Fixpoint Games

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Solving one Fixpoint Equation

We are interested in techniques for solving (systems of) fixpoint equations over a lattice

One-equation case

Solve the equation E given as

$$x =_\eta f(x)$$

where

- $f \colon L \to L$ is a monotone function over a complete lattice $(L, \sqsubseteq)$
- $\eta \in \{\mu, \nu\}$, indicating whether we are interested in the least (μ) or greatest (ν) fixpoint

The solution of E is denoted by $sol(E)$

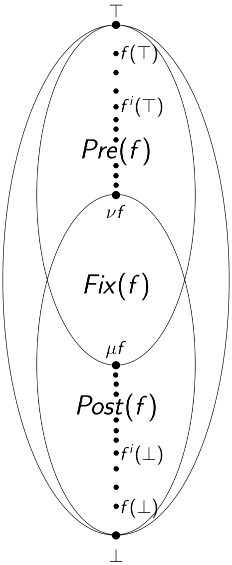
Applications in concurrency theory, model checking, program analysis

Solving one Fixpoint Equation

Solution techniques

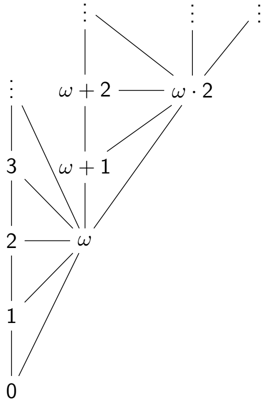
- The Knaster-Tarski theorem guarantees the existence of least and greatest fixpoints for monotone functions
- Kleene iteration: whenever f is (co-)continuous
  - $\eta = \mu$ (least fixpoint): $sol(E) = \bigsqcup_{i \in \mathbb{N}} f^i(\bot)$
  - $\eta = \nu$ (greatest fixpoint): $sol(E) = \bigsqcap_{i \in \mathbb{N}} f^i(\top)$
- In order to check whether $l \sqsubseteq sol(E)$ for some $l \in L$:
  - $\eta = \mu$ (least fixpoint): use ranking functions
  - $\eta = \nu$ (greatest fixpoint): construct a postfix-point l' ($l' \sqsubseteq f(l')$) such that $l \sqsubseteq l'$

Solving one Fixpoint Equation



If f is *not* (co-)continuous:

$\rightsquigarrow$ Kleene iteration over the ordinals (beyond ω)



Solving one Fixpoint Equation

Examples

- Bisimilarity characterized as a greatest fixpoint
- Behavioural metric characterized a a least fixpoint

Solving (Systems of) Fixpoint Equations

System of fixpoint equations

Let L be a lattice. A system of equations E over L is of the following form, where $f_i\colon L^m \to L$ are monotone functions and $\eta_i \in \{\mu, \nu\}$.

$$\begin{aligned} x_1 &=_{\eta_1} f_1(x_1, \dots, x_m) \\ &\dots \\ x_m &=_{\eta_m} f_m(x_1, \dots, x_m) \end{aligned}$$

The solution of E, denoted $sol(E) \in L^m$, is defined inductively as follows:

$$\begin{aligned} sol(\emptyset) &= () \\ sol(E) &= (sol(E[x_m := s_m]),\ s_m) \end{aligned}$$

where $s_m = \eta_m(\lambda x.\, f_m(sol(E[x_m := x]), x))$

Solving (Systems of) Fixpoint Equations

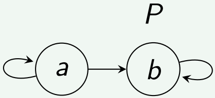
Remarks:

- $E[x_m := x]$ is a system of $m-1$ equations that one obtains by fixing the value of x_m as x and removing the last equation.
- Intuitively we fix the value of x_m as x, solve the remaining equation systems parameterized over x and then perform a fixpoint iteration (least or greatest) over x.
- The order of the equations matters.
- The solution is a fixpoint of the equation system (one of typically many fixpoints).

Solving (Systems of) Fixpoint Equations

Example: μ-calculus model checking

We consider the modal μ-calculus with $\Box$ ("all successor states satisfy ..."), $\Diamond$ ("some successor state satisfies ..."), least and greatest fixpoints.



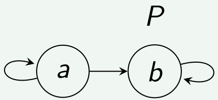
$$\nu x_2.(\mu x_1.(\Diamond x_1 \vee (P \wedge \Diamond x_2)) \wedge \Box x_2)$$

Equations over the powerset lattice of states:

$$\begin{aligned} x_1 &=_\mu \Diamond x_1 \cup (P \cap \Diamond x_2) \\ x_2 &=_\nu x_1 \cap \Box x_2 \end{aligned}$$

Solving (Systems of) Fixpoint Equations

Example: μ-calculus model checking



Equations over the powerset lattice of states:

$$x_1 \quad =_\mu \quad \Diamond x_1 \cup (P \cap \Diamond x_2)$$

x_1: "there exists a path such that eventually P holds and x_2 holds for some successor"

$$x_2 \quad =_\nu \quad x_1 \cap \Box x_2$$

x_2: "x_1 holds and all successors satisfy x_2"

