Formal Reasoning for Quantum Programs

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Thanks to Yuan Feng and Mingsheng Ying

Outline

Background

Preliminaries on quantum mechanics

Equivalences for quantum processes

Symbolic semantics

An algorithm for ground bisimulation

Hoare logic

Summary

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- On August 16, 2016, China launched the first satellite using quantum technology to send communications back to earth.
- A 2000-km quantum communication main network between Beijing and Shanghai will be fully operational later this year.

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- In 2011, the Canadian company D-Wave Systems claimed to have created the first commercial 128-qubit quantum computer, D-wave One.
- In December 2015, Google announced that, in solving a specific optimization problem, their 512-qubit D-Wave 2X is 100 million times faster than conventional single-core computers.

Quantum programming

"the real challenge will be the software .... Programming this thing [D-Wave] is ridiculously hard; it can take months to work out how to phrase a problem so that the computer can understand it."

— G. Rose
Founder and CTO at D-Wave Systems

[N. Jones. The Quantum Company. *Nature* 498:286-288, 2013.]

Quantum programming languages

- "Quantum data, classical control" [Selinger]
- Sequential languages
  - Quipper [Dalhousie Univ.]
  - LIQUi$|>$ [Microsoft]
  - Scaffold [Princeton]
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  - ...
- Concurrent languages (quantum process algebras) Aiming to specify and verify quantum protocols.
  - QPAlg [Jorrand and Lalire]
  - CQP [Gay and Nagarajan]
  - qCCS [Feng et al.]

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Dirac-notation

Let $\mathcal{H}$ be a Hilbert space.

- 'ket' $|\psi\rangle$ stands for a (normalized) vector in $\mathcal{H}$.
- 'bra' $\langle\psi|$ stands for the adjoint (dual vector) of $|\psi\rangle$.
- Generally, $A^\dagger$ stands for the adjoint of A, such that

$$(A^\dagger|\psi\rangle, |\phi\rangle) = (|\psi\rangle, A|\phi\rangle).$$

In particular, $(|\psi\rangle)^\dagger = \langle\psi|$.

Quantum states

- Associated to any quantum system is a Hilbert space known as the state space.

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- The state of a closed quantum system is described by a unit vector, say $|\psi\rangle$, in its state space.

Quantum states(Cont'd)

- $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$: lies in the state $|\psi_k\rangle$ with probability p_k, $\sum_k p_k = 1$.
  - ρ is a positive operator
  - $\mathrm{tr}(\rho) = 1$

Quantum states(Cont'd)

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  - ρ is a positive operator
  - $\mathrm{tr}(\rho) = 1$
- These two conditions characterize exactly the set of density operators.

Quantum dynamics

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A super-operator $\mathcal{E}$ over Hilbert space $\mathcal{H}$ is a linear map on the space of linear operators on $\mathcal{H}$.

- $\mathcal{E}$ is trace-preserving, if $\mathrm{tr}(\mathcal{E}(A)) = \mathrm{tr}(A)$ for any positive operator A.
- $\mathcal{E}$ is completely positive, if for any auxiliary space $\mathcal{H}'$ and any positive operator σ on the tensor Hilbert space $\mathcal{H}' \otimes \mathcal{H}$, $(\mathcal{I}_{\mathcal{H}'} \otimes \mathcal{E})(\sigma)$ is also a positive operator on $\mathcal{H}' \otimes \mathcal{H}$.

Quantum dynamics

- The evolution of a quantum system is described by a super-operator

$$\rho' = \mathcal{E}(\rho)$$

Quantum measurements

- An observable A is a Hermitian operator, $A^\dagger = A$. Let

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- The measurement disturbs the system, leaving it in a state $P_k \rho P_k / p_k$ determined by the outcome.

Syntax of qCCS

The syntax of qCCS:

$$\mathbf{nil} \mid pref.P \mid P + Q \mid P \| Q \mid P \backslash L \mid \mathbf{if}\ b\ \mathbf{then}\ P \mid A(\tilde{q}; \tilde{x})$$

where

$$pref ::= \tau \mid c?x \mid c!e \mid \underline{c}?q \mid \underline{c}!q \mid \mathcal{E}[\widetilde{q}] \mid M[\widetilde{q}; x]$$

Further requirements

- $c?x.d!x.d!x.0$

$\not\Rightarrow$ $\underline{c}?r.\underline{d}!r.\underline{d}!r.0$

- Quantum no-cloning theorem!

