\mathsf{H}}^{(i)}$ satisfy the identity

$$\widehat{\mathsf{H}}^{(1)}\widehat{\mathsf{H}}^{(2)}\cdots\widehat{\mathsf{H}}^{(N)} = \mathsf{I}, \tag{5}$$

where I is the unit $D\times D$ matrix. As a result, we obtain $N-1$ independent matrices $\widehat{\mathsf{H}}^{(i)}$. In the bigravity case (for $N = 2$), we have two matrices

$$\mathsf{H}^{(1)\mu}_{\nu} = \mathsf{g}^{(1)\mu\beta_1}\mathsf{g}^{(2)}_{\beta_1\nu}, \qquad \mathsf{H}^{(2)\mu}_{\nu} = \mathsf{g}^{(2)\mu\alpha_1}\mathsf{g}^{(i)}_{\alpha_1\nu},$$

which are mutually inverse, $\widehat{\mathsf{H}}^{(1)}\widehat{\mathsf{H}}^{(2)} = \mathsf{I}$ (see identity (5)), and it hence suffices to consider one of these matrices (see, e.g., [20]). It is therefore reasonable to define the N^2 matrices $\hat{\mathsf{p}}^{(i,j)}$:

$$\hat{\mathsf{p}}^{(i,j)\mu}{}_{\nu} = \mathsf{g}^{(i)\mu\rho}\mathsf{g}^{(j)}_{\rho\nu}, \tag{6}$$

where $i, j = 1, 2, \dots, N$. The matrices $\hat{\mathsf{p}}^{(i,j)}$ obviously satisfy the relations

$$\hat{\mathsf{p}}^{(i,j)}\hat{\mathsf{p}}^{(j,k)} = \hat{\mathsf{p}}^{(i,k)}, \tag{7}$$

$$\hat{\mathsf{p}}^{(i,j)}\hat{\mathsf{p}}^{(j,i)} = \hat{\mathsf{p}}^{(i,i)} = \mathsf{I}. \tag{8}$$

Product (7) is associative and invertible (see equality (8)), but it is not defined for all elements, and the set of p-variables is therefore a partial group [35]. We note that we have $N(N-1)/2$ independent p-matrices, which commute in the case of diagonal metrics (4). In the bigravity case (for $N = 2$), we have

$$\widehat{\mathsf{H}}^{(1)} = \hat{\mathsf{p}}^{(1,2)}, \qquad \widehat{\mathsf{H}}^{(2)} = \hat{\mathsf{p}}^{(2,1)}.$$

We construct the matrices $\widehat{\mathsf{H}}^{(i)}$ from six matrices $\hat{\mathsf{p}}^{(i,j)}$ (among which three are independent) in the case of ternary gravity (for $N = 3$):

$$\widehat{\mathsf{H}}^{(1)} = \hat{\mathsf{p}}^{(1,3)}\hat{\mathsf{p}}^{(1,2)},$$

$$\widehat{\mathsf{H}}^{(2)} = \hat{\mathsf{p}}^{(2,1)}\hat{\mathsf{p}}^{(2,3)},$$

$$\widehat{\mathsf{H}}^{(3)} = \hat{\mathsf{p}}^{(3,2)}\hat{\mathsf{p}}^{(3,1)}.$$

The matrices $\widehat{\mathsf{H}}^{(i)}$ satisfy the identity

$$\widehat{\mathsf{H}}^{(1)}\widehat{\mathsf{H}}^{(2)}\widehat{\mathsf{H}}^{(3)} = \mathsf{I}.$$

Taking equality (4) into account, we can write the eigenvalues of the matrices $\mathsf{H}_{\nu}^{(i)\mu}$ using metric eigenvalues,

$$\widehat{\mathsf{H}}^{(i)} = \operatorname{diag}\biggl(\frac{(\lambda_0^{(i)})^N}{R_0}, \frac{(\lambda_1^{(i)})^N}{R_1}, \dots, \frac{(\lambda_{D-1}^{(i)})^N}{R_{D-1}}\biggr), \tag{9}$$

where $R_a = \prod_{i=1}^{N} \lambda_a^{(i)}$. It then follows from (9) that

$$\det \widehat{\mathsf{H}}^{(i)} = \frac{(\det \mathsf{g}^{(i)})^N}{\prod_{j=1}^{N} \det \mathsf{g}^{(j)}}, \tag{10}$$

and obviously $\prod_{j=1}^{N} \det \widehat{\mathsf{H}}^{(j)} = 1$ (see identity (5)).

