Invariant-Based Verification and Synthesis for Hybrid Systems

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IFIP WG 2.2 Scientific Meeting, IMS, Singapore

Sept. 12-16, 2016

Outline

- Background
- Invariant and Verification
- Invariant-Based Synthesis
- Case Studies
- Conclusion

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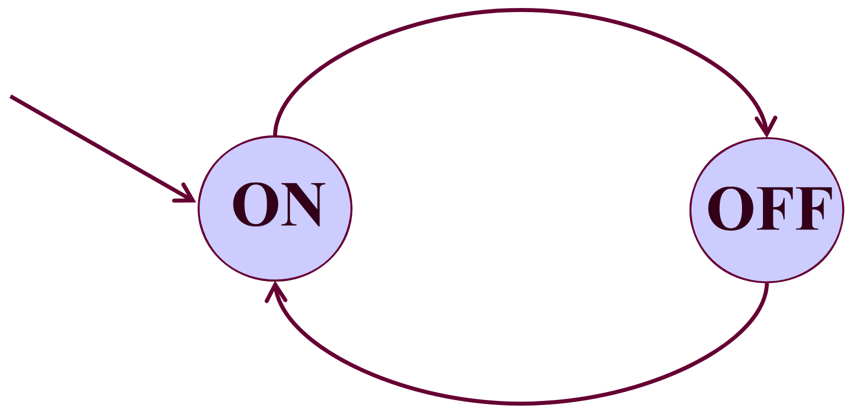
- **Background**
- **Invariant and Verification**
- **Invariant-Based Synthesis**
- **Case Studies**
- **Conclusion**

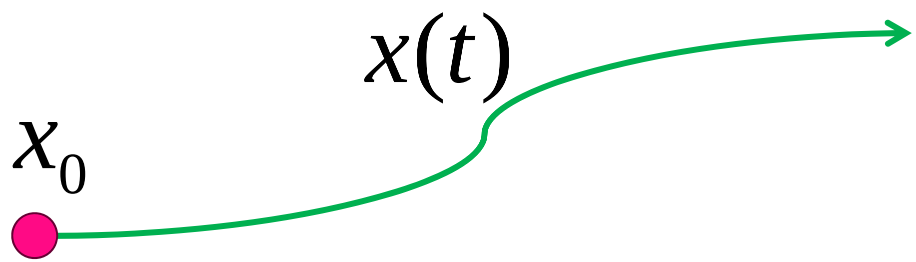
Classification of Dynamical Systems

- **Discrete**



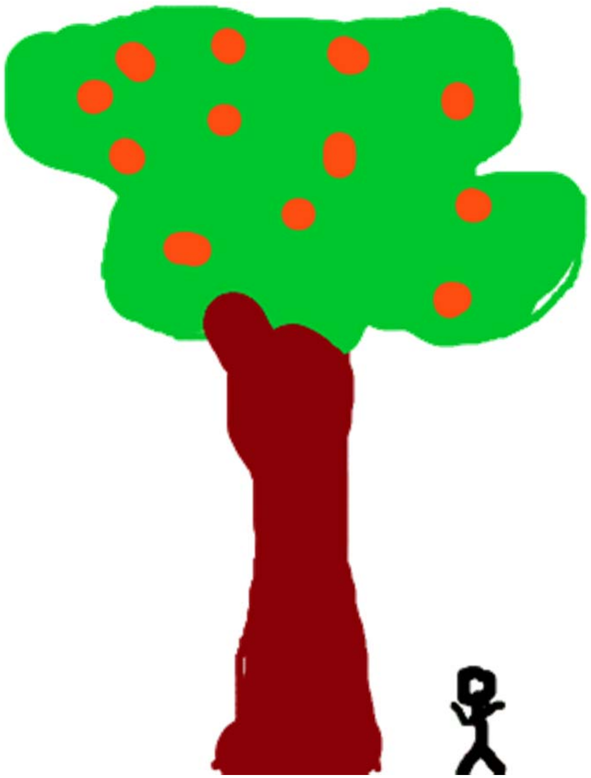
- **Continuous**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$



Hybrid System

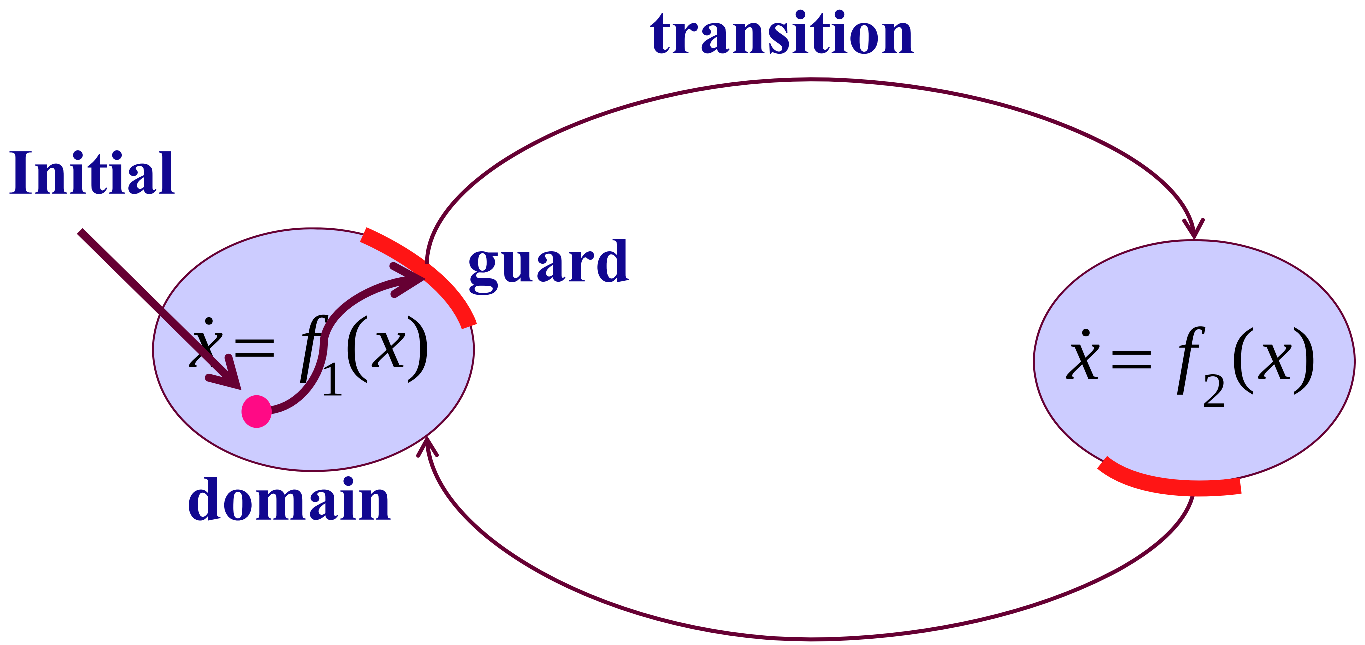
- **Continuous + Discrete**



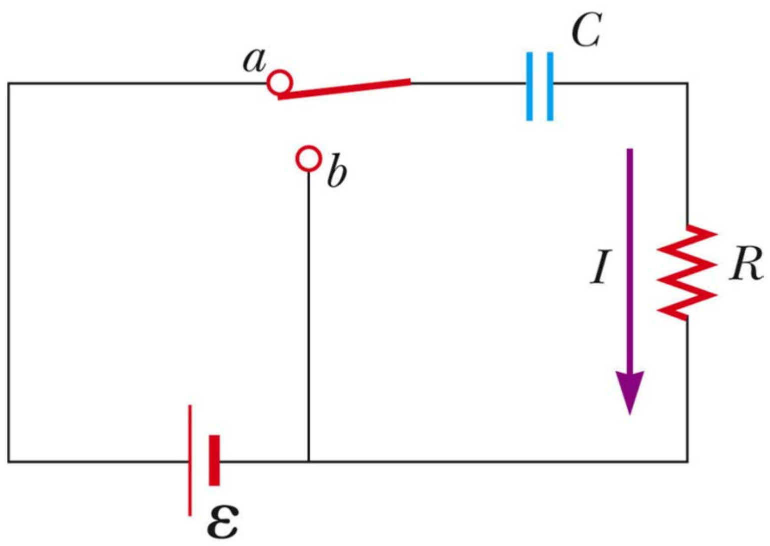
Universal Law of Gravitation

by Heer Rami
http://www.benettonplay.com/toys/flipbookdeluxe/player.php?id=294504