Syntax of qCCS, cont'd

For a process to be legal, we require

1. $q \notin qv(P)$ in the process $\underline{c}!q.P$;
2. $qv(P) \cap qv(Q) = \emptyset$ in the process $P \,||\, Q$.

Operational Semantics of qCCS

A pair of the form

$$\langle P, \rho \rangle$$

is a configuration, where P is a closed quantum process and ρ is a density operator. The set of configurations is denoted by Con. We let $\mathcal{C}, \mathcal{D}, \ldots$ range over Con.

Operational Semantics of qCCS

Let

$$\begin{aligned} Act &= \{\tau\} \cup \{c?v, c!v \mid c \text{ classical channel}, v \text{ real number}\} \cup \\ &\quad \{\underline{c}?r, \underline{c}!r \mid \underline{c} \text{ quantum channel}, r \text{ quantum variable}\}, \end{aligned}$$

and $D(Con)$ be the set of finite-support probability distributions over Con.

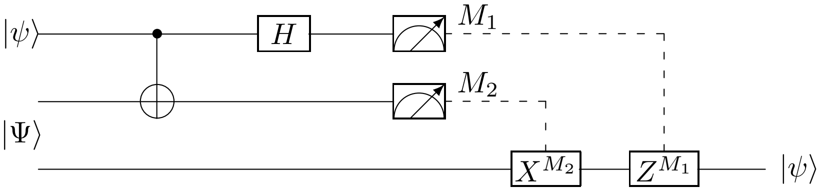
The semantics of qCCS is given by the probabilistic labeled transition system $(Con, Act, \rightarrow)$, where $\rightarrow \subseteq Con \times Act \times D(Con)$ is the smallest relation satisfying some rules.

An example: Teleportation

Quantum teleportation [Bennett, Brassard, Crepeau, Jozsa, Peres, and Wootters, PRL 1993] makes use of a maximally entangled state to teleport an unknown quantum state by sending only *classical* information.

It serves as a key ingredient in many other quantum communication protocols.

An example: Teleportation



Let

$$
\begin{aligned}
Alice &:= CNot[q, q_1].H[q].M[q, q_1; x].c!x.\mathbf{nil} \\
Bob &:= c?x.U_x[q_2].\mathbf{nil} \\
Telep &:= (Alice \| Bob) \backslash \{c\}
\end{aligned}
$$

Here $M = \sum_{i=0}^{3} \lambda_i |\tilde{i}\rangle\langle\tilde{i}|$, and

$$
\begin{aligned}
U_x[q_2].\mathbf{nil} \; &:= \; \mathbf{if}\ x = \lambda_0\ \mathbf{then}\ \sigma_0[q_2].\mathbf{nil} + \mathbf{if}\ x = \lambda_1\ \mathbf{then}\ \sigma_1[q_2].\mathbf{nil} \\
&+ \; \mathbf{if}\ x = \lambda_2\ \mathbf{then}\ \sigma_3[q_2].\mathbf{nil} + \mathbf{if}\ x = \lambda_3\ \mathbf{then}\ \sigma_2[q_2].\mathbf{nil}.
\end{aligned}
$$

$$\langle Telep,\ [(\alpha|0\rangle + \beta|1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)]\rangle$$

$$\big\downarrow \tau$$

$$\langle (H[q].M[q,q_1;x].c!x.\mathbf{nil} \| Bob)\backslash\{c\},\ [\tfrac{1}{\sqrt{2}}(\alpha(|000\rangle + |011\rangle) + \beta(|110\rangle + |101\rangle))]\rangle$$

$$\big\downarrow \tau$$

$$\langle (M[q,q_1;x].c!x.\mathbf{nil} \| Bob)\backslash\{c\},\ [\tfrac{1}{2}(\alpha(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + \beta(|010\rangle - |110\rangle + |001\rangle - |101\rangle))]\rangle$$

$$\big\downarrow \tau$$

$$\bullet \xrightarrow{1/4} \langle (c!\lambda_0.\mathbf{nil} \| Bob)\backslash\{c\},\ [\alpha|000\rangle + \beta|001\rangle]\rangle \xrightarrow{\tau} \langle (\mathbf{nil} \| \sigma_0[q_2].\mathbf{nil})\backslash\{c\},\ [|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)]\rangle \xrightarrow{\tau} \langle (\mathbf{nil} \| \mathbf{nil})\backslash\{c\},\ [|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)]\rangle$$

