

Multi-Pushdown Automata with Data

Parosh Aziz Abdulla
joint work with



M. Faouzi Atig



Aiswarya Cyriac

Outline

- **Background**
- **Model**
- **Gap-Order Constraints**
- **Signatures**
- **Reachability Algorithm**
- **Applications**

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Background

**Boolean
recursive
programs**

Pushdown Processes

Background

concurrent
recursive
programs

**Multiple-Stack
Processes**

Boolean
recursive
programs

Pushdown Processes



Background

concurrent
recursive
programs

undecidable
in general

“decidable under
bounded context
switch”

**Multiple-Stack
Processes**

Boolean
recursive
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Pushdown Processes



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**Multiple-Stack
Processes**

Boolean
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Pushdown Processes

recursive
programs
with data

**Pushdown Processes
+ Data**



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**Multiple-Stack
Processes**

Boolean
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Pushdown Processes

undecidable
in general

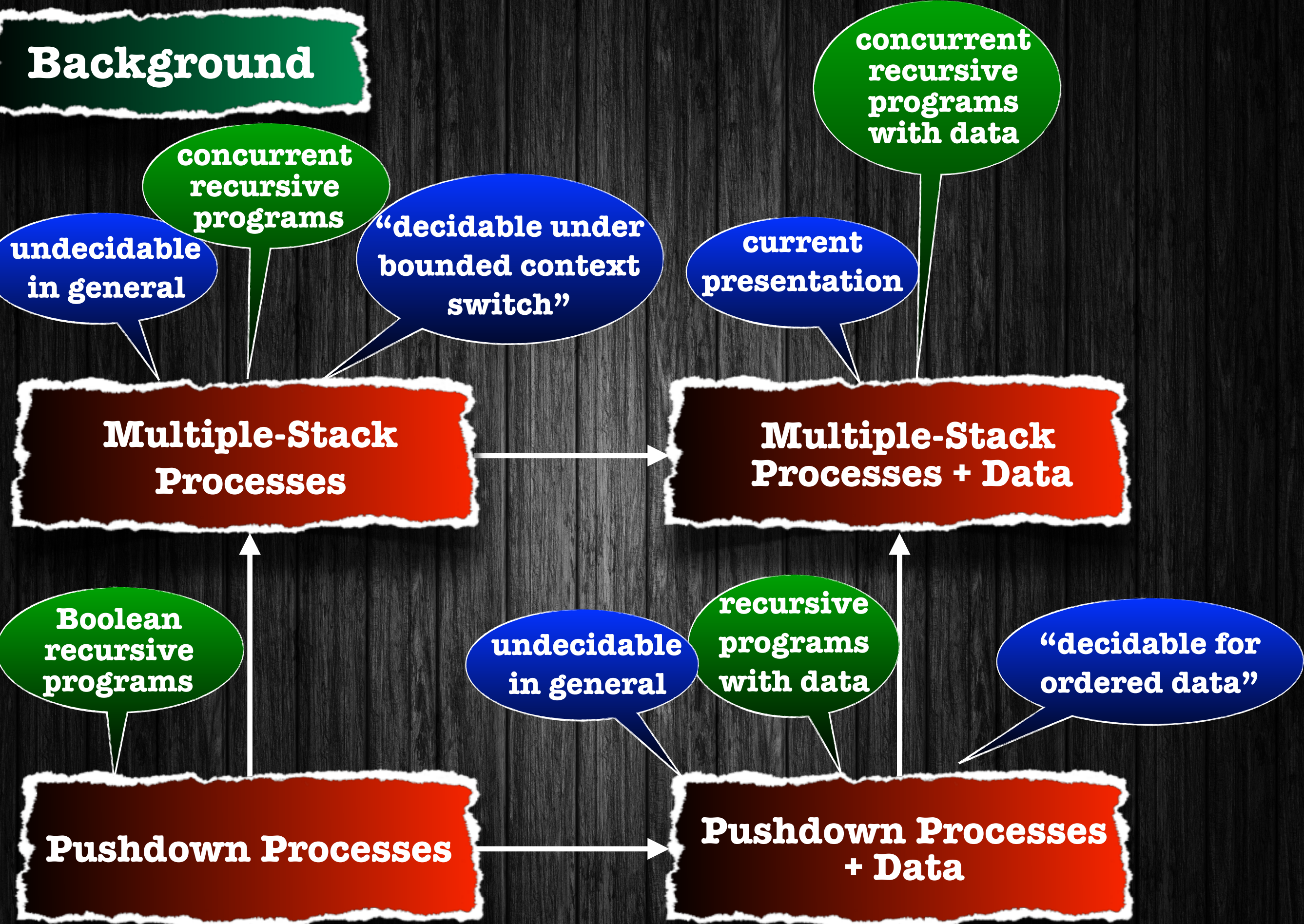
recursive
programs
with data

“decidable for
ordered data”