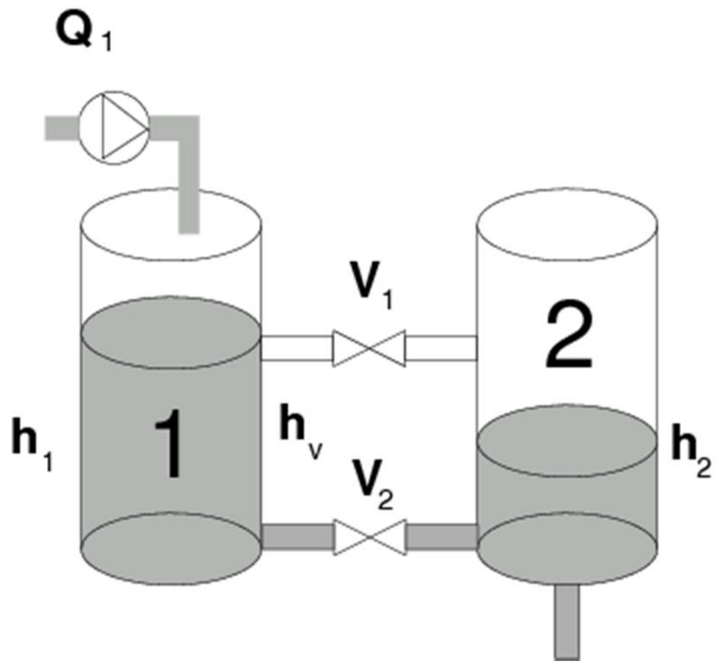
Hybrid Automata



HSs in Engineering



Electrical Circuits



Chemical Process

http://people.ee.ethz.ch/~mpt/2/docs/demos/twotanks.php

Embedded Control Systems

actuator

logic/computation **physical discipline**

sensor

Safety Critical Systems

Motivation

- Develop formal methods for enhancing the trustworthiness of safety critical embedded systems
  - Problems: Verification and Design
  - System Requirements: mainly safety
  - Techniques: symbolic/rigorous computation

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Deductive Verification

- **Program**

  x:=1;

  while (x<=1000000000)

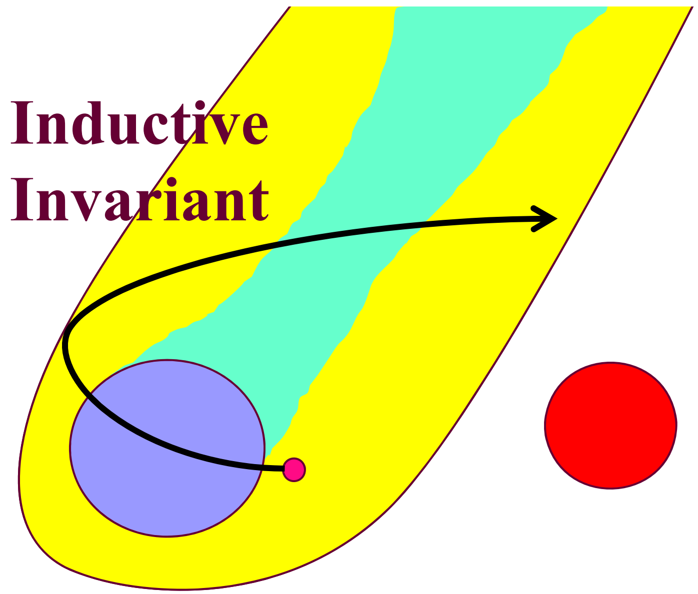
  { x:=x+1; }

  x≦0

- **Inductive Invariant**
  - x=1 ➔ x≧1
  - x≧1 ➔ x+1≧1
  - x≧1 ➔ ¬(x≦0)

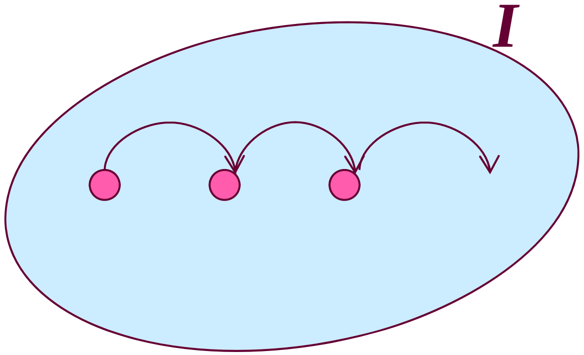
- **Continuous system**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$