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$$\bullet \xrightarrow{1/4} \langle (c!\lambda_2.\mathbf{nil} \| Bob)\backslash\{c\},\ [\alpha|100\rangle - \beta|101\rangle]\rangle \xrightarrow{\tau} \langle (\mathbf{nil} \| \sigma_3[q_2].\mathbf{nil})\backslash\{c\},\ [|10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)]\rangle \xrightarrow{\tau} \langle (\mathbf{nil} \| \mathbf{nil})\backslash\{c\},\ [|10\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)]\rangle$$

$$\bullet \xrightarrow{1/4} \langle (c!\lambda_3.\mathbf{nil} \| Bob)\backslash\{c\},\ [\alpha|111\rangle - \beta|110\rangle]\rangle \xrightarrow{\tau} \langle (\mathbf{nil} \| \sigma_2[q_2].\mathbf{nil})\backslash\{c\},\ [|11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)]\rangle \xrightarrow{\tau} \langle (\mathbf{nil} \| \mathbf{nil})\backslash\{c\},\ [|11\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)]\rangle$$

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Lifted relation

Lift $\mathcal{R} \subseteq S \times S$ to $\mathcal{R}^\circ \subseteq Dist(S) \times Dist(S)$:

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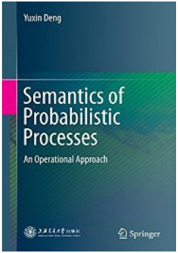
Lifted relation

Lift $\mathcal{R} \subseteq S \times S$ to $\mathcal{R}^\circ \subseteq \mathit{Dist}(S) \times \mathit{Dist}(S)$:

1. $s \mathcal{R} t$ implies $\overline{s}\ \mathcal{R}^\circ\ \overline{t}$;
2. $\Delta_i\ \mathcal{R}^\circ\ \Theta_i$ for all $i \in I$ implies $(\sum_{i \in I} p_i \cdot \Delta_i)\ \mathcal{R}^\circ\ (\sum_{i \in I} p_i \cdot \Theta_i)$ for any $p_i \in [0, 1]$ with $\sum_{i \in I} p_i = 1$, where I is a countable index set.

There are alternative formulations; related to the Kantorovich metric and the network flow problem. See e.g.

http://www.springer.com/978-3-662-45197-7



Four criteria to judge equivalence

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A relation $\mathcal{R}$ is

- barb-preserving if $\mathcal{C}\mathcal{R}\mathcal{D}$ implies that $\mathcal{C}\Downarrow_c^{\geq p}$ iff $\mathcal{D}\Downarrow_c^{\geq p}$ for any $p \in [0,1]$ and any classical channel c, where $\mathcal{C}\Downarrow_c^{\geq p}$ holds if $\mathcal{C} \stackrel{\hat{\tau}}{\Longrightarrow} \Delta$ for some Δ with

$$\sum\{\Delta(\mathcal{C}') \mid \mathcal{C}' \xrightarrow{c!v} \text{ for some } v\} \ \geq \ p;$$

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$$\sum \{ \Delta(\mathcal{C}') \mid \mathcal{C}' \xrightarrow{c!v} \text{ for some } v \} \;\geq\; p;$$

- reduction-closed if $\mathcal{C} \mathcal{R} \mathcal{D}$ implies
  - whenever $\mathcal{C} \stackrel{\hat{\tau}}{\Longrightarrow} \Delta$, there exists Θ such that $\mathcal{D} \stackrel{\hat{\tau}}{\Longrightarrow} \Theta$ and $\Delta \ \mathcal{R}^{\circ} \ \Theta$,
  - whenever $\mathcal{D} \stackrel{\hat{\tau}}{\Longrightarrow} \Theta$, there exists Δ such that $\mathcal{C} \stackrel{\hat{\tau}}{\Longrightarrow} \Delta$ and $\Delta \ \mathcal{R}^{\circ} \ \Theta$;

Four criteria to judge equivalence, cont.

- compositional if $\mathcal{C}\mathcal{R}\mathcal{D}$ implies $(\mathcal{C}||R)\mathcal{R}(\mathcal{D}||R)$ for any process R with $qv(R)$ disjoint from $qv(\mathcal{C}) \cup qv(\mathcal{D})$,

Four criteria to judge equivalence, cont.

