Differential Logical Relations

Joint work with Francesco Gavazzo and Akira Yoshimizu

Ugo Dal Lago

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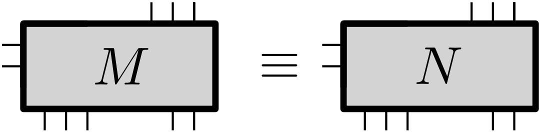
Comparing Interacting Programs



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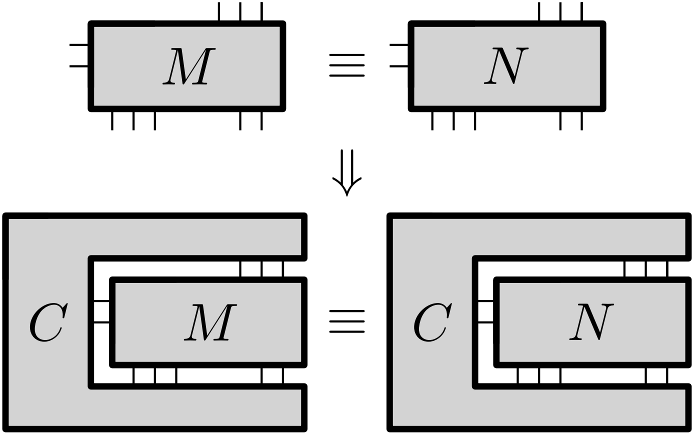

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- What if $\mathbb{X}$ is a metric space?

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- In *probabilistic* computation, one is naturally lead to observe a quantitative property, and $\mathbb{X}$ is simply $\mathbb{R}$.
- But even when computation is *deterministic*, one could well work with $\mathbb{X} = \mathbb{R}$ when real numbers are part of the underlying language [ReedPierce2010,AGHKC2017].

An Example

1

A Survey Of Techniques for Approximate Computing

Sparsh Mittal, Oak Ridge National Laboratory

Approximate computing trades off computation quality with the effort expended and as rising performance demands confront with plateauing resource budgets, approximate computing has become, not merely attractive, but even imperative. In this paper, we present a survey of techniques for approximate computing (AC). We discuss strategies for finding approximable program portions and monitoring output quality, techniques for using AC in different processing units (e.g., CPU, GPU and FPGA), processor components, memory technologies etc., and programming frameworks for AC. We classify these techniques based on several key characteristics to emphasize their similarities and differences. The aim of this paper is to provide insights to researchers into working of AC techniques and inspire more efforts in this area to make AC the mainstream computing approach in future systems.

Categories and Subject Descriptors: [**General and reference**]: Surveys and overviews; [**Hardware**]: Power and energy; [**Computer systems organization**]: Processors and memory architectures

General Terms: Design, Performance

Additional Key Words and Phrases: Review, classification, approximate computing technique (ACT), approximate storage, quality configurability, CPU, GPU, FPGA, neural networks

ACM Reference Format:
S. Mittal, "A Survey Of Techniques for Approximate Computing", 20xx. *ACM Comput. Surv.* a, b, Article 1 (2015), 34 pages.
DOI:http://dx.doi.org/10.1145/0000000.0000000

1. INTRODUCTION

As large-scale applications such as scientific computing, social media and financial analysis gain prominence, the computational and storage demands of modern systems

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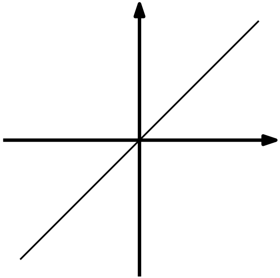

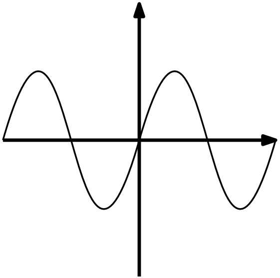

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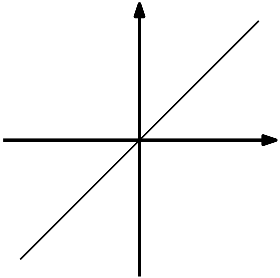

