S–Matrix in the Generalized Quantization Method

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Abstract. The formalism of the S–matrix for interacting electromagnetic field and half–spin particle field is considered. Particle field quantization is carried out according to a scheme suggested in the works of (Green (1953)) and the author (Volkov (1959)). It is shown that the basic concepts of the conventional theory of S–matrices (N–product, Wick's theorem, Feynman graphs) allow a simple generalization within the framework of the quantization scheme considered.

1. The customary methods of quantization of wave fields use as commutation relations commutators of anticommutators based on a choice of completely symmetric or completely antisymmetric wave functions in the configuration space of many identical particles. The confinement to symmetric of antisymmetric wave functions corresponds to the experimental data known at present as regards the statistics of elementary particles, but is evidently not rigorously established from the theoretical point of view. The problem as to why other possibilities are not realized in nature, "equally valid in the sense of the correspondence principle, in which " lies the essence of this limited choice of nature" (Pauli (1947)), has been discussed in lively fashion in the literature in the period of the development of quantum mechanics (see, for example, reference (Pauli (1947)).

With the development of methods of quantum theory, great progress has been achieved in the understanding of the connection of symmetric and antisymmetric wave functions with the value of the spin of particles (Pauli (1957)) and with the TCP invariance (Schwinger (1951), Pauli (1958)). However, consideration of the problems mentioned has always been carried out within the framework of the following alternative: either symmetric or antisymmetric wave functions; all other possibilities have been entirely neglected.

In this connection it is of interest to attempt to formulate this old problem, which arises in nonrelativistic quantum mechanics, in terms of the theory of wave fields.

The generalization of the existing methods of quantum field theory, which takes into consideration the presence not only of symmetric and antisymmetric wave functions, but which is also compatible with the fundamental premises of relativistic quantum theory, was carried out in the work of (Green (1953)) and later in a research of the author (Volkov (1959)) [1].

In references 1 and 2, however, questions connected with interaction were not considered. At the same time the possibility was not excluded that precisely the interaction between fields could be decisive for explanation of the separation of the existing methods of quantization [2].

In the present article we consider the formalism of the scattering matrix (S–matrix) for interacting electromagnetic field and the field of half–spin charged particles. Quantization of the field of the particles is carried out on the basis of transformed commutation relations (see below, Eq. (3)). It is shown that, in spite of the change of the quantization rules, there exists a unique procedure of expansion of the S–matrix in a series of normal derivatives (analogous to the usual technique of (Wick (1950))), which makes it possible to isolate the vacuum effects in the S–matrix. The results obtained without any essentioal change are applicable also to other local variants of interacting fields.

2. The scattering matrix for the case under consideration has the form

$$S = T(\exp(-\mathrm{i}\int H(x)\ \mathrm{d}^1 x)) \tag{1}$$

where $H(x)$ is the Hamiltonian density in the interaction representation

$$H(x) = \mathrm{i}\ e[\bar{\psi}(x), \gamma_\mu \psi(x)] A_\mu(x), \tag{2}$$

$\psi(x)$ and $\bar{\psi}(x) = \psi^+(x)\gamma_4$ are the field operators of particles satisfying the Dirac equation without interaction in the commutation representation[3]

$$\begin{aligned}
&\psi_\alpha(x)\psi_\beta(x^{'})\psi_\gamma(x^{''}) + \psi_\gamma(x^{''})\psi_\beta(x^{'})\psi_\alpha(x) = 0\\
&\psi_\alpha(x)\bar{\psi}_\beta(x^{'})\psi_\gamma(x^{''}) + \psi_\gamma(x^{''})\bar{\psi}_\beta(x^{'})\psi_\alpha(x) = -\mathrm{i}S_{\alpha\beta}(x-x^{'})\psi_\gamma(x^{''})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \mathrm{i}S_{\gamma\beta}(x^{''}-x^{'})\psi_\alpha(x)\\
&\bar{\psi}_\alpha(x)\bar{\psi}_\beta(x^{'})\psi_\gamma(x^{''}) + \psi_\gamma(x^{''})\bar{\psi}_\beta(x^{'})\bar{\psi}_\alpha(x) = -\mathrm{i}S_{\gamma\beta}(x^{''}-x^{'})\psi_\alpha(x).
\end{aligned} \tag{3}$$

$A_\mu(x)$ are the operators of the electromagnetic field, which satisfy the usual rules of commutation.