We note that for the metric $\mathsf{g}_{\mu\nu}^{(i)}$ with the signature $(+, \overbrace{-, \dots, -}^{D-1})$, the signs of the eigenvalues are

$$\lambda_0^{(i)} > 0, \quad \lambda_1^{(i)} < 0, \quad \dots, \quad \lambda_{D-1}^{(i)} < 0$$

(see, e.g., [4]). By virtue of relations (9) and (5), we find that all eigenvalues of the matrices $\widehat{\mathsf{H}}^{(i)}$ are nonzero positive. We can then define the new variables

$$\mu_a^{(i)} = \log \frac{(\lambda_a^{(i)})^N}{R_a}, \quad a = 0, 1, \dots, D-1, \quad i = 1, 2, \dots, N, \tag{11}$$

satisfying D identities

$$\sum_{i=1}^{N} \mu_a^{(i)} = 0, \quad a = 0, 1, \dots, D-1. \tag{12}$$

As a result, the number of independent μ-variables is $D(N-1)$. We can therefore take a smooth function of μ-variables as the scalar interaction potential,

$$V(\mathsf{g}^{(i)}, \mathsf{g}^{(2)}, \dots, \mathsf{g}^{(N)}) = \tilde{v}(\{\mu_a^{(i)}\}).$$

Following [20] (where the particular case $N=2$ and $D=4$ was considered), we choose a more convenient basis in the form of symmetric polynomials,

$$\sigma_k^{(i)} = \sum_{a=0}^{D-1} (\mu_a^{(i)})^k, \quad k = 1, 2, \dots, D, \tag{13}$$

connected by D relations following from identities (12). We can therefore write the scalar interaction potential for multigravity in the form

$$V(\mathsf{g}^{(i)}, \mathsf{g}^{(2)}, \dots, \mathsf{g}^{(N)}) = v(\{\sigma_k^{(i)}\}), \quad k = 1, 2, \dots, D, \quad i = 1, 2, \dots, N, \tag{14}$$

where v is a scalar function of $D(N-1)$ independent polynomials $\sigma_k^{(i)}$.

Naturally assuming the absence of interaction in the case of flat spaces, we obtain the "boundary condition"

$$v(0, 0, \dots, 0) = 0. \tag{15}$$

We explicitly express scalar interaction potential (14) as a combination of invariants of the matrices $\widehat{\mathsf{H}}^{(i)}$. From relations (10), (11), and (13), we have

$$\sigma_k^{(i)} = \operatorname{tr}(\log \widehat{\mathsf{H}}^{(i)})^k.$$

We parameterize the metrics as

$$\mathsf{g}_{\mu\nu}^{(i)} = \eta_{\mu\nu} + \mathsf{h}_{\mu\nu}^{(i)}, \tag{16}$$

where $\mathsf{h}_{\mu\nu}^{(i)}$ are excitations over a flat background. Keeping only terms quadratic in the excitations $\mathsf{h}_{\mu\nu}^{(i)}$, which correspond to the massive case and the absence of self-action, for $\sigma_1^{(i)}$ and $\sigma_2^{(i)}$, we obtain the expressions

$$\sigma_1^{(i)} = \sum_{\substack{1\le j\le N,\\ j\ne i}} \bigl[(h^{(i)} - h^{(j)}) - \bigl((h_{\mu\nu}^{(i)})^2 - (h_{\mu\nu}^{(j)})^2\bigr)\bigr], \tag{17}$$

$$\sigma_2^{(i)} = (N-1)^2 (h_{\mu\nu}^{(i)})^2 + \sum_{\substack{1\le j\le N,\\ j\ne i}} (h_{\mu\nu}^{(j)})^2 + \\ + 2 \sum_{\substack{1\le k,j\le N,\\ j\ne k,\, k\ne i,\, j\ne i}} \mathsf{h}^{(j)\mu}{}_{\nu} \mathsf{h}^{(k)\nu}{}_{\mu} - 2(N-1) \mathsf{h}^{(i)\mu}{}_{\nu} \sum_{\substack{1\le j\le N,\\ j\ne i}} \mathsf{h}^{(j)\nu}{}_{\mu}, \tag{18}$$

where $h^{(i)} \stackrel{\text{def}}{=} \mathsf{h}_{\mu\nu}^{(i)} \eta^{\mu\nu}$ and $(h_{\mu\nu}^{(i)})^2 \stackrel{\text{def}}{=} \mathsf{h}_{\mu\nu}^{(i)} \mathsf{h}^{(i)\mu\nu}$. We note that $\sigma_k^{(i)} \sim O((h^{(i)})^k)$. Hence, if we keep only quadratic terms, then we cannot consider expressions with powers $k \ge 3$.

