Goldstone and gauge fields for symmetry groups, containing Poincaire group*

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October 20, 1972

Abstract

The scheme of Goldstone and gauge fields introduction for groups of symmetry, containing Poincaire group as subgroup, is considered. The scheme is illustrated by examples of Goldstone field with spin 1/2 and gauge fields for $SU(6, C)$ symmetry group.

1. In elementary particles theory the particles, the existence of which is directly stipulated, and the interaction is defined by the invariance demands relative interactions of some symmetry group, play a special role. Thus these particles are first of all particles, connected with gauge fields, among them photon, graviton, various vector bosons, as well as Goldstone particles, which appeared in systems with degenerated vacuum.

Under the given group of inner symmetry theoretical description of gauge and Goldstone fields can be directly performed by standard procedure [1-4]. Inner quantum numbers of particles under this are simply defined by the group of symmetry under consideration, and their spin equals 0 in Goldstone case and 1 in gauge fields case.

The study of gauge and Goldstone fields with other spin values is of great interest and demands the groups of symmetry consideration, which obtain space-time coordinates interaction. At present such consideration is performed only for Poincaire groups and conformal groups.

Consideration of more complicated groups, connecting nintrivially inner symmetry group with Poincaire group, meets a number of difficulties, stipulated in such a way, that under such groups transformations the space-time coordinates are jointly transformed, while their role in equations of motion is essentially different.

*A KFTI preprint (covering ideas of the period of 1970-1972) which was submitted somewhere for publication, but *not* ever published. Translated into English by Tatiana Kudryashova. Prepared for publication by Steven Duplij under permission of Stalina Ivanovna Volkova.

In [4] the removal way of the mentioned difficulties by change to such formulation of variational principle for integral action, under which space-time coordinates stop playing the underlined role, and motion equations define some extremal 4-dimentional surface in homogeneous space of the group. The change to standard formalism by parametrization of the points of extremal surface under study by space-time coordinates.

Analogous procedure can be used for gauge fields consideration as well. In this if gauge fields are introduced on some subgroup H of the considered transformation group G, then variational principle for action integral defines some extremal 4-dimentional surface in fibred space, the basis of which is a homogeneous space of group G, and the fibres – space of frames, corresponding to subgroup H.

The consideration of Goldstome and gauge fields with half-spin [5] leads to additional difficulties, connected with Fermi particle statistics and with the existence of anticommutators in commutation relations. The mentioned difficulty can be removed by the use of symmetry groups, containing both commuting and anticommuting parameters. Such groups of symmetry were considered recently in the work by Berezin and Kac [6].

2. First of all let us consider a general scheme of space invariant action integral for Goldstone fields for arbitrary symmetry group, containing Poincaire group as a subgroup. Under this we will follow the work account [5].

Let G be such symmetry group. Let us imagine arbitrary group element G as:

$$G = e^{iPx} K(a) H(b_1 l) \tag{1}$$

where P is the energy-momentum operator, x are space-time coordinates and H is the subgroup of transformations, leaving the vacuum invariant. The subgroup transformations H include Lorentz group transformations with parameters l.

From the representation (1) by Poncarae group transformation coordinates x are transformed in a common way. Under the action on their representation (1) on the left side by arbitrary group element G coordinates x are transformed together with parameters a, which according to definition are compared with Goldstone particles fields.

For finding the action of invariant integral it is convenient to use differential Cartan forms, defined by the decomposition of the expression $G^{-1}dG$ on group generators G.

For generators, corresponding to two first multiplies (1) the Cartan forms have the following form:

$$\omega_i = \omega_i(x, a, dx, da, b, l) \tag{2}$$

and are the group G transformation invariant under the condition, that all arguments of the forms under study are transformed. Under fixed parameters b and l meanings the forms ω_i are transformed according to the subgroup H representations. Invariant relative to subgroup H transformations the combinations of the form ω_i do not depend on parameters b and l and are invariants of the group G.