Inductiveness

- **Discrete**



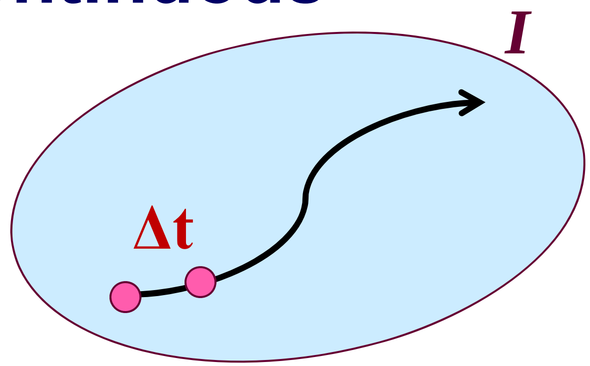
- **Inductiveness**

$$x_k \in I \rightarrow x_{k+1} \in I$$

- **Transition relation**

$$x_{k+1} = \varphi(x_k)$$

- **Continuous**



- **Inductiveness**

$$x(t) \in I \rightarrow x(t + \Delta t) \in I$$

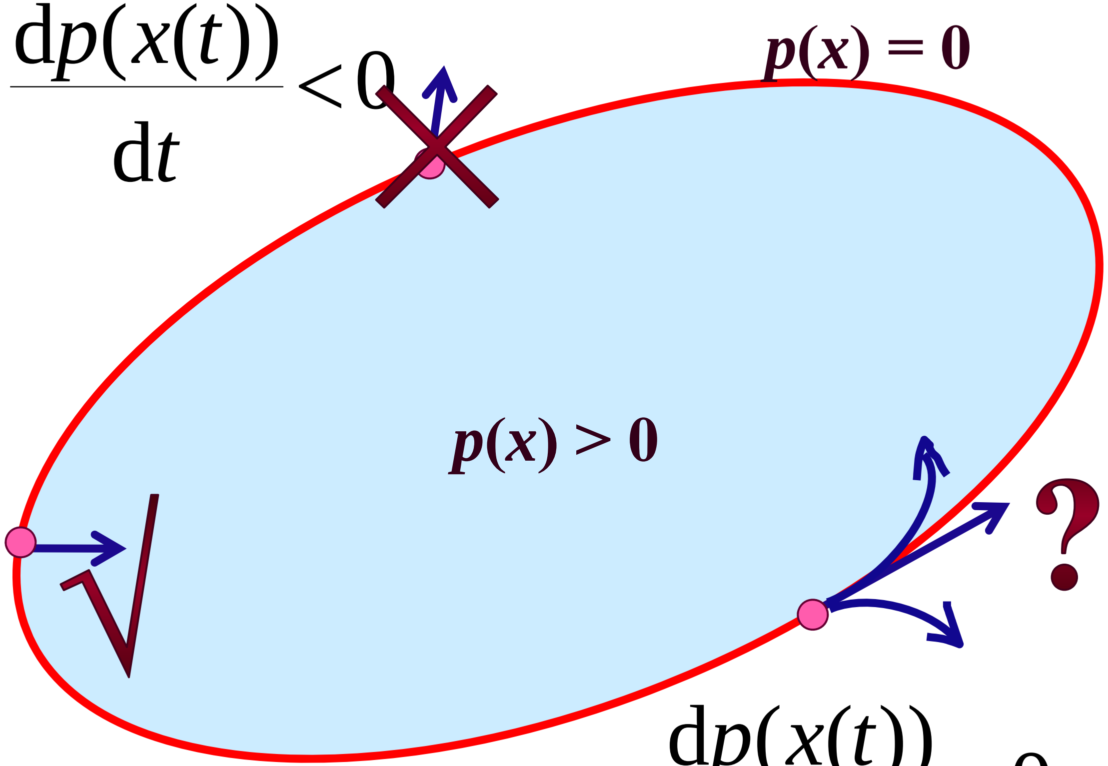
- **Transition relation**

$$x(t + \Delta t) = x(t) + x'(t) \cdot \Delta t$$

Lie Derivatives and Invariant

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)}$$

$$\frac{\mathrm{d}p(x(t))}{\mathrm{d}t} < 0$$



$$\frac{\mathrm{d}p(x(t))}{\mathrm{d}t} > 0$$

$$\frac{\mathrm{d}p(x(t))}{\mathrm{d}t} = 0$$

Higher-Order Lie Derivatives

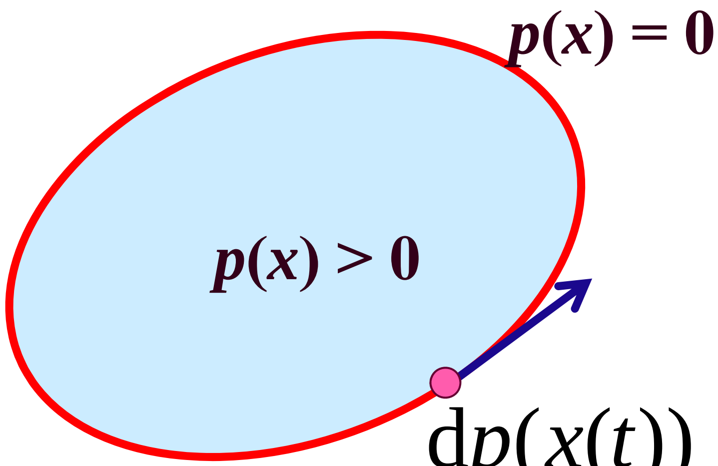
$$\frac{d^1 p}{dt^1} > 0$$

$$\vee \frac{d^1 p}{dt^1} = 0 \wedge \frac{d^2 p}{dt^2} > 0$$

$$\vee \frac{d^1 p}{dt^1} = 0 \wedge \frac{d^2 p}{dt^2} = 0 \wedge \frac{d^3 p}{dt^3} > 0$$

$$\vee \frac{d^1 p}{dt^1} = 0 \wedge \frac{d^2 p}{dt^2} = 0 \wedge \frac{d^3 p}{dt^3} = 0 \wedge \cdots$$

?



$$\frac{dp(x(t))}{dt} = 0$$

Criterion for Invariant

- $f(x)$ and $p(x)$ are polynomials
- Compute an upper bound N s.t.
- $p(x) \geq 0$ is an inductive invariant of $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x)$

iff

$$p = 0 \Rightarrow \left(\begin{array}{l} \dfrac{\mathrm{d}^1 p}{\mathrm{d}t^1} > 0 \vee \\ \dfrac{\mathrm{d}^1 p}{\mathrm{d}t^1} = 0 \wedge \dfrac{\mathrm{d}^2 p}{\mathrm{d}t^2} > 0 \vee \\ \cdots \cdots \vee \\ \dfrac{\mathrm{d}^1 p}{\mathrm{d}t^1} = 0 \wedge \dfrac{\mathrm{d}^2 p}{\mathrm{d}t^2} = 0 \wedge \cdots \wedge \dfrac{\mathrm{d}^N p}{\mathrm{d}t^N} \geq 0 \end{array} \right)$$

Main Result

- **Semi-algebraic set**

$$\bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} p_{ij}(\mathbf{x}) \triangleright 0, \quad \triangleright \in \{\geq, >\}$$

- **First-order theory of real numbers is decidable**
  - **Quantifier Elimination**

Checking whether a semi-algebraic set is an inductive invariant of a polynomial continuous dynamical systems is decidable

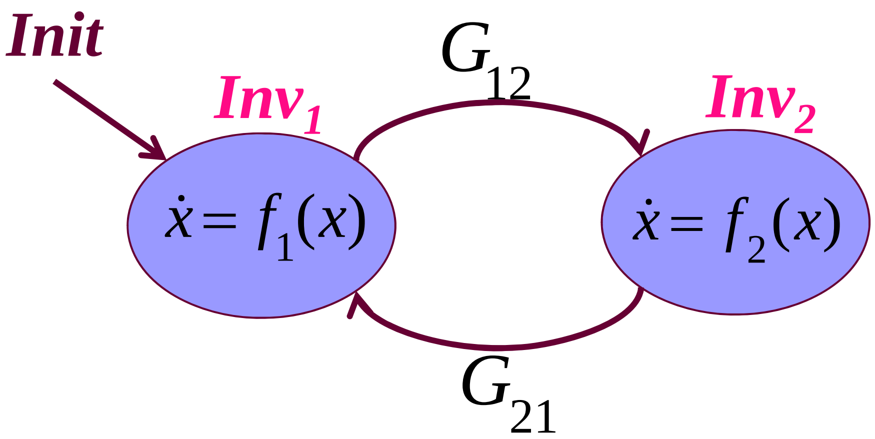
Parametric Case

- **Parametric polynomials *p(u,x)***
- ***p(u,x)* ≥ 0 is an inductive invariant of** $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ **iff *u* satisfies**

$$p(u,x)=0 \Rightarrow \left(\frac{\mathrm{d}^1 p}{\mathrm{d}t^1} > 0 \vee \frac{\mathrm{d}^1 p}{\mathrm{d}t^1} = 0 \wedge \frac{\mathrm{d}^2 p}{\mathrm{d}t^2} > 0 \vee \cdots\cdots \vee \frac{\mathrm{d}^1 p}{\mathrm{d}t^1} = 0 \wedge \frac{\mathrm{d}^2 p}{\mathrm{d}t^2} = 0 \wedge \cdots \wedge \frac{\mathrm{d}^N p}{\mathrm{d}t^N} \geq 0 \right)$$

Use parametric polynomials and quantifier elimination (or other compuation techniques) to automatically discovering inductive invariants

Inductive Invariant of HSs



$$Init \Rightarrow Inv_1$$

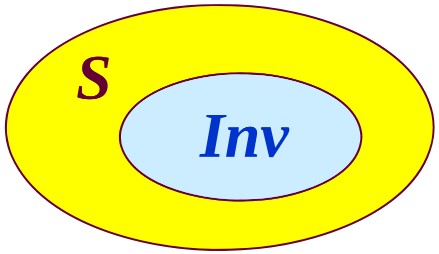
$$Inv_1,\ Inv_2$$

$$Inv_1 \wedge G_{12} \Rightarrow Inv_2$$

$$Inv_2 \wedge G_{21} \Rightarrow Inv_1$$

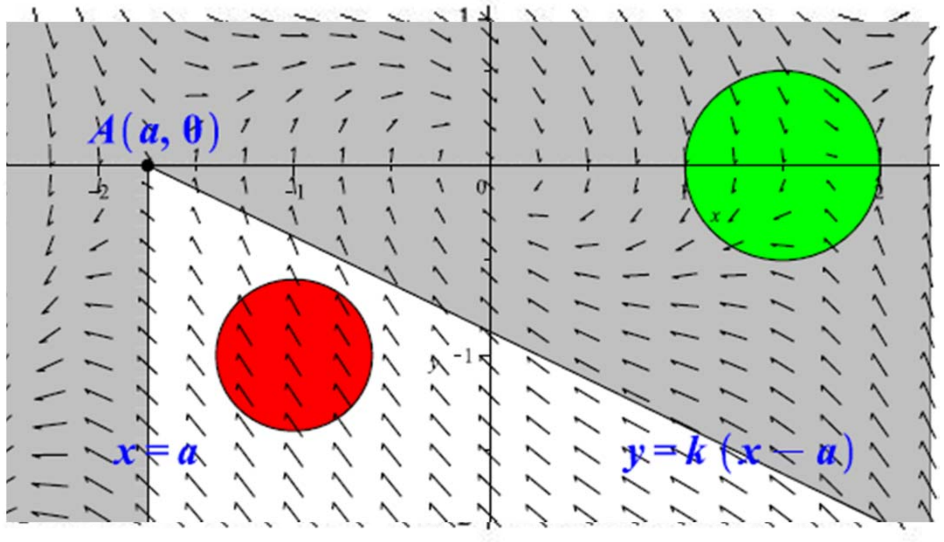
Safety Verification

- **Try to generate an invariant that implies the safety property**



- **Example**

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} y \\ -x + \frac{x^3}{3} - y \end{pmatrix}$$