We can therefore represent the scalar interaction potential in the multigravity in the quadratic approximation in the form

$$V(\mathsf{g}^{(i)}) = \sum_{i=1}^{N} [a_i \sigma_1^{(i)} + b_i (\sigma_1^{(i)})^2 + c_i \sigma_2^{(i)}], \tag{19}$$

where a_i, b_i, and c_i are arbitrary real constants. Formula (17) implies that

$$\sum_{i=1}^{N} \sigma_1^{(i)} = 0,$$

which must also follow from identities (12).

4. The Pauli–Fierz model in bigravity

As an example, we consider bigravity ($N = 2$) and obtain the Pauli–Fierz model from general principles. Instead of relations (17) and (18), we have

$$\sigma_1^{(1)} = -\sigma_1^{(2)} = h^{(1)} - h^{(2)} - \big((h^{(1)}_{\mu\nu})^2 - (h^{(2)}_{\mu\nu})^2\big) \equiv \sigma_1,$$
$$\sigma_2^{(1)} = \sigma_2^{(2)} = (h^{(1)}_{\mu\nu})^2 + (h^{(2)}_{\mu\nu})^2 - 2\mathsf{h}^{(1)\mu}{}_\nu \mathsf{h}^{(2)\nu}{}_\mu \equiv \sigma_2$$

(up to terms quadratic in the excitations $\mathsf{h}^{(1)}_{\mu\nu}$ and $\mathsf{h}^{(2)}_{\mu\nu}$). For the scalar interaction potential, sum (19) (with condition (15) taken into account) becomes

$$V(\mathsf{g}^{(1)}, \mathsf{g}^{(2)}) = a\sigma_1 + b\sigma_1^2 + c\sigma_2, \tag{20}$$

where a, b, and c are arbitrary real constants with the dimension of the fourth power of mass. The total bigravity action is then

$$S_2 = -M_1^2 \int d^4x\, R_1 \sqrt{g_1} - M_2^2 \int d^4x\, R_2 \sqrt{g_2} + \int d\Omega^{(2)}_{\rm int}\, V(\mathsf{g}^{(1)}, \mathsf{g}^{(2)}), \tag{21}$$

where $M_{1,2}$ are constants with the dimension of mass and $d\Omega^{(2)}_{\rm int}$ is invariant volume (2) of interaction for bigravity, which in this case becomes

$$d\Omega^{(2)}_{\rm int} = d^4x\, \sqrt[4]{g_1 g_2}\, \frac{1}{\alpha+\beta+\gamma}\left[\frac{\alpha}{2}\left(\sqrt{\frac{g_1}{g_2}} + \sqrt{\frac{g_2}{g_1}}\right) + \beta + 2\gamma\left(\sqrt{\frac{g_1}{g_2}} + \sqrt{\frac{g_2}{g_1}}\right)^{-1}\right], \tag{22}$$

where α, β, and γ are dimensionless parameters, $\alpha + \beta + \gamma \neq 0$. We note that parameterization (16) of equality (22) results in the expression

$$d\Omega^{(2)}_{\rm int} = d^4x\, \sqrt[4]{g_1 g_2} + \ldots,$$

where the ellipsis denotes terms quadratic in the excitations $\mathsf{h}^{(1)}_{\mu\nu}$ and $\mathsf{h}^{(2)}_{\mu\nu}$. These terms do not contribute to (21), because we restrict ourself to the second order and scalar interaction potential (20) does not contain terms without $\mathsf{h}^{(1)}_{\mu\nu}$ and $\mathsf{h}^{(2)}_{\mu\nu}$. Using expansion (16) and applying it to action (21), we obtain

