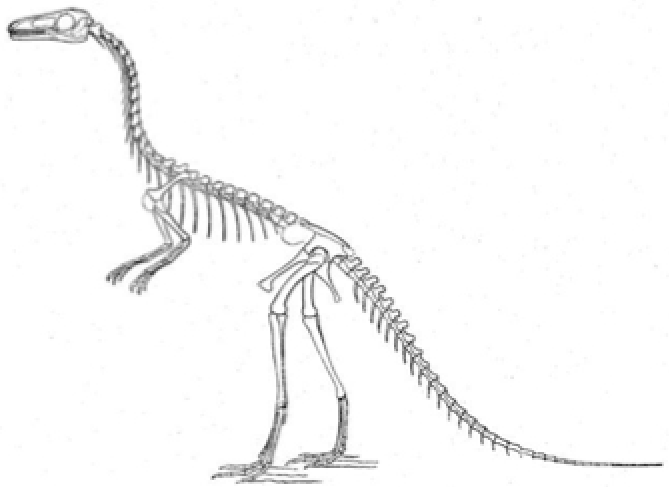


Semantics of signal flow

Pawel Sobocinski
IFIP WG 2.2 Munich

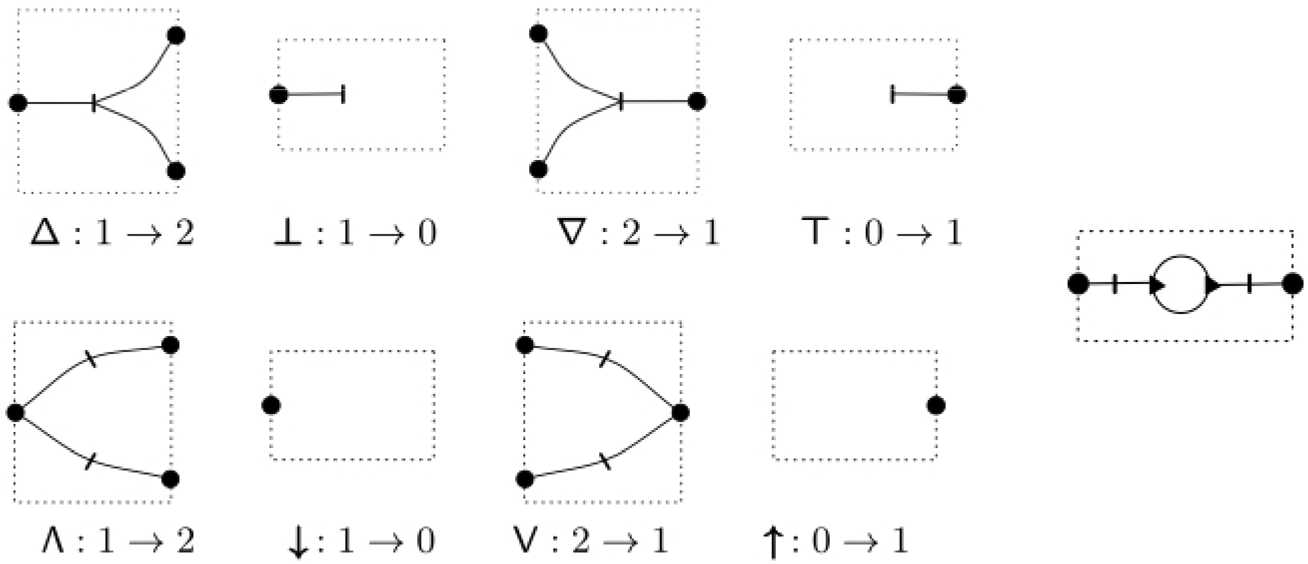
(joint work with Filippo Bonchi and Fabio Zanasi)

Emerging field of (Categorical) Network Theory

- Graphical structures as first-class entities
  - uses the theory of *symmetric monoidal categories*
  - emphasis on **compositionality** and **interconnections**
- Name dropping: Samson Abramsky *(economics, quantum entanglement*), John Baez (*control, biology, climate change*), Bob Coecke *(quantum circuits, linguistics*), Dusko Pavlovic (*complexity, computability, security*), …
- Connections with the **behavioural approach** (causality considered harmful) in control theory - Jan Willems et al

Last time

- "Network theoretic" approaches to Petri nets (algebras of Petri Nets with Boundaries)



- We will see a very similar set of generators today…

Signal-flow graphs (SFG)

Older than IFIP WG 2.2!