For generators of the subgroup H of Cartan form

$$\Theta_\alpha = \Theta_\alpha(x, a, dx, da, b, l, db, dl) \tag{3}$$

contains additional dependence from differentials db and dl , as a consequence, such differential forms cannot directly be used under construction of invariant action integral as independent invariants. Though together with other differential forms, such that differentials of the form ω_i (2) or differentials of the fields of dependent on the right side from parameters b and l, the forms Θ_α (3) are used under the definition of covariant differentiation operation:

$$D\psi(b, l) = d\psi(b, l) + \Theta_\alpha \Upsilon_\alpha \psi(b, l) \tag{4}$$

where Υ_α are subgroup H generators.

For the construction of invariant action integral it is sufficient to consider the combinations of ω forms , ψ fields and their covariant differentials, invariant relative subgroup H transformations and representable under condition, that a parameters and ψ fields are x functions, as:

$$S = \int L(a, \psi, \frac{\partial a}{\partial x}, \frac{\partial \psi}{\partial x}) d^4x \tag{5}$$

Presence of volume element d^4x in the expression (5) put essential constraints on the structure of possible combinations. In the simplest way when the considered combination is the product of some ω forms and their covariant differentials, the general number of differentials according to (5) must be 4. besides, the demand of element volume factorization d^4x in (5) will be satisfied under the condition, that the product of ω form and differentials of these forms are external products and external differentials.

In the general case the admitted invariant combinations are homogeneous relative differentials functions, in which each from differentials belongs to some antisymmetrized differentials of 4th degree. Antisymmetrized fourfolds of differentials define infinitesimal volumes on some 4-dimentional surface. Under parametrization of the surface coordinates by space-time coordinates x each from the considered fourfolds, and due to homogeneity each from admitted invariants are transformed to the type, corresponding to subintegral term of the expression (5) in the expression (5) the a variables correspond to Goldstone particles fields, ψ variables correspond to all other fields.

Note, that the general construction of invariant integral action scheme, considered above, does not completely solve all the problems of phenomenological description of Goldstone particles. Under consideration of every special case it is necessary to make sure, that the action integral (5) satisfies all other necessary demands of correct field theory. In presence in Goldstone particles the spin, the most essential from such demands is the absence of "ghost" states, connected with longitudinal and scalar components of Lorentz group representations, for Goldstone particles fields.

3. As an example, illustration the general scheme, let us consider the action integral for Goldstone particles with spin 1/2 [5]. In the connection with the fact, that neutrino, as well as other leptons, have spin 1/2 and are the easiest from known elementary particles possibilities to interpret these particles as Goldstone particles, is no doubt of great interest from physical point of view as well.

Let us consider as G group the space-time coordinate x transformation group, ψ spinor and ψ^+ Hermitian conjugate spinor, which besides Poincaire group transformations contain the following transformations with ς parameters

$$\begin{aligned} \psi &\rightarrow \psi' = \psi + \varsigma \\ \psi^+ &\rightarrow \psi^{+\prime} = \psi^+ + \varsigma^+ \\ x_\mu &\rightarrow x'_\mu = x_\mu - \frac{a}{2i}(\varsigma^+ \sigma_\mu \psi - \psi^+ \sigma_\mu \varsigma), \end{aligned} \tag{6}$$

where σ_r are relativistic Pauli matrices and all spinor values are implied anticommuting. The constant a is arbitrary and has dimension of 4th length order.

For the phenomenological integral action construction let us consider differential forms, invariant relatively transformations (6)

$$\omega_r = dx_\mu + \frac{a}{2i}(\psi^+ \sigma_\mu d\psi - d\psi^+ \sigma_\mu \psi) \tag{7}$$

Admitting, that the interaction terms contain minimal derivatives number, compatible with invariance demand, action integral is homogeneously defined and has the following form:

$$S = \frac{1}{a} \int \omega_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3, \tag{8}$$

where the sign $\wedge$ means the external product.