Thanks to the commutability of the operators $H(x)$ and $H(x^{'})$, the T–product in Eq.(1) outside the light cone is determined in a unique, relativistically–invariant fashion.

The operators of the electromagnetic field and the field of particles commute with one another; therefore the T–product in Eq.(1) can be represented

[1] The work of Green was not known to the author during preparation of reference 2 for publication.

[2] The possible connection of the symmetry of a wave function with a definite type of interaction in nonrelativistic quantum mechanics has been investigated by (Yaffe (1930)).

[3] We use the notation of reference 2, which is cited below as I.

in the form of the products of two independent one of which contains only the field operators of the particles, while the other contains only the operators of the electromagnetic field. The latter of these T–products will not be considered, since it has the same form as in ordinary theory.

The absence in the quantization method under consideration of simple commutation rules between the two operators makes difficult the separation of the vacuum effects in the T–product, which depend on the field operators of the particles, and requires a generalization of the concept of normal product.

In order to make clear the idea of such a generalization, let us look first at the simplest case, in which there are two operators: a_k is the destruction of a particle in the state k and a_l^+ that of the creation of a particle in the state l [or, similarly the operators b_k (b_k^+) of destruction (creation) of antiparticles].

The fundamental properties of these vectors are defined by the relations (9), (13), and (14) of I.

Let us determine the normal product $N(a_l^+ a_k)$ of the operators a_l^+ and a_k by the direct action of the N–product on the arbitrary basis vector[4]:

$$\begin{aligned} N(a_i^+ a_k) a_1^+ a_2^+ \dots a_n^+ \Phi_0 &= \delta_{k1} a_l^+ a_2^+ \dots a_n^+ \Phi_0 + \delta_{k2} a_1^+ a_l^+ \dots a_n^+ \Phi_0 + \dots \\ &\quad + \delta_{kn} a_1^+ a_2^+ \dots a_l^+ \Phi_0 + \dots \\ &= \sum_{j=1}^{n} \delta_{kj} a_1^+ \dots a_{j-1}^+ a_l^+ a_{j+1}^+ \dots a_n^+ \Phi_0, \end{aligned} \tag{4}$$

where Φ_0 is the vector of the vacuum state for nonintersecting fields and the indices $1, 2, \dots n$ determine the state of the particle.

As is seen directly from the definition (4), the N–product in the first place preserves the symmetry of the wave function, which is important in the establishment of the connection with nonrelativistic theory, and, in the second place, does not contain the vacuum effects, which are connected with the possibility of the destruction by the operator a_k of a particle previously created by the operator a_l^+.

We note that in the quantization with anticommutators, the determination just considered of the normal derivative coincides with the usual one.

Making use of the commutation relations for the operators a and a^+ (9, I), it is easy to find an explicit expression for the normal product $N(a_l^+ a_k)$ in terms of the operators a_l^+ and a_k:

$$N(a_l^+ a_k) = a_l^+ a_k - a_k a_l^+ - \delta_{lk} \tag{5}$$

The normal product of the operators b_k^+ and b_l is determined in similar fashion:

[4] In Eq.(4) [and in the subsequent formula (6)] the operators b^+ (or a^+) which can enter into the determination of the basis vector are omitted. Such operators, if there are any, do not affect the action of the N–products considered in (4) and (6), and without change in their position go over into the righthand parts of the corresponding equations.

