Gauge Fields on Superspaces with Different Holonomy Groups

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Abstract. We discuss variants of a generally–covariant theory of superfields with nonzero values of the torsion tensors and the curvature tensor.

1. A generally-covariant theory of superfields was recently proposed (Arnowitt, Nath, Zumino (1975), Arnowitt and Nath (1975)) for a space with coordinates $Z^A = (\chi^\mu, \phi^\alpha, \phi^{\dot\alpha})$, ($\chi^\mu$ are the usual spatial coordinates, and ϕ^α and $\phi^{\dot\alpha}$ are anticommuting spinor coordinates).[1]

The generalized Einstein equation for the superspace takes the form (Arnowitt and Nath (1975))

$$R_{AB} = 0 \tag{1}$$

where $R_{AB} = R^C_{AC;B}$ and $R^D_{AB;C}$ is the curvature tensor of the superspace.

In the expansion of the superfield of the metric tensor g_{AB} in terms of ordinary fields, Eq. (1), leads to second–order equations both for the fields with integer spins and for fields with half–integer spin. The latter circumstance is due to the absence from the structure constants of a maximal holonomy group[2] of Riemannian superspace satisfying Eq. (1), of quantities that can play the role of the matrices γ_μ in the equations for fields with half–integer spin, and is a shortcoming of the theory.

We wish to call attention in this paper to the existence of generally–covariant theories free of the foregoing shortcoming.

2. The Cartan equation for a superspace with coordinates z_A can be written in the form (Volkov, Soroka (1972))

$$\mathrm{d}\omega^A(\delta) + \omega^B(\delta) \wedge \mathit{\Gamma}_B{}^A(\mathrm{d}) = \frac{1}{2}\omega^B(\delta) \wedge \omega^C(\mathrm{d}) T_{CB}{}^A \ , \tag{2}$$

$$\mathrm{d}\mathit{\Gamma}_A{}^B(\delta) + \mathit{\Gamma}_A{}^C(\delta) \wedge \mathit{\Gamma}_C{}^B(\mathrm{d}) = \frac{1}{2}\omega^C(\delta) \wedge \omega^D(\mathrm{d}) R_{DC;A}{}^B \ , \tag{3}$$

where T^A_{CB} and $R^B_{DC;A}$ are the torsion and curvature tensors. The differentiations and the products of the forms in expressions (2) and (3) are external

[1] For a detailed bibliography on supersymmetry see Zumino (1974), where no reference is made, however, to the pioneering paper Golfand and Likhtman (1971).

[2] The holonomy group is a group of transformations of a reference frame in parallel transfer of the latter along an infinitesimal closed contour.

and are determined, just for ordinary spaces, by alternating the differentials d and δ.

We choose as the holonomy group of the considered superspace the Poincare group supplemented by translation of the spinor variables. In this case only the components $\Gamma^{\beta}_{\alpha}(\mathrm{d}), \Gamma^{\dot\beta}_{\dot\alpha}(\mathrm{d})$, and $\Gamma^{\nu}_{\mu}(\mathrm{d})$ of the differential form of connectivity differ from zero and satisfy the relations

$$2g^{\mu\rho}\Gamma^{\nu}_{\rho}(\mathrm{d}) = \Gamma^{\alpha}_{\beta}(\mathrm{d})(\sigma^{\mu\nu})^{\beta}_{\alpha} + \Gamma^{\dot\alpha}_{\dot\beta}(\mathrm{d})(\sigma^{\mu\nu})^{\dot\beta}_{\dot\alpha} \ , \tag{4}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2}(\sigma^{\mu}\sigma^{\nu} - \sigma^{\nu}\sigma^{\mu}) \ .$$

The invariant action integral for the superfields that determine the differential forms $\omega^{A}(\mathrm{d})$ and $\Gamma^{A}_{B}(\mathrm{d})$, can be represented in the form

$$\int L(R,T)W\prod_{A}\mathrm{d}z^{A} \tag{5}$$

where L is an invariant function of the curvature and torsion tensors, and

$$W = \det|\omega_{A}{}^{B}| \ , \tag{6}$$

where the matrix ω^{B}_{A} is determined by the form coefficients[3] $\omega^{B}(\mathrm{d}) = \mathrm{d}z^{A}\omega^{B}_{A}$.