**Pushdown Processes
+ Data**



Background



Background

concurrent
recursive
programs

undecidable
in general

“decidable under
bounded context
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**Multiple-Stack
Processes**

concurrent
recursive
programs
with data

current
presentation

“decidable for
ordered data under
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**Multiple-Stack
Processes + Data**

Boolean
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Pushdown Processes

**Pushdown Processes
+ Data**

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- **Syntax**
- **Configurations**
- **Runs**
- **Reachability**

Model

finite set of
variables over $\mathbb{N}$

x, y, z

finite set of
stacks

σ_1, σ_2

x, y, z

Automaton

σ_1

$(b, 0)$
 $(a, 3)$

σ_2

$(b, 2)$

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Automaton

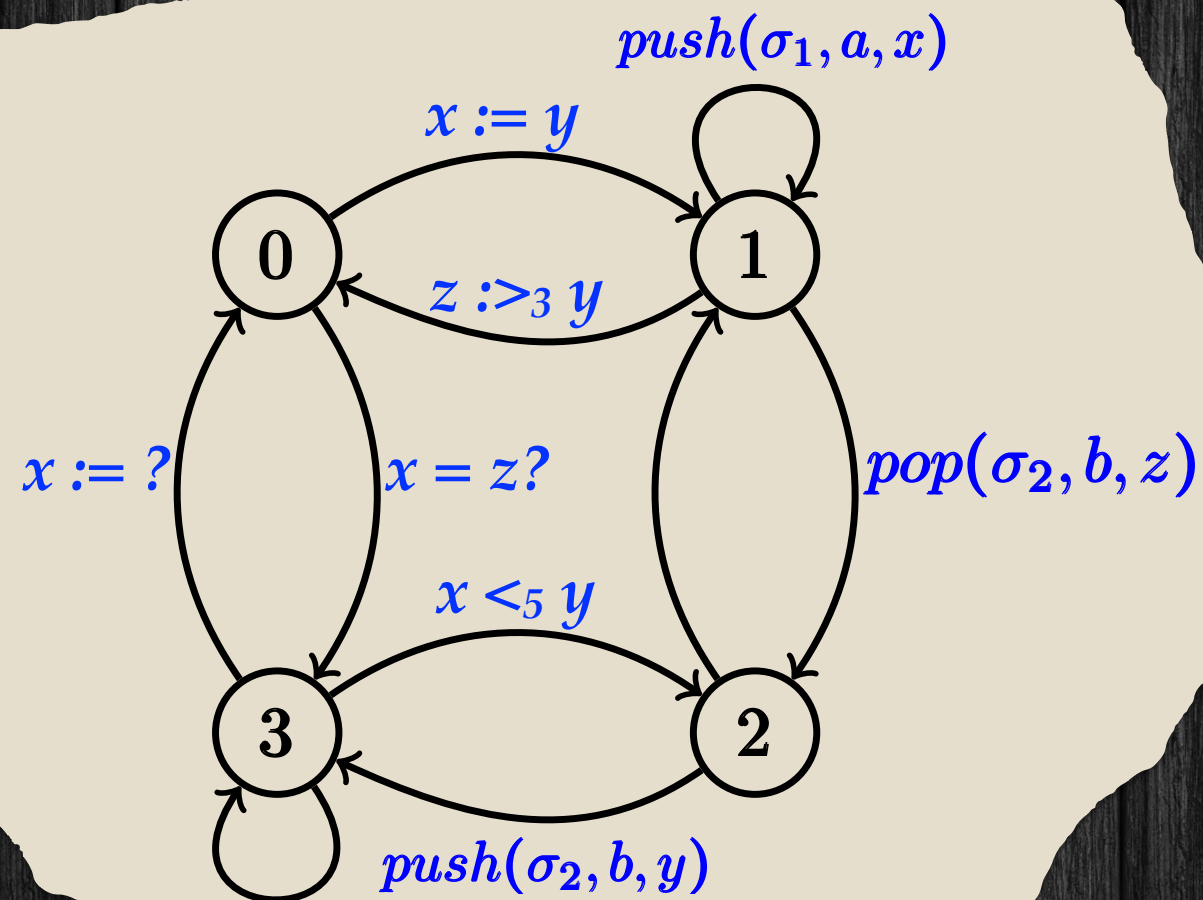
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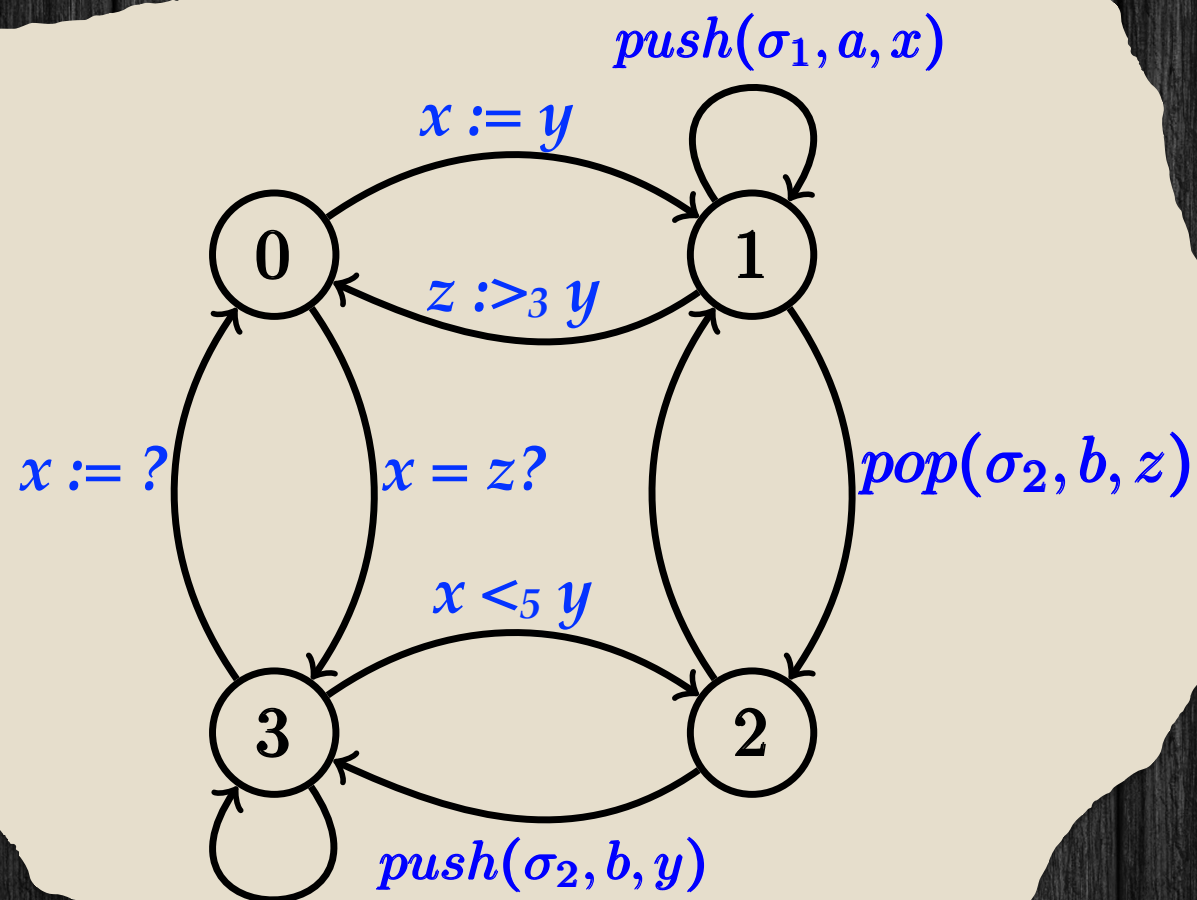
finite set of
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finite set of
stacks