- Originally discovered by Shannon in the early 1940s, then forgotten
- Re-discovered by Mason and others in the 1950s
  - sometimes SFGs are called Mason diagrams
- Discrete case: take a stream of signals as input and produce a stream of signals as output
  - closely related to sets of **recurrence formulas**
- Feature a restricted form of feedback
- **Slogan**: "electrical-engineer's deterministic finite automata"

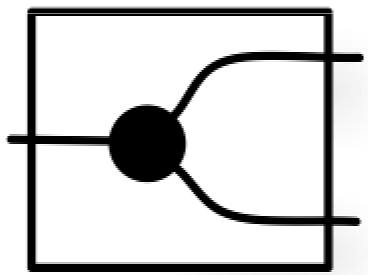
Plan of the talk

1. Denotational semantics of signal flow with sound and complete axiomatisation
    - I will give operational intuitions, but these will be informal
2. Operational semantics of signal flow and a full abstraction theorem
    - I will formalise the operational intuitions

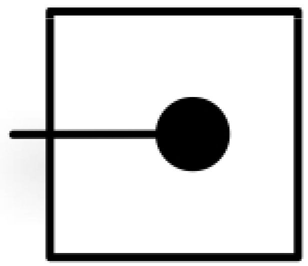
Unidirectional signal flow

Basic signal flow (left-to-right)

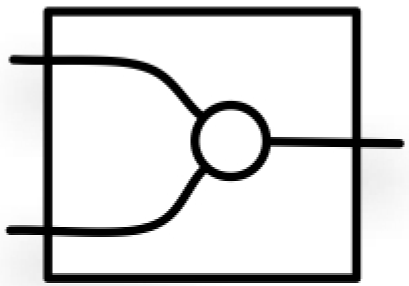
- **signals** are assumed to take values in some field (say the rationals, for concreteness)



Copy



Discard



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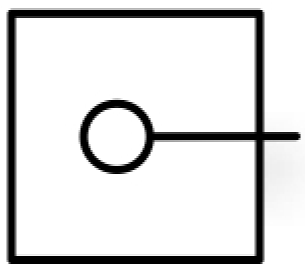
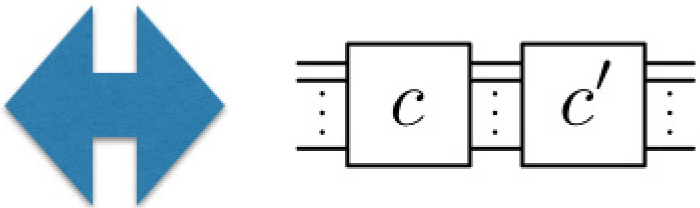


Output Zero

Interlude: general operations for composing circuits

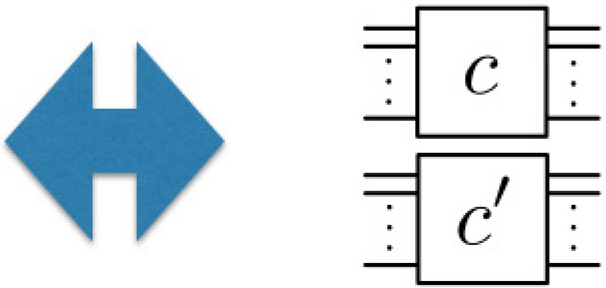
- Gates can be pasted together in two ways
  - side-by-side by connecting ports