$$N(b_l b_k^+) b_1^+ b_2^+ \dots b_n^+ \Phi_0 = -\sum_{j=1}^{n} \delta_{jl} b_1^+ \dots b_{j-1}^+ b_k^+ b_{j+1}^+ \dots b_n^+ \Phi_0 \qquad (6)$$

where

$$N(b_l b_k^+) = b_l b_k^+ - b_k^+ b_l + \delta_{kl} \qquad (7)$$

For the case of two particle and antiparticle creation operators, and correspondingly for two destruction operators, we determine the normal product with the aid of the following relations:

$$N(a_l^+ b_k^+) = a_l^+ b_k^+ - b_k^+ a_l^+, \qquad (8)$$

$$N(b_l a_k) = b_l a_k - a_k b_l \qquad (9)$$

The relations (5), (7)—(9) make it possible to write down the current operator in the form of a normal product. Actually, if the wave functions of the particle and antiparticle in the state k are connected by the relation $v_k = C\bar{u}_k$, where C is the charge–conjugation matrix, u_k and v_k are the coefficients in the expansion (7,I), then $\bar{u}_k \gamma_\mu u_k = \bar{v}_k \gamma_\mu v_k$, as a consequence of which,

$$ie[\bar{\psi}(x), \gamma_\mu \psi(x)] = ieN[\bar{\psi}(x)\gamma_\mu \psi(x)]. \qquad (10)$$

In the general case, the N–product depends on an arbitrary number of pairs of operators[5] and is determined by the following relations:

$$N(a_1^+ a_2; \dots; a_{1'}^+ b_{2'}^+; \dots; b_{1''} a_{2''}; \dots; b_{1'''} b_{2'''}^+; \dots) = \\ N(a_{1'}^+ b_{2'}^+) \dots N(a_1^+ a_2; \dots) N(b_{1'''} b_{2'''}^+; \dots) N(b_{1''} a_{2''}) \dots \qquad (11)$$

The order of arrangement of pairs of operators under the sign of the N–product in this formula is arbitrary.

The normal products $N(a_1^+ a_2; \dots)$ and $N(b_1 b_2^+; \dots)$ in Eq. (11) depend only on pairs of operators of the form $a^+ a$ and bb^+, respectively. The N–products of such a type are determined, similarly to (4) and (6), by the action of these products on the basis vectors:

$$N(a_k^+ a_l; a_m^+ a_r; \dots) a_1^+ a_2^+ \dots a_n^+ \Phi_0 = \\ \sum_{i,j\dots=1}^{n} \delta_{li} \delta_{rj} \dots a_1^+ \dots a_{i-1}^+ a_k^+ a_{i+1}^+ \dots a_{j-1}^+ a_m^+ a_{j+1}^+ \dots a_n^+ \Phi_0 \qquad (12)$$

$$N(b_k b_l^+; b_m b_r^+; \dots) b_1^+ b_2^+ \dots b_n^+ \Phi_0 = (-1)^P \sum_{i,\dots=1}^{n} \delta_{ki} \delta_{mj} \\ \dots b_1^+ \dots b_{i-1}^+ b_l^+ b_{i+1}^+ \dots b_{j-1}^+ b_r^+ b_{j+1}^+ \dots b_n^+ \Phi_0 \qquad (13)$$

[5] We limit ourselves her to a consideration of the N–products only of an even number of field operators of the particles. Such a limitation is not essential in what follows, since an even number of particle field operators always enters into the S–matrix and into all observable physical quantities.

summation in (12) and (13) is carried out over all non–coinciding indices;P is the number of pairs of operators of the form bb^+ (see the last footnote but one).

Equations (8),(9),and (11)–(13) determine the N–product for an arbitrary even number of operators and make it possible to represent any product of N–products (including the T–product) in the form of a sum of normal products.