- compositional if $\mathcal{C}\mathcal{R}\mathcal{D}$ implies $(\mathcal{C}||R)\mathcal{R}(\mathcal{D}||R)$ for any process R with $qv(R)$ disjoint from $qv(\mathcal{C}) \cup qv(\mathcal{D})$,
- closed under super-operator application, if $\mathcal{C}\mathcal{R}\mathcal{D}$ implies $\mathcal{E}(\mathcal{C})\mathcal{R}\mathcal{E}(\mathcal{D})$ for any $\mathcal{E} \in \mathcal{SO}(\mathcal{H}_{\overline{qv(\mathcal{C})}})$.

Reduction barbed congruence

Originated in [Honda & Tokoro 1995].

Let reduction barbed congruence, written $\approx_r$, be the largest relation over configurations which is

- barb-preserving,
- reduction-closed,
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Let reduction barbed congruence, written $\approx_r$, be the largest relation over configurations which is

- barb-preserving,
- reduction-closed,
- compositional,
- closed under super-operator application,
- and furthermore, if $\mathcal{C} \approx_r \mathcal{D}$ then $qv(\mathcal{C}) = qv(\mathcal{D})$ and $\mathrm{env}(\mathcal{C}) = \mathrm{env}(\mathcal{D})$.

Open bisimulation

Inspired by [Sangorigi 1996].

A relation $\mathcal{R} \subseteq \mathit{Con} \times \mathit{Con}$ is an open simulation if $\mathcal{C}\mathcal{R}\mathcal{D}$ implies that

- $qv(\mathcal{C}) = qv(\mathcal{D})$, and $\mathrm{env}(\mathcal{C}) = \mathrm{env}(\mathcal{D})$,
- for any $\mathcal{E} \in \mathcal{SO}(\mathcal{H}_{\overline{qv(\mathcal{C})}})$, whenever $\mathcal{E}(\mathcal{C}) \xrightarrow{\alpha} \Delta$, there is some Θ with $\mathcal{E}(\mathcal{D}) \stackrel{\hat{\alpha}}{\Longrightarrow} \Theta$ and $\Delta\ \mathcal{R}^\circ\ \Theta$.

A relation $\mathcal{R}$ is an open bisimulation if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are open simulations. We let $\approx_o$ be the largest open bisimulation.

Theorem : Congruence

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- The relation $\approx_o$ between processes is preserved by all the constructors of qCCS except for summation.

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- The relation $\approx_o$ between processes is preserved by all the constructors of qCCS except for summation.
- $\mathcal{C} \approx_o \mathcal{D}$ if and only if $\mathcal{C} \approx_r \mathcal{D}$.

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An equivalence for super-operators

Let $\sqsubseteq$ be the Löwner preorder defined on operators: $A \sqsubseteq B$ if and only if $B - A$ is positive semi-definite.

For two super-operators $\mathcal{A}, \mathcal{B}$ on $\mathcal{H}$, let $\mathcal{A} \lesssim_V \mathcal{B}$ if for any $\rho \in \mathcal{D}(\mathcal{H})$, $\mathrm{tr}_{\overline{V}}(\mathcal{A}(\rho)) \sqsubseteq \mathrm{tr}_{\overline{V}}(\mathcal{B}(\rho))$, where $\overline{V}$ is the complement set of V in $qVar$.

Let $\eqsim_V$ be $\lesssim_V \cap \gtrsim_V$ and we abbreviate $\lesssim_\emptyset$ and $\eqsim_\emptyset$ to $\lesssim$ and $\eqsim$, respectively.

Super-operator valued distributions

A super-operator valued distribution Δ over S is a function from S to $\mathcal{SO}(\mathcal{H})$ such that $\sum_{s \in S} \Delta(s) \precsim \mathcal{I}_{\mathcal{H}}$.

Let $\mathcal{D}ist_{\mathcal{H}}(S)$ be the set of finite-support super-operator valued distributions over S.

Symbolic semantics

Inspired by [Hennessy & Lin 1995]

A pair of the form $(\!|t, \mathcal{E}|\!)$, where $t \in \mathcal{T}$ and $\mathcal{E} \in \mathcal{SO}_t(\mathcal{H})$, is called a snapshot. The set of snapshots is denoted by SN.

The symbolic semantics of qCCS is given by the qLTS $(SN, BAct_s, \rightarrow)$ on snapshots, where $\rightarrow \subseteq SN \times BAct_s \times \mathcal{D}ist_{\mathcal{H}}(SN)$ is the smallest relation satisfying a few rules.

Symbolic semantics

E.g.

