Fixpoint Theory – Upside Down

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Upper and Lower Bounds for Fixpoints

Let $f \colon \mathbb{L} \to \mathbb{L}$ be a monotone function over a complete lattice $\mathbb{L}$. By Knaster-Tarski it has a least fixpoint μf and a greatest fixpoint νf.

Any pre-fixpoint ($\ell \in \mathbb{L}$ with $f(\ell) \sqsubseteq \ell$) is an upper bound for μf and any post-fixpoint ($\ell \in \mathbb{L}$ with $\ell \sqsubseteq f(\ell)$) is a lower bound for νf.

Challenge

Can we find suitable witnesses guaranteeing that $\ell \in \mathbb{L}$ is a lower bound for μf or an upper bound for νf?

Applications: termination probability, behavioural distances, bisimilarity, stochastic games . . .

Fixpoint Theory

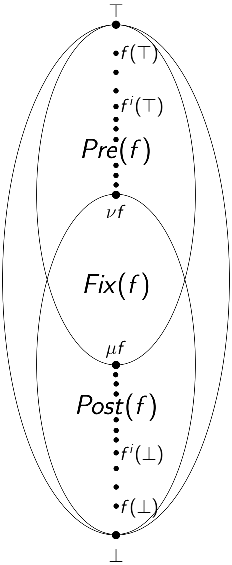
Solution techniques

- The Knaster-Tarski theorem guarantees the existence of least and greatest fixpoints for monotone functions
- We have the following proof rules for upper and lower bounds:

$$\frac{f(\ell) \sqsubseteq \ell}{\mu f \sqsubseteq \ell} \qquad \frac{\ell \sqsubseteq f(\ell)}{\ell \sqsubseteq \nu f}$$

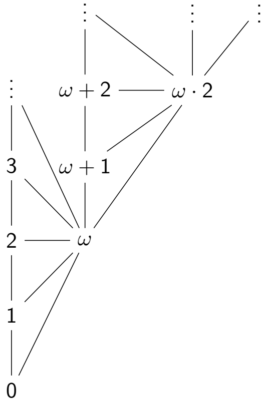
- Kleene iteration: whenever f is (co-)continuous
  - $\mu f = \bigsqcup_{i \in \mathbb{N}} f^i(\bot)$ (least fixpoint)
  - $\nu f = \bigsqcap_{i \in \mathbb{N}} f^i(\top)$ (greatest fixpoint)

Fixpoint Theory



If f is *not* (co-)continuous:

$\rightsquigarrow$ Kleene iteration over the ordinals (beyond ω)



Fixpoint Theory

The following proof rules (based on Kleene iteration) provide guarantees for the opposite bounds. By i we denote some ordinal.

$$\frac{\ell \sqsubseteq f^i(\bot)}{\ell \sqsubseteq \mu f} \qquad \frac{f^i(\top) \sqsubseteq \ell}{\nu f \sqsubseteq \ell}$$

This is related to ranking functions that are e.g. used in termination analysis.

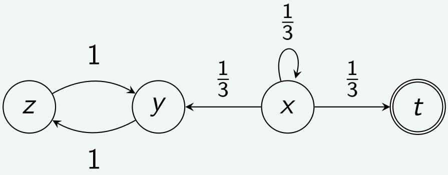
Problems: there is no straightforward witness that guarantees these bounds, (ordinals are involved)

Our aim: provide proof rules of the form

$$\frac{\ell \sqsubseteq f(\ell) + \text{extra conditions}}{\ell \sqsubseteq \mu f} \qquad \frac{f(\ell) \sqsubseteq \ell + \text{extra conditions}}{\nu f \sqsubseteq \ell}$$

Termination Probability

What is the probability of terminating from state x?



Termination Probability

Markov chain

$(S, T, (p_s)_{s \in S \setminus T})$ where

- S is the finite state space,
- $T \subseteq S$ are the terminal states and
- $p_s \colon S \to [0,1]$ is a probability distribution

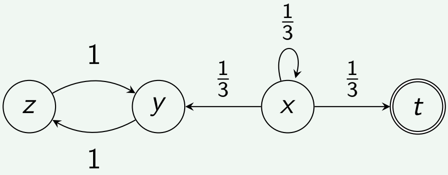
Termination probability as least fixpoint

Termination probability given by μf where $f \colon [0,1]^S \to [0,1]^S$ and for $a \colon S \to [0,1]$, $s \in S$:

$$f(a)(s) = \begin{cases} 1 & \text{if } s \in T \\ \sum_{t \in S} p_s(t) \cdot a(t) & \text{otherwise} \end{cases}$$

Termination Probability

What is the probability of terminating from state x?