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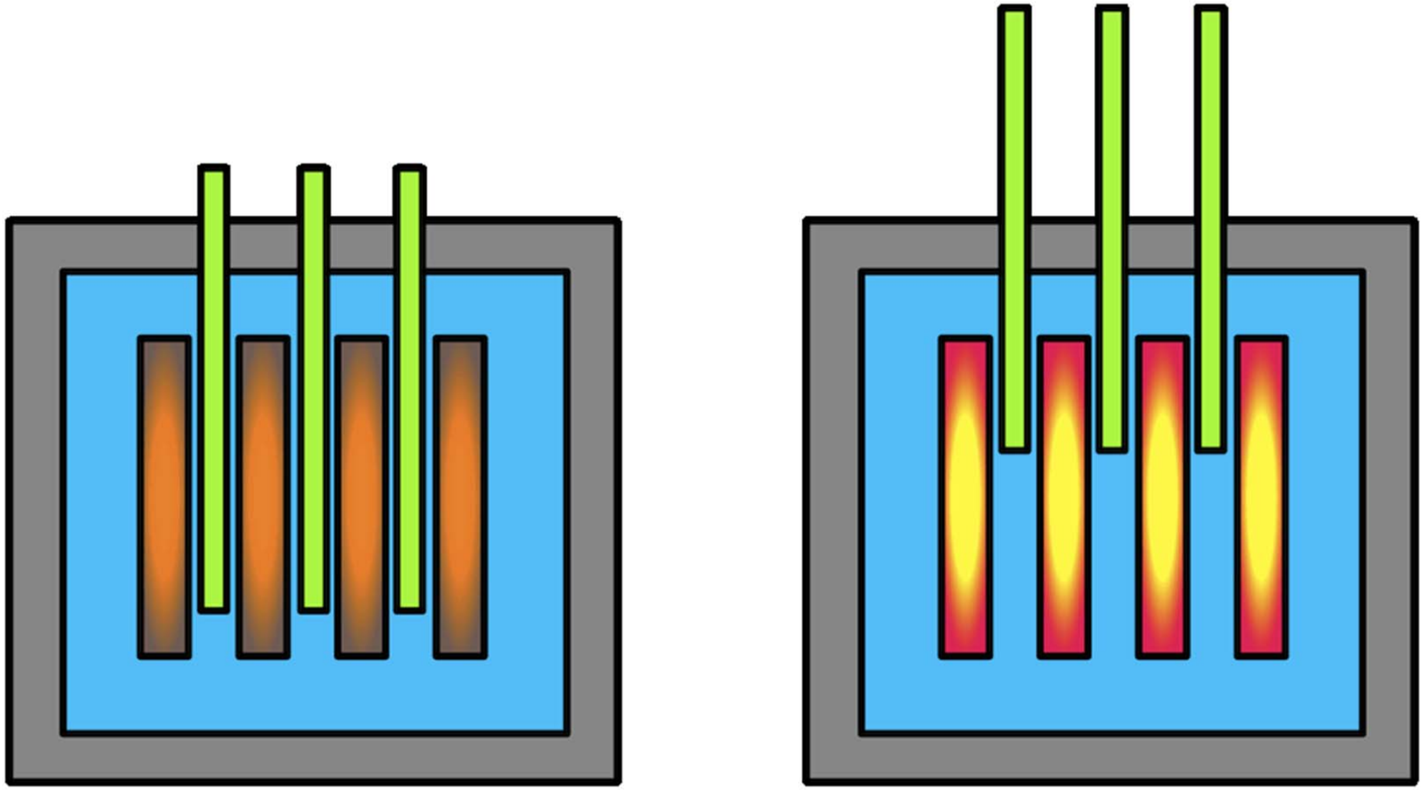
Problem Description

- Given an initial specification of a hybrid system and a safety requirement, construct a refined hybrid system such that the safety requirement is satisfied
  - domains
  - guards