Combined: "from all reachable states there is a path along which P holds infinitely often"

Solving (Systems of) Fixpoint Equations

Efficient algorithms for μ-calculus model-checking

n: number of states d: alternation depth of formula

- Naive approach: use the definition $\rightsquigarrow O(n^d)$
- Reduce model-checking problem to a parity game and determine whether the existential player has a winning strategy
  - Local on-the fly algorithms [Stevens, Stirling] that perform an on-the fly search for a winning strategy of the existential player (proving that a given state satisfies a formula)
  - Progress measures [Jurdzinski] $\rightsquigarrow O(n^{\frac{d}{2}})$
  - Quasi-polynomial algorithms [Calude, Jain, Khoussainov, Bakhadyr, Li, Stephan] $\rightsquigarrow O(n^{\lceil \log d \rceil + c})$

Solving (Systems of) Fixpoint Equations

Example: lattice-valued μ-calculi

Variants: Non-boolean μ-calculi that do not check whether a formula holds in a state, but measure the "degree" with respect to which a formula is satisfied:

$$x \models \varphi \text{ is replaced by } [\![\varphi]\!]: X \to L$$

- Latticed μ-calculus [Kupferman, Lustig]
  $\leadsto$ over a lattice L
- Quantitative probabilistic μ-calculus [Huth, Kwiatkowska]
  $\leadsto$ over the real interval $L = [0, 1]$
- Łukasiewicz μ-calculus [Mio, Simpson]
  $\leadsto$ over the real interval $L = [0, 1]$

$\leadsto$ we require methods and techniques for solving fixpoint equations over general lattices (as opposed to powerset lattices)

Fixpoint Games

Aim: consider a game perspective for solving systems of fixpoint equations for general lattices

Let E be a system of m equations over a lattice L with a basis B_L ($B_L \subseteq L$ such that every $l \in L$ can be obtained as $l = \bigsqcup B'$ where $B' \subseteq B_L$). Let $sol(E) = (s_1, \dots, s_m)$ be the solution.

- Given $b \in B_L$, $i \in \{1, \dots, m\}$ the existential player ($\exists$, Eve) wants to prove that $b \sqsubseteq s_i$.
- The universal player ($\forall$, Adam) is the adversary of $\exists$ and wants to show that $b \not\sqsubseteq s_i$.

Precursor games:

- Parity games
- Unfolding games [Venema]
  - are being played on a powerset lattice
  - single fixpoint equation

Fixpoint Games

Fixpoint game (first version)

Position	Player	Moves
(b, i)	$\exists$	$(l_1, \ldots, l_m)$ such that $b \sqsubseteq f_i(l_1, \ldots, l_m)$
$(l_1, \ldots, l_m)$	$\forall$	(b', j) such that $b' \sqsubseteq l_j$

$b, b' \in B_L$, $\bot \notin B_L$, $(l_1, \ldots, l_m) \in L^m$

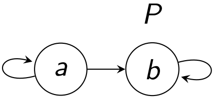
Winning condition ("parity condition")

	$\exists$	$\forall$
Finite game	$\forall$ unable to move	$\exists$ unable to move
Infinite game	$\eta_h = \nu$	$\eta_h = \mu$

Where $h \in \{1, \ldots, m\}$ is the highest equation index occurring infinitely often.

Fixpoint Games

We play the game on the powerset lattice $L = \mathcal{P}(\{a,b\})$ with basis $B_L = \{\{a\},\{b\}\}$ for $b = \{a\}$, $i = 2$:



$$\begin{aligned} x_1 &=_\mu \Diamond x_1 \cup (P \cap \Diamond x_2) = f_1(x_1,x_2) \\ x_2 &=_\nu x_1 \cap \Box x_2 = f_2(x_1,x_2) \end{aligned}$$

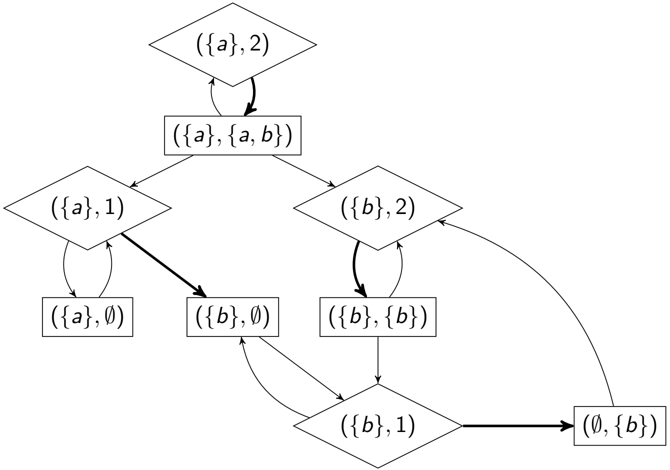
Remember: the second component of the solution contains all states such that "from all reachable states there is a path along which P holds infinitely often"

