Phenomenological Lagrangian for Spin Waves

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Abstract. A phenomenological Lagrangian describing the interaction of long–wavelength spin waves is obtained. The derivation of the Lagrangian is based on nonlinear realizations of the symmetry group of the Heisenberg Hamiltonian and on the assumption that the spin-wave excitation mechanism on ferromagnets, ferrites, and antiferromagnets is of the Goldstone type.

1. At low temperatures, the excitation spectrum of solids is governed by collective excitations, such as phonons, spin waves, etc. The collective excitations have been extensively studied using various methods. In our view, it is of great interest to establish rigorously the existence of a special type of collective excitation, i.e., Goldstone's particles (or quasiparticles Goldstone (1961)). The energy spectrum of such excitations has a special form (the excitation frequency $\omega(k)$ vanishes when the quasimomentum k tends to zero[1]), and the excitations are related to the symmetry properties of systems with a large number of degrees of freedom.

The fact that spin waves in ferromagnets, antiferromagnets, and ferrites correspond to Goldstone excitations is well known (see, for example, Hugenholtz (1967)). The following qualitative excitation mechanism of spin waves was proposed: the existence of a preferred direction of the magnetization breaks the symmetry of the original microscopic Hamiltonian of the system and, as a result, collective excitation (i.e., spin waves) are created, which tend to restore the broken symmetry.

A similar excitation mechanism is assumed in the case of general Goldstone particles and they are regarded as a specific reaction of the system to the symmetry breaking of the ground state. In connection with the discovery of the approximate "chiral" symmetries in elementary particle physics and of the properties of π mesons (pseudoscalar meson octet) with the properties of Goldstone particles of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ $[\mathrm{SU}(3)\times\mathrm{SU}(3)]$ symmetry group, a new phenomenological method of description of the interaction of Goldstone particles was recently proposed (Nambu and Jona–Lasinio (1961); Weinberg (1967); Schwinger (1967)). This method is based on the assumption that the symmetry of the system of Goldstone particles is fully restored.

[1] It will be shown that this relationship between the frequency and the quasimomentum $[\omega(k)\to 0$ if $|k|\to 0]$, which is obeyed by relativistic Goldstone particles, may not be satisfied in the case of essentially nonrelativistic particles (see Sec.3).

The mathematical formulation of this assumption is equivalent to the requirement that the phenomenological Lagrangian describing the interaction of Goldstone particles is strictly invariant under the transformations of the symmetry group in question. In fact, the Lagrangian can be expressed in terms of variables with a nonlinear transformation law (under the transformations of the group in question) which correspond to the Goldstone fields. The most general formulation of the Lagrangian for relativistic Goldstone particles with a dispersion law $\omega^2 - k^2 = 0$, which is valid for antisymmetry group, has been given by Coleman et al. (1969); Callan et al. (1969), and by Volkov (1969).

Our aim is to describe the interaction of an arbitrary number of "soft" spin waves, using the phenomenological Lagrangian method. This is equivalent to the approximation in which only the lowest powers of the magnon energy and momentum are retained in all the interaction matrix elements. In Sec. 2, the case of antiferromagnetic spin waves is discussed. There is a complete correspondence between this simplest case and the situation encountered in the elementary particle physics. In Secs. 4 and 5, the cases of more complicated excitation spectra are considered. The most general case is discussed in Sec. 5.

We shall try to make our treatment essentially independent of other approaches but, for brevity, we have to omit the proofs of important concepts in the phenomenological Lagrangian method. For example, we shall omit the proof that the matrix elements of the S matrix on the mass surface are independent of the actual choice of the parametrization of the spin-wave operators. Likewise we omit the justification of the term which describes the interaction of spin waves with other quasiparticles, etc, (see, for example, Coleman et al. (1969); Callan et al. (1969), Volkov (1969)).

2. To illustrate the general method of phenomenological Lagrangians, we shall make use of the well-known boson creation and annihilation operators of spin waves, which were introduced by Holstein and Primakoff (1941) and by Dyson (1956). We shall consider the Heisenberg Hamiltonian describing the exchange interaction