$$c\,;c'$$



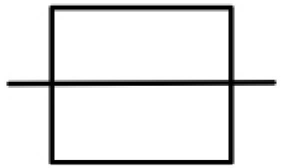
  - stacking one on the other

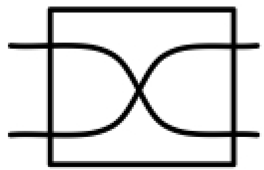
$$c \bigoplus c'$$



Interlude: auxiliary tiles

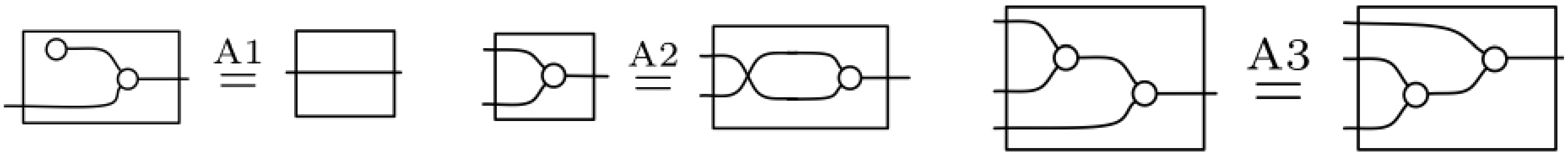
- The following tiles are also available



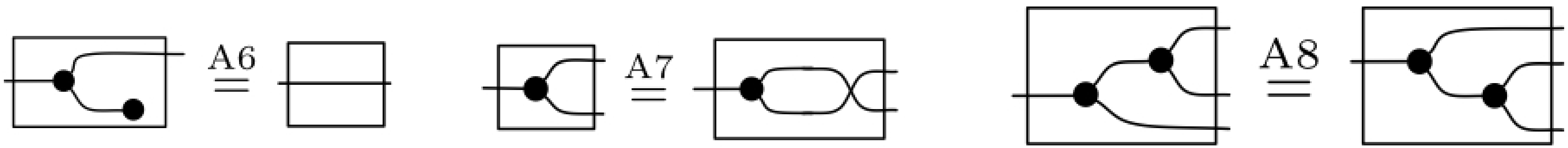


- Compound **circuits** are equated using the laws of symmetric monoidal categories (forming what is called a PROP)
    - “circuits that look the same” are the same (associativity, identities, middle-four interchange)
    - permuting wires behaves as expected (e.g. wires do not get “tangled”)
    - naturality: any gate can be slid past a symmetry

Imposing equations

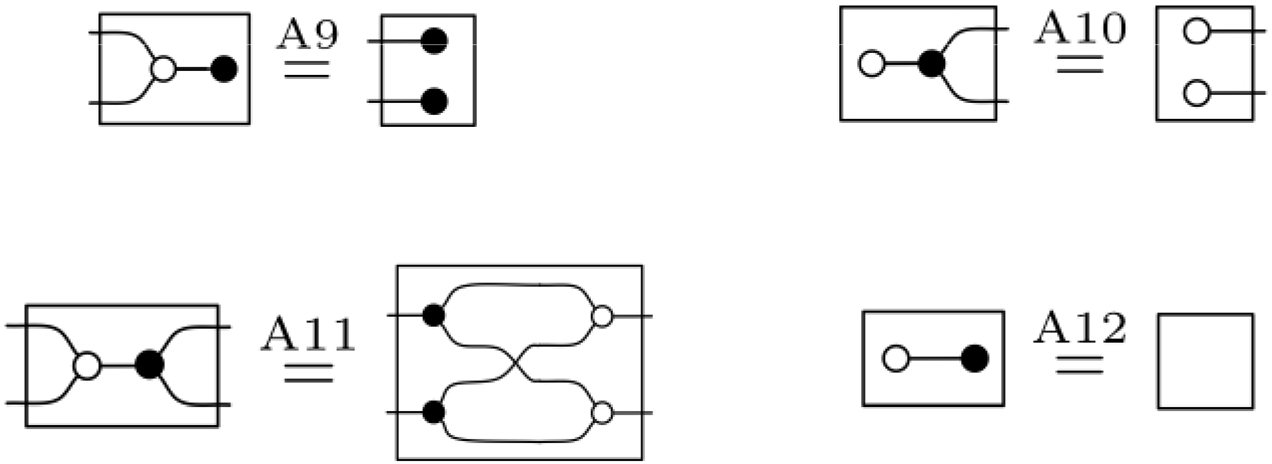


Addition is a commutative monoid



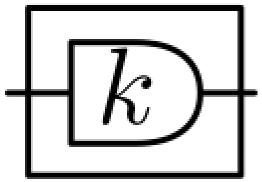
Copying is a cocommutative comonoid

Interactions between black and white (copy and add)