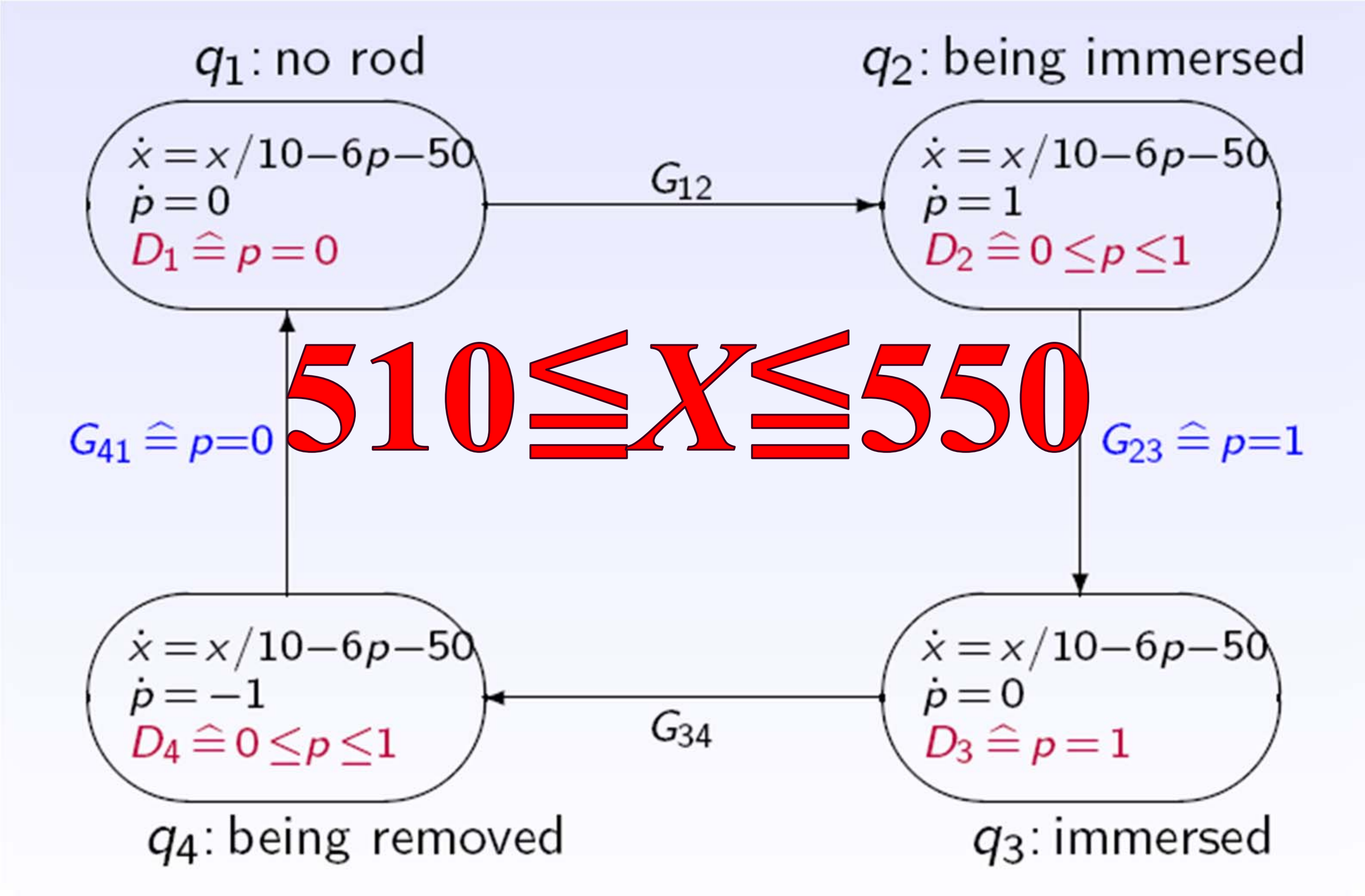
Nuclear Reactor



http://commons.wikimedia.org/wiki/File:Control_rods_schematic.svg

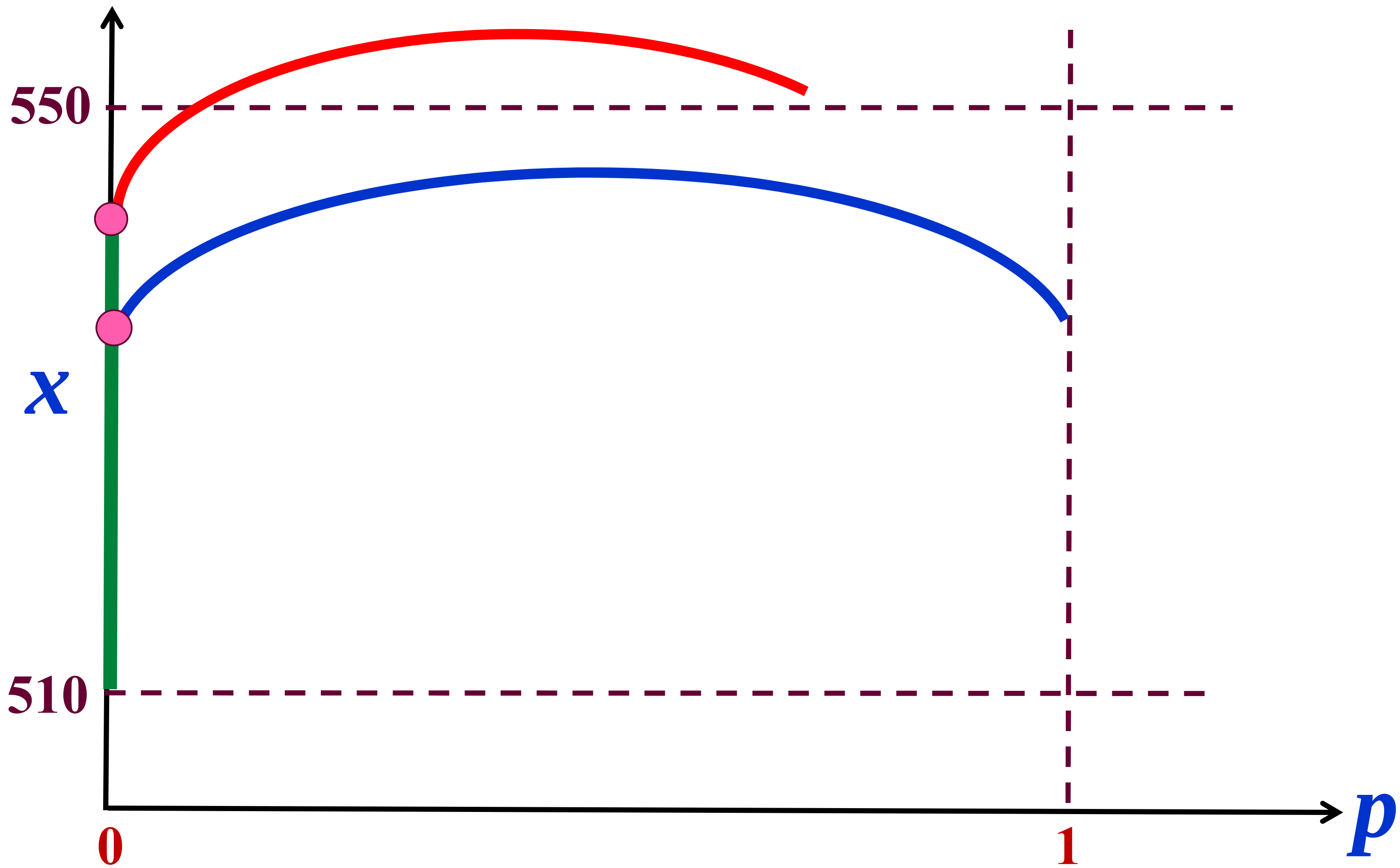
Hybrid Automata Model

- ***x***: **temperature of the reactor**
- ***p***: **fraction of the rod immersed into the reactor**