The expression (8) corresponds to some invariant 4-dimentional volume in group parameters space. Under the 4-volumes definition by function $\psi = \psi(x)$ set the action integral (8) can be written in the next more common form:

$$S = \frac{1}{a} \int \mid W \mid d^4x$$

where $\mid W \mid$ is determinant of matrix W

$$\begin{aligned} W_{\mu\nu} &= \delta_{\mu\nu} + aT_{\mu\nu} \\ T_{\mu\nu} &= \frac{1}{2i}(\psi^+ \sigma_\mu \partial_\nu \psi - \partial_\nu \psi - \partial_\nu \psi^+ \sigma_\mu \psi) \end{aligned} \tag{9}$$

From (6) and (7) it follows, that action integral as tensor function T has the following form:

$$S = \int [\frac{1}{a} + T_{\mu\mu} + \frac{a}{2}(T_{\mu\mu} + T_{\nu\nu} - T_{\mu\nu}T_{\nu\mu}) + \frac{a^3}{3!}\sum_{\rho}(-1)^p T_{\mu\mu}T_{\nu\nu}T_{\rho\rho} + \frac{a^3}{4!}\sum(-1)^p T_{\mu\mu}T_{\nu\nu}T_{\rho\rho}T_{\sigma\sigma}, \quad (10)$$

where $\sum_{\rho}$ corresponds to the sum in second indices transpositions in T tensor products.

The $T_{\mu\mu}$ term corresponds to the kinetic term for massless spinor Goldstone field. The additions, containing two, three and four T tensor products describe interaction with the presence of corresponding four, six, eight fields. The derivative level from interaction fields equals T comultiplies number.

Goldstone particles interaction with other fields and be defined by invariant relative transformations (6) method.

4. The general procedure, described in item 2 can be applied both for gauge fields consideration.

For such generalization consideration let us notice, that differential forms (2) and (3) have simple geometrical sense and define some linear connection in space with x and a coordinates. At the same time (2) define infinitesimal shifts, corresponding to dx and da differentials, in some basis in x and a points and with b and l parameters. The latter are connected with H subgroup transformation and define the orientation of the considered basis relative to some fixed basis in the same point. The forms (3) are proper forms of connectiveness and define the operation of covariant differentiation for all subgroup representations.

Linear connectiveness forms are not simply defined. The linear forms (2) and (3) combinations of the form

$$\omega_i^{'} = \omega_i + a_{ik}\omega_k \quad (11)$$

$$\Theta_\alpha^{'} = \Theta_\alpha + b_{\alpha k}\omega_k \quad (12)$$

also define some linear connectiveness. Under this, if a_{ik} and $b_{\alpha k}$ coefficients are constant, then (2') and (3') forms, as well as (2) and (3) forms are invariant relatively to G group transformations.

The most common point of view, corresponding to gauge fields consideration, comprises a_{ik} and $b_{\alpha k}$ coefficients comparison in (2') and (3') independent variables, which under G group transformations are transformed in such a way that (2') and (3') forms be transformed the same way as (2) and (3). a_{ik} and $b_{\alpha k}$ derivatives dependence from x is defined in this case from variational principle for invariant integral action, expressed through (2') and (3') forms and their covariant differentials.

The simplest example for action integral, comprising (2') and (3') forms, is the following external product

$$d^4 I_1 = \frac{1}{2_i}(D\Theta^A_B \omega^B_{\dot{C}} \omega^{\dot{C}}_A - D\Theta^{\dot{A}}_{\dot{B}} \omega^{\dot{B}}_C \omega^C_{\dot{A}}), \tag{13}$$

which is invariant under $SL(n, C)$ group transformations. Under parity transformations, consisting in index exchange without points on index with points and vice versa, the expression (9') changes the sign and, consequently, is a pseudoscalar.

$D\Theta^A_B$ in (9') defines the covariant external differential and is defined by the expression

$$D\Theta^A_B = d\Theta^A_B(\delta) - \delta\Theta^A_B(d) + \Theta^A_C(d)\Theta^C_B(\delta) - \Theta^A_C(\delta)\Theta^C_B(d) \tag{14}$$

To action integral (9') the terms, not containing derivatives can be added, the simplest of which is the of 4 forms external product

$$d^4 I_\alpha = \omega^A_{\dot{B}} \omega^{\dot{B}}_C \omega^C_{\dot{D}} \omega^{\dot{D}}_A \tag{15}$$

The action integral (10') and (11) describes gauge field multiplet with spin 2^+ and 1^- .

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