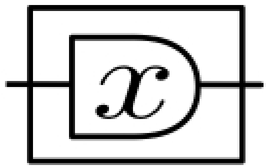


these are known as the equations of (commutative/cocommutative) **bialgebra**

Two more gates



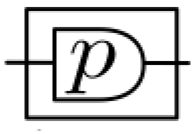
amplifier - the signal is multiplied by k



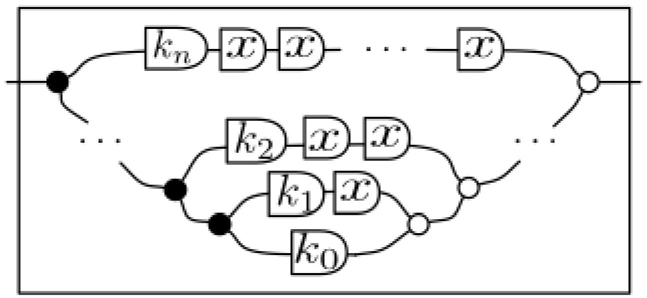
delay - a register, that initially holds 0,
and acts as a synchronous 1-space buffer

Polynomials

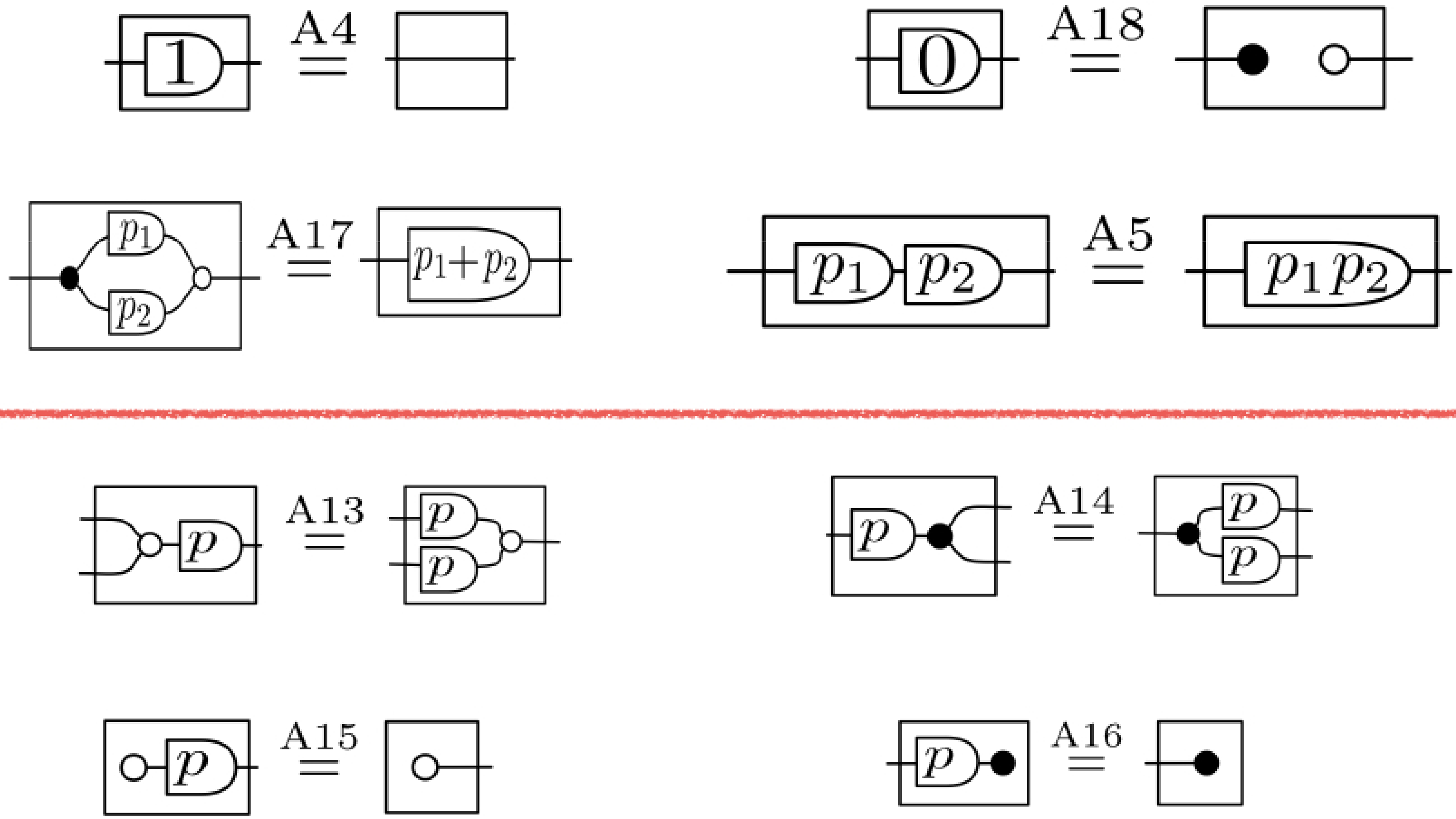
$$p = k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n$$