Fixpoint Games

Notation:

- Game positions (nodes) of $\exists$: $\diamond$
- Game positions (nodes) of $\forall$: $\Box$

Fixpoint Games



Only minimal moves of ∃ are given. Thick arrows: winning strategy of ∃

Fixpoint Games

Is the game correct and complete for all lattices?
("$\exists$ has a winning strategy for $(b, i) \iff b \sqsubseteq s_i$")

Counterexample

$L = \mathbb{N} \cup \{\omega\}$, $B_L = L \backslash \{0\}$
$f : L \to L$, $f(n) = n + 1$, $f(\omega) = \omega$

$$x \quad =_\mu \quad f(x)$$

We play a game to check whether ω is below the solution (= least fixpoint):

$$\omega \overset{\exists}{\leadsto} \omega \overset{\forall}{\leadsto} \omega \dots$$

$\forall$ would win this game ... This means that something is wrong!

Fixpoint Games

In this case $\omega \sqsubseteq \bigsqcup_{i\in\mathbb{N}} f^i(0)$, but $\omega \not\sqsubseteq f^i(0)$ for all $i \in \mathbb{N}$.

However, in order to win, $\exists$ has to descend in the lattice in order to reach $\bot = 0$ and enforce a finite game. ($\exists$ has to be able to go beyond the "limit ordinals" in the fixpoint iteration.)

Solution: play with basis $B_L = \mathbb{N}\backslash\{0\}$. This forces $\forall$ to pick some $n \in \mathbb{N}$.

What are the restrictions on the basis in general?

Way-Below Relation, Algebraic and Continuous Lattices

Way-below relation (definition)

Given two elements $l, l' \in L$ we say that l is way-below l', written $l \ll l'$ when for all directed set $D \subseteq L$, if $l' \sqsubseteq \bigsqcup D$ then there exists $d \in D$ such that $l \sqsubseteq d$.

- It holds that $\omega \not\ll \omega$, since $\omega \sqsubseteq \bigsqcup \mathbb{N}$, but ω is not below any element of the directed set $\mathbb{N}$.
- For two sets $Y, Y' \in \mathcal{P}(X)$ it holds that $Y \ll Y'$ iff $Y \subseteq Y'$ and Y finite.
- For $x, x' \in [0, 1]$ it holds that $x \ll x'$ iff $x < x'$ or $x = 0$.

Way-Below Relation, Algebraic and Continuous Lattices

Algebraic lattice (definition)

An element $l \in L$ is compact if $l \ll l$.
A lattice L is algebraic if the compact elements form a basis.

- Every powerset lattice is algebraic.
- $\mathbb{N} \cup \{\omega\}$ is algebraic.
- $[0,1]$ is *not* algebraic. (Only 0 is compact.)

Soundness and completeness of the fixpoint game (first version)

The game is

- always correct ("$\exists$ has a winning strategy for $(b,i) \Rightarrow b \sqsubseteq s_i$")
- and complete ("$b \sqsubseteq s_i \Rightarrow \exists$ has a winning strategy for (b,i)")

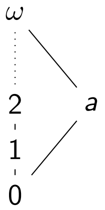
iff B_L consists of compact elements (and hence L is algebraic).

Way-Below Relation, Algebraic and Continuous Lattices

Continuous Lattice [Scott]

A lattice L is continuous if for all $l \in L$ it holds that $l = \bigsqcup\{l' \in L \mid l' \ll l\}$.

- Every algebraic lattice is continuous.
- $[0,1]$ is a continuous lattice.
- The lattice to the right is not continuous: $a \not\ll a$, so $\bigsqcup\{l \in L \mid l \ll a\} = 0 \neq a$.



Way-Below Relation, Algebraic and Continuous Lattices

Fixpoint game (second version)

Position	Player	Moves
(b, i)	$\exists$	$(l_1, \ldots, l_m)$ such that $b \sqsubseteq f_i(l_1, \ldots, l_m)$
$(l_1, \ldots, l_m)$	$\forall$	(b', j) such that $b' \ll l_j$

$b, b' \in B_L$, $\bot \notin B_L$, $(l_1, \ldots, l_m) \in L^m$

The winning conditions stay unchanged.

Soundness and completeness of the fixpoint game (second version)

The game is

- always complete
- and correct iff L is continuous.

Conclusion

Further contributions

- Progress measures: computing the strategy of the existential player (global algorithm)
- Local algorithm for checking whether a lattice element is below the solution
- Integration with up-to techniques for stopping earlier
- Variant of the game: play on the powerset of the basis (sound and complete for all complete lattices)

Conclusion

Open question

Does the theory developed here help to solve fixpoint equations over the reals, metrics and other infinite lattices?

- ▷ initial experiments with SMT solvers
- ▷ methods for approximating the solution