As an example, let us consider the product $N[\bar{\psi}(1)\psi(2)]\ N[\bar{\psi}(3)\psi(4)]$ (the numbers 1,2,3,4 indicate the dependence of the operators on the coordinates and spinor indices). Making use of the commutation relations for the operators a, a^+, b and b^+ [Eqs. (3) and (9,I)] and the determination of the normal products, we obtain

$$\begin{aligned} &N(\bar{\psi}(1)\psi(2))N(\bar{\psi}(3)\psi(4)) = \\ &\quad N(\bar{\psi}(1)\psi(2);\bar{\psi}(3)\psi(4) - \mathrm{i}S^+(2,3)N(\bar{\psi}(1)\psi(4)) \\ &\quad + \mathrm{i}S^-(4,1)N(\bar{\psi}(3)\psi(2)) - 2S^+(2,3)S^-(4,1), \end{aligned} \tag{14}$$

where S^+ and S^- are the usual (+)- and (−)- fold commutation functions:

$$\begin{aligned} S^+(x) &= -\frac{\mathrm{i}}{(2\pi)^3}(\gamma\frac{\partial}{\partial x} - m)\int_{p_0>0}\delta(p^2+m^2)\mathrm{e}^{\mathrm{i}px}\mathrm{d}^4p, \\ S^-(x) &= \frac{\mathrm{i}}{(2\pi)^3}(\gamma\frac{\partial}{\partial x} - m)\int_{p_0<0}\delta(p^2+m^2)\mathrm{e}^{\mathrm{i}px}\mathrm{d}^4p. \end{aligned} \tag{15}$$

A similar formula holds for the T–product

$$\begin{aligned} &T[N(\bar{\psi}(1)\psi(1'))N(\bar{\psi}(2)\psi(2'))] = \\ &\quad N(\bar{\psi}(1)\psi(1');\bar{\psi}(2)\psi(2')) + S^F(1',2)N(\bar{\psi}(1)\psi(2')) + \\ &\quad S^F(2',1)N(\bar{\psi}(2)\psi(1')) - 2S^F(1',2)S^F(2',1), \end{aligned} \tag{16}$$

where

$$S^F(x) = \frac{\mathrm{i}}{(2\pi)^4}(\gamma\frac{\partial}{\partial x} - m)\int_{p_0<0}\delta(p^2+m^2-\mathrm{i}\varepsilon)\mathrm{e}^{\mathrm{i}px}\mathrm{d}^4p;\ \varepsilon\to 0. \tag{17}$$

The prime indicates the possible difference of spinor indices in the corresponding operators.

In the general case of a T—product from an arbitrary number of N–products, the following rule holds, similar to the rule of Wick in the ordinary quantization theory.

In order to expand a T–product of the form

$$T[N(\bar{\psi}(1)\bar{(}1'))N(\bar{\psi}(2)\psi(2'))\ldots N(\bar{\psi}(n)\psi(n'))$$

in a sum of normal products, it is necessary to consider all possible couplings of operators $\psi(a'($ and $\bar{\psi}(b)$, which do not enter into the composition of one and the same normal product, and to substitute these couplings in the functions $S^F(a',b)$. As a result of the superposition of the couplings, all the

N–products located under the sign of the T–product are united in groups which are uncoupled among themselves, and which either contain no operators (closed loop) or contain two operators $\bar{\psi}(a)$ and $\psi(b')$ (open lines). In the latter case, it is necessary to join the two disconnected operators in a pair and to put under the sign N–products of the form $N(\bar{\psi}(a)\psi(b')\ldots)$. In the presence of closed loops, each of them must be multiplied by an additional factor of 2[6].

We note that the rule formulated above has the usual graphical interpretation i terms of a Feynman diagram.

3. The relations considered in the preceding section make it possible to investigate in a simple fashion the matrix element of the scattering matrix corresponding to some particular process. As an illustration, we consider the process of scattering of two particles.

In order to determine completely the state of the two particles (in the given system of quantization), it is necessary, in addition to the quantities that characterize the individual states (spin, momentum), also to give the symmetry of the wave function (in the case of pure states) or the relative weights[7] of the symmetric and antisymmetric functions (for mixed states).

The basic orthonormalized vectors of states for different types of symmetry have the form

$$\frac{1}{\sqrt{2}}(a_k^+ a_l^+ \pm a_l^+ a_k^+)\Phi_0;\ k \neq l,$$

k and l are indices characterizing the spin and angular momentum of the individual states.