$$H_H = \sum \mathcal{J}_{ik} \mathbf{S}_i \mathbf{S}_k \ . \qquad (1)$$

The transition from the Hamiltonian defined by Eq. (1) to the Holstein–Primakoff-Dyson Hamiltonian, which describes the spin–wave interaction, can be accomplished by expressing the spin operators $\mathbf{S}_i$ in terms of the creation and the annihilation operators a_i^+, a_i. In the phenomenological Lagrangian method, the actual form of the representation of the operators $\mathbf{S}_i$ in terms of the operators a_i^+, a_i is not relevant. The crucial fact is that, irrespective of the representation in question, the rotational symmetry of the Hamiltonian in the spin space is conserved, i.e., the Hamiltonian is invariant under the transformations of the SO(3) symmetry group. The only property which depends on the actual form of the representation of $\mathbf{S}_i$ is the form of the transformations of the operators a_i, a_i^+, which realize the transformations of the SO(3) group. Since the relationship between the operators $\mathbf{S}_i$ and the operators a_i^+, a_i is nonlinear, the transformation of the operators a_i^+, a_i under the action of the SO(3) group is also nonlinear.

However, the direct determination of the symmetry properties of the spin–wave interaction Hamiltonian in the Holstein-Primakoff-Dyson form is very complicated and, as far as we know, has not been carried out successfully.

Therefore, we shall try to deduce the transformation properties of the spin–wave operators under the action of the SO(3) group independently of the form of the Hamiltonian defined by Eq. (1) and of the actual form of the representation of the operators $\mathbf{S}_i$ in terms of a_i, a_i^+. On the basis of the transformation properties of the spin–wave operators and the invariance of the spin–wave interaction Lagrangian under the transformations of SO(3) group, we shall obtain, in the "soft–magnon' limit, the interaction Lagrangian, which depends only on several phenomenological parameters.

It should be noted that there is an important difference between the Holstein-Primakoff-Dyson approach and the phenomenological Lagrangian method. In fact, in the former case, the operators a_i, a_i^+ correspond to the nonrenormalized magnon operators. Therefore, in the calculation of the interaction between real magnons, it is necessary to take into account all the Feynman diagrams, including the closed loops. In the long–wavelength limit, these diagrams lead to a renormalization of the energy spectrum and of the interaction constants. On the other hand, in the phenomenological Lagrangian method, all such effects are assumed to have been taken into account, i.e., the local fields in question correspond to renormalized phenomenological magnon fields. On other words, the phenomenological Lagrangian method assumes that, in the "soft–magnon" limit, the matrix elements of the S matrix are smooth functions of the real–magnon momenta (with the exception of poles which are due to magnon exchange). The terms containing poles, which can be deduced from the Lagrangian derived solely on the basis of symmetry considerations, correspond to "tree–like" diagrams (diagrams without loops).

The proposed program constitutes the basis of the phenomenological Lagrangian method.

3. We shall demonstrate the derivation of the phenomenological Lagrangian in the simplest case of antiferromagnetic spin waves. The spin–wave spectrum in an antiferromagnet is doubly degenerate and, for small k, the magnon frequency is a linear function of k . Therefore, noninteracting antiferromagnetic spin waves can be described by two local–field operators $A_i(\mathbf{x}, t)$ $(i = 1, 2)$ which satisfy the following equation:

$$\partial^2 A_i(\mathbf{x}, t) - c^2 \nabla^2 A_i(\mathbf{x}, t) = o, \tag{2}$$

where ∂ and ∇ denote, respectively, the derivatives with respect to time and space. Equation (2) can be derived from the Lagrangian

$$L = \frac{1}{2}\{\partial A_i(\mathbf{x},t)\partial A_i(\mathbf{x},t) - c^2\nabla A_i(\mathbf{x},t)\nabla A_i(\mathbf{x},t)\} \ . \tag{3}$$

To obtain the Lagrangian which describes the interaction of spin waves, we shall assume that the transformation of the local fields $A_i(\mathbf{x},t)$ under the action of the SO(3) group is governed by the following equations:

$$\delta A_i(\mathbf{x},t) = \varepsilon_3\varepsilon_{ik3}A_i(\mathbf{x},t) \ , \tag{4}$$

$$\delta A_i(\mathbf{x},t) = \varepsilon_i(1 - f^2A^2(\mathbf{x},t)) + f^2\cdot 2(\varepsilon_k A_k)A_i \ , \tag{5}$$

where $f^2 \neq 0$. Here, ε_3 and ε_i are the parameters of infinitesimal transformations of the SO(3) group and $\varepsilon_3\varepsilon_{ik3}$ is an antisymmetric tensor. Equations (4) and (5) describe the infinitesimal transformations of the three–parameter SO(3) group . These equations have the following simple geometrical meaning: the fields $A_k(\mathbf{x},t)$ represent the coordinates in a plane corresponding to the stereographic projection of a sphere of radius $1/f$ on the plane; the coordinates in the plane transform according to Eqs. (4) and (5) under infinitesimal rotations of the sphere. Therefore, Eqs. (4) and (5) correspond to the introduction of a coordinate system on a sphere which has the property that each coordinate corresponds to a local field A_i; they also describe the transformations of these fields under infinitesimally small rotations of the sphere. For $f^2 = 0$, the transformations (4) and (5) correspond to the transformation group of a plane. Clearly, the Lagrangian defined by Eq. (3) is invariant under such transformations.