$v(x)$: current
value of x

x, y, z

σ_1, σ_2



Model

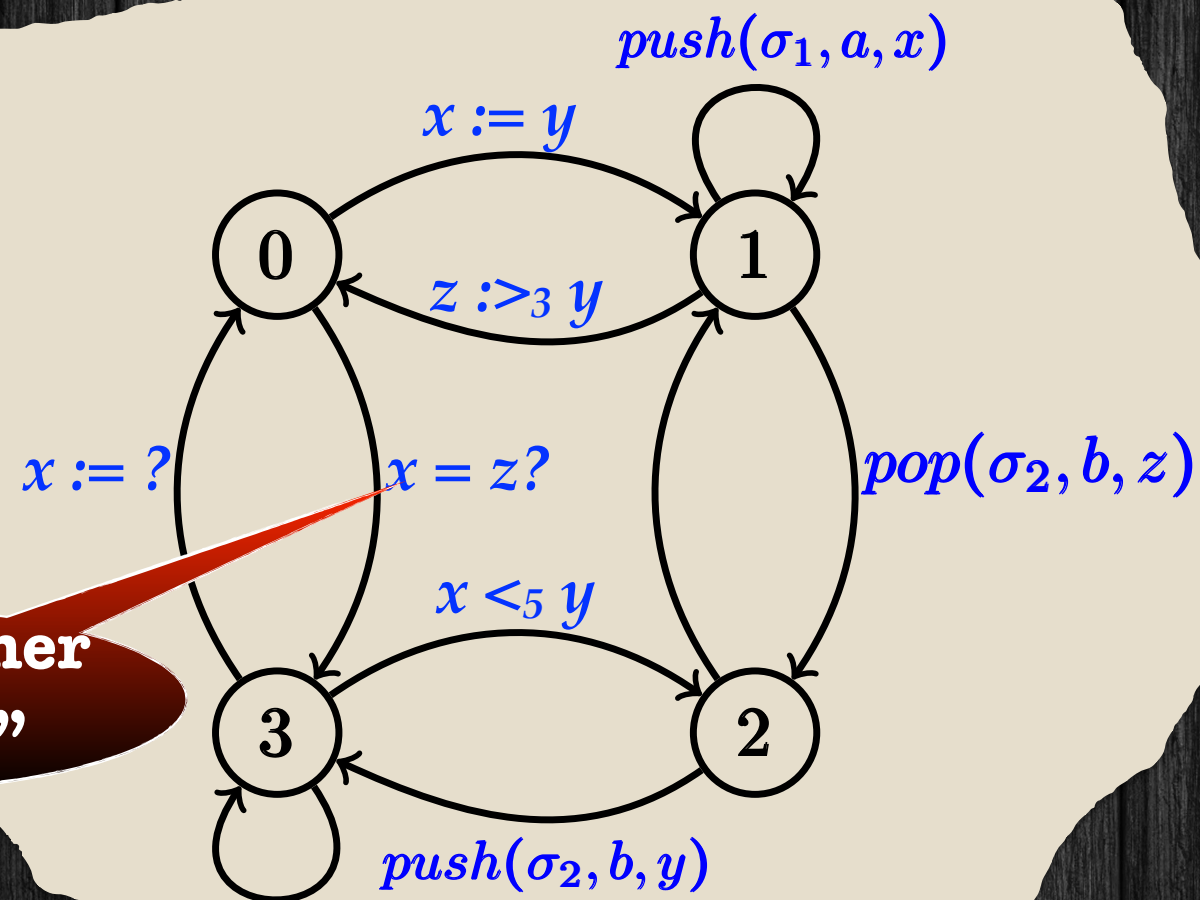
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“test whether
 $v(x) = v(z)$ ”