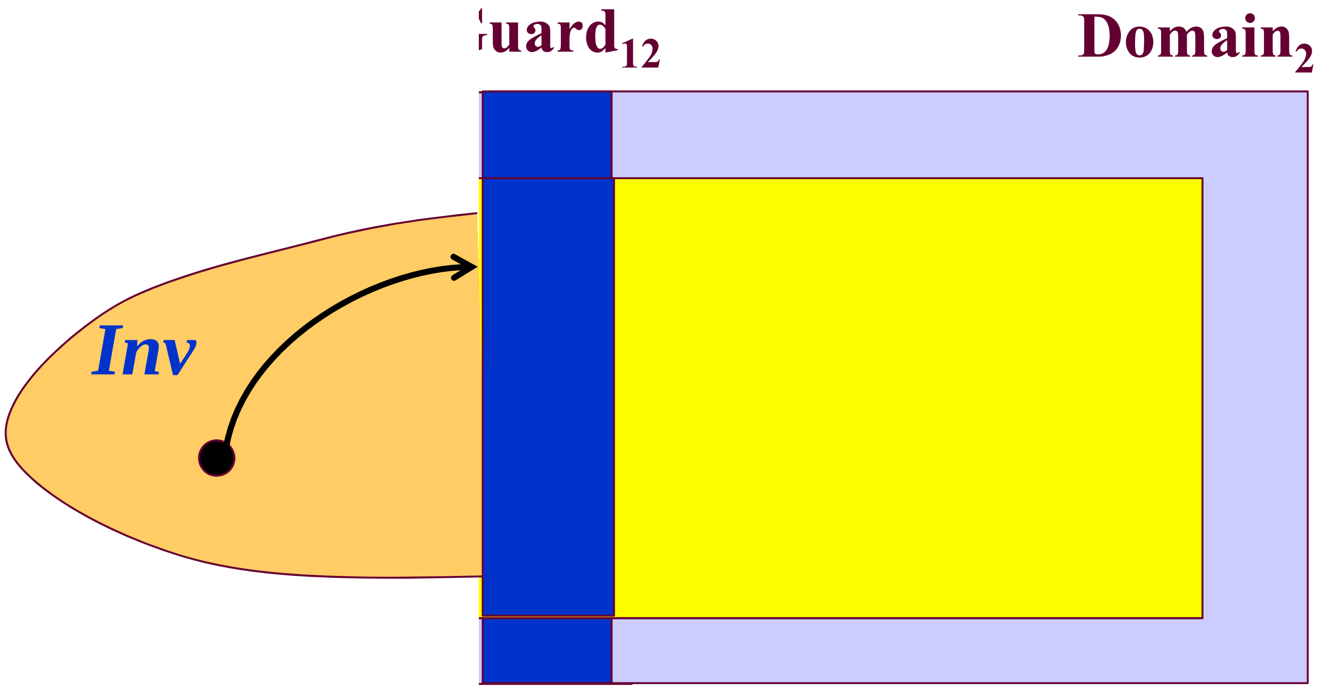


Violation of Safety

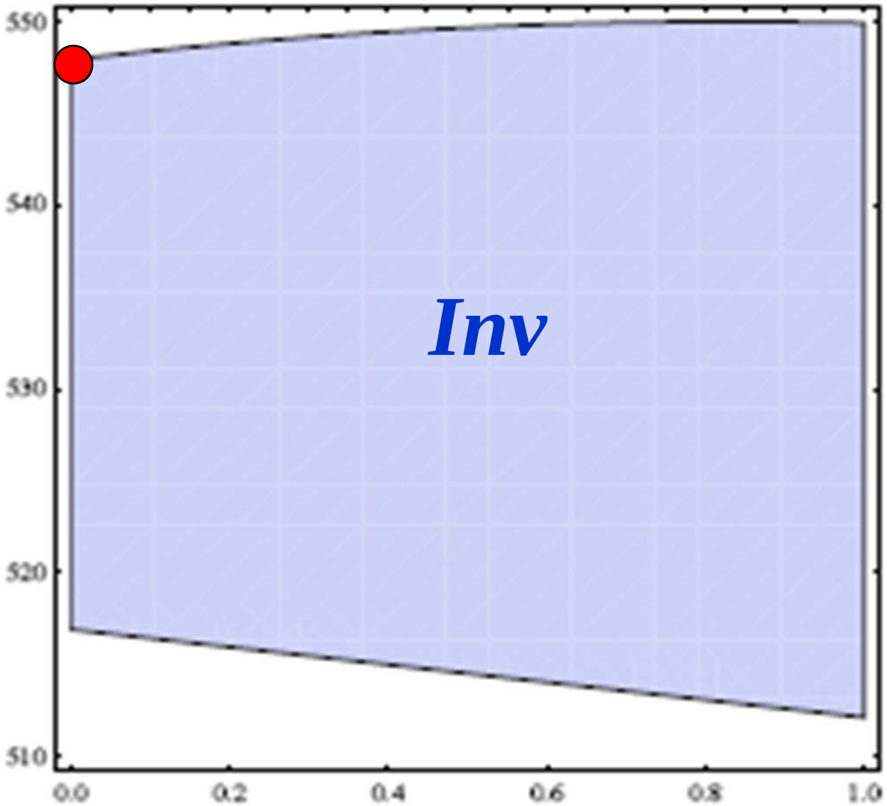
- 510≦x≦550



Invariant for Refinement



Result



$$x \le \frac{6575}{12} \approx 547.92$$

Optimization

- **Further refine the hybrid system according to certain optimization criteria**
- **polynomial objective function + semi-algebraic feasible region**
  - **Symbolic optimization**

$$c_3 = \inf_{\mathbf{u}_3} \sup_{\mathbf{u}_2} \min_{\mathbf{u}_1} g_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \quad \text{over } D_3(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$$

$$\exists \mathbf{u}_3.((\exists \mathbf{u}_1 \mathbf{u}_2 . D_3) \wedge \forall \mathbf{u}_2.(\exists \mathbf{u}_1 . D_3 \Rightarrow \exists \mathbf{u}_1.(D_3 \wedge g_3 \leq z))) \Leftrightarrow z \rhd c_3$$

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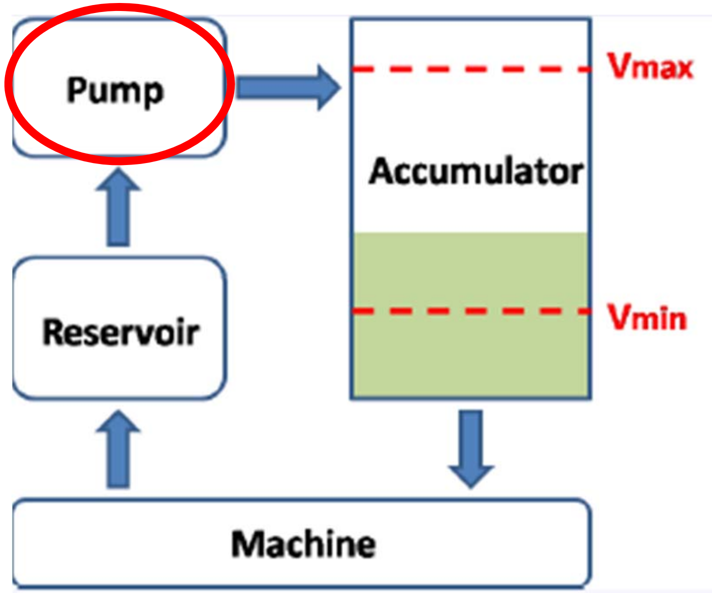
- **Case Studies**
  - Oil pump
  - Lunar lander
- Conclusion

Oil Pump Switching

- First studied in [Cassez et al. HSCC09, 45% improvement]
- Provided by the German company HYDAC
- Determine the time points to switch the pump on/off s.t.

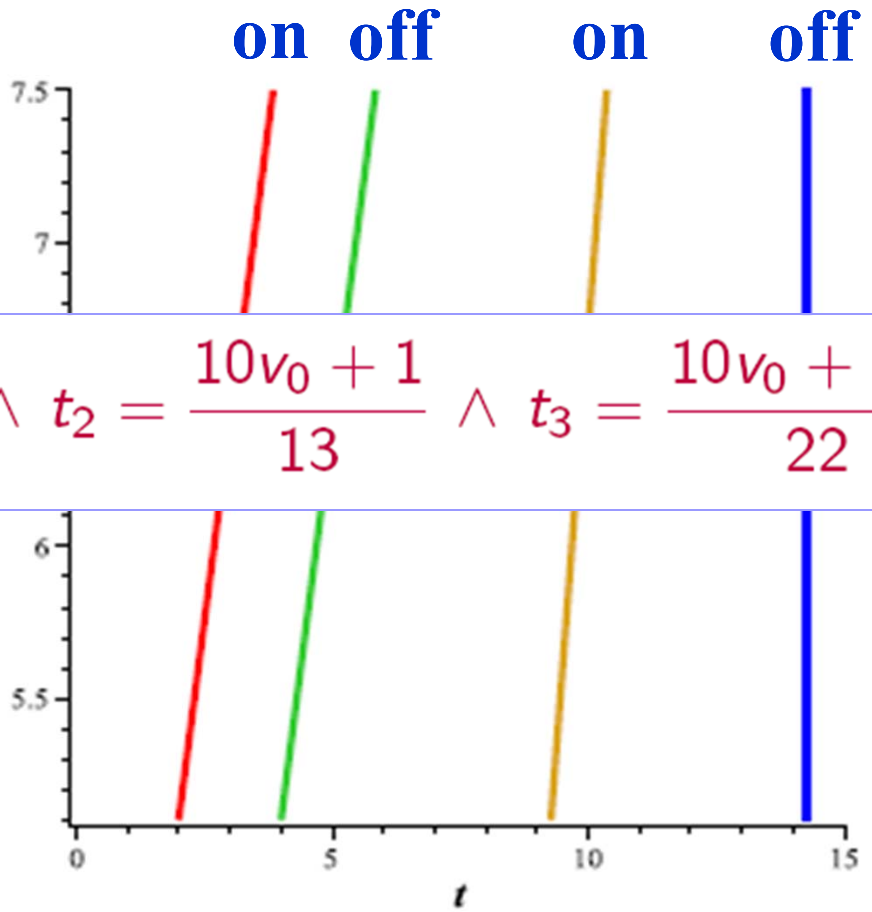


  - **Safety:** $v(t) \in [V_{\mathrm{min}}, V_{\mathrm{max}}], \quad \forall t \in [0, \infty)$
  - **Optimality:**

minimize $\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} v(t) \mathrm{d}t$