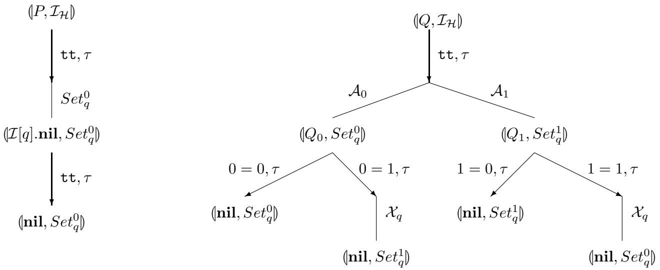
$$Meas_s \quad \frac{M = \sum_{i\in I} \lambda_i |\phi_i\rangle\langle\phi_i|}{(\!|M[\widetilde{q}; x].t, \mathcal{E}|\!) \xrightarrow{\mathtt{tt},\tau} \sum_{i\in I} \mathcal{A}_{\widetilde{r}}^{\phi_i} \bullet (\!|t\{\lambda_i/x\}, Set_{\widetilde{r}}^{\phi_i}\mathcal{E}|\!)}$$

where

$$\mathcal{A}_{\widetilde{r}}^{\phi_i} \quad : \quad \rho \mapsto |\phi_i\rangle_{\widetilde{r}}\langle\phi_i|\rho|\phi_i\rangle_{\widetilde{r}}\langle\phi_i| \tag{1}$$

$$Set_{\widetilde{r}}^{\phi_i} \quad : \quad \rho \mapsto \sum_{j\in I} |\phi_i\rangle_{\widetilde{r}}\langle\phi_j|\rho|\phi_j\rangle_{\widetilde{r}}\langle\phi_i|. \tag{2}$$

Symbolic semantics



Symbolic bisimulation

Definition

Let $\mathfrak{S} = \{\mathcal{S}^b : b \in BExp\}$ be a family of equivalence relations on SN. $\mathfrak{S}$ is called a symbolic (strong open) bisimulation if for any $b \in BExp$, $(\!|t, \mathcal{E}|\!)\mathcal{S}^b(\!|u, \mathcal{F}|\!)$ implies that

1. $qv(t) = qv(u)$ and $\mathcal{E} \eqsim_{\overline{qv(t)}} \mathcal{F}$, if b is satisfiable;
2. for any $\mathcal{G} \in \mathcal{SO}_t(\mathcal{H}_{\overline{qv(t)}})$, whenever $(\!|t, \mathcal{G}\mathcal{E}|\!) \xrightarrow{b_1,\gamma} \Delta$ with $bv(\gamma) \cap fv(b, t, u) = \emptyset$, there exists a collection of booleans B such that $b \wedge b_1 \to \bigvee B$ and $\forall\ b' \in B$, $\exists b_2, \gamma'$ with $b' \to b_2$, $\gamma =_{b'} \gamma'$, $(\!|u, \mathcal{G}\mathcal{F}|\!) \xrightarrow{b_2,\gamma'} \Xi$, and $(\mathcal{G}\mathcal{E} \bullet \Delta)\mathcal{S}^{b'}(\mathcal{G}\mathcal{F} \bullet \Xi)$.

Ground bisimulation

Definition

A family of equivalence relations $\{\mathcal{S}^b : b \in BExp\}$ is called a symbolic *ground* bisimulation if for any $b \in BExp$, $(\!|t, \mathcal{E}|\!)\mathcal{S}^b(\!|u, \mathcal{F}|\!)$ implies that

1. $qv(t) = qv(u)$ and $\mathcal{E} \approx_{\overline{qv(t)}} \mathcal{F}$, if b is satisfiable,
2. whenever $(\!|t, \mathcal{E}|\!) \stackrel{b_1,\gamma}{\rightarrow} \Delta$ with $bv(\gamma) \cap fv(b, t, u) = \emptyset$, there exists a collection of booleans B such that $b \wedge b_1 \rightarrow \bigvee B$ and $\forall\ b' \in B, \exists b_2, \gamma'$ with $b' \rightarrow b_2$, $\gamma =_{b'} \gamma'$, $(\!|u, \mathcal{F}|\!) \stackrel{b_2,\gamma'}{\rightarrow} \Xi$, and $(\mathcal{E} \bullet \Delta)\mathcal{S}^{b'}(\mathcal{F} \bullet \Xi)$.