We shall require the interaction Lagrangian to be invariant under the transformations (4) and (5) for $f^2 \neq 0$. In the "soft–magnon" limit, the invariant Lagrangian is defined uniquely as follows:

$$L = \frac{1}{2}\frac{\partial A_i\partial A_i - c^2\nabla\partial A_i\nabla\partial A_i}{(1+f^2A_k^2)^2} \ . \tag{6}$$

The constant f^2 which appears in Eq. (6) is a phenomenological constant which governs the interaction strength of spin waves.

The interaction of an arbitrary number of "soft" magnons can be described by means of an S matrix, corresponding to the Lagrangian defined by Eq. (6), in the approximation of "tree–like" diagrams.

For two–magnon scattering processes, the matrix element is governed by the first term in the expansion of Eq. (6) in powers of f^2 :

$$L' = -f^2(\partial A_i\partial A_i - c^2\nabla\partial A_i\nabla\partial A_i)A_k^2 \tag{7}$$

and is identical with the matrix element which was discussed in (Cartan (1910)).

The many–magnon scattering processes are governed by higher–order terms in the expansion of the S matrix in powers of f^2. Because of the nonlinearity of the transformations (4-5), the symmetry properties make it possible to obtain relationships between the matrix elements corresponding to different numbers of magnons.

It should be noted that the form of the Lagrangian defined by Eq. (6) depends on the form of the realization of the SO(3) group transformations. If we perform in Eqs. (4) and (5) the following transformation to new local fields $A'(x)$ (which corresponds to the choice of new coordinates on a sphere):

$$A'x = A(x) + \eta\ (A(x)),$$

where η is an arbitrary analytic function such that $\eta(0) = 0$, Eqs. (4) and (5) and the Lagrangian defined by Eq. (6) will have a different form in these new fields. Nevertheless, the form of the S matrix on the mass surface will be the same for the original and the transformed Lagrangians (see Coleman et al. (1969); Callan et al. (1969), Volkov (1969)).

We shall write the Lagrangian in a form which is convenient in many problems and which corresponds to normal coordinates on a sphere, i.e.,

$$L = \frac{1}{2} g_{\alpha\beta}(A_\gamma)(\partial A_\alpha \partial A_\beta - c^2 \nabla A_\alpha \nabla A_\beta)\ , \tag{8}$$

where

$$g_{\alpha\beta}(A_i) = \delta_{\alpha\beta} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+2)!} (m^k)_{\alpha\beta} \tag{9}$$

and

$$m_{\alpha\beta} = \tilde{J}^2 (A_k^2 \delta_{\alpha\beta} - A_\alpha A_\beta)\ ; \tag{10}$$

$(m^k)_{\alpha\beta}$ in Eq. (9) denotes the k-th power of the matrix $m_{\alpha\beta}$ defined by Eq. (10).

Equation (9) can be also written in the form

$$g_{\alpha\beta}(A) = \delta_{\alpha\beta} + \frac{m_{\alpha\beta}}{A^4 \tilde{f}^4} \left\{ \frac{1 - \cos 2\sqrt{\tilde{f}^2 A_k^2}}{2} - \tilde{J}^2 A_k^2 \right\}\ . \tag{11}$$

It should be noted that Eq. (8) is invariant under the transformations (4) and also under the transformations

$$\delta A_\alpha = (\sqrt{m}\ \mathrm{ctg} \sqrt{m})_{\alpha\beta}\ \varepsilon_\beta\ , \tag{12}$$

which realize the transformations of the SO(3) group. The quantities ε_β in Eq. (12) are the parameters of an infinitesimal transformation.