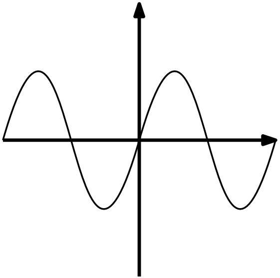

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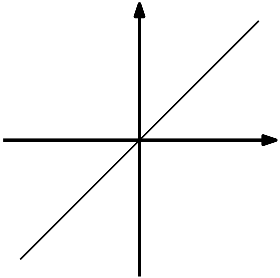

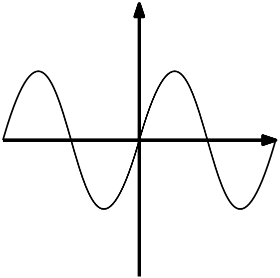

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What if the *environment* feeds the function with values close to 0, only?

An Example

A Semantics for Approximate Program Transformations

Edwin Westbrook and Swarat Chaudhuri
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Houston, TX 77005
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Abstract

An approximate program transformation *is a transformation that can change the semantics of a program within a specified empirical error bound. Such transformations have wide applications: they can decrease computation time, power consumption, and memory usage, and can, in some cases, allow implementations of incomputable operations. Correctness proofs of approximate program transformations are by definition quantitative. Unfortunately, unlike with standard program transformations, there is as of yet no modular way to prove correctness of an approximate transformation itself. Error bounds must be proved for each transformed program individually, and must be reproved each time a program is modified or a different set of approximations are applied.*

as floating-point numbers, lossy compression, and approximation algorithms for NP-hard problems. Such techniques are often used to trade off accuracy of the result for **reduced resource usage**, for resources such as computation time, power, and memory. In addition, some approximation techniques are also used to ensure **computability**. For example, true representations of real numbers (e.g., [7], [1]), require some operations, such as comparison, to be incomputable; floating-point comparison, in contrast, is efficiently decidable on modern computers.

Recently, there has been a growing interest in *language-based approximations*, where *approximate program transformations* are performed by the programming language environment [21], [12], [19], [18], [4], [3], [16]. Such approaches allow the user to give an *exact program* as a specification, and then apply some

Types

$$\tau, \rho ::= REAL \,\big|\, \tau \to \rho \,\big|\, \tau \times \rho.$$

A Toy Language

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Typing Rules

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{}{\Gamma \vdash r : REAL} \qquad \frac{f_n \in \mathcal{F}_n}{\Gamma \vdash f_n : REAL^n \to REAL} \qquad \frac{\Gamma, x : \tau \vdash M : \rho}{\Gamma \vdash \lambda x.M : \tau \to \rho}$$

$$\frac{\Gamma \vdash M : \tau \to \rho \quad \Gamma \vdash N : \tau}{\Gamma \vdash MN : \rho} \qquad \frac{\Gamma \vdash M : \tau \quad \Gamma \vdash N : \rho}{\Gamma \vdash \langle M, N \rangle : \tau \times \rho} \qquad \frac{}{\Gamma \vdash \pi_1 : \tau \times \rho \to \tau} \qquad \frac{}{\Gamma \vdash \pi_2 : \tau \times \rho \to \rho}$$

$$\frac{\Gamma \vdash M : \tau \quad \Gamma \vdash N : \tau}{\Gamma \vdash \texttt{iflz}\ M\ \texttt{else}\ N : REAL \to \tau} \qquad \frac{\Gamma \vdash M : \tau \to \tau \quad \Gamma \vdash N : \tau}{\Gamma \vdash \texttt{iter}\ M\ \texttt{base}\ N : REAL \to \tau}$$

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Denotational Semantics

$$[\![REAL]\!] = \mathbb{R}; \qquad [\![\tau \to \rho]\!] = [\![\tau]\!] \to [\![\rho]\!]; \qquad [\![\tau \times \rho]\!] = [\![\tau]\!] \times [\![\rho]\!].$$

Distance Spaces

$(\!|REAL|\!) = \mathbb{R}^{\infty}_{\geq 0}$; $\quad (\!|\tau \to \rho|\!) = [\![\tau]\!] \times (\!|\tau|\!) \to (\!|\rho|\!)$; $\quad (\!|\tau \times \rho|\!) = (\!|\tau|\!) \times (\!|\rho|\!)$

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The distance between two programs of type $\tau \to \rho$ is a function which:

- Given an input in $[\![\tau]\!]$...
- And a distance in $(\!|\tau|\!)$...
- Returns a distance in $(\!|\rho|\!)$.