$$S_2 = \int d^4x (L_{\rm kin} + L_{\rm int}), \tag{23}$$

where

$$L_{\rm kin} = \frac{1}{4} M_1^2 [\partial^\rho \mathsf{h}^{(1)}_{\mu\nu}\, \partial_\rho \mathsf{h}^{(1)\mu\nu} - \partial^\mu h^{(1)}\, \partial_\mu h^{(1)} + 2\partial_\mu \mathsf{h}^{(1)\mu\nu}\, \partial_\nu h^{(1)} - 2\partial_\mu \mathsf{h}^{(1)\mu\nu}\, \partial_\rho \mathsf{h}^{(1)\rho}_\nu] +$$
$$+ \frac{1}{4} M_2^2 [\partial^\rho \mathsf{h}^{(2)}_{\mu\nu}\, \partial_\rho \mathsf{h}^{(2)\mu\nu} - \partial^\mu h^{(2)}\, \partial_\mu h^{(2)} + 2\partial_\mu \mathsf{h}^{(2)\mu\nu}\, \partial_\nu h^{(2)} - 2\partial_\mu \mathsf{h}^{(2)\mu\nu}\, \partial_\rho \mathsf{h}^{(2)\rho}_\nu],$$
$$L_{\rm int} = a(h^{(1)} - h^{(2)})^2 + b(\mathsf{h}^{(1)}_{\mu\nu} - \mathsf{h}^{(2)}_{\mu\nu})(\mathsf{h}^{(1)\mu\nu} - \mathsf{h}^{(2)\mu\nu}) +$$
$$+ c(\mathsf{h}^{(2)}_{\mu\nu}\mathsf{h}^{(2)\mu\nu} - \mathsf{h}^{(1)}_{\mu\nu}\mathsf{h}^{(1)\mu\nu}) + \frac{c}{4}\big((h^{(1)})^2 - (h^{(2)})^2\big).$$

We further apply the (3+1)-expansion [31] to total action (23). Segregating the spatial and temporal components in L_{int}, we obtain

$$
\begin{aligned}
L_{\text{int}} = {} & a(h_{00}^{(1)} - h_{00}^{(2)} - h_{ii}^{(1)} + h_{ii}^{(2)})^2 + b(h_{00}^{(1)} - h_{00}^{(2)})(h_{00}^{(1)} - h_{00}^{(2)}) - \\
& - 2b(h_{0i}^{(1)} - h_{0i}^{(2)})(h_{0i}^{(1)} - h_{0i}^{(2)}) + b(h_{ij}^{(1)} - h_{ij}^{(2)})(h_{ij}^{(1)} - h_{ij}^{(2)}) + \\
& + c(h_{00}^{(2)} h_{00}^{(2)} - 2h_{0i}^{(2)} h_{0i}^{(2)} + h_{ij}^{(2)} h_{ij}^{(2)} - h_{00}^{(1)} h_{00}^{(1)} + 2h_{0i}^{(1)} h_{0i}^{(1)} - h_{ij}^{(1)} h_{ij}^{(1)}) + \\
& + \frac{c}{4}\big((h_{00}^{(1)} - h_{ii}^{(1)})^2 - (h_{00}^{(2)} - h_{ii}^{(2)})^2\big).
\end{aligned}
$$

We restrict our consideration to only the scalar sector because it suffices for eliminating ghost modes from the spectrum (see [31] for the case of standard gravity). We write the (3+1)-expansion using the parameterization

$$
h_{00}^{(r)} = 2\varphi_r, \qquad h_{0i}^{(r)} = \partial_i B_r, \qquad h_{ij}^{(r)} = -2(\psi_r \delta_{ij} - \partial_i \partial_j E_r),
$$

where φ_r, ψ_r, B_r, and E_r are the scalar fields for the perturbed metric $\mathsf{h}_{\mu\nu}^{(r)}$, $r = 1, 2$. From formula (23), we obtain expressions for the kinetic and interaction terms:

$$
\begin{aligned}
L_{\text{kin}} = {} & M_1^2[-2\psi_1\, \partial_k\partial_k\psi_1 - 6\dot\psi_1^2 - 4\varphi_1\, \partial_k\partial_k\psi_1 - 4\dot\psi_1\, \partial_k\partial_k B_1 + 4\dot\psi_1\, \partial_k\partial_k \dot E_1] + \\
& + M_2^2[-2\psi_2\, \partial_k\partial_k\psi_2 - 6\dot\psi_2^2 - 4\varphi_2\, \partial_k\partial_k\psi_2 - 4\dot\psi_2\, \partial_k\partial_k B_2 + 4\dot\psi_2\, \partial_k\partial_k \dot E_2],
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
L_{\text{int}} = {} & a\big(2(\varphi_1 - \varphi_2) + 6(\psi_1 - \psi_2) - 2\Delta(E_1 - E_2)\big)^2 + \\
& + b\big(4(\varphi_1 - \varphi_2)^2 + 2(B_1 - B_2)(\Delta B_1 - \Delta B_2) + 12(\psi_1 - \psi_2)^2 + 4(\Delta E_1 - \Delta E_2)^2 - \\
& - 8(\psi_1 - \psi_2)(\Delta E_1 - \Delta E_2)\big) + \\
& + c\big(4(\varphi_2^2 - \varphi_1^2) + 12(\psi_2^2 - \psi_1^2) + B_2\Delta B_2 - B_1 \Delta B_1 + 4((\Delta E_2)^2 - (\Delta E_1)^2) + \\
& + 8(\psi_1 \Delta E_1 - \psi_2 \Delta E_2)\big) + c\big((\varphi_1 + 3\psi_1 - \Delta E_1)^2 - (\varphi_2 + 3\psi_2 - \Delta E_2)^2\big).
\end{aligned}
\tag{25}
$$