For a two–magnon process, we obtain ($\tilde{f} \equiv 2f$)

$$L = -\frac{(2f)^2}{3!} \{ A_\alpha^2 [(\partial A_\alpha)^2 - c^2 (\nabla A_\alpha)^2] - [(A_\alpha \partial A_\alpha)^2 - c^2 (A_\alpha \nabla A_\alpha)^2] \}\ . \tag{13}$$

Let us compare this matrix element with that defined by Eq. (7). To simplify Eq. (13), we shall make use of the laws of conservation of energy and momentum:

$$\partial[A_\alpha^2(A_\beta \partial A_\beta)] = c^2 \nabla[A_\alpha^2(A_\beta \nabla A_\beta)] = 0 \tag{14}$$

which implies

$$2[(A_\alpha \partial A_\alpha)^2 - c^2((A_\alpha \nabla A_\alpha)^2] + A_\alpha^2[(\partial A_\alpha)^2 - c^2(\nabla A_\alpha)^2] =$$
$$= -A_\alpha^2[(A_\alpha \partial^2 A_\alpha) - c^2(A_\alpha \nabla^2 A_\alpha)] \ . \tag{15}$$

Unless stated otherwise, we shall always use the abbreviations $(A_\alpha \partial A_\alpha)$, $(A_\alpha \nabla A_\alpha)$, etc., for $\sum_{\alpha=1}^{2}(A_\alpha \partial A_\alpha), \sum_{\alpha=1}^{2}(A_\alpha \partial A_\alpha)$, etc.

Therefore, on the mass surface, Eq. (13) becomes identical with L':

$$L' = L = -f^2 A_\alpha^2[(\partial A_\alpha)^2 - c^2(\nabla A_\alpha)^2]. \tag{16}$$

However, it should be noted that this fact is trivial since, for the Lagrangian defined be Eq. (8), where $g_{\alpha\beta}(A)$ is an arbitrary function, the matrix element of the scattering on the mass surface can be always brought to the form (16) by the transformation (15), irrespective of the symmetry properties of the Lagrangian.

4. The case discussed in Sec. 3 concerns spin waves with a linear dispersion law. Let us now investigate more complicated dispersion laws. It should be noted that expressions (6) and (8) are the only invariants under the transformations (4-5) which do not contain higher than second-order derivatives. Nevertheless, it will be shown later that terms containing the first derivative of arbitrary fields with respect to time can be added to expressions (6) and (8). Such terms are not invariant under the transformations of the SO(3) group but, since the resulting contribution represents the total time derivative, these terms do not influence the physical properties of the system. On the other hand, the presence in the Lagrangian of the terms which contain first derivatives with respect to time changes the form of the magnon spectrum, which makes it possible to use the phenomenological Lagrangian method to describe the interaction of spin waves in ferrites and ferromagnets.

Let us consider the following expression, which is linear in the first derivative with respect to the field A_α:

$$\omega_3 = 2\,\frac{A_1 \partial A_2 - A_2 \partial A_1}{1 + f^2 A_\alpha^2} \ . \tag{17}$$

Using the explicit form of the transformation (4), it can be shown that Eq. (17) is invariant under this transformation.

The change in ω_3 as a result of the transformation (5) has the form

$$\delta\omega_3 = 2\partial(\epsilon_1 A_2 - \epsilon_2 A_1) \ , \tag{18}$$

i.e., it is a total time derivative.

Adding to the Lagrangian defined by Eq. (6) the expression (17) multiplied by an arbitrary phenomenological constant, we obtain a modified expression for the Lagrangian whose invariance properties are defined up to terms containing total time derivatives.

It can be shown that, in the case of two spin wave branches and for a given choice of the coordinates on a sphere, the term (17) is defined uniquely. This problem will be discussed in detail in the next section.

Let us investigate the form of the spectrum generated by the Lagrangian defined by Eq. (6) with the additional term (17). Retaining in Eqs. (6) and (17) only the terms bilinear in the fields, we obtain the following expressions in the momentum representation:

$$\left.\begin{aligned} (\omega^2 - c^2\mathbf{k}^2)A_1 + 2ib\omega A_2 = 0, \\ (\omega^2 - c^2\mathbf{k}^2)A_2 - 2ib\omega A_1 = 0, \end{aligned}\right\} \tag{19}$$

where $1/2\, b$ is the phenomenological constant in front of the term (17) in the total Lagrangian.

It follows from Eq. (19) that the spectrum of spin waves has the form

$$\omega = \sqrt{b^2 + c^2k^2} \pm b \ , \tag{20}$$

which is identical with the spectrum of magnons in ferrites and, for large b, with the spectrum of ferromagnetic magnons (Akhiezer et al. (1968)). It should be noted that the parameter b is proportional to the internal field in the system. In fact, this follows either from a comparison of Eq. (20) with the standard expressions for the spin–wave spectra in ferrites, or directly from the transformation properties of Eq. (17) under the reflection of the spatial coordinates. The Lagrangian of a ferromagnet corresponds to the sum of Eq. (17) and the second term in Eq. (6).