To determine the probability amplitudes of the scattering process under consideration, we compute the matrix elements of the N–product between the different basis vectors.

Separating in the N–products the terms giving non-vanishing contributions to the matrix element

$$N(\bar{\psi}(1)\gamma_\mu\psi(1);\bar{\psi}(2)\gamma_\mu\psi(2)) =$$
$$2N(a_{k'}^+ a_k; a_{l'}^+ a_l)\bar{u}_{k'}(1)\gamma_\mu u_k(1)\bar{u}_{l'}(2)\gamma_\mu u_l(2) +$$

[6] The fundamental difference between the ordinary technique of (Wick (1950)) and its generalization considered here consists in the appearance of this factor takes place not only for virtual processes, but also for processes which occur with the creation of pairs of real particles and antiparticles (as a consequence of the normalization of the operator wave function). In the case of more complicated schemes of quantization (Green (1953)), which lead in the general case to statistics of particles with maximal occupation number m for each of the individual states, the expansion of the T–product in a sum of N–products takes place in precisely the same fashion, but in this case each closed loop acquires an additional factor of m.

[7] A more detailed realization of the state (furnishing of coefficients in the expansion of the wave function over symmetric and antisymmetric states) has no meaning because of the identity of the particles.

$$2N(a_{l'}^{+}a_k;a_{k'}^{+}a_l)\bar{u}_{l'}(1)\gamma_\mu u_k(1)\bar{u}_{k'}(2)\gamma_\mu u_l(2), \tag{18}$$

where the primed indices characterize the state of the particles in the final states, while $u_k(1)$ etc are wave function of single particle states, and noting that as a consequence of (12),

$$N(a_{k'}^{+}a_k;a_{l'}^{+}a_l)\frac{1}{\sqrt{2}}(a_k^{+}a_l^{+} \pm a_l^{+}a_k^{+})\Phi)0 = \frac{1}{\sqrt{2}}(a_{k'}^{+}a_{l'}^{+} \pm a_{l'}^{+}a_{k'}^{+})\Phi)0$$

$$N(a_{l'}^{+}a_k;a_{k'}^{+}a_l)\frac{1}{\sqrt{2}}(a_k^{+}a_l^{+} \pm a_l^{+}a_k^{+})\Phi)0 = \frac{1}{\sqrt{2}}(a_{l'}^{+}a_{k'}^{+} \pm a_{k'}^{+}a_{l'}^{+})\Phi)0 \tag{19}$$

we get the following expression for the non–vanishing matrix elements:

$$\Phi_0^*\frac{1}{\sqrt{2}}(a_{l'}a_{k'} \pm a_{k'}a_{l'})N(\bar{\psi}(1)\gamma_\mu\psi(1);\bar{\psi}(2)\gamma_\nu\psi(2))\frac{1}{\sqrt{2}}(a_k^{+}a_l^{+} \pm a_l^{+}a_k^{+})\Phi_0 =$$
$$2(\bar{u}_{k'}(1)\gamma_m u u_k(1)\bar{u}_{l'}(2)\gamma_\nu u_l(2) \pm \bar{u}_{l'}(1)\gamma_\mu u_k(1)\bar{u}_{k'}(2)\gamma_\nu u_l(2)). \tag{20}$$

Taking into account the sign $(-)$ in Eq. (20), we obtain the well–known formula of Møller. The sign $(+)$ in Eq. (20) leads to the following expression for the scattering cross section (in the center–of–mass system):

$$\mathrm{d}\sigma_{(+)} = \frac{r_0^2}{\varepsilon^2(\varepsilon^2-1)^2}\left\{\frac{(2\varepsilon^2-1)^2}{\sin^4\theta} - \frac{4\varepsilon^4-5\varepsilon^2+\frac{5}{4}}{\sin^2\theta} + \frac{(\varepsilon^2-1)^2}{4}\right\}.$$

For the case of a mixed state, the scattering cross sections $\mathrm{d}\sigma^{+}$ and $\mathrm{d}\sigma^{-}$ are averaged with the corresponding weighting factors.

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