Synthesized Switching Controller

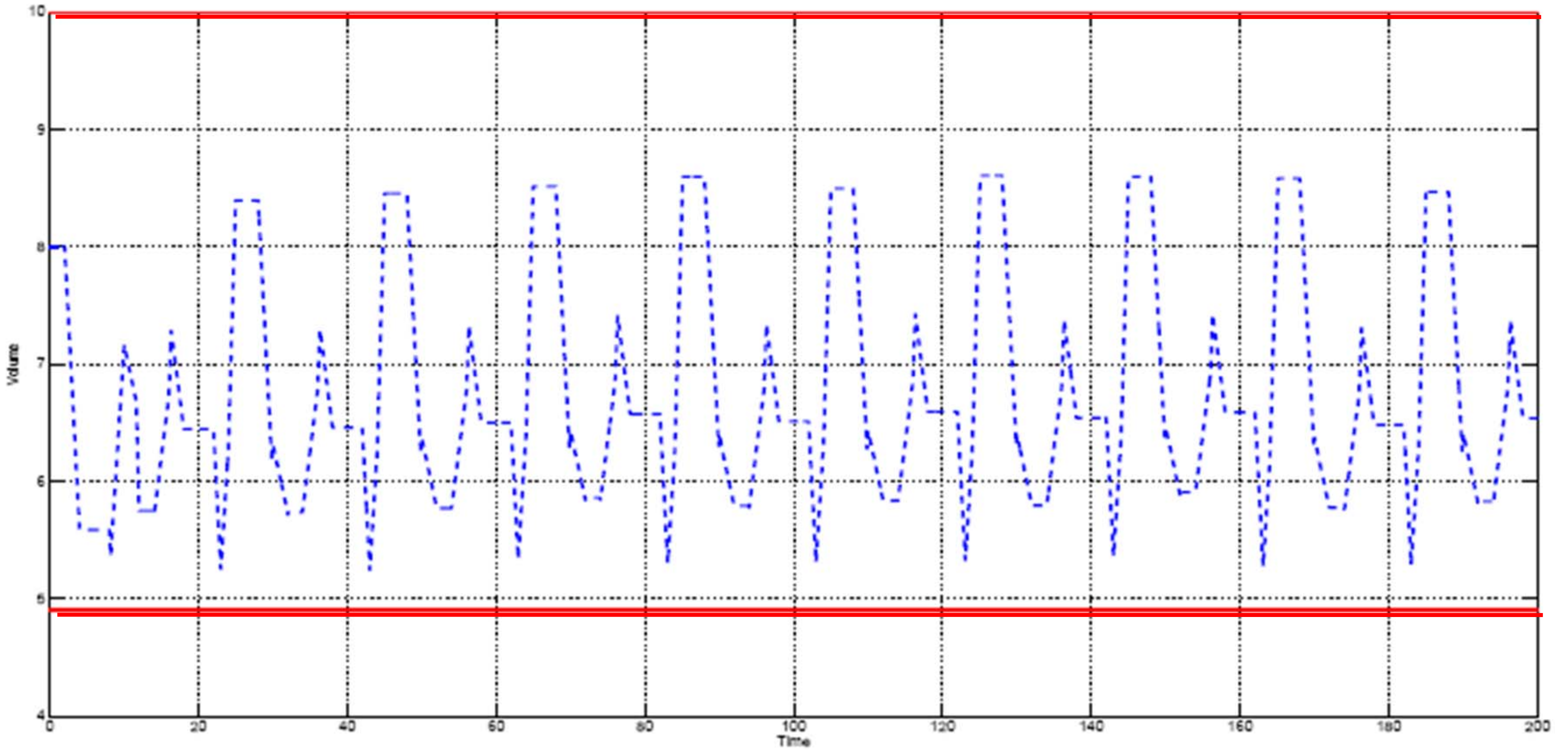
- ***v_0*** **is the initial volume of oil**



$$t_1 = \frac{10v_0 - 25}{13} \wedge t_2 = \frac{10v_0 + 1}{13} \wedge t_3 = \frac{10v_0 + 153}{22} \wedge t_4 = \frac{157}{11}$$

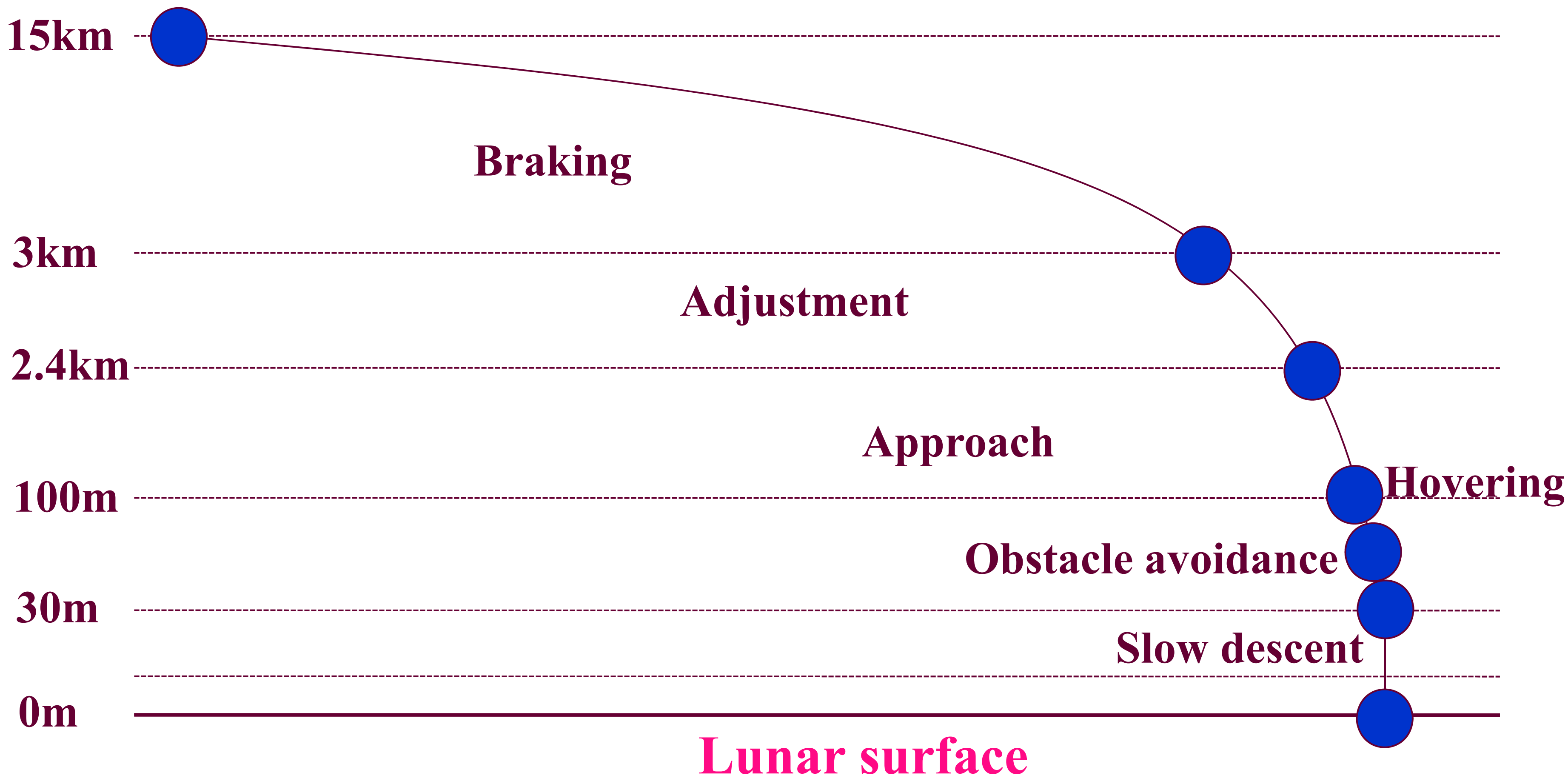
Performance

- **Safety**



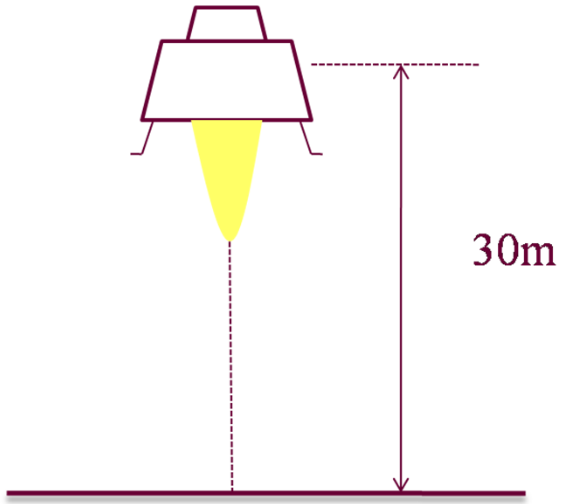
- **Improve the optimal value of** [HSCC09] **by 7.5%**
- **The synthesized controller is correct, also optimal**

