Reward Testing Equivalences for Processes

Rob van Glabbeek

Data61, CSIRO, Sydney, Australia

University of New South Wales, Sydney, Australia

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The corresponding paper is dedicated to Rocco De Nicola, on the occasion of his 65^{th} birthday.

Rocco's work has been a source of inspiration to my own.

The general theory of testing

It assumes

- a set of processes $\mathbb{P}$,
- a set of tests $\mathbb{T}$, which can be applied to processes,
- a set of outcomes $\mathbb{O}$ of applying a test to a process, and
- a function $\mathcal{A}pply : \mathbb{T} \times \mathbb{P} \rightarrow \mathscr{P}^{+}(\mathbb{O})$, representing the possible results of applying a specific test to a specific process,
- a partial order $\leq$ on $\mathbb{O}$: some outcomes are better than others.

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$O_1 \sqsubseteq_{\mathrm{Ho}} O_2$ if $\forall o_1 \in O_1\ \exists o_2 \in O_2$ such that $o_1 \leq o_2$
$O_1 \sqsubseteq_{\mathrm{Sm}} O_2$ if $\forall o_2 \in O_2\ \exists o_1 \in O_1$ such that $o_1 \leq o_2$.

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$P \sqsubseteq_{\mathrm{may}} Q$ iff $\mathcal{A}pply(T, P) \sqsubseteq_{\mathrm{Ho}} \mathcal{A}pply(T, Q)$ for every test T.
$P \sqsubseteq_{\mathrm{must}} Q$ iff $\mathcal{A}pply(T, P) \sqsubseteq_{\mathrm{Sm}} \mathcal{A}pply(T, Q)$ for every test T.

CCS

$$\alpha.E \xrightarrow{\alpha} E \qquad \frac{E_j \xrightarrow{\alpha} E'_j}{\sum_{i\in I} E_i \xrightarrow{\alpha} E'_j}\ (j \in I)$$

$$\frac{E \xrightarrow{\alpha} E'}{E|F \xrightarrow{\alpha} E'|F} \qquad \frac{E \xrightarrow{a} E',\ F \xrightarrow{\bar{a}} F'}{E|F \xrightarrow{\tau} E'|F'} \qquad \frac{F \xrightarrow{\alpha} F'}{E|F \xrightarrow{\alpha} E|F'}$$

$$\frac{E \xrightarrow{\alpha} E'}{E\backslash L \xrightarrow{\alpha} E'\backslash L}\ (\alpha, \bar{\alpha} \notin L) \qquad \frac{E \xrightarrow{\alpha} E'}{E[f] \xrightarrow{f(\alpha)} E'[f]} \qquad \frac{\mathbf{fix}(\!|S_X{:}S|\!) \xrightarrow{\alpha} E}{\mathbf{fix}(\!|X{:}S|\!) \xrightarrow{\alpha} E}$$

α ranges over $Act = \mathscr{A} \uplus \bar{\mathscr{A}} \uplus \{\tau\}$.

May and Must Testing for CCS

$Act^{\omega} := Act \cup \{\omega\}$. $\omega \notin Act$ is a special action reporting success.

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A *computation* $\pi \in \mathbb{T}^*_{\mathrm{CCS}}$ is a sequence $T_0, T_1, T_2, \ldots$ of tests, s.t.
(i) if T_n is the final element, then $T_n \stackrel{\tau}{\not\longrightarrow}$, and
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It is *successful* if it contains a state T with $T \stackrel{\omega}{\longrightarrow}$.

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Comp(T, P) is the set of computations whose starting from $T|P$.

$$\mathcal{A}pply(T,P) := \{\top \mid \exists \text{ successful } \pi \in Comp(T,P)\} \cup \{\bot \mid \exists \text{ unsuccessful } \pi \in Comp(T,P)\}.$$

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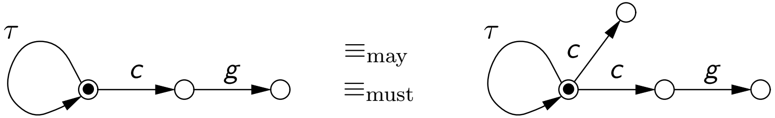
Now $P \sqsubseteq_{\mathrm{may}} Q$ holds unless $\exists T$ such that $T|P$ has a successful computation but Q has not.
Likewise $P \sqsubseteq_{\mathrm{must}} Q$ holds unless there is a test T such that $T|P$ has only successful computations but Q has not.

