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SEMIGROUPS OF SUPERMATRICES AND ONE-PARAMETER IDEMPOTENT SUPEROPERATORS

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Supermatrix semigroups and their different reductions are introduced and investigated. One-parameter semigroups of antitriangle idempotent supermatrices and corresponding superoperator semigroups are defined and their features are studied. It is shown that t-linear idempotent superoperators and the usual exponential superoperators are mutually dual in some sense. The first one gives an additional (odd) solution (to the standard exponential operator) of the initial Cauchy problem. The corresponding functional equation and an analog of resolvent are found. Differential and functional equations for idempotent (super)operators are derived for their general t power-type dependence.

KEYWORDS : supermatrix, reduction, one-parameter semigroup, idempotent, Cauchy problem, resolvent

Supermatrix groups [1, 2, 3] play indispensable role in modern supersymmetric models construction [4, 5, 6]. Further mathematical development [7] needs thorough consideration of their inner properties and include noninvertibility in a strong way [8, 9], i.e. by exploiting of the semigroup theory methods [10, 11, 12]. Usually matrix semigroups are defined over the field $\mathbb{K}$ [13] (on some nonsupersymmetric generalizations of $\mathbb{K}$-representations see [14, 15]). But modern realistic supersymmetric unified particle theories [16] are considered in superspace [17, 18]. So all variables and functions are defined not over the field $\mathbb{K}$, but over Grassmann-Banach superalgebras over $\mathbb{K}$ [19, 20, 21], they become in general noninvertible and therefore they should be considered by the semigroup theory, which was claimed in [22, 23]. Some new semigroups having nontrivial abstract properties were found in [24]. Also, it was shown that supermatrices of the special (antitriangle) shape can form various strange and sandwich semigroups not known before [25, 8].

From another side operator semigroups [26] are very much important in mathematical physics [27, 28, 29] viewed as a general theory of evolution systems [30, 31, 32]. Its development covers many new fields [33, 34, 35, 36], but one of vital for modern theoretical physics directions — supersymmetry and related mathematical structures [37, 38]— was not considered before in application to the general operator semigroup theory. The main difference between previous considerations is the fact that among building blocks (e.g. elements of corresponding matrices) there exist noninvertible objects (divisors of zero and nilpotents) which by themselves can form another semigroup. Therefore, we have to take into account this fact and investigate properly such a possibility as well, which can be called a $semigroup \times semigroup$ method.

Here we study continuous supermatrix representations of idempotent operator semigroups previously introduced for bands in [25, 39], then consider one-parametric semigroups (for general theory see [27, 30, 40]) of antitriangle supermatrices and corresponding superoperator semigroups [41]. The first ones continuously represent idempotent semigroups and second ones lead to new superoperator semigroups with nontrivial properties.

Let Λ be a commutative $\mathbb{Z}_2$-graded superalgebra [1] over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0,1\} = \mathbb{Z}_2 | \, a \in \Lambda_i\}$. The even homomorphism $\mathfrak{m}_b : \Lambda \to \mathbb{B}$ is called a *body map* and the odd homomorphism $\mathfrak{m}_s : \Lambda \to \mathbb{S}$ is called a *soul map* [42], where $\mathbb{B}$ and $\mathbb{S}$ are purely even and odd algebras over $\mathbb{K}$ and $\Lambda = \mathbb{B} \oplus \mathbb{S}$. It can be thought that, if we have the Grassmann algebra Λ with generators $\xi_i, \ldots, \xi_n$ $\xi_i\xi_j + \xi_j\xi_i = 0$, $1 \leq i, \, j \leq n$, in particular $\xi_i^2 = 0$ (n can be infinite, and only this case is nontrivial [43, 44] and interesting [45]), then any even x and odd $\varkappa$ elements have the expansions (which can be infinite)

$$x = x_{body} + x_{soul} = x_{body} + y_{12}\xi_1\xi_2 + y_{13}\xi_1\xi_3 + \ldots = x_{body} + \sum_{1 \leq r \leq n} \sum_{1 < i_1 < \ldots < i_{2r} \leq n} y_{i_1 \ldots i_{2r}} \xi_{i_1} \ldots \xi_{i_{2r}} \qquad (1)$$

$$\varkappa = \varkappa_{soul} = y_1\xi_1 + y_2\xi_2 + \ldots + y_{123}\xi_1\xi_2\xi_3 + \ldots = \sum_{1 \leq r \leq n} \sum_{1 < i_1 < \ldots < i_r \leq n} y_{i_1 \ldots i_{2r-1}} \xi_{i_1} \ldots \xi_{i_{2r-1}} \qquad (2)$$

where $y_{i_1 \ldots i_r} \in \mathbb{K}$. So we obviously have $\mathfrak{m}_b(x) = x_{body}$, $\mathfrak{m}_b(\varkappa) = 0$ and $\mathfrak{m}_s(x) = x_{soul}$, $\mathfrak{m}_s(\varkappa) = \varkappa_{soul}$.

From (1)-(2) it follows

Corollary 1. The equations $x^2 = 0$ and $x\varkappa = 0$ have nonzero nontrivial solutions (zero divisors and even nilpotents, while odd objects are always nilpotent).

Conjecture 2. *If zero divisors and nilpotents will be included in the following analysis as elements of matrices, then one can find new and unusual properties of corresponding matrix semigroups.*

From this viewpoint we consider general properties of supermatrices [1] and introduce their additional reduction [25].

IDEAL STRUCTURE OF $(1+1)\times(1+1)$-SUPERMATRICES

Let us consider $(p|q)$-dimensional linear model superspace $\Lambda^{p|q}$ over Λ (in the sense of [1, 2]) as the even sector of the direct product $\Lambda^{p|q}=\Lambda_0^p\times\Lambda_1^q$ [42, 21]. The even morphisms $\mathrm{Hom}_0\left(\Lambda^{p|q},\Lambda^{m|n}\right)$ between superlinear spaces $\Lambda^{p|q}\to\Lambda^{m|n}$ are described by means of $(m+n)\times(p+q)$-supermatrices [1, 2] (for some nontrivial properties see [46, 47]). In what follows we will treat noninvertible morphisms [48, 49] on a par with invertible ones [25].

We consider $(1+1)\times(1+1)$-supermatrices[1] describing the elements from $\mathrm{Hom}_0\left(\Lambda^{1|1},\Lambda^{1|1}\right)$ in the standard $\Lambda^{1|1}$ basis [1]

$$M\equiv\begin{pmatrix} a & \alpha \\ \beta & b\end{pmatrix}\in\mathrm{Mat}_\Lambda\left(1|1\right) \tag{3}$$

where $a,b\in\Lambda_0$, $\alpha,\beta\in\Lambda_1$, $\alpha^2=\beta^2=0$ (in the following we use Latin letters for elements from Λ_0 and Greek letters for ones from Λ_1, and all odd elements are nilpotent of index 2).

The supertrace and Berezinian (superdeterminant) are defined by [1]

$$\mathrm{str}M=a-b, \tag{4}$$

$$\mathrm{Ber}M=\frac{a}{b}+\frac{\beta\alpha}{b^2}. \tag{5}$$

Observe that first term corresponds to triangle supermatrices, second term - to antitriangle ones (which we use below).