As in the preceding section, the expansion of the Lagrangian defined by Eqs. (6), (17) in powers of f^2 makes it possible to obtain the matrix elements of the S matrix for an arbitrary number of "soft magnons." By analogy with the relativistic theory, it is assumed in the discussion of the Goldstone excitations in solids that ω vanishes in the limit $\mathbf{K} \to 0$. It follows from our discussion that, as far as the breaking and the subsequent restoration of the symmetry is concerned, the two branches in Eq. (2) are completely equivalent.

Furthermore, only in the limit $b \to \infty$ (ferromagnetic case) can the restoration of the symmetry be achieved by considering only one excitation branch. As already noted, this special feature of the present model (in contrast to the relativistic case) is due to the fact that additional terms linear in the field derivatives can be added to the Lagrangian.

As before, the S matrix elements describing many-magnon scattering processes can be obtained by expanding the phenomenological Lagrangian in powers of the field A_i and using the standard perturbation theory in which only the "tree–like" diagrams are considered. As a result of such an expansion of Eqs. (17) or (19), the following products appear in the S matrix:

$$\epsilon_{ikl} H_i A_k \partial A_l f(A_i^2) \ . \tag{21}$$

These terms depend explicitly on the internal magnetic field, and, therefore, define a direction in the spin–wave space. In the derivation of Eq. (21) from Eqs. (17) or (19), we have taken into account the fact that the parameter b is proportional to the internal magnetic field and we have used the tensor notation (assuming that the magnetic field is in the direction of the third axis).

On the other hand, we assume that the parameter b and the corresponding magnetic field are invariant under the transformations (4),(15) of the SO(3) group. This last requirement is compatible with Eq. (24), provided the magnetic field is orthogonal to the surface of the sphere at every point. In this case, the magnetic field is in the direction of the third axis only if the expansion in powers of A_i is carried out at the origin. If, in the derivation of the S matrix, the Lagrangian is expanded in powers of $A_i - A_{0i}$, where A_{0i} is a fixed point on the sphere, the magnetic field will be orthogonal to the sphere at this point. Physically, this ambiguity in the direction of the magnetic field is related to the fact that the vacuum state of the system is infinitely degenerate. Before a certain vacuum state is chosen, the system is completely symmetric. The apparent symmetry breaking occurs only in the derivation of the S matrix when the states involved in the scattering are considered, i.e., a well-defined vacuum state is chosen. As already discussed, such a choice of the vacuum state corresponds to the expansion of the Lagrangian in powers of the field in the neighborhood of a fixed point. For any scattering process involving a finite number of magnons, the expansion in question is determined by differential surface elements of finite order, and, therefore, the states corresponding to different vacuum states (i.e., expansions at different points) are independent. Therefore, the phenomenological Lagrangian under study can be used to demonstrate the relationship between the Goldstone particles and the symmetry of the system as well as the vacuum degeneracy.

We shall conclude this section by giving the explicit form of the additional term in the Lagrangian defined by Eq. (8) and its variation under the transformations (12):

$$\omega_3 = \frac{A_1 \partial A_2 - A_2 \partial A_1}{f^2 A^2} \{1 - \cos \sqrt{f^2 A^2}\} \ , \tag{22}$$

$$\delta\omega_3 = \frac{\partial}{\partial t} \left\{ (e_1 A_2 - e_2 A_1) \frac{\mathrm{tg} \frac{\sqrt{f^2 A^2}}{2}}{\sqrt{A^2 f^2}} \right. \ . \tag{23}$$

Here, $A^2 = A_1^2 + A_2^2$. The matrix elements of the S matrix on the mass surface corresponding to the Lagrangians defined by Eqs. (6),(17) and (8),(22) are identical for any number of interacting magnons.