Testing does not capture conditional liveness



Processes identified by may and must testing, but with different conditional liveness properties

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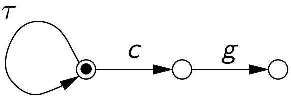

$\equiv_{\text{may}}$

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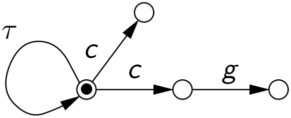

Processes identified by may and must testing, but with different conditional liveness properties

A conditional liveness property says that

> *under certain conditions something good will eventually happen.*

Reward Testing

A *reward test* is a CCS process, but with α ranging over $Act \times \mathbb{R}$. Such a *valued action* is tagged with a real number, the *reward* for executing this action. A negative reward is a penalty.

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A *reward computation* π is a sequence $T_0, r_1, T_1, r_2, T_2, \ldots$, s.t.:
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The *reward* of π: $\sum_{i=1}^n r_i$ or $\displaystyle\inf_{n\to\infty} \sum_{i=1}^{n} r_i \in \mathbb{R} \cup \{-\infty, \infty\}$.

Let $\mathcal{A}pply(T,P) := \{reward(\pi) \mid \pi \in Comp^R(T,P)\}$.

Must versus may

$P \sqsubseteq^{\text{must}}_{\text{reward}} Q$ holds iff under any reward test, the worst possible reward for Q is better than the worst possible reward for P.

$P \sqsubseteq^{\text{may}}_{\text{reward}} Q$ holds iff under any reward test, the best possible reward for Q is better than the best possible reward for P.

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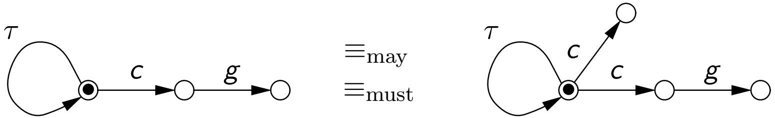
Theorem: $P \sqsubseteq^{\text{may}}_{\text{reward}} Q$ iff $Q \sqsubseteq^{\text{must}}_{\text{reward}} P$.

Reward testing captures conditional liveness



Processes identified by may and must testing, but with different conditional liveness properties

They are distinguished by reward testing

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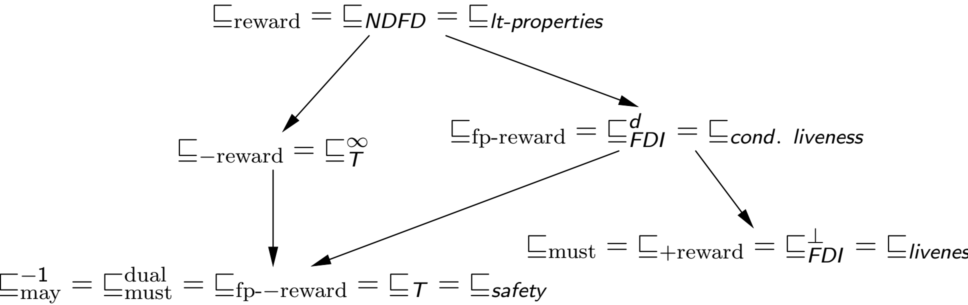
Theorem: Reward testing respects conditional liveness properties.

A spectrum of testing preorders



Congruence properties

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Let *stable* be the predicate that holds for a process P iff $P \stackrel{\tau}{\not\longrightarrow}$.

Write $P \sqsubseteq_X^\tau Q$ iff $P \sqsubseteq_X Q \wedge (\mathit{stable}(P) \Rightarrow \mathit{stable}(Q))$.

$\sqsubseteq_{\mathrm{reward}}^\tau$ is a congruence for all operators of CCS.

Axioms

$$
\left\{
\begin{array}{rcl}
\tau.X + Y & \equiv & \tau.X + \tau.(X + Y) \\
\alpha.X + \tau.(\alpha.Y + Z) & \equiv & \tau(\alpha.X + \alpha.Y + Z) \\
\alpha.(\tau.X + \tau.Y) & \equiv & \alpha.X + \alpha.Y \\
\tau.X + Y & \sqsubseteq & \tau.(X + Y) \\
\tau.X + Y & \sqsubseteq & X \\
\tau.\Delta X + Y & \equiv & \Delta(X + Y)
\end{array}
\right\}
$$

$\Delta P := \mathbf{fix}(\!|X\colon X \stackrel{def}{=} \tau.X + P\ |\!)$

Must testing: $\Delta X = \Delta Y$.

Congruence for recursion?

An equivalence $\sim$ is a (full) congruence for recursion if

$$\frac{S_Y \sim T_Y \quad \text{for all } Y \in dom(S)}{\mathbf{fix}(\!|X{:}S|\!) \sim \mathbf{fix}(\!|X{:}T|\!)} \qquad \frac{E \sim F}{\mu X.E \sim \mu X.F}$$

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$\uparrow$

strong bisimilarity

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Conclusion

I presented a new theory of testing yielding a finer equivalence, that respects conditional liveness properties.