For sets of matrices we use corresponding bold symbols, e.g. $\mathbf{M}\overset{def}{=}\left\{M\in\mathrm{Mat}_\Lambda\left(1|1\right)\right\}$, and the set product is standard $\mathbf{M}\cdot\mathbf{N}\overset{def}{=}\left\{\cup MN\,|\,M,N\in\mathrm{Mat}_\Lambda\left(1|1\right)\right\}$. Denote a set of invertible elements of $\mathbf{M}$ by $\mathbf{M}^*$, and $\mathbf{I}=\mathbf{M}\setminus\mathbf{M}^*$. In [1] it was proved that $\mathbf{M}^*=\left\{M\in\mathbf{M}|\,\mathfrak{m}_b\left(a\right)\neq0\wedge\mathfrak{m}_b\left(b\right)\neq0\right\}$. Consider the invertibility structure of $\mathrm{Mat}_\Lambda\left(1|1\right)$ in more detail. Let us denote

$$\begin{array}{ll}\mathbf{M}'=\left\{M\in\mathbf{M}|\,\mathfrak{m}_b\left(a\right)\neq0\right\}, & \mathbf{I}'=\left\{M\in\mathbf{M}|\,\mathfrak{m}_b\left(a\right)=0\right\},\\ \mathbf{M}''=\left\{M\in\mathbf{M}|\,\mathfrak{m}_b\left(b\right)\neq0\right\}, & \mathbf{I}''=\left\{M\in\mathbf{M}|\,\mathfrak{m}_b\left(b\right)=0\right\}.\end{array} \tag{6}$$

Then $\mathbf{M}=\mathbf{M}'\cup\mathbf{I}'=\mathbf{M}''\cup\mathbf{I}''$ and $\mathbf{M}'\cap\mathbf{I}'=\varnothing$, $\mathbf{M}''\cap\mathbf{I}''=\varnothing$, therefore $\mathbf{M}^*=\mathbf{M}'\cap\mathbf{M}''$. The Berezinian $\mathrm{Ber}\,M$ is well-defined for the supermatrices from $\mathbf{M}''$ only and is invertible when $M\in\mathbf{M}^*$, but for the supermatrices from $\mathbf{M}'$ the inverse $\left(\mathrm{Ber}\,M\right)^{-1}$ is well-defined and is invertible when $M\in\mathbf{M}^*$ too [1].

Under the ordinary supermatrix multiplication the set $\mathbf{M}$ is a semigroup of all $(1|1)$ supermatrices [50], and the set $\mathbf{M}^*$ is a subgroup of $\mathbf{M}$. In the standard basis $\mathbf{M}^*$ represents the general linear group $GL_\Lambda\left(1|1\right)$ [1]. A subset $\mathbf{I}\subset\mathbf{M}$ is an ideal of the semigroup $\mathbf{M}$ [51].

Proposition 3. *1) The sets $\mathbf{I}$, $\mathbf{I}'$ and $\mathbf{I}'$ are isolated ideals of $\mathbf{M}$.*

2) The sets $\mathbf{M}^*$, $\mathbf{M}^{'}$ and $\mathbf{M}^{''}$ are filters of the semigroup $\mathbf{M}$.

3) The sets $\mathbf{M}'$ and $\mathbf{M}''$ are subsemigroups[2] *of $\mathbf{M}$, which are $\mathbf{M}'=\mathbf{M}^*\cup\mathbf{J}'$ and $\mathbf{M}''=\mathbf{M}^*\cup\mathbf{J}''$ with the isolated ideals $\mathbf{J}'=\mathbf{M}'\setminus\mathbf{M}^*=\mathbf{M}'\cap\mathbf{I}''$ and $\mathbf{J}''=\mathbf{M}''\setminus\mathbf{M}^*=\mathbf{M}''\cap\mathbf{I}'$ respectively.*

4) The ideal of the semigroup $\mathbf{M}$ is $\mathbf{I}=\mathbf{I}'\cup\mathbf{J}'=\mathbf{I}''\cup\mathbf{J}''$.

Proof. Let $M_3=M_1M_2$, then $a_3=a_1a_2+\alpha_1\beta_2$ and $b_3=b_1b_2+\beta_1\alpha_2$. Taking the body part we derive $\mathfrak{m}_b\left(a_3\right)=\mathfrak{m}_b\left(a_1\right)\mathfrak{m}_b\left(a_2\right)$, and $\mathfrak{m}_b\left(b_3\right)=\mathfrak{m}_b\left(b_1\right)\mathfrak{m}_b\left(b_2\right)$. Then use the definitions. □

TWO TYPES OF SUPERMATRIX REDUCTION

From (5) we obviously have different **two dual types** of supermatrices [25].

Definition 4. *Even-reduced supermatrices* are elements from $\mathrm{Mat}_\Lambda\left(1|1\right)$ of the form

$$M_{even}\equiv\begin{pmatrix} a & \alpha \\ 0 & b\end{pmatrix}\in\mathrm{RMat}_\Lambda^{even}\left(1|1\right)\subset\mathrm{Mat}_\Lambda\left(1|1\right). \tag{7}$$

Odd-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda\left(1|1\right)$ of the form

$$M_{odd}\equiv\begin{pmatrix} 0 & \alpha \\ \beta & b\end{pmatrix}\in\mathrm{RMat}_\Lambda^{odd}\left(1|1\right)\subset\mathrm{Mat}_\Lambda\left(1|1\right). \tag{8}$$

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[1] Sometimes we restrict ourselves to this simple case for clearness taking into account that the most of properties and conclusions hold valid for general block $(p+q)\times(p+q)$-supermatrices as well.

[2] But not subgroups as it was incorrectly translated in the English edition [1], see pp. 95, 103, which correspond to the original Russian edition, Moscow, MGU, 1983, pp. 89, 93, where the sets $\mathbf{M}'$ and $\mathbf{M}''$ denoted as $G'Mat\left(1,1|\Lambda\right)$ and $G''Mat\left(1,1|\Lambda\right)$ are correctly called semigroups. This can partially explain the fact, why semigroups were not intensively developed in supermathematics before, while in ordinary mathematics this question was answered positively [52] (see also for numerous applications the references in [26, 51, 53, 11, 27, 13, 30] and even in [54]).

Conjecture 5. *The odd-reduced supermatrices have a nilpotent (but nonvanishing in general case) Berezinian*

$$\mathrm{Ber} M_{odd} = \frac{\beta\alpha}{b^2} \neq 0, \quad (\mathrm{Ber} M_{odd})^2 = 0. \tag{9}$$

REMARK. Indeed this property (9) prevented one in the past from the use of this type (odd-reduced) of supermatrices in physics. All previous applications (excluding [25, 39, 55, 9]) were connected with triangle (even-reduced, similar to Borel [56]) ones and first term in Berezinian $\mathrm{Ber} M_{even} = \frac{a}{b}$ (5).

The even- and odd-reduced supermatrices are *mutually dual* in the sense of the Berezinian addition formula [25]

$$\mathrm{Ber} M = \mathrm{Ber} M_{even} + \mathrm{Ber} M_{odd}. \tag{10}$$

Obviously, the even-reduced matrices $\mathbf{M}_{even}$ form a semigroup $\mathfrak{M}_{even}(1|1)$ which is a subsemigroup of $\mathfrak{M}(1|1)$, because of $\mathbf{M}_{even} \cdot \mathbf{M}_{even} \subseteq \mathbf{M}_{even}$ and the unity is in $\mathfrak{M}_{even}(1|1)$. This trivial observation leads to general structure (Borel) theory of ordinary matrices [56]: triangle matrices form corresponding substructures, subgroups and subsemigroups (see for general theory e.g. [57]). It was believed before that in case of supermatrices this situation should not be changed, because supermatrix multiplication is the same [1]. But they did not take into account *zero divisors and nilpotents* appearing naturally and inevitably in supercase [9].

Conjecture 6. *Standard (lower or upper) triangle supermatrices are not the only substructures due to unusual properties of zero divisors and nilpotents appearing among elements (see (1)-(2) and* **Corollary 1***).*

It means that in such consideration we have additional (to triangle) class of subsemigroups. Then we can formulate the following general

Problem 1. For a given n, m, p, q to describe and classify **all** possible substructures (subgroups and **subsemigroups**) of $(m+n) \times (p+q)$-supermatrices.

An example of such new substructures are Γ-matrices considered below.

Conjecture 7. *These new substructures lead to corresponding* ***new superoperators*** *which are represented by one-parameter substructures of supermatrices.*

We first consider possible (not triangle) subsemigroups of supermatrices.

ODD-REDUCED SUPERMATRIX SEMIGROUPS

In general, the odd-reduced matrices M_{odd} do not form a semigroup, since their multiplication is not closed in general $\mathbf{M}_{odd} \cdot \mathbf{M}_{odd} \subset \mathbf{M}$. Nevertheless, some subset of $\mathbf{M}_{odd}$ *can form* a semigroup [25]. That can happen due to the existence of zero divisors in Λ, and so we have $\mathbf{M}_{odd} \cdot \mathbf{M}_{odd} \cap \mathbf{M}_{odd} = \mathbf{M}_{odd}^{smg} \neq \varnothing$.