Closure under super-operator application

Definition

A relation $\mathcal{S}$ on SN is said to be closed under super-operator application if $(\!|t, \mathcal{E}|\!)\mathcal{S}(\!|u, \mathcal{F}|\!)$ implies $(\!|t, \mathcal{G}\mathcal{E}|\!)\mathcal{S}(\!|u, \mathcal{G}\mathcal{F}|\!)$ for any $\mathcal{G} \in \mathcal{SO}_t(\mathcal{H}_{\overline{qv(t)}})$.

Theorem

A family of equivalence relations $\{\mathcal{S}^b : b \in BExp\}$ is a symbolic bisimulation if and only if it is both a ground bisimulation and closed under super-operator application.

Special case

Theorem

If t and u are both free of quantum input, then $(\!|t, \mathcal{E}|\!) \sim_s^b (\!|u, \mathcal{F}|\!)$ if and only if $(\!|t, \mathcal{E}|\!) \sim_g^b (\!|u, \mathcal{F}|\!)$.

Symbolic bisimilarity

Theorem

1. *For each* $b \in BExp$, $\sim_s^b$ *is an equivalence relation.*
2. *The family* $\{\sim_s^b : b \in BExp\}$ *is a symbolic bisimulation.*

Symbolic vs open bisimulation

Theorem

1. *$t \sim_s^b u$ if and only if for any evaluation ψ, $\psi(b) = tt$ implies $t\psi \sim_o u\psi$.*
2. *$t \sim_s u$ if and only if $t \sim_o u$.*

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The algorithm

Bisim$(\mathfrak{t}, \mathfrak{u})$ = **Match**$(\mathfrak{t}, \mathfrak{u}, \mathtt{tt}, \emptyset)$

Match$(\mathfrak{t}, \mathfrak{u}, b, W) =$ where $\mathfrak{t} = (\!|t, \mathcal{E}|\!)$ and $\mathfrak{u} = (\!|u, \mathcal{F}|\!)$
if $(\mathfrak{t}, \mathfrak{u}) \in W$ **then**
  $(\theta, T) := (\mathtt{tt}, \emptyset)$
else
  for $\gamma \in Act(\mathfrak{t}, \mathfrak{u})$ **do**
    $(\theta_\gamma, T_\gamma) :=$ **MatchAction**$(\gamma, \mathfrak{t}, \mathfrak{u}, b, W)$
  end
  $(\theta, T) := (\bigwedge_\gamma \theta_\gamma, \ \bigsqcup_\gamma (T_\gamma \sqcup \{(\mathfrak{t}, \mathfrak{u}) \mapsto (b \wedge \bigwedge_\gamma \theta_\gamma)\}))$
end
return $(\theta \wedge (qv(t) = qv(u)) \wedge (\mathcal{E} \approx_{\overline{qv(t)}} \mathcal{F}), T)$

MatchAction$(\gamma, \mathfrak{t}, \mathfrak{u}, b, W) =$
...
case τ
  for $\mathfrak{t} \xrightarrow{b_i, \tau} \Delta_i$ *and* $\mathfrak{u} \xrightarrow{b'_j, \tau} \Theta_j$ **do**
    $(\theta_{ij}, T_{ij}) :=$ **MatchDistribution**$(\Delta_i, \Theta_j, b \wedge b_i \wedge b'_j, \{(\mathfrak{t}, \mathfrak{u})\} \cup W)$
  end
  return $(\bigwedge_i (b_i \rightarrow \bigvee_j (b'_j \wedge \theta_{ij})) \wedge \bigwedge_j (b'_j \rightarrow \bigvee_i (b_i \wedge \theta_{ij})), \ \bigsqcup_{ij} T_{ij})$
endsw
...

MatchDistribution(Δ, Θ, b, W)=
for $\mathfrak{t}_i \in \lceil \Delta \rceil$ *and* $\mathfrak{u}_j \in \lceil \Theta \rceil$ **do**
  $(\theta_{ij}, T_{ij}) :=$ **Match**$(\mathfrak{t}_i, \mathfrak{u}_j, b, W)$
end
$\mathcal{R} := \{(\mathfrak{t}, \mathfrak{u}) \mid b \rightarrow (\bigsqcup_{ij} T_{ij})(\mathfrak{t}, \mathfrak{u})\}^*$
return (**Check**$(\Delta, \Theta, \mathcal{R})$, $\bigsqcup_{ij} T_{ij}$)