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DLRs as Ternary Relations

$$\begin{aligned}
\delta_{REAL}(M, r, N) &\Leftrightarrow |NF(M) - NF(N)| \leq r; \\
\delta_{\tau \times \rho}(M, (d_1, d_2), N) &\Leftrightarrow \delta_\tau(\pi_1 M, d_1, \pi_1 N) \wedge \delta_\rho(\pi_2 M, d_2, \pi_2 N) \\
\delta_{\tau \to \rho}(M, d, N) &\Leftrightarrow (\forall V \in CV(\tau).\ \forall x \in (\!|\tau|\!).\ \forall W \in CV(\tau). \\
&\qquad \delta_\tau(V, x, W) \Rightarrow \delta_\rho(MV, d([\![V]\!], x), NW) \wedge \delta_\rho(MW, d([\![V]\!], x), NV)).
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Theorem (Fundamental Lemma, Version I)

For every $\vdash M : \tau$, *there is* $d \in (\!|\tau|\!)$ *such that* $\delta_\tau(M, d, M)$.

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$$\Downarrow$$

$$(C[M], d([\![M]\!], e), C[N]) \in \delta_{REAL}$$

Back to the Example

Claim

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Proof.

Consider any pairs of real numbers $r, s \in \mathbb{R}$ such that $|r - s| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}^{\infty}_{\geq 0}$. We have that:

$$\begin{aligned} |\sin r - s| &= |\sin r - r + r - s| \leq |\sin r - r| + |r - s| \\ &\leq |\sin r - r| + \varepsilon = f(r, \varepsilon) \\ |\sin s - r| &= |\sin s - \sin r + \sin r - r| \\ &\leq |\sin s - \sin r| + |\sin r - r| \leq |s - r| + |\sin r - r| \\ &\leq \varepsilon + |\sin r - r| = f(r, \varepsilon). \end{aligned}$$

where $f = \lambda\!\!\!\lambda \langle x, y\rangle . y + |x - \sin x|$. □

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- Of course the context C needs to be taken into account, but **once and for all**: the functional F can be built without knowing either M_{SIN} or M_{ID}.

Hereditarily Null Distances

$$(\!|REAL|\!)^0 = \{0\} \qquad (\!|\tau \times \rho|\!)^0 = (\!|\tau|\!)^0 \times (\!|\rho|\!)^0$$

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This is much larger than $\{\lambda\!\!\lambda\langle x, y\rangle.0\}$

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Lemma

Whenever $\vdash M, N : \tau$, M *is logically related to* N *iff* $\delta_\tau(M, d, N)$ *where* $d \in (\!|\tau|\!)^0$.

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$$(\!|\tau \to \rho|\!)^{<\infty} = \{f \in (\!|\tau \to \rho|\!) \mid \forall x \in [\![\tau]\!].\forall t \in (\!|\tau|\!)^{<\infty}.f(x,t) \in (\!|\rho|\!)^{<\infty}\}.$$

Lemma

Whenever $\vdash M, N : \tau$, M *is logically related to* N *iff* $\delta_\tau(M, d, N)$ *where* $d \in (\!|\tau|\!)^0$.

Theorem (Fundamental Lemma, Version II)

For every $\vdash M : \tau$, *there is* $d \in (\!|\tau|\!)^{<\infty}$ *such that* $\delta_\tau(M, d, M)$.

Conclusions

- **Other Interesting Results**
  - Differential logical relations are examples of generalized metric domains, which (contrarily to metric spaces) form a *cartesian closed category*.
  - Not only logical relations, but also metric logical relations can be seen as special kinds of distances.

Conclusions

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$$\delta(x,0,y) \Rightarrow x = y$$
$$\delta(x,d,y) \Rightarrow \delta(y,d,x)$$
$$\delta(x,d,y) \wedge \delta(y,e,y) \wedge \delta(y,f,z) \Rightarrow \delta(x,d+e+f,z)$$

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- **Further Work**
  - Capture more expressive constructs and calculi (e.g. *recursion* by way of step-indexing).
  - Higher-order continuity?
  - Abstracting functional distances by way of step functions.
  - Make behavioral metrics *context-dependent*.

Thank You!

Questions?