Model

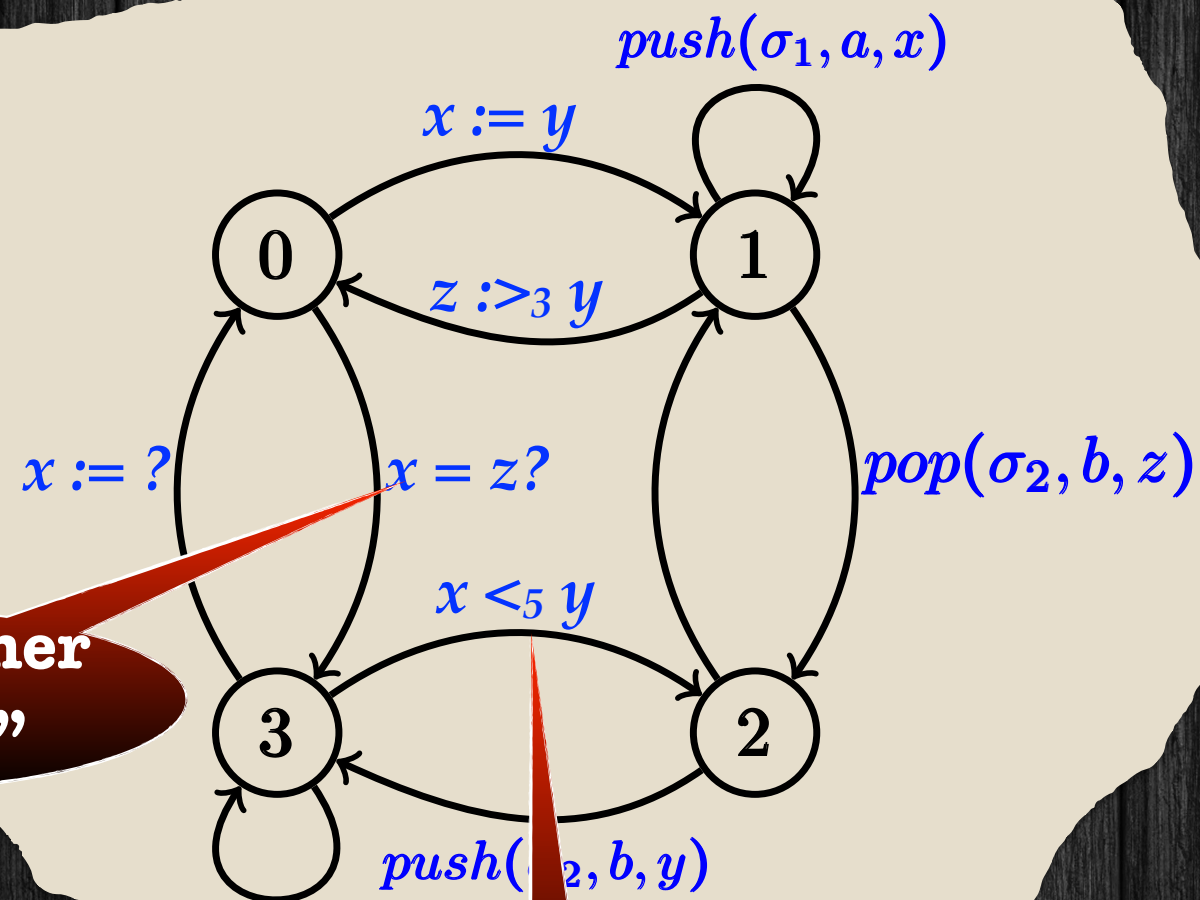
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finite set of
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$v(x)$: current
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“test whether
 $v(x) = v(z)$ ”

“test whether
 $v(x) + 5 < v(y)$ ”