Check$(\Delta, \Theta, \mathcal{R}) =$
$\theta := \mathtt{tt}$
for $S \in \lceil \Delta \rceil \cup \lceil \Theta \rceil / \mathcal{R}$ **do**
  $\theta := \theta \wedge (\Delta(S) \approx \Theta(S))$

Correctness

Theorem

For two snapshots $\mathfrak{t}$ *and* $\mathfrak{u}$*, the function* $\textbf{Bisim}(\mathfrak{t}, \mathfrak{u})$ *terminates.*
Moreover, if $\textbf{Bisim}(\mathfrak{t}, \mathfrak{u}) = (\theta, T)$ *then* $T(\mathfrak{t}, \mathfrak{u}) = \theta = mgb(\mathfrak{t}, \mathfrak{u})$.

Complexity

Assume the ability of real computation, the worst case time complexity of executing **Bisim**$(\mathfrak{t}, \mathfrak{u})$ is $O(n^5 / \log n)$. To implement the algorithm, we have to approximate super-operators using matrices of algebraic or even rational numbers, thus increase the complexity.

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Quantum while-language [Ying 2011]

- Fix the alphabet of quantum **while**-language: A countably infinite set $qVar$ of quantum variables. Symbols $q, q', q_0, q_1, q_2, \ldots$ denote quantum variables.
- Each quantum variable $q \in qVar$ has a type $\mathcal{H}_q$ (a Hilbert space).
- For simplicity, we only consider two basic types:

$$\textbf{Boolean} = \mathcal{H}_2, \qquad \textbf{integer} = \mathcal{H}_\infty.$$

- A quantum register is a finite sequence $\overline{q} = q_1, \ldots, q_n$ of distinct quantum variables. Its state Hilbert space:

$$\mathcal{H}_{\overline{q}} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i}.$$

Quantum programs

$$
\begin{aligned}
S ::= \ & \mathbf{skip} \mid q := |0\rangle \mid \bar{q} := U[\bar{q}] \mid S_1; S_2 \\
& \mid \mathbf{if}\ (\square m \cdot M[\bar{q}] = m \rightarrow S_m)\ \mathbf{fi} \\
& \mid \mathbf{while}\ M[\bar{q}] = 1\ \mathbf{do}\ S\ \mathbf{od}.
\end{aligned}
$$

Notations

- A positive operator ρ is called a *partial density operator* if $tr(\rho) \leq 1$.
- Write $\mathcal{D}(\mathcal{H})$ for the set of partial density operators in $\mathcal{H}$.
- Write $\mathcal{H}_{all}$ for the tensor product of the state Hilbert spaces of all quantum variables:
$$\mathcal{H}_{all} = \bigotimes_{q \in qVar} \mathcal{H}_q.$$
- Let $\overline{q} = q_1, ..., q_n$ be a quantum register. An operator A in the state Hilbert space $\mathcal{H}_{\overline{q}}$ of $\overline{q}$ has a cylindrical extension $A \otimes I$ in $\mathcal{H}_{all}$.
- We will use E to denote the empty program; i.e. termination.
- A configuration is a pair $\langle S, \rho \rangle$, where:
  1. S is a quantum program or the empty program E;
  2. $\rho \in \mathcal{D}(\mathcal{H}_{all})$, denoting the (global) state of quantum variables.
- A transition between quantum configurations:

Operational semantics (selected rules)

(SC) $$\frac{\langle S_1, \rho\rangle \to \langle S_1', \rho'\rangle}{\langle S_1; S_2, \rho\rangle \to \langle S_1'; S_2, \rho'\rangle}$$

where $E; S_2 = S_2$.

(IF) $$\overline{\langle \mathbf{if}\ (\square m \cdot M[\overline{q}] = m \to S_m)\ \mathbf{fi}, \rho\rangle \to \langle S_m, M_m \rho M_m^\dagger\rangle}$$

for each possible outcome m of measurement $M = \{M_m\}$.