Termination Probability

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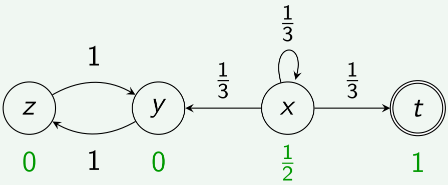
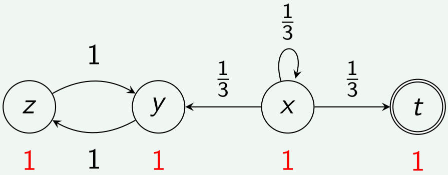


Least fixpoint, giving the termination probability for x

Termination Probability

What is the probability of terminating from state x?



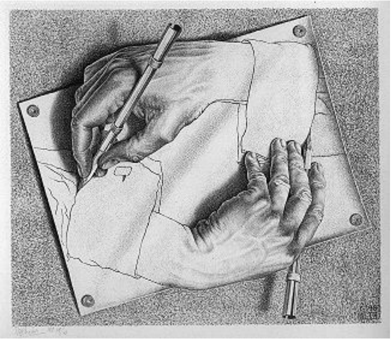
A different fixpoint, not providing a lower bound for the termination probability of x

Termination Probability

We can not trust a fixpoint or pre-fixpoint to give us a lower bound on the termination probability (given by a *least* fixpoint).

▷ Can we detect those fixpoints that are not least fixpoints? Where is the culprit?

In the example: y and z convince each other incorrectly (!) that they have termination probability 1 $\rightsquigarrow$ vicious cycle



Termination Probability

Idea: compute the set of states that still has some "wiggle room" or "slack". That is, those states that can say:

> *"If all my successors would reduce their value by δ, I could also reduce my value by δ."*

This can be computed as a greatest fixpoint on a finite set $\mathcal{P}(S)$ (instead of the infinite lattice that we considered before).

If the function is sufficiently well-behaved (monotone and non-expansive) and this greatest fixpoint is empty
$\leadsto$ we know that we have reached the least fixpoint (respectively a post-fixpoint below the least fixpoint).

Setting

Requirements

The lattice is of the form $\mathbb{L} = \mathbb{M}^Y$ (set of functions of the form $Y \to \mathbb{M}$), where

- Y finite
- $\mathbb{M}$ is a complete MV-chain (total order)
  For instance: $\mathbb{M} = [0, 1]$ or $\mathbb{M} = \{0, \dots, k\}$ with truncated addition ($\oplus$) and subtraction ($\ominus$)
- $f : \mathbb{M}^Y \to \mathbb{M}^Y$ monotone and non-expansive wrt. to the supremum metric (applying the function does not increase the distance)

Galois Connections and Fixpoints

$$f_{\#}^{a,\delta} = \gamma^{a,\delta} \circ f \circ \alpha^{a,\delta} \circlearrowright \mathcal{P}(Y) \underset{\gamma^{a,\delta}}{\overset{\alpha^{a,\delta}}{\rightleftarrows}} \mathbb{M}^Y \circlearrowleft f$$

- f: fixpoint function
- Galois connection:
  - $\alpha^{a,\delta}(Y')$ transforms a, decreasing values $a(y)$, $y \in Y'$ by δ
  - $\gamma^{a,\delta}(b)$ returns those $y \in Y$ which satisfy $a(y) \ominus b(y) \geq \delta$
- $f_{a,\delta}^{\#}$: approximation, computing the "wiggle room"

Observations:

- For small δ the function $f_{\#}^{a,\delta}$ does not depend on $\delta \rightsquigarrow f_{\#}^{a}$
- From results on Galois connections and fixpoints: $a \sqsubseteq f(a)$, $\mu f_{\#}^{a} = \emptyset$ imply $a \sqsubseteq \mu f$.

Applications

There are compositionality results for constructing the approximations, so that they can be obtained from basic functions (constants, min, max, average, reindexing), composition and disjoint union.

This allows us to provide witnesses for:

- lower bounds of termination probabilities
- lower bounds for payoffs in stochastic games
  (we also provide new strategy iteration algorithms for non-stopping stochastic games)
- non-bisimilarity of states
- lower bounds for behavioural distances

The technique can also be used to μf from above (and to iterate to νf from below). (See [Fu], [Bacci, Bacci, Larsen, Mardare, Tang, van Breugel] for the case of behavioural metrics.)

Future Work

- Is it possible to lift some of the restrictions? In particular:
  - is it possible to handle partial (instead of total) orders?
  - what about infinite carrier sets Y?
- Does it make sense to generalize the Galois connection? (multiplicative instead of additive variants?)
- Further case studies: energy games, coalgebraic behavioural metrics, ...