We consider the part of the total Lagrangian that contains the scalar fields φ_1 and φ_2:

$$
\begin{aligned}
L(\varphi) = {} & -4M_1^2\varphi_1\Delta\psi_1 - 4M_2^2\varphi_2\Delta\psi_2 + \varphi_1^2(4a + 4b - 3c) + \varphi_2^2(4a + 4b + 3c) + \\
& + \varphi_1(24a(\psi_1 - \psi_2) - 8a(\Delta E_1 - \Delta E_2) + 6c\psi_1 - 2c\Delta E_1) + \\
& + \varphi_2(-24a(\psi_1 - \psi_2) + 8a(\Delta E_1 - \Delta E_2) - 6c\psi_2 + 2c\Delta E_2) - 8\varphi_1\varphi_2(a + b).
\end{aligned}
$$

Obviously, the Lagrangian does not contain terms quadratic in the fields φ_1 and φ_2 if

$$
4a + 4b - 3c = 0, \qquad 4a + 4b + 3c = 0, \qquad a + b = 0,
\tag{26}
$$

i.e., scalar fields become nondynamical (see the details in [31]). System (26) is equivalent to the equations

$$
a + b = 0, \qquad c = 0.
\tag{27}
$$

We note that we can express the Lagrangian in terms of the differences of the corresponding fields only if the above relations for the parameters are satisfied. Introducing the variables

$$
\varphi = \varphi_1 - \varphi_2, \qquad B = B_1 - B_2,
\tag{28}
$$

$$
\psi = \psi_1 - \psi_2\,, \qquad E = E_1 - E_2,
\tag{29}
$$

we can write interaction Lagrangian (25) in the form

$$L^{(2)}_{\rm int} = 4a\left[6\psi^2 + 6\varphi\psi - 2\varphi\Delta E - 4\psi\Delta E - \frac{1}{2}B\Delta B\right]. \tag{30}$$

Expression (30) coincides with the massive Pauli–Fierz Lagrangian in the (3+1)-expansion of standard gravity [31]. To prove the equivalence of bigravity (21) and the Pauli–Fierz theory, we must also consider the kinetic part. We note that we can represent kinetic term (24) in terms of fields (28) and (29) only if we use the equations of motion. For this, we write total Lagrangian (24), (25) taking (28) into account. We have

$$\begin{aligned} L^{(2)}_{\rm kin} + L^{(2)}_{\rm int} = {} & M_1^2[-2\psi_1\,\partial_k\partial_k\psi_1 - 6\dot\psi_1^2 - 4\varphi_1\,\partial_k\partial_k\psi_1 - 4\dot\psi_1\,\partial_k\partial_k B_1 + 4\dot\psi_1\,\partial_k\partial_k\dot E_1] + \\ & + M_2^2[-2\psi_2\,\partial_k\partial_k\psi_2 - 6\dot\psi_2^2 - 4\varphi_2\,\partial_k\partial_k\psi_2 - 4\dot\psi_2\,\partial_k\partial_k B_2 + 4\dot\psi_2\,\partial_k\partial_k\dot E_2] + \\ & + 24a(\psi_1-\psi_2)^2 + 4a[6(\varphi_1-\varphi_2)(\psi_1-\psi_2) - 2(\varphi_1-\varphi_2)\Delta(E_1-E_2)] - \\ & - 16a(\psi_1-\psi_2)\Delta(E_1-E_2) - 2a(B_1-B_2)\Delta(B_1-B_2). \end{aligned} \tag{31}$$