denotes

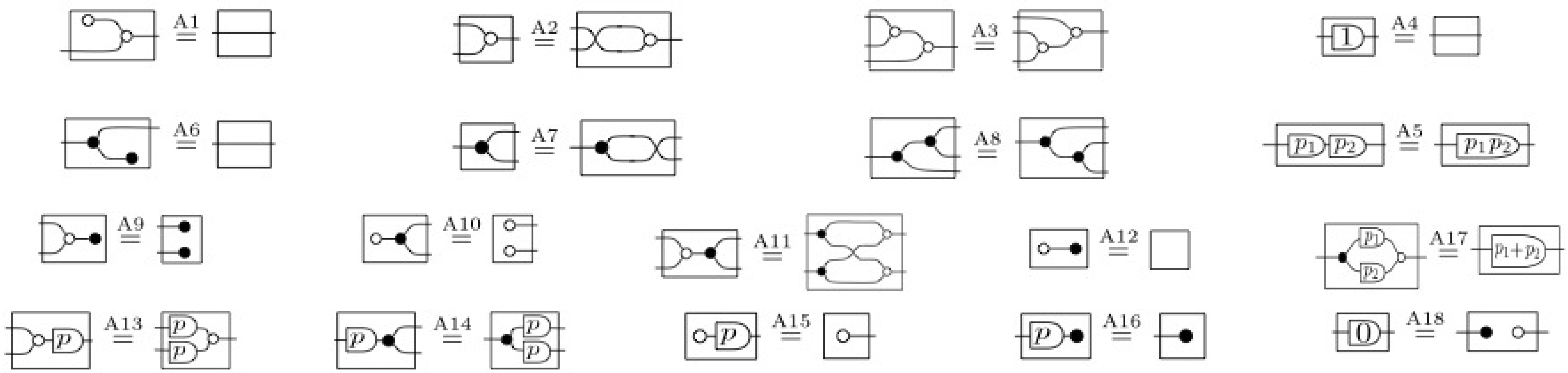


A few more equations



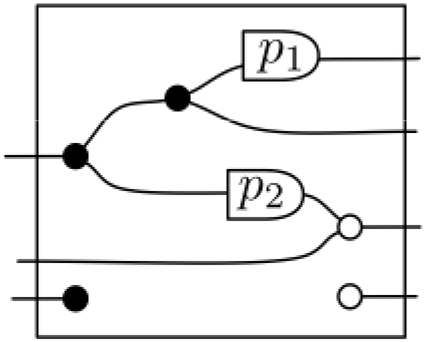
Diagrams = Matrices

- **Theorem.** There is an isomorphism of categories between the cat of diagrams modulo equations



and the category where morphisms m to n are nxm matrices over the polynomial ring *k*[x]. In particular, composition of diagrams is matrix multipication.