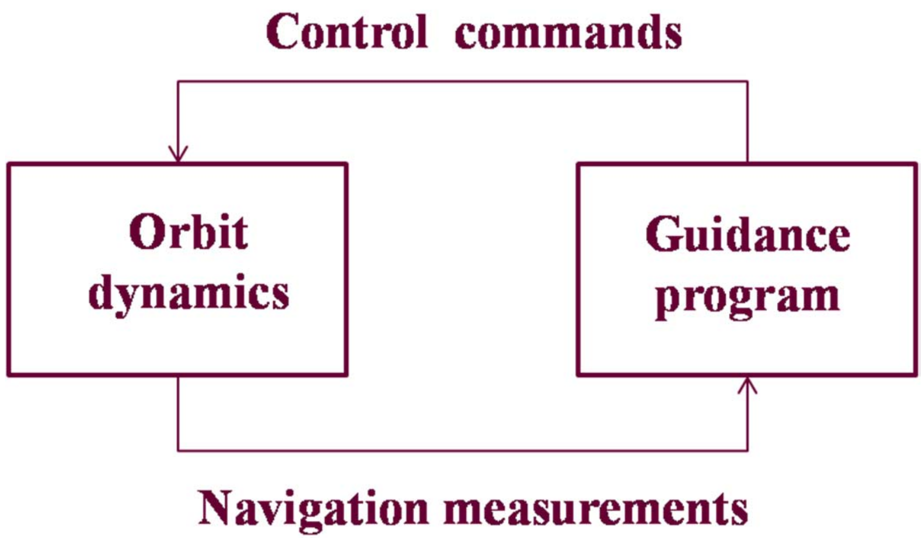
Soft Landing



Slow Descent Phase

- Trajectory control

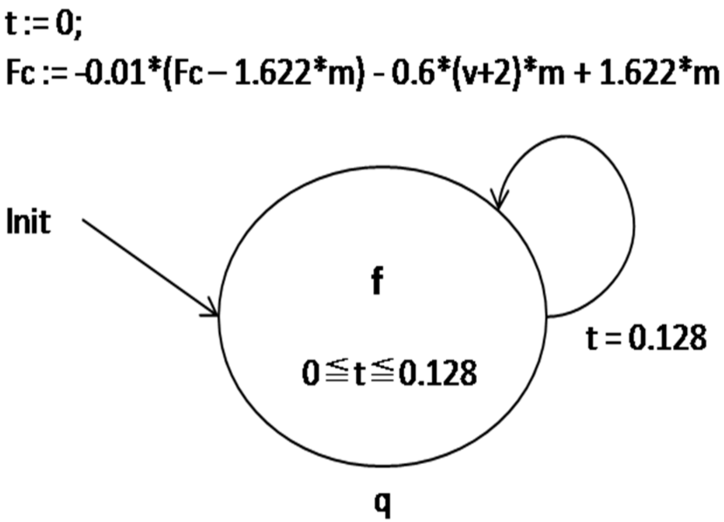




- Sampling period： **ΔT = 0.128s**
- Control objective: **v = -2m/s**

Hybrid Automata Model

- **Dynamics**



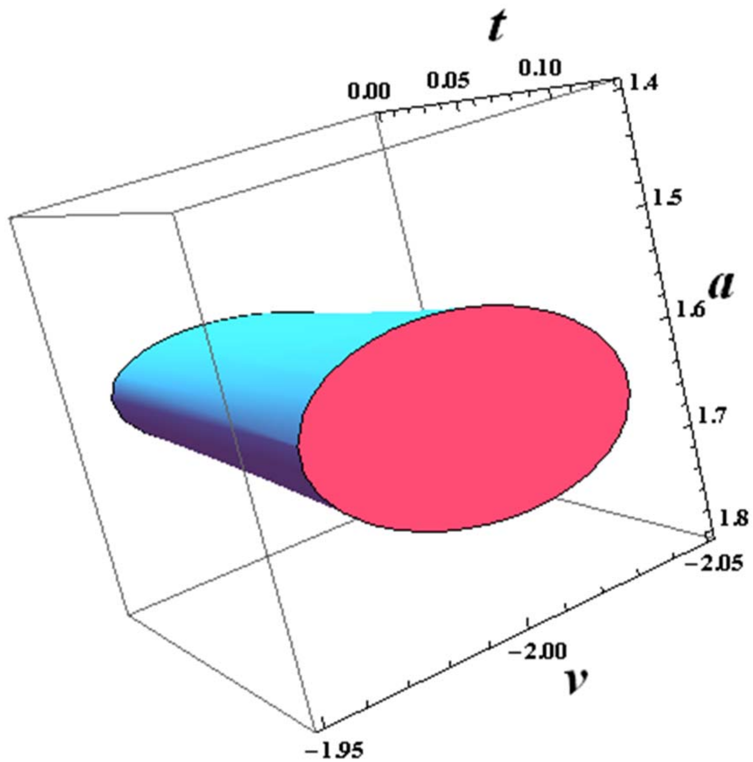
$$\mathrm{f} \widehat{=} \begin{cases} \dot{v} &= \dfrac{F_c}{m} - 1.622 \\ \dot{m} &= -\dfrac{F_c}{2500} \\ \dot{F}_c &= 0 \\ \dot{t} &= 1 \end{cases}$$

- **Replace the non-polynomial term by a new variable: *a = Fc/m***

Verification

- **Safety requirement: $|v - (-2)| \leqq 0.05$**

- **Generated Invariant:**



Kong, H., He, F., Song, X., Hung,W., Gu, M.: Exponential-condition-based barrier certificate generation for safety verification of hybrid systems. In: CAV'13. pp. 242–257 (2013)

Conclusion

- **Hybrid systems attracts more and more interests with the development of safety critical embedded systems**
- **Invariant plays an important role in the study (formal verification, controller synthesis) of hybrid systems**
- **Semi-algebraic inductive invariant checking for polynomial continuous/hybrid systems is decidable**

Conclusion

- **Use parametric polynomials and symbolic computation to automatically discover invariants, and to perform optimization**
  - **rigorous**
  - **high complexity (may be combined with numeric computation)**
  - **Non-polynomial systems transformed to polynomials ones**
- **Case studies show good prospect of proposed methods**

Related references

- Hengjun Zhao, Mengfei Yang, Naijun Zhan, Bin Gu, Liang Zou and Yao Chen (2014): *Formal verification of a descent guidance control program of a lunar lander*, in Proc. of FM 2014, *Lecture Notes in Computer Science 8442*, pp.733-748.
- Hengjun Zhao, Naijun Zhan and Deepak Kapur (2013): *Synthesizing switching controllers for hybrid systems by generating invariants*, in Proc. of the Jifeng Festschrift, *Lecture Notes in Computer Science* 8051, pp.354-373.
- Hengjun Zhao, Naijun Zhan, Deepak Kapur, and Kim G. Larsen (2012): *A "hybrid" approach for synthesizing optimal controllers of hybrid systems: A Case study of the oil pump industrial example*, in Proc. of FM 2012, *Lecture Notes in Computer Science* 7436, pp.471-485, 2012.
- Jiang Liu, Naijun Zhan and Hengjun Zhao (2011):*Computing semi-algebraic invariants for polynomial dynamical systems*, in Proc. of EMSOFT 2011, pp.97-106, ACM Press.

Thanks!

Questions?