The Euler–Lagrange system of equations for the fields B_1 and B_2 is then

$$\begin{aligned} 4M_1^2\Delta\dot\psi_1 + 4a(\Delta B_1 - \Delta B_2) &= 0, \\ 4M_2^2\Delta\dot\psi_2 + 4a(\Delta B_2 - \Delta B_1) &= 0, \end{aligned} \tag{32}$$

where we represent the relevant part of the Lagrangian in the form

$$L(B) = 4M_1^2\,\partial_k\dot\psi_1\,\partial_k B_1 + 4M_2^2\,\partial_k\dot\psi_2\,\partial_k B_2 + 2a(\partial_k B_1 - \partial_k B_2)(\partial_k B_1 - \partial_k B_2). \tag{33}$$

By virtue of (28), system (32) transforms into

$$M_1^2\Delta\dot\psi_1 = -a\Delta B, \qquad M_2^2\Delta\dot\psi_2 = a\Delta B, \tag{34}$$

which implies the equality

$$M_1^2\psi_1 = -M_2^2\psi_2. \tag{35}$$

For the field ψ (see definitions (29)), we obtain

$$\psi = \psi_1 - \psi_2 = \psi_1 + \frac{M_1^2}{M_2^2}\psi_1 = \frac{M_1^2+M_2^2}{M_2^2}\psi_1 = -\frac{M_2^2}{M_1^2}\psi_2 - \psi_2 = -\frac{M_1^2+M_2^2}{M_1^2}\psi_2.$$

Taking Eqs. (34) into account, we obtain

$$B = \frac{M_1^2}{-a}\dot\psi_1 = \frac{M_1^2M_2^2}{-a(M_1^2+M_2^2)}\,\dot\psi.$$

The part $L(B)$ of the Lagrangian given by (33) then becomes

$$L(B) = 2\frac{M_1^4M_2^4}{a(M_1^2+M_2^2)^2}\,\dot\psi\Delta\dot\psi.$$

Varying expression (31) in the fields φ_1 and φ_2, we obtain the system

$$-M_1^2\Delta\psi_1+6a(\psi_1-\psi_2)-2a(\Delta E_1-\Delta E_2)=0,$$
$$-M_2^2\Delta\psi_2-6a(\psi_1-\psi_2)+2a(\Delta E_1-\Delta E_2)=0,$$

which by virtue of (29) and (35) is equivalent to the equation

$$\Delta E=-\frac{M_1^2M_2^2}{2a(M_1^2+M_2^2)}\Delta\psi+3\psi.$$

As a result, we can rewrite the part of Lagrangian that contains the fields E_1 and E_2 in the form

$$\begin{aligned}L(E)&=4M_1^2\dot\psi_1\Delta\dot E_1+4M_2^2\dot\psi_2\Delta\dot E_2-8a\varphi\Delta E-16a\psi\Delta E=\\&=4\frac{M_1^2M_2^2}{M_1^2+M_2^2}\dot\psi\left(-\frac{M_1^2M_2^2}{2a(M_1^2+M_2^2)}\Delta\dot\psi+3\dot\psi\right)-\\&-8a(\varphi+2\psi)\left(-\frac{M_1^2M_2^2}{2a(M_1^2+M_2^2)}\Delta\psi+3\psi\right).\end{aligned}$$

We also express the remaining terms in the kinetic term of total Lagrangian (31) in terms of the field ψ:

$$\begin{aligned}L_{\rm k}(\psi)=&-2M_1^2\psi_1\Delta\psi_1-2M_2^2\psi_2\Delta\psi_2-\\&-6M_1^2\dot\psi_1^2-6M_2^2\dot\psi_2^2-4M_1^2\varphi_1\Delta\psi_1-4M_2^2\varphi_2\Delta\psi_2=\\=&-2\frac{M_1^2M_2^2}{M_1^2+M_2^2}(\psi\Delta\psi+3\dot\psi^2+2\varphi\Delta\psi).\end{aligned}$$

Total Lagrangian (31) is

$$\begin{aligned}L^{(2)}_{\rm kin}+L^{(2)}_{\rm int}&=L_{\rm k}(\psi)+L(B)+L(E)+24a\psi^2+24a\varphi\psi=\\&=6\frac{M_1^2M_2^2}{M_1^2+M_2^2}(\dot\psi^2+\psi\Delta\psi)-24a\psi^2.\end{aligned}$$