Model

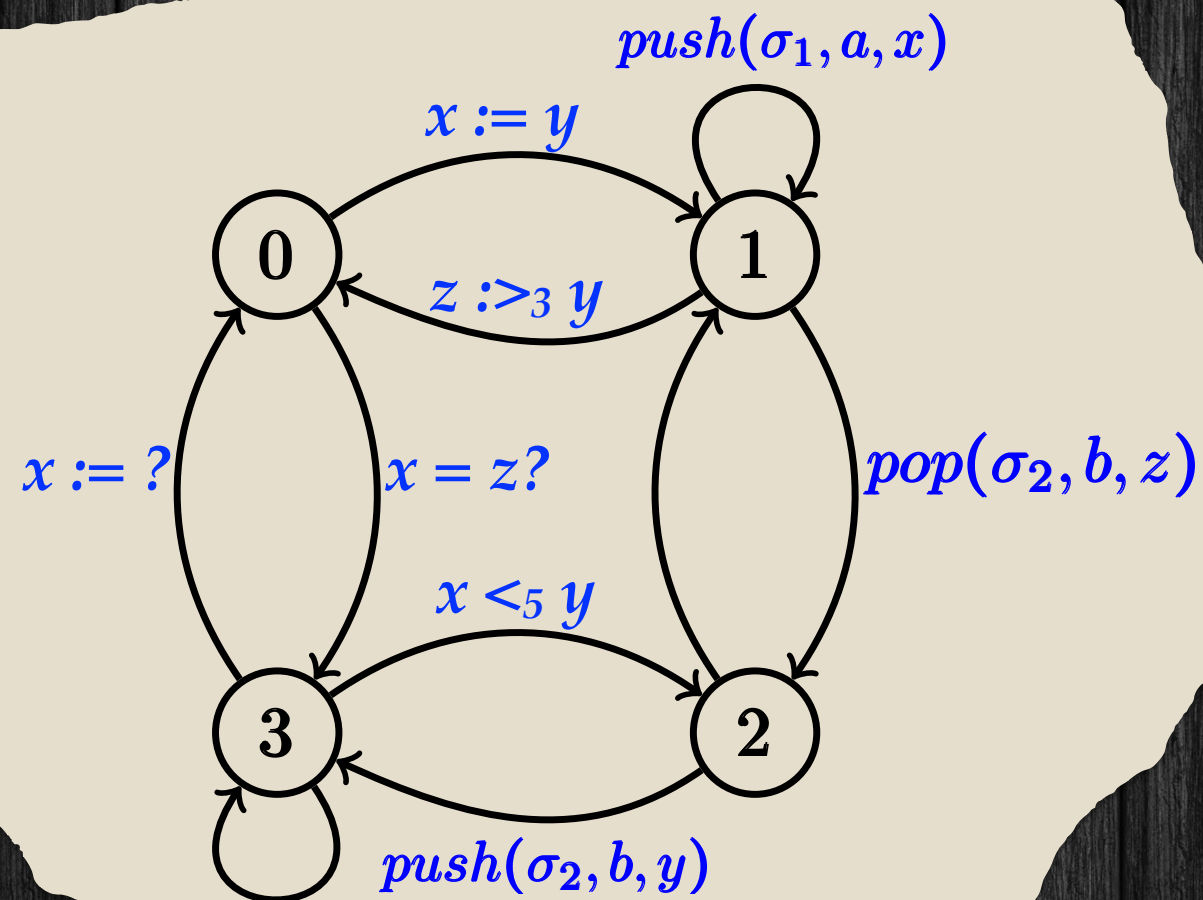
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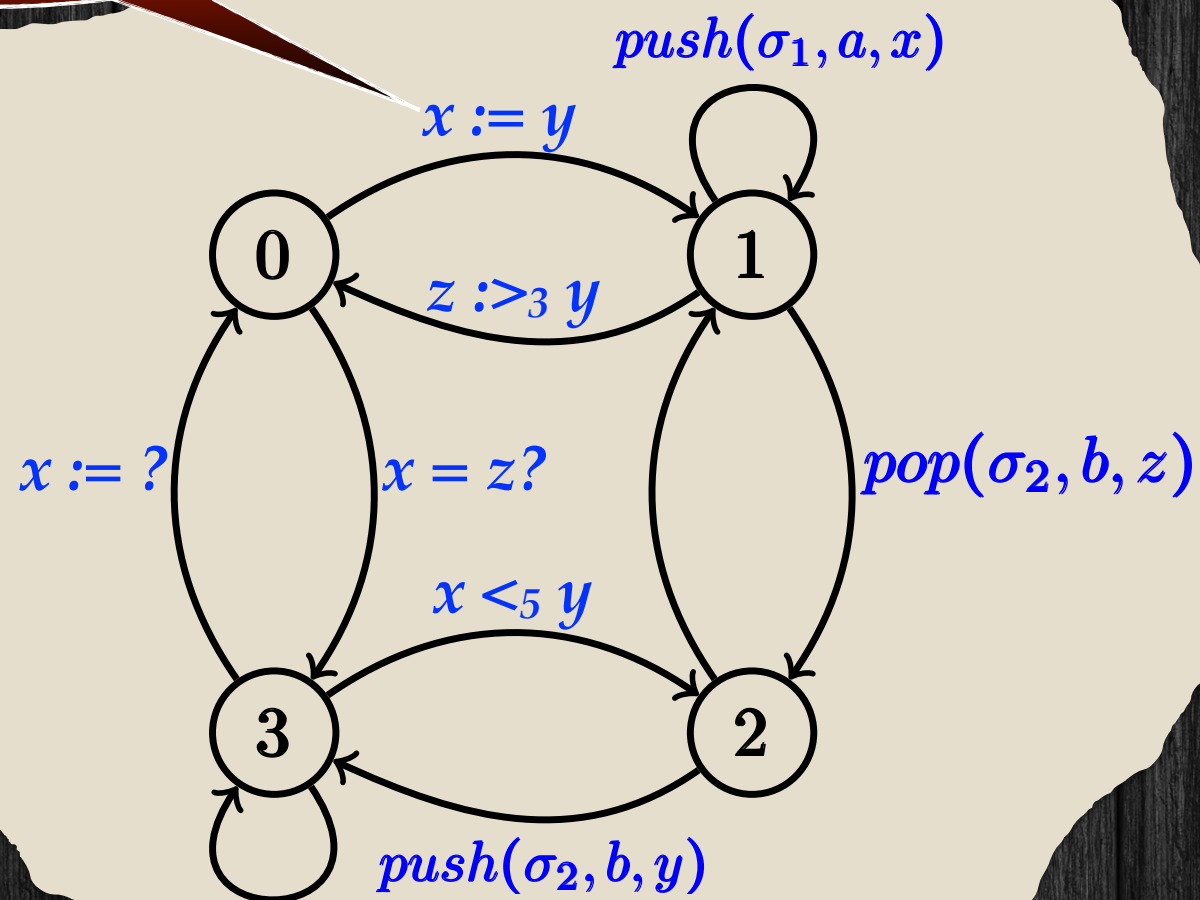
finite set of
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$v(x)$: current
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“assign $v(y)$ to x ”