Example



$$\begin{pmatrix} p_1 & 0 & 0 \\ 1 & 0 & 0 \\ p_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

What do polynomial matrices have to do with streams?

Stream semantics 1

- Formal Power Series (fps)
  - possibly infinite polynomial

$$k_0 + k_1 x + k_2 x^2 + \ldots$$

  - in combinatorics, number theory and other areas of maths power series are ubiquitous
  - often used as “coat-hangers” for various infinite sequences of numbers

Stream semantics 2

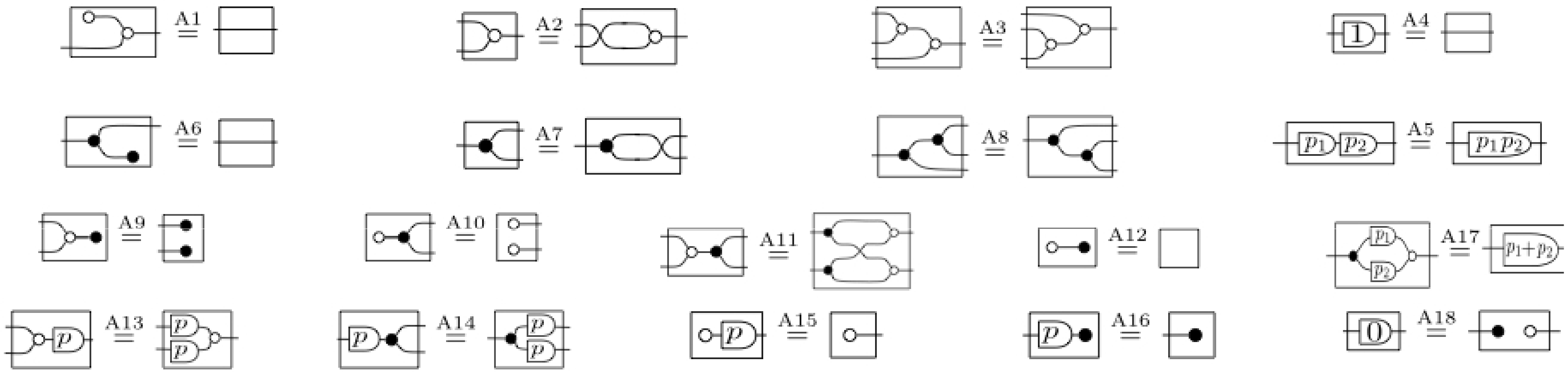
- Our diagrams m to n diagrams are thus stream transformers with m inputs and n outputs.
- Given an m vector v of input streams (fps), the vector of output streams is simply