We represent the constant a in terms of the new constant $m_{\rm g}^2$:

$$a=\frac{1}{4}\frac{M_1^2M_2^2}{M_1^2+M_2^2}\,m_{\rm g}^2.$$

The scalar sector of bigravity then becomes

$$L=6\frac{M_1^2M_2^2}{M_1^2+M_2^2}(\dot\psi^2+\psi\Delta\psi-m_{\rm g}^2\psi^2)=6\frac{M_1^2M_2^2}{M_1^2+M_2^2}(\partial_\mu\psi\,\partial^\mu\psi-m_{\rm g}^2\psi^2),$$

where $m_{\rm g}$ is the graviton mass. Taking conditions (27) into account, we can then write action (21) as

$$S_{\rm g}=-M_1^2\int d^4x\,R_1\sqrt{-g_1}-M_2^2\int d^4x\,R_2\sqrt{-g_2}-\frac{1}{4}\frac{M_1^2M_2^2}{M_1^2+M_2^2}\int d^4x\,(g_1g_2)^{1/4}(\sigma_2-\sigma_1^2).$$

Hence, only the total action of bigravity results in the Pauli–Fierz theory. We note that the interaction term was obtained in [20] based on semiheuristic reasonings, while we have obtained it in the quadratic approximation framework using the (3+1)-expansion.

5. Conclusions

We have constructed the invariant volume of interaction of multigravity in the general form. We used a particular case of the volume taken as the sum of three different means (only the geometric mean was used in [20]) to analyze the bigravity model. In the framework of the (3+1)-expansion formalism, we rigorously (in the quadratic approximation) proved that the total bigravity Lagrangian (with kinetic terms of the Einstein type taken into account) is equivalent to the massive Pauli–Fierz theory.

Appendix: Expansion of $\sqrt{g}$ in small excitations

Standardly expanding $\sqrt{g}$ in the small excitations $h_{\mu\nu}$, we use the expression $\log(\det \mathsf{g}_{\mu\nu}) = \mathrm{tr}(\log \mathsf{g}_{\mu\nu})$ and obtain

$$\sqrt{g} = \exp\left(\frac{1}{2}\,\mathrm{tr}(\log \mathsf{g}_{\mu\nu})\right).$$

Over the flat background metric $\mathsf{g}_{\mu\nu} = \eta_{\mu\nu} + \mathsf{h}_{\mu\nu}$, we obtain

$$\sqrt{g} = 1 + \frac{1}{2}h - \frac{1}{4}\mathsf{h}_{\mu\alpha}\mathsf{h}^{\mu\alpha} + \frac{1}{8}h^2 \tag{A.1}$$

up to $O(h^2)$, where $h = \mathsf{h}_{\mu\nu}\eta^{\mu\nu}$.

We present the method for calculating the expansion of $\sqrt{g}$, which can be used for any background metric ${}^{(0)}\mathsf{g}_{\mu\nu}$. The general formulas for expanding $\sqrt{g}$ were presented in [34] up to the first order and in [36] up to the second order. We have

$$\mathsf{g}_{\mu\nu} = {}^{(0)}\mathsf{g}_{\mu\nu} + h_{\mu\nu}.$$

Hence (in the case $D = 4$), we obtain

$$\det({}^{(0)}\mathsf{g}_{\mu\nu} + \mathsf{h}_{\mu\nu}) = \varepsilon^{\alpha\beta\rho\sigma}({}^{(0)}\mathsf{g}_{0\alpha} + \mathsf{h}_{0\alpha})({}^{(0)}\mathsf{g}_{1\beta} + \mathsf{h}_{1\beta})({}^{(0)}\mathsf{g}_{2\rho} + \mathsf{h}_{2\rho})({}^{(0)}\mathsf{g}_{3\sigma} + \mathsf{h}_{3\sigma}),$$

where $\varepsilon^{0123} = +1$. Up to $O(h^2)$, we have

$$\det({}^{(0)}\mathsf{g}_{\mu\nu} + \mathsf{h}_{\mu\nu}) = \det({}^{(0)}\mathsf{g}_{\mu\nu}) + \mathsf{h}_{\mu\nu}\mathsf{K}^{\mu\nu}({}^{(0)}\mathsf{g}) + \mathsf{h}_{\mu\nu}\mathsf{h}_{\alpha\beta}\mathsf{F}^{\mu\nu\alpha\beta}({}^{(0)}\mathsf{g}),$$