To find $\mathbf{M}_{odd}^{smg}$ we consider a $(1+1) \times (1+1)$ example. Let $\alpha, \beta \in \Gamma_{set}$, where $\Gamma_{set} \subset \Lambda_1$. We denote $\mathrm{Ann}\,\alpha \stackrel{def}{=} \{\gamma \in \Lambda_1 \,|\, \gamma \cdot \alpha = 0\}$ and $\mathrm{Ann}\,\Gamma_{set} = \bigcap_{\alpha} \in \Gamma \;\; \mathrm{Ann}\,\alpha$ (here the intersection is crucial). Then we define *left* and *right* Γ-*matrices*

$$\mathbf{M}_{odd(L)}^{\Gamma} \stackrel{def}{=} \begin{pmatrix} 0 & \Gamma_{set} \\ \mathrm{Ann}\,\Gamma_{set} & b \end{pmatrix}, \quad \mathbf{M}_{odd(R)}^{\Gamma} \stackrel{def}{=} \begin{pmatrix} 0 & \mathrm{Ann}\,\Gamma_{set} \\ \Gamma_{set} & b \end{pmatrix}. \tag{11}$$

Proposition 8. *The Γ-matrices $\mathbf{M}_{odd(L,R)}^{\Gamma} \subset \mathbf{M}_{odd}$ form subsemigroups of $\mathfrak{M}(1|1)$ under the standard supermatrix multiplication, if $b\Gamma \subseteq \Gamma$.*

Definition 9. Γ-*semigroups* $\mathfrak{M}_{odd(L,R)}^{\Gamma}(1|1)$ are subsemigroups of $\mathfrak{M}(1|1)$ formed by the Γ-matrices $\mathbf{M}_{odd(L,R)}^{\Gamma}$ under supermatrix multiplication.

Corollary 10. The Γ-matrices are additional to triangle substructures of supermatrices which form semigroups.

Let us consider general square antitriangle $(p+q) \times (p+q)$-supermatrices (having even parity in notations of [1]) of the form

$$M_{odd}^{p|q} \stackrel{def}{=} \begin{pmatrix} 0_{p\times p} & \Gamma_{p\times q} \\ \Delta_{q\times p} & B_{q\times q} \end{pmatrix}, \tag{12}$$

where ordinary matrix $B_{q\times q}$ consists of even elements and matrices $\Gamma_{p\times q}$ and $\Delta_{q\times p}$ consist of odd elements [1, 2] (we drop their indices below). The Berezinian of $M_{odd}^{p|q}$ can be obtained from the general formula by reduction and in case of invertible B (which is implied here) is (cf. (9))

$$\mathrm{Ber}\, M_{odd}^{p|q} = -\frac{\det\left(\Gamma B^{-1} \Delta\right)}{\det B}. \tag{13}$$

Assertion 11. *A set of supermatrices* $\mathbf{M}_{odd}^{p|q}$ *form a semigroup* $\mathfrak{M}_{odd}^{\Gamma}(p|q)$ *of* $\Gamma^{p|q}$*-matrices, if* $\Gamma_{set}\Delta_{set} = 0$*, i.e. antidiagonal matrices are orthogonal, and* $\Gamma_{set}\mathbf{B} \subset \Gamma_{set}$*,* $\mathbf{B}\Delta_{set} \subset \Delta_{set}$*.*

Proof. Consider the product

$$M_{odd_1}^{p|q} M_{odd_2}^{p|q} = \begin{pmatrix} \Gamma_1\Delta_2 & \Gamma_1 B_2 \\ B_1\Delta_2 & B_1B_2 + \Delta_1\Gamma_2 \end{pmatrix} \tag{14}$$

and observe the condition of vanishing even-even block, which gives $\Gamma_1\Delta_2 = 0$, and others are obvious. □

From (14) it follows

Corollary 12. Two $\Gamma^{p|q}$-matrices satisfy the band relation $M_1M_2 = M_1$, iff

$$\Gamma_1 B_2 = \Gamma_1, \quad B_1\Delta_2 = \Delta_2, \quad B_1B_2 + \Delta_1\Gamma_2 = B_1. \tag{15}$$

Definition 13. We call a set of $\Gamma^{p|q}$-matrices satisfying additional condition $\Delta_{set}\Gamma_{set} = 0$, a set of *strong* $\Gamma^{p|q}$-matrices.

Strong $\Gamma^{p|q}$-matrices have some extra nice features and all supermatrices considered below are of this class.

Corollary 14. Idempotent strong $\Gamma^{p|q}$-matrices are defined by relations

$$\Gamma B = \Gamma, \quad B\Delta = \Delta, \quad B^2 = B. \tag{16}$$

The product of n strong $\Gamma^{p|q}$-matrices M_i has the following form

$$M_1M_2\ldots M_n = \begin{pmatrix} 0 & \Gamma_1 A_{n-1} B_n \\ B_1 A_{n-1}\Delta_n & B_1 A_{n-1} B_n \end{pmatrix}, \tag{17}$$

where $A_{n-1} = B_2B_3\ldots B_{n-1}$, and its Berezinian is

$$\mathrm{Ber}\,(M_1M_2\ldots M_n) = -\frac{\det(\Gamma_1 A_{n-1}\Delta_n)}{\det(B_1 A_{n-1} B_n)}. \tag{18}$$

ONE-(EVEN)-PARAMETER SUPERMATRIX IDEMPOTENT SEMIGROUPS

Here we investigate one-(even)-parameter subsemigroups of Γ-semigroups and as a particular example for clearness of statements consider $\mathfrak{M}_{odd}(1|1)$, where all characteristic features taking place in general $(p+q)\times(p+q)$ as well can be seen. These formulas will be applied for establishing the corresponding superoperator semigroup properties.

A simplest semigroup can be constructed from antidiagonal nilpotent supermatrices of the shape

$$Y_\alpha(t) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 0 \end{pmatrix}. \tag{19}$$

where $t \in \Lambda^{1|0}$ is an even parameter of the Grassmann algebra Λ which continuously "numbers" elements $Y_\alpha(t)$ and $\alpha \in \Lambda^{0|1}$ is a fixed odd element of Λ which "numbers" the sets $\mathbf{Y}_\alpha = \bigcup_t Y_\alpha(t)$.

Definition 15. The supermatrices $Y_\alpha(t)$ together with a null supermatrix $Z \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ form a *continuous null semigroup* $\mathfrak{Z}_\alpha(1|1) = \{\mathbf{Y}_\alpha \cup Z; \cdot\}$ having the null multiplication

$$Y_\alpha(t)\,Y_\alpha(u) = Z. \tag{20}$$

Assertion 16. *For any fixed* $t \in \Lambda^{1|0}$ *the set* $\{Y_\alpha(t), Z\}$ *is a 0-minimal ideal in* $\mathfrak{Z}_\alpha(1|1)$*.*

REMARK. If we consider, for instance, a one-(even)-parameter odd-reduced supermatrix of another shape $R_\alpha(t) = \begin{pmatrix} 0 & \alpha \\ \alpha & t \end{pmatrix}$, then multiplication of $R_\alpha(t)$ is not closed since $R_\alpha(t)\,R_\alpha(u) = \begin{pmatrix} 0 & \alpha u \\ \alpha t & tu \end{pmatrix} \notin \mathbf{R}_\alpha = \bigcup_t R_\alpha(t)$. Note that any other possibility except ones considered below also do not give closure of multiplication.

Thus the only nontrivial closed systems of one-(even)-parameter odd-reduced (antitriangle) $(1+1)\times(1+1)$ supermatrices are $\mathbf{P}_\alpha = \bigcup_t P_\alpha(t)$ where

$$P_\alpha(t) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix}, \quad P_\alpha^2(t) = P_\alpha(t), \quad \mathrm{Ber}\,P_\alpha(t) = 0 \tag{21}$$

and $\mathbf{Q}_\alpha = \bigcup_t Q_\alpha(u)$ where

$$Q_\alpha(u) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix} \quad Q_\alpha^2(u) = Q_\alpha(u) \quad \mathrm{Ber}\,Q_\alpha(u) = 0. \tag{22}$$

We establish multiplication properties of the idempotent noninvertible supermatrices $P_\alpha(t)$ and $Q_\alpha(u)$.