Model

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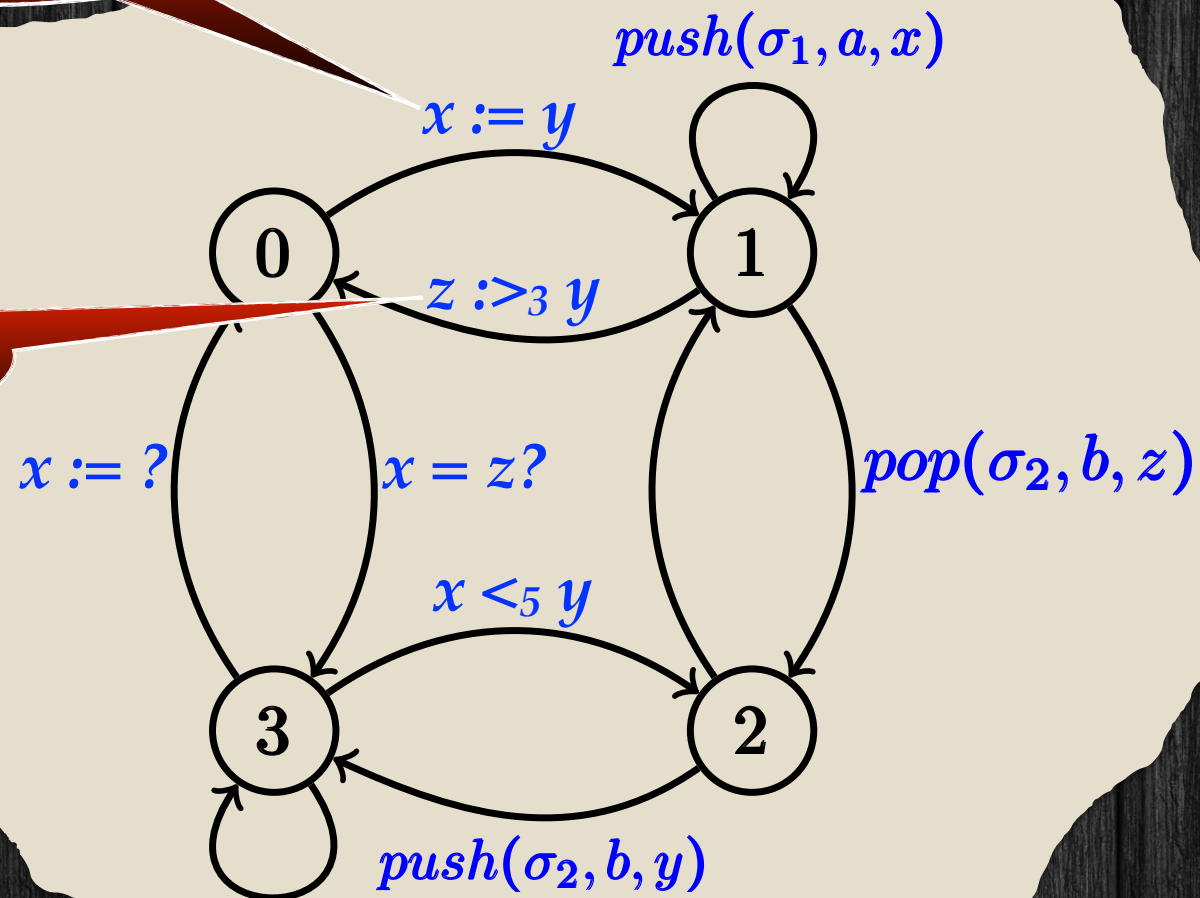
$v(x)$: current
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x, y, z