where

$$\mathsf{K}^{\mu\nu} = \varepsilon^{\alpha\beta\rho\sigma}(\delta^\mu_0\delta^\nu_\alpha{}^{(0)}\mathsf{g}_{1\beta}{}^{(0)}\mathsf{g}_{2\rho}{}^{(0)}\mathsf{g}_{3\sigma} + \delta^\mu_1\delta^\nu_\beta{}^{(0)}\mathsf{g}_{0\alpha}{}^{(0)}\mathsf{g}_{2\rho}{}^{(0)}\mathsf{g}_{3\sigma} + \\ + \delta^\mu_2\delta^\nu_\rho{}^{(0)}\mathsf{g}_{0\alpha}{}^{(0)}\mathsf{g}_{1\beta}{}^{(0)}\mathsf{g}_{3\sigma} + \delta^\mu_3\delta^\nu_\sigma{}^{(0)}\mathsf{g}_{0\alpha}{}^{(0)}\mathsf{g}_{1\beta}{}^{(0)}\mathsf{g}_{2\rho}),$$

$$F^{\mu\nu\alpha\beta} = \varepsilon^{\chi\omega\rho\sigma}(\delta^\mu_0\delta^\nu_\chi\delta^\alpha_1\delta^\beta_\omega{}^{(0)}\mathsf{g}_{2\rho}{}^{(0)}\mathsf{g}_{3\sigma} + \delta^\mu_0\delta^\nu_\chi\delta^\alpha_2\delta^\beta_\rho{}^{(0)}\mathsf{g}_{1\omega}{}^{(0)}\mathsf{g}_{3\sigma} + \\ + \delta^\mu_0\delta^\nu_\chi\delta^\alpha_3\delta^\beta_\sigma{}^{(0)}\mathsf{g}_{1\omega}{}^{(0)}\mathsf{g}_{2\rho} + \delta^\mu_1\delta^\nu_\omega\delta^\alpha_2\delta^\beta_\rho{}^{(0)}\mathsf{g}_{0\chi}{}^{(0)}\mathsf{g}_{3\sigma} + \\ + \delta^\mu_1\delta^\nu_\omega\delta^\alpha_3\delta^\beta_\sigma{}^{(0)}\mathsf{g}_{0\chi}{}^{(0)}\mathsf{g}_{2\rho} + \delta^\mu_2\delta^\nu_\rho\delta^\alpha_3\delta^\beta_\sigma{}^{(0)}\mathsf{g}_{0\chi}{}^{(0)}\mathsf{g}_{1\omega}).$$

The general expression for the expansion of $\sqrt{g}$ then becomes

$$\sqrt{g} = \sqrt{{}^{(0)}g} - \frac{\mathsf{h}_{\mu\nu}\mathsf{K}^{\mu\nu}({}^{(0)}\mathsf{g}) + \mathsf{h}_{\mu\nu}\mathsf{h}_{\alpha\beta}\mathsf{F}^{\mu\nu\alpha\beta}({}^{(0)}\mathsf{g})}{2\sqrt{{}^{(0)}g}} - \frac{\big(\mathsf{h}_{\mu\nu}\mathsf{K}^{\mu\nu}({}^{(0)}\mathsf{g})\big)^2}{8\sqrt{({}^{(0)}g)^3}}, \tag{A.2}$$

where ${}^{(0)}g = |\det{}^{(0)}\mathsf{g}_{\mu\nu}|$.

In the standard case of expansion over a flat metric ${}^{(0)}\mathsf{g}_{\mu\nu} = \eta_{\mu\nu}$ considered in the paper, the expressions for ${}^{(0)}g\ K^{\mu\nu}$ and $F^{\mu\nu\alpha\beta}$ become

$$\det({}^{(0)}\mathsf{g}_{\mu\nu}) = {}^{(0)}g = -1, \qquad \mathsf{K}^{\mu\nu} = -\eta^{\mu\nu}, \qquad \mathsf{F}^{\mu\nu\alpha\beta} = \frac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\mu\nu}\eta^{\alpha\beta}),$$

and we have

$$\sqrt{g} = 1 - \frac{-h + \mathsf{h}_{\mu\nu}\mathsf{h}_{\alpha\beta}(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\mu\nu}\eta^{\alpha\beta})/2}{2} - \frac{(-h)^2}{8} = 1 + \frac{1}{2}h - \frac{1}{4}\mathsf{h}_{\mu\nu}\mathsf{h}^{\mu\nu} + \frac{1}{8}h^2$$

for (A.2). It is important that this expression coincides with (A.1).

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