5. Let us now consider the most general expressions for the phenomenological spin–wave Lagrangian. We shall base our discussion solely on the symmetry properties of the Lagrangian under the transformation of the SO(3) group and make no assumptions about the spin–wave spectrum. The discussion presented in Secs. 4 and 5 concerns the special cases corresponding to a special choice of phenomenological parameters. The method of the Cartan differential forms (see, for example, Cartan (1910)) can used to take into account the symmetry requirements. To introduce the Cartan forms, we shall consider an arbitrary parametrization of the SO(3) group elements

$$g = g(\alpha_i), \quad g \in \mathrm{SO}(3); \qquad i = 1, 2, 3 \tag{24}$$

and define three Cartan forms $\omega_i(\alpha_i, \mathrm{d}\alpha_k)$ by the following equation:

$$g^{-1}(\alpha)\mathrm{d}g(\alpha) = \mathrm{i}\omega_k(\alpha, \mathrm{d}\alpha)\frac{\sigma_k}{2} \ , \tag{25}$$

where $\sigma_k/2$ are the generators of the SO(3) group , i.e., the Pauli matrices. Clearly, the Cartan forms are invariant under the action of the elements of the SO(3) group on the left, i.e.,

$$g(\tilde{\alpha}_i)g(\alpha) = g(\alpha') \tag{26}$$

where $g(\tilde{\alpha}_i)$ is an arbitrary element of the SO(3) group.

In fact, we obtain

$$\mathrm{i}\omega_i(\alpha', \mathrm{d}\alpha')\frac{\sigma_i}{2} = g^{-1}(\alpha')\mathrm{d}g(\alpha') = g^{-1}(\alpha)g^{-1}(\tilde{\alpha})g(z)\mathrm{d}g(\alpha) = \mathrm{i}\omega_i(\alpha, \mathrm{d}\alpha)\frac{\sigma_i}{2} \ . \tag{27}$$

Using the linear independence and completeness of the differential forms $\omega_i(\alpha, \mathrm{d}\alpha)$, it can be shown that any invariant form $\Omega(\alpha, \mathrm{d}\alpha)$ can be represented in the form

$$\Omega(\alpha, \mathrm{d}\alpha) = d_i\omega_i(\alpha, \mathrm{d}\alpha) \ . \tag{28}$$

where the coefficients d_i are constant, i.e., the forms $\omega_i(\alpha, \mathrm{d}\alpha)$ represent a complete system of the invariants of the SO(3) group. To derive the most general expression for the phenomenological Lagrangian of spin waves, we shall assume that to each parameter α_i there corresponds a local field $A_i(\mathbf{x}, t)$ and that the differentials $\mathrm{d}\alpha_i$ correspond either to the time or the space derivatives of the field. The Lagrangian which contains no higher than the second–order derivatives with respect to the field and is invariant under the thansformations of the SO(3) group [assuming that the local fields $A_i(\mathbf{x}, t)$ transform as the group parameters] has the form

$$L = \frac{1}{2}a_{ik}\omega_i(A, \partial A)\omega_k(A, \partial A) + 2b_{\mathrm{i}}\omega_i(A, \partial A)$$

$$-\frac{1}{2}C_{ik;\ lm}\omega_i(A, \nabla_i A)\omega_k(A, \nabla_m A) + 2d_{i;\ l}\omega_i(A, \nabla_l A) \ . \tag{29}$$

Let us first determine the values of the phenomenological parameters and the choice of the coordinates in the group space which correspond to the examples discussed in the previous sections.

The following parametrization of the SO(3) group elements corresponds to the Lagrangian defined by Eq. (6):

$$g(\alpha) = \left[\frac{1 + \mathrm{i}f(\alpha_1\sigma_1 + \alpha_2\sigma_2)}{1 - \mathrm{i}f(\alpha_1\sigma_1 + \alpha_2\sigma_2)}\right]^{1/2} \mathrm{e}^{\mathrm{i}\alpha_3 \frac{\sigma_3}{2}} \quad . \tag{30}$$

The parametrization (30) and Eq. (25) imply that ω_i have the form

$$\left.\begin{aligned} \omega_1 &= \tfrac{2}{1+f^2\alpha_i^2}[\partial\alpha_1 \cos\alpha_3 - \partial\alpha_2 \sin\alpha_3] \\ \omega_2 &= \tfrac{2}{1+f^2\alpha_i^2}[\partial\alpha_1 \sin\alpha_3 - \partial\alpha_2 \cos\alpha_3] \end{aligned}\right\} \tag{31}$$

and

$$\omega_3 = \frac{2(\alpha_1\partial\alpha_2 - \alpha_2\partial\alpha_1)}{1 + f^2\alpha_i^2} + \partial\alpha_3 \quad . \tag{32}$$

Here, $\alpha_i^2 = \alpha_1^2 + \alpha_2^2$.