Assertion 17. *Sets of idempotent supermatrices $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ form left zero and right zero semigroups respectively with multiplication*

$$P_\alpha(t)\,P_\alpha(u) = P_\alpha(t)\,, \tag{23}$$
$$Q_\alpha(t)\,Q_\alpha(u) = Q_\alpha(u)\,. \tag{24}$$

if and only if $\alpha^2 = 0$.

Proof. It simply follows from supermatrix multiplication law and general previous considerations. □

Corollary 18. The sets $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are rectangular bands since

$$P_\alpha(t)\,P_\alpha(u)\,P_\alpha(t) = P_\alpha(t)\,, \tag{25}$$
$$P_\alpha(u)\,P_\alpha(t)\,P_\alpha(u) = P_\alpha(u) \tag{26}$$

and

$$Q_\alpha(u)\,Q_\alpha(t)\,Q_\alpha(u) = Q_\alpha(u)\,, \tag{27}$$
$$Q_\alpha(t)\,Q_\alpha(u)\,Q_\alpha(t) = Q_\alpha(t) \tag{28}$$

with components $t = t_0 + \operatorname{Ann}\alpha$ and $u = u_0 + \operatorname{Ann}\alpha$ correspondingly.

They are orthogonal in sense of

$$Q_\alpha(t)\,P_\alpha(u) = E_\alpha, \tag{29}$$

where

$$E_\alpha \overset{def}{=} \begin{pmatrix} 0 & \alpha \\ \alpha & 1 \end{pmatrix}, \qquad E_\alpha^2 = E_\alpha, \qquad \operatorname{Ber} E_\alpha = 0 \tag{30}$$

is a right "unity" and left "zero" in the semigroup $\mathbf{P}_\alpha$, because

$$P_\alpha(t)\,E_\alpha = P_\alpha(t)\,, \qquad E_\alpha P_\alpha(t) = E_\alpha \tag{31}$$

and a left "unity" and right "zero" in the semigroup $\mathbf{Q}_\alpha$, because

$$Q_\alpha(t)\,E_\alpha = E_\alpha, \qquad E_\alpha Q_\alpha(t) = Q_\alpha(t)\,. \tag{32}$$

It is important to note that $P_\alpha(1) = Q_\alpha(1) = E_\alpha$, and so $\mathbf{P}_\alpha \cap \mathbf{Q}_\alpha = E_\alpha$. Therefore, almost all properties of $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are similar, and we will consider only one of them in what follows. For generalized Green's relations and more detail properties of odd-reduced supermatrices of higher dimension see [39, 8, 9].

ODD-REDUCED SUPERMATRIX OPERATOR SEMIGROUPS

Let us consider a semigroup $\mathcal{P}$ of superoperators $\mathsf{P}(t)$ (see for general theory [27, 28, 30]) represented by the one-even-parameter semigroup $\mathbf{P}_\alpha$ of odd-reduced supermatrices $P_\alpha(t)$ (21) which act on $(1|1)$-dimensional superspace $\mathbb{R}^{1|1}$ as follows $P_\alpha(t)\,\mathtt{X}$, where $\mathtt{X} = \begin{pmatrix} x \\ \varkappa \end{pmatrix} \in \mathbb{R}^{1|1}$, where x is even coordinate, $\varkappa$ is odd coordinate $\left(\varkappa^2 = 0\right)$ having expansions (1) and (2) respectively (see **Corollary 1**). We have a representation $\rho : \mathcal{P} \to \mathbf{P}_\alpha$ with correspondence $\mathsf{P}(t) \to P_\alpha(t)$ or $\mathsf{P}(t) \approx P_\alpha(t)$, but (as is usually made, e.g. [30]) we identify space of superoperators with the space of corresponding matrices[3].

Definition 19. An odd-reduced "dynamical" system on $\mathbb{R}^{1|1}$ is defined by an odd-reduced supermatrix-valued function $\mathsf{P}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{odd}(1|1)$ and "time evolution" of the state $\mathtt{X}(0) \in \mathbb{R}^{1|1}$given by the function $\mathtt{X}(t) : \mathbb{R}_+ \to \mathbb{R}^{1|1}$, where

$$\mathtt{X}(t) = \mathsf{P}(t)\,\mathtt{X}(0) \tag{33}$$

and can be called as orbit of $\mathtt{X}(0)$ under $\mathsf{P}(\cdot)$.

REMARK. In general the definition, the continuity, the functional equation and most of conclusions below hold valid also for $t \in \mathbb{R}^{1|0}$ (as e.g. in [30, p. 9]) including "nilpotent time" directions (see **Corollary 1**).

[3]For convenience we preserve operator notations and use somewhere the representation sign $\approx$ for clearness

From (23) it follows that

$$\mathsf{P}(t)\,\mathsf{P}(s)=\mathsf{P}(t)\,, \tag{34}$$

and so superoperators $\mathsf{P}(t)$ are idempotent. Also they form a rectangular band, because of

$$\mathsf{P}(t)\,\mathsf{P}(s)\,\mathsf{P}(t)=\mathsf{P}(t)\,, \tag{35}$$

$$\mathsf{P}(s)\,\mathsf{P}(t)\,\mathsf{P}(s)=\mathsf{P}(s)\,. \tag{36}$$

We observe that

$$\mathsf{P}(0)\approx\begin{pmatrix}0 & 0\\ \alpha & 1\end{pmatrix}\neq\mathsf{I}\approx\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{37}$$

as *opposite* to the standard case [27]. A “generator” $\mathsf{A}=\mathsf{P}'(t)$ is

$$\mathsf{A}\approx\begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix}, \tag{38}$$

and so the standard definition of generator [27]

$$\mathsf{A}=\lim_{t\to 0}\frac{\mathsf{P}(t)-\mathsf{P}(0)}{t}. \tag{39}$$

holds and for difference we have the standard relation

$$\mathsf{P}(t)-\mathsf{P}(s)=\mathsf{A}\cdot(t-s)\,. \tag{40}$$

The following properties of the generator A take place

$$\mathsf{P}(t)\,\mathsf{A}=\mathsf{Z}, \tag{41}$$

$$\mathsf{A}\mathsf{P}(t)=\mathsf{A}, \tag{42}$$

where “zero operator” Z is represented by the null supermatrix, $\mathsf{A}^2=\mathsf{Z}$, and therefore generator A is a nilpotent of second degree.

From (39) it follows that

$$\mathsf{P}(t)=\mathsf{P}(0)+\mathsf{A}\cdot t. \tag{43}$$

Definition 20. We call operators which can be presented as a linear supermatrix function of t a t-*linear superoperators*.

From (43) it follows that $\mathsf{P}(t)$ is a t-linear superoperator.

Proposition 21. *Superoperators* $\mathsf{P}(t)$ ***cannot be presented*** *as an exponent (as for the standard operator semigroups* $\mathsf{T}(t)=e^{\mathsf{A}\cdot t}$ *[27]).*

Proof. In our case

$$\mathsf{T}(t)=e^{\mathsf{A}\cdot t}=\mathsf{I}+\mathsf{A}\cdot t\approx\begin{pmatrix}1 & \alpha t\\ 0 & 1\end{pmatrix}\notin\mathbf{P}_{\alpha}. \tag{44}$$

□

REMARK. Exponential superoperator $\mathsf{T}(t)=e^{\mathsf{A}\cdot t}$ is represented by *even-reduced* supermatrices $\mathsf{T}(\cdot):\mathbb{R}_+\to\mathfrak{M}_{even}(1|1)$ [30], but idempotent superoperator $\mathsf{P}(t)$ is represented by *odd-reduced* supermatrices $\mathsf{P}(\cdot):\mathbb{R}_+\to\mathfrak{M}_{odd}(1|1)$ (see **Definition 4**).

Nevertheless, the superoperator $\mathsf{P}(t)$ satisfies *the same* linear differential equation

$$\mathsf{P}'(t)=\mathsf{A}\cdot\mathsf{P}(t) \tag{45}$$

as the standard exponential superoperator $\mathsf{T}(t)$ (the initial value problem [30])

$$\mathsf{T}'(t)=\mathsf{A}\cdot\mathsf{T}(t)\,. \tag{46}$$

This leads to the following

Corollary 22. In case initial state does not equal unity $\mathsf{P}(0)\neq\mathsf{I}$, there exists an *additional class* of solutions of the initial value problem (45)-(46) among odd-reduced (antidiagonal) idempotent t-linear (*nonexponential*) superoperators.