In the simplest case, L is a function of the following invariant quantities

$$R_1 = \mathrm{i}[R_{\mu\nu;\alpha}{}^{\beta}(\sigma^{\mu\nu})^{\alpha}_{\beta} - R_{\mu\nu;\dot\alpha}{}^{\dot\beta}(\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta}] \ , \tag{7a}$$

$$R_2 = \mathrm{i}(R_{\alpha\beta;\gamma}{}^{\beta}\,\epsilon^{\alpha\gamma} - R_{\dot\alpha\dot\beta;\dot\gamma}{}^{\dot\beta}\,\epsilon^{\dot\alpha\dot\gamma}) \ , \tag{7b}$$

$$T_1 = \mathrm{i}T_{\alpha\dot\beta;}{}^{\mu}\,(\sigma_{\mu})^{\dot\beta\alpha} \ , \tag{7c}$$

$$T_2 = \mathrm{i}[T_{\mu\dot\alpha;}{}^{\beta}\,(\sigma^{\mu})^{\dot\alpha}_{\beta} - T_{\mu\alpha;}{}^{\dot\beta}\,(\sigma^{\mu})^{\alpha}_{\dot\beta}] \ . \tag{7d}$$

3. We consider the case when

$$L = a_1R_1 + a_2R_2 \ . \tag{8}$$

The variation of the superfields for individual terms of the action integral (5) and (8) takes the form

$$(\widetilde{R_{DC;A}{}^{B}W}) = \left\{\left[-\omega^{F}_{C}R_{FD;A}{}^{B} + \frac{1}{2}(-)^{F}\tilde\omega^{F}_{F}R_{CD;A}{}^{B} - (-)^{F}T_{CF;}{}^{F}\tilde\Gamma^{A}_{DB} + \right.\right.$$
$$\left.\left. + \frac{1}{2}T_{CD}{}^{F}\tilde\Gamma^{B}_{FA}\right] - (-)^{CD}(C\leftrightarrow D)\right\}W \ , \tag{9}$$

[3] The holonomy group is a group of transformations of a reference frame in parallel transfer of the latter along an infinitesimal closed contour.

where the tilde over a quantity denotes its variation, and the factors of the form $(-)^F$ determine the signs of the corresponding terms, depending on their graduation.

The equations for the superfields are established by substituting the expression (9) in (7a,b) and by equating to zero the coefficients of the independent variations $\tilde{\omega}^B_A, \tilde{\Gamma}^B_{A\alpha}$, and $\tilde{\Gamma}^{\dot{B}}_{A\alpha}$.

An essential difference between the equations obtained from the Lagrangian (8) and the generalized Einstein equations considered in (Arnowitt, Nath, Zumino (1975), Arnowitt and Nath (1975)) is the fact that as a result of the weakening of the holonomy group not all the components of the torsion tensor are equal to zero[4]. In particular, the Lagrangian (8) contains a solution corresponding to superspace with constant torsion tensor

$$T^{\mu}_{\alpha\dot{\beta}} = \mathrm{i}t(\sigma^{\mu})_{\alpha\dot{\beta}} \tag{10}$$

for which

$$\Gamma^{\beta}_{A\alpha} = \Gamma^{\dot{\beta}}_{A\dot{\alpha}} = 0, \quad \omega^{\alpha} = d\phi^{\dot{\alpha}}, \quad \omega^{\dot{\alpha}} = \mathrm{d}\phi^{\dot{\alpha}}$$

$$\omega^{\mu}(\mathrm{d}) = \mathrm{d}x^{\mu} + \frac{t}{2\mathrm{i}}(\phi\sigma^{\mu}\mathrm{d}\phi^{+} - \mathrm{d}\phi\sigma^{\mu}\phi^{+}) \ . \tag{11}$$

Such a superspace, first introduced in (Volkov, Akulov (1972, 1974), Salam and Strathdee (1974)), constitutes the basis of different variants of supersymmetry theory.

For solutions that differ little from (11), the Lagrangian leads to typically supersymmetric structure of the superfields.

4. In the presence of invariant combinations of the torsion tensor in the Lagrangian, the variations of such combinations can be expressed in terms of the quantity

$$\begin{aligned}(\widetilde{T^A_{BC;}W}) = \Bigg\{ \Bigg[&-\tilde{\omega}^F_B T^A_{FC;} + \frac{1}{2}(-)^F \tilde{\omega}^F_F T^A_{BC;} + (-)^{AC+F}\omega^A_B T^F_{FC} \\ &+ \frac{1}{2}(-)^{(A+F)(B+C+F)}\tilde{\omega}^A_F T^F_{BC;} + \tilde{\Gamma}^A_{BC:} \Bigg] - (-)^{BC}(B \leftrightarrow C) \Bigg\} W\end{aligned}$$

$$\tilde{W} = (-)^F \tilde{\omega}^F_F W \ . \tag{12}$$

The requirement that the space with a constant vector torsion be edmitted by the equations of motion leads in this case to definite relations between the constants at different degrees of the torsion tensor (7c) in the Lagrangian.

[4] We note that if we require in addition that all the components of the torsion tensor be equal to zero, then the latter holds for the Lagrangian (8) only under the condition that all the components of the curvature tensor also vanish.

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