Equations (30-32) contain three group parameters $\alpha_1, \alpha_2, \alpha_3$, whereas Eq. (6) involves only two local fields. To eliminate one of the components, we shall choose the parameters a_{ik} and $c_{ik,\, lm}$ so that the Lagrangian is independent of A_3. This requirement can be satisfied if

$$a_{11} = a_{22}; \qquad c_{11,\, lm} = c_{22,\, lm} \tag{33}$$

and all the other constants vanish. The Lagrangian defined by Eq. (6) satisfies the conditions (33) and the additional requirement

$$c_{11,\, lm} = c_{11} \cdot \delta_{lm} \quad , \tag{34}$$

which is equivalent to the requirement that the dependence of the frequencies on the wave vector is anisotropic. It is impossible to eliminate the components A_3 from the terms in Eq. (29) which are linear in the derivatives. However, since ω_3 only as a factor under the total derivative sign, we can retain this term in the Lagrangian defined by Eq. (29), i.e.,

$$b_3 \neq 0 \quad , \tag{35}$$

which corresponds, in the isotropic case $d_{3l} \neq 0$, to the addition of the expression (17) to the Lagrangian defined by Eq. (6). The Lagrangian defined by Eqs. (8-10) and the expression (22) corresponds to the following parametrization of the group elements:

$$g(\alpha) = \mathrm{e}^{\mathrm{i}(\frac{\sigma_1}{2}\alpha_1 + \frac{\sigma_2}{2}\alpha_2)f} \mathrm{e}^{\mathrm{i}\frac{\sigma_3}{2}\alpha_3} \quad , \tag{36}$$

which leads to the following differential forms:

$$\left.\begin{aligned} \omega_1 &= F_1 \cos\alpha_3 - F_2 \sin\alpha_3 , \\ \omega_2 &= F_1 \sin\alpha_3 + F_2 \cos\alpha_3 , \\ F_i = \partial\alpha_i \frac{\sin\sqrt{\alpha^2 f^2}}{\sqrt{\alpha^2 f^2}} &+ \alpha_i \frac{(\alpha_k \partial\alpha_k)}{f^2\alpha^2}\left[1 - \frac{\sin\sqrt{\alpha^2 f^2}}{\sqrt{\alpha^2 f^2}}\right] . \end{aligned}\right\} . \tag{37}$$

Here, $\alpha^2 = \alpha_1^2 + \alpha_2^2 (\alpha_k \partial\alpha_k) = \alpha_1\partial\alpha_1 + \alpha_2\partial\alpha_2$ and

$$\omega_3 = \frac{(\alpha_1\partial\alpha_2 - \alpha_2\partial\alpha_1)}{\alpha^2 f^2}(1 - \cos\sqrt{\alpha^2 f^2}) + \partial\alpha_3 \ . \tag{38}$$

In the isotropic case, the elimination of the third component corresponds to Eqs. (33-35). Assuming that the frequency depends only on $\mathbf{k}^2$ (isotopic case), we shall discuss the form of the excitation spectrum, which follows from the Lagrangian defined by Eq. (29). It is convenient to consider the highest–symmetry parametrization of the SO(3) group elements, i.e.,

$$g(\alpha) = \mathrm{e}^{\mathrm{i}\frac{\sigma}{2}\alpha}; \quad f = 1 \ . \tag{39}$$

Using eq .(25), we obtain, with the accuracy up to terms quadratic in the group parameters, the following expression for the forms $\omega_i (i = 1, 2, 3)$:

$$\omega_i = \mathrm{d}\alpha_i + \frac{1}{2}\varepsilon_{ikl}\alpha_k \mathrm{d}\alpha_l + \ldots \ . \tag{40}$$

Substituting eq/(40) in Eq. (25) and retaining in Eq. (25) only the terms quadratic in the field, we obtain the Lagrangian of noninteracting spin waves,

$$L = \frac{1}{2} a_{ik}\partial A_i \partial A_k + b_i \varepsilon_{ikl} A_k \partial A_l - \frac{1}{2} c_{ik} \nabla A_i \nabla A_k \ . \tag{41}$$

The Lagrangian defined by Eq. (41) leads to the following equations of motion in the momentum representation:

$$\omega^2 a_{ik} A_k + 2b_i \mathrm{i}\varepsilon_{ikl} A_k \omega - \mathbf{k}^2 c_{ik} A_k = 0 \ . \tag{42}$$