Problem 2. To find among general $(p+q)\times(m+n)$-supermatrices **all possible nonexponential classes** which solve the initial value problem (46).

Let us compare behavior of superoperators $\mathsf{P}(t)$ and $\mathsf{T}(t)$. First of all, their generators coincide

$$\mathsf{P}'(0)=\mathsf{T}'(0)=\mathsf{A}. \tag{47}$$

But powers of $\mathsf{P}(t)$ and $\mathsf{T}(t)$ are different $\mathsf{P}^n(t)=\mathsf{P}(t)$ and $\mathsf{T}^n(t)=\mathsf{T}(nt)$. In their common actions the superoperator which is from the left transfers its properties to the right hand side as follows

$$\mathsf{T}^n(t)\,\mathsf{P}(t)=\mathsf{P}((n+1)\,t)\,, \tag{48}$$
$$\mathsf{P}^n(t)\,\mathsf{T}(t)=\mathsf{P}(t)\,. \tag{49}$$

Their commutator is nonvanishing

$$[\mathsf{T}(t)\,\mathsf{P}(s)]=\mathsf{P}'(0)\,t=\mathsf{T}'(0)\,t=\mathsf{A}t, \tag{50}$$

which can be compared with the pure exponential commutator (for our case) $[\mathsf{T}(t)\,\mathsf{T}(u)]=0$ and idempotent commutator

$$[\mathsf{P}(t)\,\mathsf{P}(s)]=\mathsf{P}'(0)\,(t-s)=\mathsf{A}\,(t-s)\,. \tag{51}$$

Assertion 23. *All superoperators $\mathsf{P}(t)$ and $\mathsf{T}(t)$ commute in case of "nilpotent time" and*

$$t\in\operatorname{Ann}\alpha. \tag{52}$$

REMARK. The uniqueness theorem [30, p. 3] holds valid only for $\mathsf{T}(t)$, because of the nonvanishing commutator $[\mathsf{A},\mathsf{P}(t)]=\mathsf{A}\neq 0$.

Corollary 24. The superoperator $\mathsf{T}(t)$ is an inner inverse for $\mathsf{P}(t)$, because of

$$\mathsf{P}(t)\,\mathsf{T}(t)\,\mathsf{P}(t)=\mathsf{P}(t)\,, \tag{53}$$

but it is not an outer inverse, because

$$\mathsf{T}(t)\,\mathsf{P}(t)\,\mathsf{T}(t)=\mathsf{P}(2t)\,. \tag{54}$$

Let us try to find a (possibly noninvertible) operator U which connects exponential and idempotent superoperators $\mathsf{T}(t)$ and $\mathsf{P}(t)$.

Assertion 25. *The "semi-similarity" relation*

$$\mathsf{T}(t)\,\mathsf{U}=\mathsf{U}\mathsf{P}(t) \tag{55}$$

holds if

$$\mathsf{U}\approx\begin{pmatrix}\sigma\alpha & \sigma\\ 0 & \rho\alpha\end{pmatrix} \tag{56}$$

which is noninvertible triangle and depends from two odd constants, and the "adjoint" relation

$$\mathsf{U}^*\mathsf{T}(t)=\mathsf{P}(t)\,\mathsf{U}^* \tag{57}$$

holds if

$$\mathsf{U}^*\approx\begin{pmatrix}0 & \alpha vt\\ \alpha u & v\end{pmatrix} \tag{58}$$

which is also noninvertible antitriangle and depends from two even constants and "time".

REMARK. Note that U is nilpotent of third degree, since $\mathsf{U}^2=\sigma\rho\mathsf{A}$, but the "adjoint" superoperator is not nilpotent at all if v is not nilpotent.

Both A and Z behave as zeroes, but $\mathsf{Y}(t)$ (see (19)) is a two-sided zero for $\mathsf{T}(t)$ only, since

$$\mathsf{T}(t)\,\mathsf{Y}(t)=\mathsf{Y}(t)\,\mathsf{T}(t)=\mathsf{Y}(t)\,, \tag{59}$$

while

$$\mathsf{P}(t)\,\mathsf{Y}(t)=\mathsf{Y}(0)\,,\qquad \mathsf{Y}(t)\,\mathsf{P}(t)=\mathsf{A}t. \tag{60}$$

If we add A and Z to superoperators $\mathsf{P}(t)$, then we obtain an extended odd-reduced noncommutative superoperator semigroup $\mathcal{P}_{odd} = \bigcup \mathsf{P}(t) \bigcup \mathsf{A} \bigcup \mathsf{Z}$ with the following Cayley table (for convenience we add $\mathsf{Y}(t)$ and $\mathsf{T}(t)$ as well)

The Cayley table of the superoperator semigroup $\mathcal{P}_{odd}$

$1 \setminus 2$	$\mathsf{P}(t)$	$\mathsf{P}(s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(t)$	$\mathsf{T}(s)$
$\mathsf{P}(t)$	$\mathsf{P}(t)$	$\mathsf{P}(t)$	Z	Z	$\mathsf{P}(t)$	$\mathsf{P}(t)$	$\mathsf{P}(t)$
$\mathsf{P}(s)$	$\mathsf{P}(s)$	$\mathsf{P}(s)$	Z	Z	$\mathsf{P}(s)$	$\mathsf{P}(s)$	$\mathsf{P}(s)$
A	A	A	Z	Z	Z	A	A
Z	Z	Z	Z	Z	Z	Z	Z
$\mathsf{Y}(t)$	$\mathsf{A}t$	$\mathsf{A}s$	Z	Z	Z	$\mathsf{Y}(t)$	$\mathsf{Y}(t)$
$\mathsf{T}(t)$	$\mathsf{P}(2t)$	$\mathsf{P}(t+s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(2t)$	$\mathsf{T}(t+s)$
$\mathsf{T}(s)$	$\mathsf{P}(t+s)$	$\mathsf{P}(2s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(t+s)$	$\mathsf{T}(2s)$

(61)

It is easily seen that associativity in the left upper square holds, and so the table (61) is actually represents a semigroup of superoperators $\mathcal{P}_{odd}$ (under supermatrix multiplication).

The analogs of the "smoothing operator" $\mathsf{V}(t)$ [30] are

$$\mathsf{V}_P(t) = \int_0^t \mathsf{P}(s)\, ds = \frac{t}{2}\left(\mathsf{P}(t) + \mathsf{P}(0)\right) \approx \begin{pmatrix} 0 & \alpha \dfrac{t^2}{2} \\ \alpha t & t \end{pmatrix}, \tag{62}$$

$$\mathsf{V}_T(t) = \int_0^t \mathsf{T}(s)\, ds = \frac{t}{2}\left(\mathsf{T}(t) + \mathsf{T}(0)\right) \approx \begin{pmatrix} t & \alpha \dfrac{t^2}{2} \\ 0 & t \end{pmatrix}. \tag{63}$$

Let us consider the differential sequence of sets of superoperators $\mathsf{P}(t)$

$$\mathsf{S}_n \xrightarrow{\partial} \mathsf{S}_{n-1} \xrightarrow{\partial} \ldots \mathsf{S}_1 \xrightarrow{\partial} \mathsf{S}_0 \xrightarrow{\partial} \mathsf{A} \xrightarrow{\partial} \mathsf{Z}, \tag{64}$$

where $\partial = d/dt$ and

$$\mathsf{S}_n = \bigcup_t \frac{t^n}{n(n-1)\ldots 1} \mathsf{P}\left(\frac{t}{n+1}\right), \tag{65}$$

and by definition

$$\mathsf{S}_0 = \bigcup_t \mathsf{P}(t), \tag{66}$$

$$\mathsf{S}_1 = \bigcup_t \mathsf{V}_P(t). \tag{67}$$

GENERALIZED FUNCTIONAL EQUATION AND EVOLUTION

Now we construct an analog of the standard operator semigroup functional equation [27, 30]

$$\mathsf{T}(t+s) = \mathsf{T}(t)\,\mathsf{T}(s). \tag{68}$$

Using the multiplication law (34) and manifest representation (21). for the idempotent superoperators $\mathsf{P}(t)$ we can formulate