(L0) $$\overline{\langle \mathbf{while}\ M[\overline{q}] = 1\ \mathbf{do}\ S\ \mathbf{od}, \rho\rangle \to \langle E, M_0 \rho M_0^\dagger\rangle}$$

(L1) $$\overline{\langle \mathbf{while}\ M[\overline{q}] = 1\ \mathbf{do}\ S\ \mathbf{od}, \rho\rangle \to \langle S; \mathbf{while}\ M[\overline{q}] = 1\ \mathbf{do}\ S\ \mathbf{od}, M_1 \rho M_1^\dagger\rangle}$$

Semantic function

- Let S be a quantum program. Then its semantic function

$$[\![S]\!] : \mathcal{D}(\mathcal{H}_{all}) \to \mathcal{D}(\mathcal{H}_{all})$$

$$[\![S]\!](\rho) = \sum \{|\rho' : \langle S, \rho\rangle \to^* \langle E, \rho'\rangle|\}$$

Quantum Predicates

- What is a quantum predicate?
- A quantum predicate should be a physical observable!
- A *quantum predicate* in a Hilbert space $\mathcal{H}$ is a Hermitian operator M in $\mathcal{H}$ with all its eigenvalues lying within the unit interval $[0, 1]$.
- The set of predicates in $\mathcal{H}$ is denoted $\mathcal{P}(\mathcal{H})$.

Satisfaction of Quantum Predicates

- $tr(M\rho)$ may be interpreted as the degree to which quantum state ρ satisfies quantum predicate M.

Correctness Formulas

- A correctness formula is a statement of the form:

$$\{P\}S\{Q\}$$

  where:
  - S is a quantum program;
  - $P, Q \in \mathcal{P}(\mathcal{H}_{all})$ are quantum predicates in $\mathcal{H}_{all}$.
  - P is called the precondition, Q the postcondition.

Partial Correctness, Total Correctness

- Two interpretations of Hoare logical formula $\{P\}S\{Q\}$:
  - *Partial correctness*: If an input to program S satisfies the precondition P, then either S does not terminate, or it terminates in a state satisfying the postcondition Q.
  - *Total correctness*: If an input to program S satisfies the precondition P, then S must terminate and it terminates in a state satisfying the postcondition Q.

Partial Correctness, Total Correctness (Continued)

- The correctness formula $\{P\}S\{Q\}$ is true in the sense of *total correctness*, written

$$\models_{tot} \{P\}S\{Q\},$$

  if:

$$tr(P\rho) \leq tr(Q[\![S]\!](\rho))$$

  for all $\rho \in \mathcal{D}(\mathcal{H}_{all})$, where $[\![S]\!]$ is the semantic function of S.

- The correctness formula $\{P\}S\{Q\}$ is true in the sense of *partial correctness*, written

$$\models_{par} \{P\}S\{Q\},$$

  if:

$$tr(P\rho) \leq tr(Q[\![S]\!](\rho)) + [tr(\rho) - tr([\![S]\!](\rho))]$$

  for all $\rho \in \mathcal{D}(\mathcal{H}_{all})$.

Hoare logic for partial correctness (selected rules)

(R – SC) $$\frac{\{P\}S_1\{Q\} \quad \{Q\}S_2\{R\}}{\{P\}S_1;S_2\{R\}}$$

(R – IF) $$\frac{\{P_m\}S_m\{Q\} \text{ for all } m}{\{\sum_m M_m^\dagger P_m M_m\}\ \textbf{if}\ (\square m \cdot M[\overline{q}] = m \rightarrow S_m)\ \textbf{fi}\{Q\}}$$

(R – LP) $$\frac{\{Q\}S\,\{M_0^\dagger P M_0 + M_1^\dagger Q M_1\}}{\{M_0^\dagger P M_0 + M_1^\dagger Q M_1\}\textbf{while}\ M[\overline{q}] = 1\ \textbf{do}\ S\ \textbf{od}\{P\}}$$

(R – Or) $$\frac{P \sqsubseteq P' \quad \{P'\}S\{Q'\} \quad Q' \sqsubseteq Q}{\{P\}S\{Q\}}$$

Soundness Theorem

For any quantum **while**-program S and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$:

$$\vdash_{qPD} \{P\}S\{Q\} \text{ implies } \models_{par} \{P\}S\{Q\}.$$

(Relative) Completeness Theorem

For any quantum **while**-program S and quantum predicates $P, Q \in \mathcal{P}(\mathcal{H}_{all})$:

$$\models_{par} \{P\}S\{Q\} \text{ implies } \vdash_{qPD} \{P\}S\{Q\}.$$

Theorem prover for quantum programs

- A theorem prover for quantum Hoare logic based on Isabelle/HOL has been implemented by Liu et al.

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- `https://arxiv.org/pdf/1601.03835.pdf`

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- Apply the open bisimulation to analyze quantum cryptographic protocols, e.g. BB84 quantum key distribution protocol
- Model checking for quantum protocols
- Termination analysis
- Invariant generation
- Fully abstract denotational semantics

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Thank you!