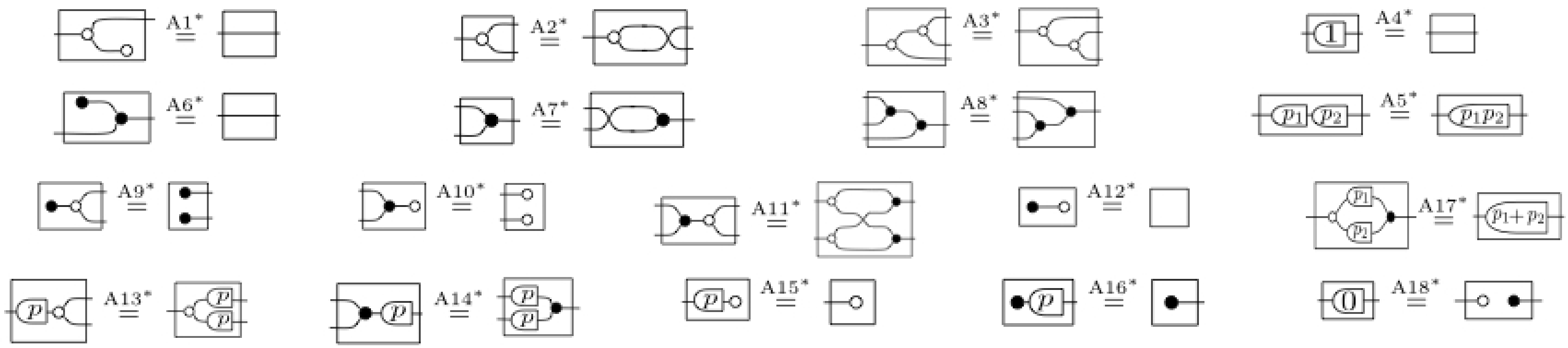
$$A\mathbf{v}$$

- (where A is the matrix corresponding to the diagram)
- Notice that multiplying an fps by x is exactly the “delay by 1” operation

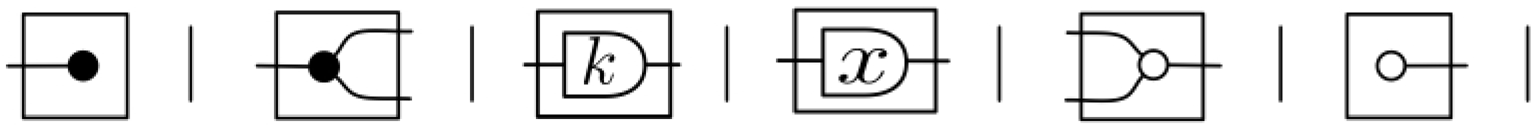
Mixed (wild!) signal flow

Reverse signal flow (right to left)

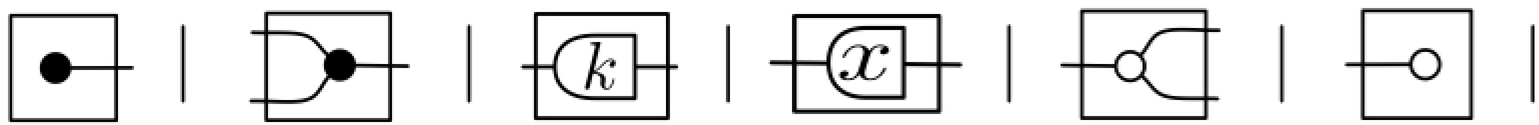




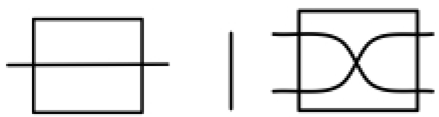
What if signal can flow in both directions?



left to right



right to left

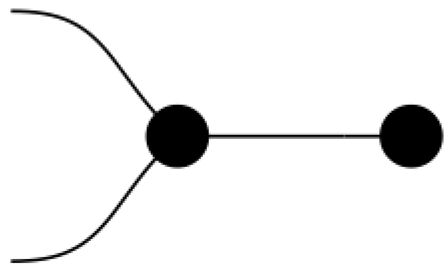


wiring

How to make sense of signal flow in both directions?