Definition 26. The *odd-reduced idempotent superoperators* $\mathsf{P}(t)$ satisfy the following generalized functional equation

$$\mathsf{P}(t+s) = \mathsf{P}(t)\,\mathsf{P}(s) + \mathsf{N}(t,s), \tag{69}$$

where

$$\mathsf{N}(t,s) = \mathsf{P}'(t)\, s.$$

The presence of second term $\mathsf{N}(t,s)$ in the right hand side of the generalized functional equation (69) can be connected with nonautonomous and deterministic properties of systems describing by it [30]. Indeed, from (33) it follows that

$$\begin{aligned} \mathtt{X}(t+s) &= \mathsf{P}(t+s)\,\mathtt{X}(0) = \mathsf{P}(t)\,\mathsf{P}(s)\,\mathtt{X}(0) + \mathsf{P}'(t)\, s\mathtt{X}(0) \\ &= \mathsf{P}(t)\,\mathtt{X}(s) + \mathsf{P}'(t)\, s\mathtt{X}(0) \neq \mathsf{P}(t)\,\mathtt{X}(s) \end{aligned} \tag{70}$$

as opposite to the always implied relation for exponential superoperators $\mathsf{T}(t)$ (translational property [27, 30])

$$\mathsf{T}(t)\,\mathtt{X}(s) = \mathtt{X}(t+s)\,, \tag{71}$$

which follows from (68). Instead of (71), using the band property (34) we obtain

$$\mathsf{P}(t)\,\mathtt{X}(s) = \mathtt{X}(t)\,, \tag{72}$$

which can be called the “moving time” property.

Problem 3. To find a “dynamical system” with time evolution satisfying the **“moving time” property** (72) instead of the translational property (71).

Assertion 27. *For “nilpotent time” satisfying* (52) *the generalized functional equation* (69) *coincides with the standard functional equation* (68)*, and therefore the idempotent operators* $\mathsf{P}(t)$ *describe autonomous and deterministic “dynamical” system and satisfy the translational property* (71)*.*

Proof. Follows from (52) and (70). □

Problem 4. To find **all maps** $\mathsf{P}(\cdot) : \mathbb{R}_{+} \to \mathfrak{M}(p|q)$ satisfying the generalized functional equation (69).

We turn to this problem later, and now consider some features of the Cauchy problem for idempotent superoperators.

ODD SOLUTION FOR THE CAUCHY PROBLEM

Let us consider an action (33) of superoperator $\mathsf{P}(t)$ in superspace $\mathbb{R}^{1|1}$ as $\mathtt{X}(t) = \mathsf{P}(t)\,\mathtt{X}(0)$, where the initial components are $\mathtt{X}(0) = \begin{pmatrix} x_0 \\ \varkappa_0 \end{pmatrix}$. From (33) the evolution of the components has the form

$$\begin{pmatrix} x(t) \\ \varkappa(t) \end{pmatrix} = \begin{pmatrix} \alpha\varkappa_0 t \\ \alpha x_0 + \varkappa_0 \end{pmatrix} \tag{73}$$

which shows that superoperator $\mathsf{P}(t)$ does not lead to time dependence of odd components. Then from (73) we see that

$$\mathtt{X}'(t) = \begin{pmatrix} \alpha\varkappa_0 \\ 0 \end{pmatrix} = const. \tag{74}$$

This is in full agreement with an analog of the Cauchy problem for our case

$$\mathtt{X}'(t) = \mathsf{A}\cdot\mathtt{X}(t)\,. \tag{75}$$

Assertion 28. *The solution of the Cauchy problem* (75) *is given by* (33)*, but the idempotent superoperator* $\mathsf{P}(t)$ ***can not be presented*** *in exponential form as in the standard case [27], but only in the* t*-linear form* $\mathsf{P}(t) = \mathsf{P}(0) + \mathsf{A}\cdot t \neq e^{\mathsf{A}\cdot t}$*, as we have already shown in* (43)*.*

This allows us to formulate

Theorem 29. *In superspace* $\mathbb{R}^{1|1}$ *the solution of the Cauchy initial problem with the same generator* A *is* <u>*two-fold*</u> *and is given by* <u>*two*</u> *different type of superoperators:*

1. *Exponential superoperator* $\mathsf{T}(t)$ *represented by the even-reduced supermatrices (**even solution**);*
2. *Idempotent* t*-linear superoperator* $\mathsf{P}(t)$ *represented by the odd-reduced supermatrices (**odd solution**).*

For comparison the standard solution of the Cauchy problem (75)

$$\mathtt{X}(t) = \mathsf{T}(t)\,\mathtt{X}(0)$$

in components is

$$\begin{pmatrix} x(t) \\ \varkappa(t) \end{pmatrix} = \begin{pmatrix} x_0 + \alpha\varkappa_0 t \\ \varkappa_0 \end{pmatrix}, \tag{76}$$

which shows that the time evolution of even coordinate is also in nilpotent even direction $\alpha\varkappa_0$ as in (73), but with addition of initial (possibly nonilpotent) x_0, while odd coordinate is (another) constant as well. That leads to

Assertion 30. *"Even" and "odd" evolutions coincide, if even initial coordinate vanishes $x_0 = 0$ or common starting point is pure odd $\mathtt{X}(0) = \begin{pmatrix} 0 \\ \varkappa_0 \end{pmatrix}$.*

A very much important formula is the condition of commutativity [27]

$$[\mathsf{A}, \mathsf{P}(t)]\, \mathtt{X}(t) = \mathsf{A}\mathtt{X}(t) = \begin{pmatrix} \alpha\varkappa(t) \\ 0 \end{pmatrix} = 0, \tag{77}$$

which satisfies, when $\alpha \cdot \varkappa(t) = 0$, while in the standard case the commutator $[\mathsf{A}, \mathsf{T}(t)]\, \mathtt{X}(t) = 0$, i.e. vanishes without any additional conditions [27].

SUPERANALOG OF RESOLVENT FOR EXPONENTIAL AND IDEMPOTENT SUPEROPERATORS

For resolvents $\mathsf{R}_P(z)$ and $\mathsf{R}_T(z)$ we use an analog of the standard formula from [27] in the form

$$\mathsf{R}_P(z) = \int\limits_0^\infty e^{-zt}\mathsf{P}(t)\, dt, \tag{78}$$

$$\mathsf{R}_T(z) = \int\limits_0^\infty e^{-zt}\mathsf{T}(t)\, dt. \tag{79}$$

Using the supermatrix representation (21) we obtain

$$\mathsf{R}_P(z) \approx \begin{pmatrix} 0 & \dfrac{\alpha}{z^2} \\ \dfrac{\alpha}{z} & \dfrac{1}{z} \end{pmatrix}, \tag{80}$$

$$\mathsf{R}_T(z) \approx \begin{pmatrix} \dfrac{1}{z} & \dfrac{\alpha}{z^2} \\ 0 & \dfrac{1}{z} \end{pmatrix}. \tag{81}$$

We observe, that $\mathsf{R}_T(z)$ satisfies the standard resolvent relation [30]

$$\mathsf{R}_T(z) - \mathsf{R}_T(w) = (w - z)\, \mathsf{R}_T(z)\, \mathsf{R}_T(w), \tag{82}$$

but its analog for $\mathsf{R}_P(z)$

$$\mathsf{R}_P(z) - \mathsf{R}_P(w) = (w - z)\, \mathsf{R}_P(z)\, \mathsf{R}_P(w) + \frac{w - z}{zw^2}\mathsf{A} \tag{83}$$

has additional term proportional to the generator A.

PROPERTIES OF t-LINEAR IDEMPOTENT (SUPER)OPERATORS

Here we consider properties of general t-linear (super)operators of the form

$$\mathsf{K}(t) = \mathsf{K}_0 + \mathsf{K}_1 t, \tag{84}$$

where $\mathsf{K}_0 = \mathsf{K}(0)$ and $\mathsf{K}_1 = \mathsf{K}'(0)$ are constant (super)operators represented by $(n \times n)$ matrices or $(p + q) \times .(p + q)$ supermatrices with t ("time") independent entries. Obviously, that the generator of a general t-linear (super)operator is

$$\mathsf{A}_K = \mathsf{K}'(0) = \mathsf{K}_1. \tag{85}$$

We will find system of equations for K_0 and K_1 for some special cases appeared in above consideration.