The condition that the determinant of the system of equations (42) vanishes yields the dispersion equation

$$\alpha\omega^6 + 3\beta\omega^4 + 3\gamma\omega^2 + \delta = 0 \ , \tag{43}$$

where the determinants α, β, γ, and δ are defined by

$$\left.\begin{aligned} &\alpha = M_1, \ \underset{-}{=} - (M_5 + M_2\mathbf{k}^2), \\ &\gamma = (M_6\mathbf{k}^2 + M_3\mathbf{k}^4), \quad \delta = -M_4\mathbf{k}^6 \end{aligned}\right\} \tag{44}$$

and

$$\left.\begin{aligned} M_1 &= \ |a_{ik}|, \quad M_4 = |c_{ik}|, \\ M_2 &= \tfrac{1}{3\,!}\ \varepsilon_{ikl}\ \varepsilon_{prs}\ a_{ip}\ a_{kr}\ c_{ls}, \\ M_3 &= \tfrac{1}{3\,!}\ \varepsilon_{ikl}\ \varepsilon_{prs}\ a_{ip}\ c_{kr}\ c_{ls}, \\ M_5 &= \tfrac{1}{3}\ b_i\ a_{ik}\ b_k, \quad M_6 = \tfrac{1}{3}\ b_i\ c_{ik}\ b_k . \end{aligned}\right\} \tag{45}$$

Since Eq. (43) is bicubic, for every root ω_i of Eq. (43), there is also a root $-\omega_i$. Therefore, using the standard method, we can separate in the local–field operators the terms corresponding to the creation and the annihilation of magnons.

The dependence of the coefficients in Eq. (43) on $\mathbf{k}^2$ [Eq. (44)] yields the frequency spectrum for small $\mathbf{k}^2$. If $\mathbf{k}^2 = 0$, we find that $\gamma = \delta = 0$ and Eq. (43) has only one nonvanishing root. The product of the roots which tend to zero in the limit of small $\mathbf{k}^2$ is proportional to δ, and, therefore, to $\mathbf{k}^6$; it follows from the form of the dependence of the coefficient γ on $\mathbf{k}^2$ that the sum of the roots is proportional to $\mathbf{k}^2$. Therefore, for small $\mathbf{k}^2$, the roots of Eq. (43) have the form

$$\left.\begin{aligned} \omega_1^2 &= A + B\mathbf{k}^2, \\ \omega_2^2 &= C\mathbf{k}^2, \\ \omega_3^2 &= D\mathbf{k}^4. \end{aligned}\right\} \tag{46}$$

Substituting Eq. (46) in Eq. (43), we obtain

$$\left.\begin{aligned} &A = \tfrac{M_5}{M_1} \ , B = \tfrac{M_2}{M_1} - \tfrac{M_6}{M_5} \ , \\ &C = \tfrac{M_6}{M_5} \ , D = \tfrac{M_4}{M_6} \ . \end{aligned}\right\} \tag{47}$$

Since $\omega_i^2 > 0$, the coefficients A, B, and D should be positive, which constrains the possible values of phenomenological constants appearing in the Lagrangian defined by Eq. (29).

If the coefficients b_i in the Lagrangian defined by Eq. (29) and in Eq. (42) vanish, then $M_5 = M_6 = 0$, which implies that all the frequencies ω_i are proportional to $\mathbf{k}^2$.

It should be noted that nine out of fifteen phenomenological constants a_{ik}, c_{ik}, and b_i are related to the structure of the equation describing noninteracting spin waves, and the remaining six govern the interaction of spin waves. In fact, the transformation in the free and in the interaction Lagrangian from the fields A_k to fields A'_k ,

$$A_k = (a^{-1/3})_{kl} A_l \ , \tag{48}$$

leads to a free Lagrangian of the type defined by Eq. (41), in which the coefficient of the first term is proportional to the unit matrix; this also represents the standard form of the free Lagrangian. In the expansion of the interaction Lagrangian in powers of the field A' , the six coefficients of the matrix $(a^{-1/2})_{kl}$ correspond to the phenomenological coupling constants and describe the interaction of an arbitrary number of "soft magnons."

The derivation of the S matrix, which is based on the expansion of the Lagrangian defined by Eqs. (6), (8) in powers of the field A_i, corresponds to a special choice of the vacuum state.

It follows from Eq. (42) that the apparent symmetry breaking of the Lagrangian is due to the fact that the quantities a_{ik}, c_{ik} are tensors and b_i

is a vector (they all depend on the macroscopic parameters of the system). However, such a symmetry violation does not contradict the invariance of Eq. (29), in which all the tensor quantities are defined in a local basis which is closely related to the properties of the SO(3) group.

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