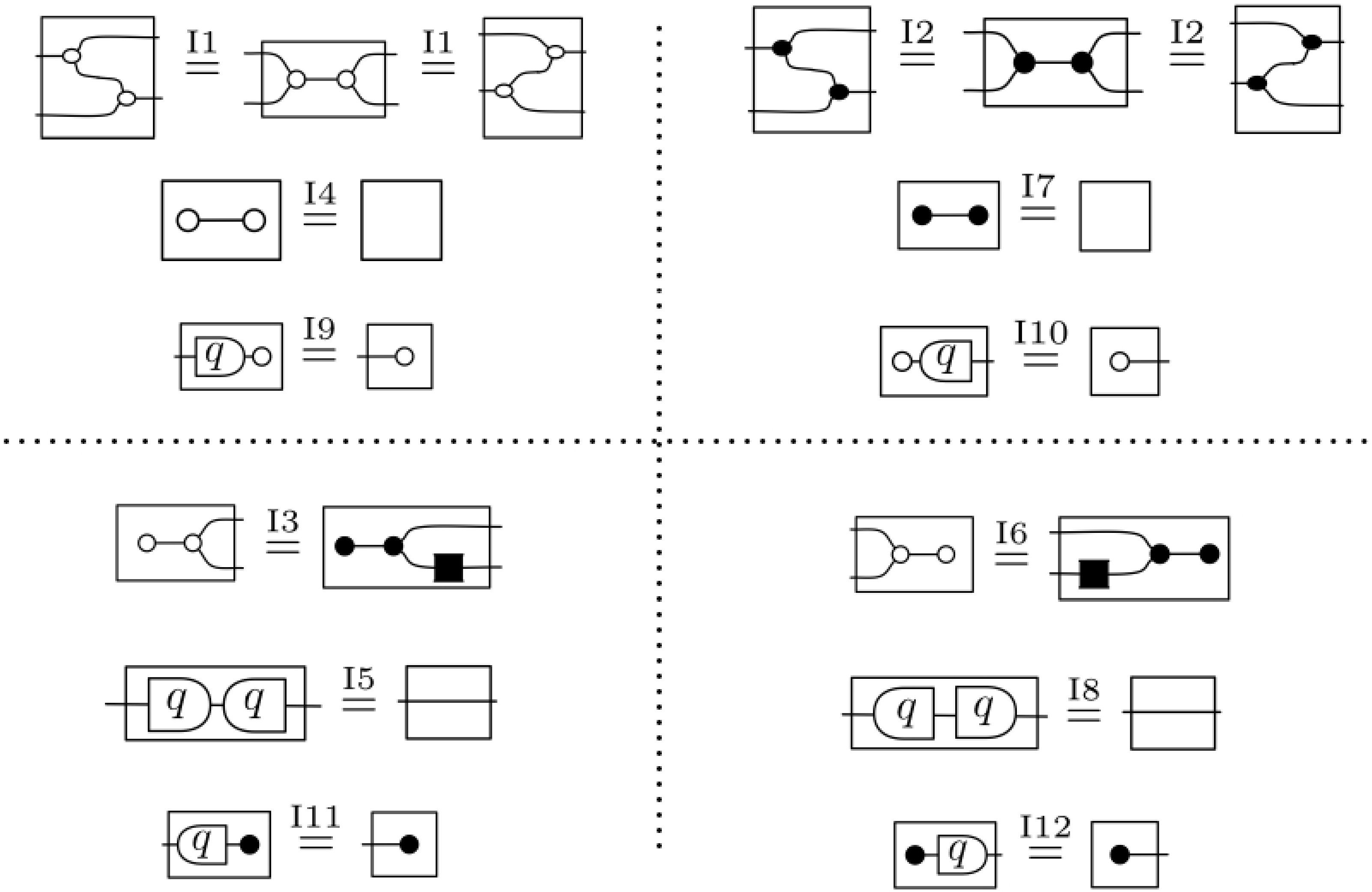
- In some circuits signal **cannot** be interpreted as simply going from left to right or right to left

  - eg



  - A: Circuits are no longer matrices, but **subspaces**. Composition is **relational**.

Combining signal flows



Diagrams = subspaces

- A **linear relation** is a relation that is also a subspace.
- **Theorem.** There is an isomorphism between diagrams modulo the equations we have seen and the category where arrows from m to n are linear relations (subspaces of $k(x)^{m+n}$) over the field $k(x)$ of polynomial fractions, and composition is relational.
- This theorem was also proved independently by Baez and Erbele (2014) who gave a different, equivalent set of axioms
- Axioms are similar to the ZX-calculus of Coecke and Duncan (2008) for quantum circuits… why?

The general case

- Bonchi, S., Zanasi. *Interacting Hopf Algebras (arxiv, 2014)*

- Starting with the graphical theory of matrices over a principal ideal domain R, we obtain the graphical theory of linear relations over the field of fractions of R

- Technically, the axioms arise canonically from **distributive laws** - see the paper above for details!