Assertion 31. *If a t-linear (super)operator $\mathsf{K}(t)$ satisfies the band equation (*34*)*

$$\mathsf{K}(t)\, \mathsf{K}(s) = \mathsf{K}(t), \tag{86}$$

then it is idempotent and the constant component (super)operators K_0 and K_1 satisfy the system of equations

$$\mathsf{K}_0^2 = \mathsf{K}_0, \qquad \mathsf{K}_1^2 = \mathsf{Z}, \tag{87}$$

$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1, \qquad \mathsf{K}_0\mathsf{K}_1 = \mathsf{Z}, \tag{88}$$

from which it follows, that K_0 is idempotent, K_1 is nilpotent, and K_1 is right divisor of zero and left zero for K_0.

For non-supersymmetric operators we have

Corollary 32. The components of t-linear operator $\mathsf{K}(t)$ have the following properties: idempotent matrix K_0 is similar to an upper triangular matrix with 1 on the main diagonal and nilpotent matrix K_1 is similar to an upper triangular matrix with 0 on the main diagonal [13, 57].

Comparing with the previous particular super case (43) we have $\mathsf{K}_0 = \mathsf{P}(0)$ and $\mathsf{K}_1 = \mathsf{A} = \mathsf{P}'(0)$.

REMARK. In case of $(p+q) \times (p+q)$ supermatrices the triangularization properties of **Corollary 32** do not hold valid due to presence divisors of zero and nilpotents among entries (see **Corollary 1**), and so the inner structure of the component supermatrices satisfying (87)-(88) can be much different from the standard non-supersymmetric case [13, 57].

Let us consider the structure of t-linear operator $\mathsf{K}(t)$ satisfying the generalized functional equation (69).

Assertion 33. *If a t-linear (super)operator $\mathsf{K}(t)$ satisfies the generalized functional equation*

$$\mathsf{K}(t+s) = \mathsf{K}(t)\,\mathsf{K}(s) + \mathsf{K}'(t)\,s, \tag{89}$$

then its component (super)operators K_0 and K_1 satisfy the system of equations

$$\mathsf{K}_0^2 = \mathsf{K}_0, \qquad \mathsf{K}_1^2 = \mathsf{Z}, \tag{90}$$

$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1, \qquad \mathsf{K}_0\mathsf{K}_1 = \mathsf{Z}, \tag{91}$$

We observe that the systems (87)-(88) and (90)-(91) are fully identical. It is important to observe the connection of the above properties with the differential equation for t-linear (super)operator $\mathsf{K}(t)$

$$\mathsf{K}'(t) = \mathsf{A}_K \cdot \mathsf{K}(t)\,. \tag{92}$$

Using (85) we obtain the equation for components

$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{93}$$

$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1. \tag{94}$$

That leads to the following

Theorem 34. *For any t-linear (super)operator $\mathsf{K}(t) = \mathsf{K}_0 + \mathsf{K}_1 t$ the next statements are equivalent:*

1. $\mathsf{K}(t)$ *is idempotent and satisfies the band equation (86);*

2. $\mathsf{K}(t)$ *satisfies the generalized functional equation (89);*

3. $\mathsf{K}(t)$ *satisfies the differential equation (92) and has idempotent time independent part $\mathsf{K}_0^2 = \mathsf{K}_0$ which is orthogonal to its generator $\mathsf{K}_0\mathsf{A} = \mathsf{Z}$.*

GENERAL t-POWER-TYPE IDEMPOTENT (SUPER)OPERATORS

Let us consider idempotent (super)operators which depend from time by power-type function, and so they have the form

$$\mathsf{K}(t) = \sum_{m=0}^{n} \mathsf{K}_m t^m, \tag{95}$$

where K_m are t-independent (super)operators represented by $(n \times n)$ matrices or $(p+q) \times .(p+q)$ supermatrices. This power-type dependence of is very much important for super case, when supermatrix elements take value in Grassmann algebra, and therefore can be nilpotent (see (1)–(2) and **Corollary 1**).

We now start from the band property $\mathsf{K}(t)\,\mathsf{K}(s) = \mathsf{K}(t)$ and then find analogs of the functional equation and differential equation for them. Expanding the band property (86) in component we obtain n-dimensional analog of (87)-(88) as

$$\mathsf{K}_0^2 = \mathsf{K}_0, \qquad \mathsf{K}_i^2 = \mathsf{Z},\ 1 \le i \le n, \tag{96}$$

$$\mathsf{K}_i\mathsf{K}_0 = \mathsf{K}_i,\ 1 \le i \le n, \quad \mathsf{K}_0\mathsf{K}_i = \mathsf{Z},\ 1 \le i \le n, \tag{97}$$

$$\mathsf{K}_i\mathsf{K}_j = \mathsf{Z},\ 1 \le i, j \le n, i \ne j. \tag{98}$$

Proposition 35. *The n-generalized functional equation for any t-power-type idempotent (super)operators (95) has the form*

$$\mathsf{K}(t+s) = \mathsf{K}(t)\,\mathsf{K}(s) + \mathsf{N}_n(t,s)\,, \text{ where } \mathsf{N}_n(t,s) = \sum_{m=1}^{n}\sum_{l=m}^{n} \mathsf{K}_l \frac{l\,(l-1)\ldots(l-m+1)}{m!} s^m t^{l-m}. \tag{99}$$

Proof. For the difference using the band property (86) we have $\mathsf{N}_n(t,s) = \mathsf{K}(t+s) - \mathsf{K}(t)\,\mathsf{K}(s) = \mathsf{K}(t+s) - \mathsf{K}(t)$. Then we expand in Taylor series around t and obtain $\mathsf{N}_n(t,s) = \sum_{m=1}^{n} \mathsf{K}^{(m)}(t)\,\dfrac{s^m}{m!}$, where $\mathsf{K}^{(m)}(t)$ denotes n-th derivative which is a finite series for the power-type $\mathsf{K}(t)$ (95). □

The differential equation for idempotent (super)operators coincide with the standard initial value problem only for t-linear operators. In case of the power-type operators (95) we have

Proposition 36. *The n-generalized differential equation for any t-power-type idempotent (super)operators (95) has the form*

$$\mathsf{K}'(t) = \mathsf{A}_K \cdot \mathsf{K}(t) + \mathsf{U}_n(t), \tag{100}$$

where

$$\mathsf{U}_n(t) = \begin{cases} 0 & n = 1 \\ \sum_{m=2}^{n} m\mathsf{K}_m t^{m-1} & n \geq 2 \end{cases}. \tag{101}$$

Proof. To find the difference $\mathsf{U}_n(t)$ we use the expansion (95) and the band conditions for components (96)–(98). □

CONCLUSION

In general one-parametric matrix semigroups and corresponding superoperator semigroups represented by antitriangle idempotent supermatrices and their generalization to higher dimensions $(p+q) \times (m+n)$ have many unusual and nontrivial properties [8, 9, 25, 39]. Here we considered only some of them related to their connection with functional and differential equations of corresponding superoperators. The stated **Problems 1-3** are worthwhile to investigate in future. It would be also interesting to generalize the above constructions to higher dimensions, to study continuity properties of the introduced idempotent superoperators, to consider multi-time evolution and to find the corresponding applications in modern supersymmetric models.

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REFERENCES

1. Berezin F. A. Introduction to Superanalysis. Dordrecht. *Reidel*, 1987.

2. Leites D. A. // *Introduction to the theory of supermanifolds*. Russian Math. Surv. 1980. V. **35**. № 1. P. 1–64.

3. Bars I. // *Supergroups and their representations*. Introduction to Supersymmetry in Particle and Nuclear Physics. New York. *Plenum Press*, 1984. P. 107–184.

4. van Nieuwenhuizen P., West P. Principles of Supersymmetry and Supergravity. Cambridge. *Cambridge Univ. Press*, 1989. 453 p.

5. Kaku M. Introduction to Superstrings and M-Theory. Berlin. *Springer-Verlag*, 1998. 587 p.

6. Bailin D., Love A. Supersymmetric Gauge Field Theory and String Theory. Bristol. *Institute of Physics*, 1994. 322 p.

7. Quantum Fields and Strings: A Cource for Mathematicians // Deligne P., Etingof P., Freed D. S., Jeffrey L. C., Kazhdan D., Morgan J. W., Morrison D. R., Witten E., *eds.* V. **1, 2** Providence. *American Mathematical Society* 1999. 1495 p.

8. Duplij S. Semigroup methods in supersymmetric theories of elementary particles. Kharkov. *Habilitation Thesis, Kharkov State University,* math-ph/9910045, 1999. 483 p.

9. Duplij S. Semisupermanifolds and semigroups. Kharkov. *Krok*, 2000. 220 p.

10. Howie J. M. An Introduction to Semigroup Theory. London. *Academic Press*, 1976. 270 p.

11. Higgins P. M. Techniques of Semigroup Theory. Oxford. *Oxford University Press*, 1992. 254 p.

12. Lawson M. V. Inverse Semigroups: The Theory of Partial Symmetries. Singapore. *World Sci.*, 1998. 412 p.

13. Okniński J. Semigroups of Matrices. Singapore. *World Sci.*, 1998. 453 p.

14. Ponizovskii J. S. // *On a type of matrix semigroups*. Semigroup Forum. 1992. V. **44**. P. 125–128.

15. Okniński J., Ponizovskii J. S. // *A new matrix representation theorem for semigroups*. Semigroup Forum. 1996. V. **52**. P. 293–305.

16. Mohapatra R. N. Unification and Supersymmetry: The Frontiers of Quark-lepton Physics. Berlin. *Springer-Verlag,* 1986. 309 p.

17. Salam A., Strathdee J. // *Supersymmetry and superfields*. Fortschr. Phys. 1978. V. **26**. P. 57–123.

18. Gates S. J., Grisaru M. T., Rocek M., et al. Superspace. Reading. *Benjamin*, 1983.

19. De Witt B. S. Supermanifolds. Cambridge. *Cambridge Univ. Press*, 2nd edition. 1992. 407 p.

20. Khrennikov A. Y. Superanalysis. M.. *Nauka*, 1997. 304 p.

21. Vladimirov V. S., Volovich I. V. // *Superanalysis*. **1**. *Differential calculus*. Theor. Math. Phys. 1984. V. **59**. № 1. P. 3–27.

22. Duplij S. // *On semigroup nature of superconformal symmetry*. J. Math. Phys. 1991. V. **32**. № 11. P. 2959–2965.

23. Duplij S. // *Ideal structure of superconformal semigroups*. Theor. Math. Phys. 1996. V. **106**. № 3. P. 355–374.

24. Duplij S. // *Some abstract properties of semigroups appearing in superconformal theories*. Semigroup Forum. 1997. V. **54**. № 2. P. 253–260.

25. Duplij S. // *On an alternative supermatrix reduction*. Lett. Math. Phys. 1996. V. **37**. № 3. P. 385–396.

26. Hille E., Phillips R. S. Functional Analysis and Semigroups. Providence. *Amer. Math. Soc.*, 1957. 808 p.

27. Davies E. B. One-Parameter Semigroups. London. *Academic Press*, 1980. 230 p.

28. Goldstein J. A. Semigroups of Linear Operators and Applications. Oxford. *Oxford University Press*, 1985. 347 p.

29. Hille E. Methods in Classical and Functional Analysis. Reading. *Addison-Wesley*, 1972. 267 p.

30. Engel K.-J., Nagel R. One-parameter semigroups for linear evolution processes. Berlin. *Springer-Verlag*, 1999. 584 p.

31. Belleni-Morante A. Applied Semigroups and Evolution Equations. Oxford. *Oxford University Press*, 1979. 341 p.

32. Daners D., Koch Medina P. Abstract Evolution Equations, Periodic Problems and Applications. New York. *Longman*, 1992. 267 p.

33. Berg C., Christensen J. P. R., Ressel P. Harmonic Analysis on Semigroups. Berlin. *Springer-Verlag*, 1984.

34. Satyanarayana M. Positively Ordered Semigroups. New York. *Dekker*, 1979.

35. Berglund J., Junghenn H. D., Milnes P. Analysis on Semigroups. New York. *Wiley*, 1989.

36. Ahmed N. U. Semigroup Theory With Application to Systems and Control. New York. *Wiley*, 1991.

37. Gawedzki K. // *Supersymmetries — mathematics of supergeometry*. Annales Poincare Phys.Theor. 1977. V. **27**. P. 335–366.

38. Choquet-Bruhat Y. // *Mathematics for classical supergravities*. Lect. Notes Math. 1987. V. **1251**. P. 73–90.

39. Duplij S. // *Supermatrix representations of semigroup bands*. Pure Math. Appl. 1996. V. **7**. № 3-4. P. 235–261.

40. Clemént P., Heijmans H. J. A. M., Angenent S., van Duijn C. J., de Pagter B. One-Parameter Semigroups. Amsterdam. *North-Holland*, 1987. 436 p.

41. Duplij S. // *On supermatrix idempotent operator semigroups*. Kharkov, 2000. 11 p. (*Preprint* / *Kharkov National Univ.*, `math.FA/0006001`).

42. Rogers A. // *A global theory of supermanifolds*. J. Math. Phys. 1980. V. **21**. № 5. P. 1352–1365.

43. Ivashchuk V. D. // *Invertibility of elements in infinite-dimensional Grassmann-Banach algebras*. Theor. Math. Phys. 1990. V. **84**. № 1. P. 13–22.

44. Pestov V. G. // *On enlargability of infinite-dimensional Lie superalgebras*. J. Geom. and Phys. 1993. V. **10**. P. 295–314.

45. Bahturin Y. A., Mikhalev A. A., Petrogradsky V. M., Zaicev M. V. Infinite-dimensional Lie Superalgebras. Berlin. *Walter de Cruyter*, 1992. 325 p.

46. Backhouse N. B., Fellouris A. G. // *Grassmann analogs of classical matrix groups*. J. Math. Phys. 1985. V. **26**. № 6. P. 1146–1151.

47. Urrutia L. F., Morales N. // *The Cayley-Hamilton theorem for supermatrices*. J. Phys. 1994. V. **A27**. № 6. P. 1981–1997.

48. Nashed M. Z. Generalized Inverses and Applications. New York. *Academic Press*, 1976. 321 p.

49. Davis D. L., Robinson D. W. // *Generalized inverses of morphisms*. Linear Algebra Appl. 1972. V. **5**. P. 329–338.

50. McAlister D. B. // *Representations of semigroups by linear transformations.* **1,2**. Semigroup Forum. 1971. V. **2**. P. 189–320.

51. Clifford A. H., Preston G. B. The Algebraic Theory of Semigroups. V. **1** Providence. *Amer. Math. Soc.*, 1961.

52. Howie J. M. // *Why study semigroups?* Math. Chronicle. 1987. V. **16**. P. 1–14.

53. Grillet P.-A. Semigroups. An Introduction to the Structure Theory. New York. *Dekker*, 1995. 416 p.

54. Boyd J. P. Social semigroups. A unified theory of scaling and blockmodelling as applied to social networks. Fairfax. *George Mason Univ. Press*, 1991. 267 p.

55. Duplij S. // *Noninvertible $N=1$ superanalog of complex structure*. J. Math. Phys. 1997. V. **38**. № 2. P. 1035–1040.

56. Borel A. Linear Algebraic Groups. Moscow. *Science*, 1972.

57. Radjavi H., Rosenthal P. Simultaneous Triangularization. Berlin. *Springer-Verlag*, 1999. 318 p.

ПОЛУГРУППЫ СУПЕРМАТРИЦ И ОДНОПАРАМЕТРИЧЕСКИХ ИДЕМПОТЕНТНЫХ СУПЕРОПЕРАТОРОВ

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В работе рассматриваются полугруппы суперматриц и исследуются их различные редукции. Определяются однопараметрические полугруппы антитреугольных суперматриц и изучаются свойства соответствующих полугрупп супероператоров. Показано, что t-линейные идемпотентные супероператоры и обычные экспоненциальные супероператоры являются дуальными в некотором смысле, и первые дают дополнительное (нечетное) решение (по отношению с стандартному экспоненциальному) проблемы Коши. Найдены соответствующее функциональное уравнение и получен аналог резольвенты. Для идемпотентных (супер)операторов с t-зависимостью степенного вида найдены дифференциальные и функциональные уравнения.

КЛЮЧЕВЫЕ СЛОВА: суперматрица, редукция, однопараметрическая полугруппа, идемпотент, проблема Коши, резольвента