σ_1, σ_2

“assign $v(y)$ to x ”

“assign some
 $u > v(y) + 3$ to z ”



Model

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finite set of
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$v(x)$: current
value of x

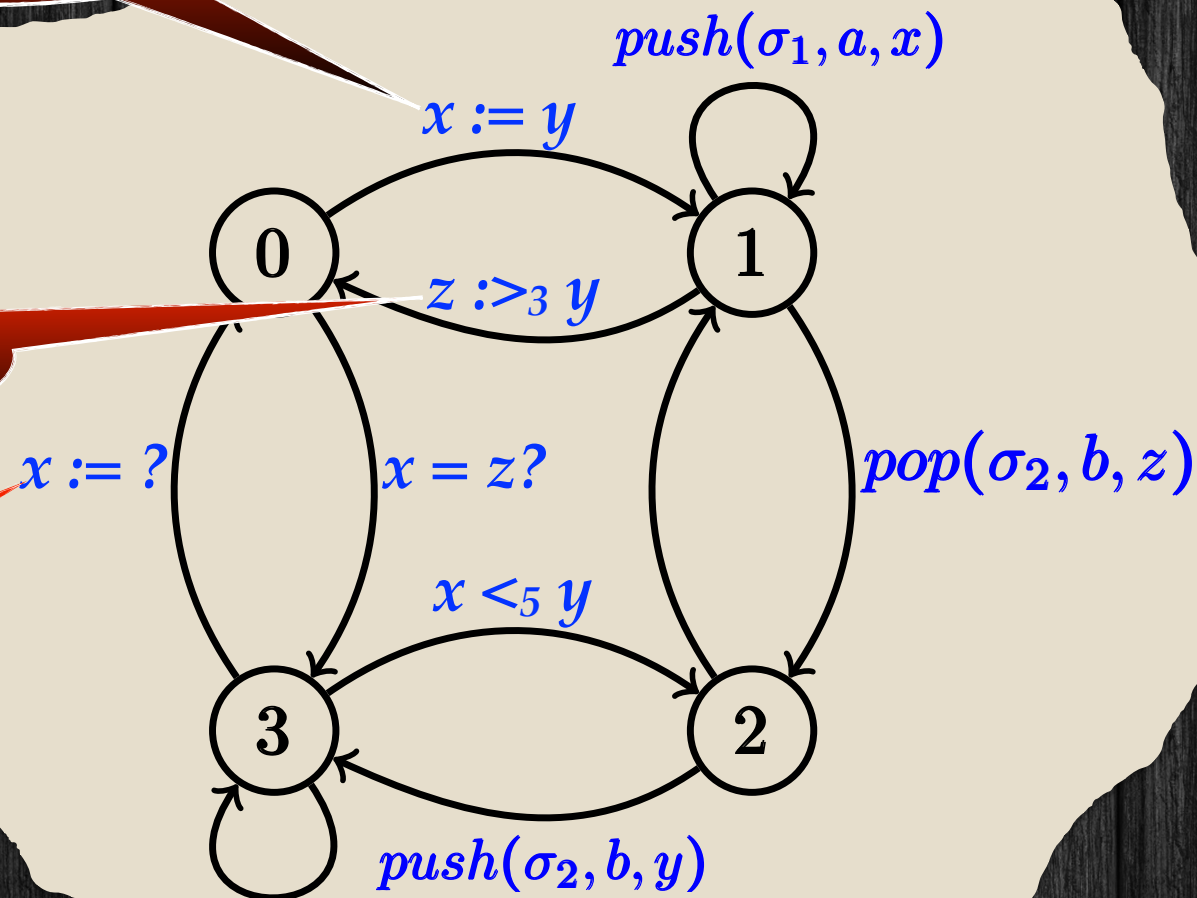
x, y, z

σ_1, σ_2

“assign $v(y)$ to x ”

“assign some
 $u > v(y) + 3$ to z ”

“assign some
 u to y ”



Model

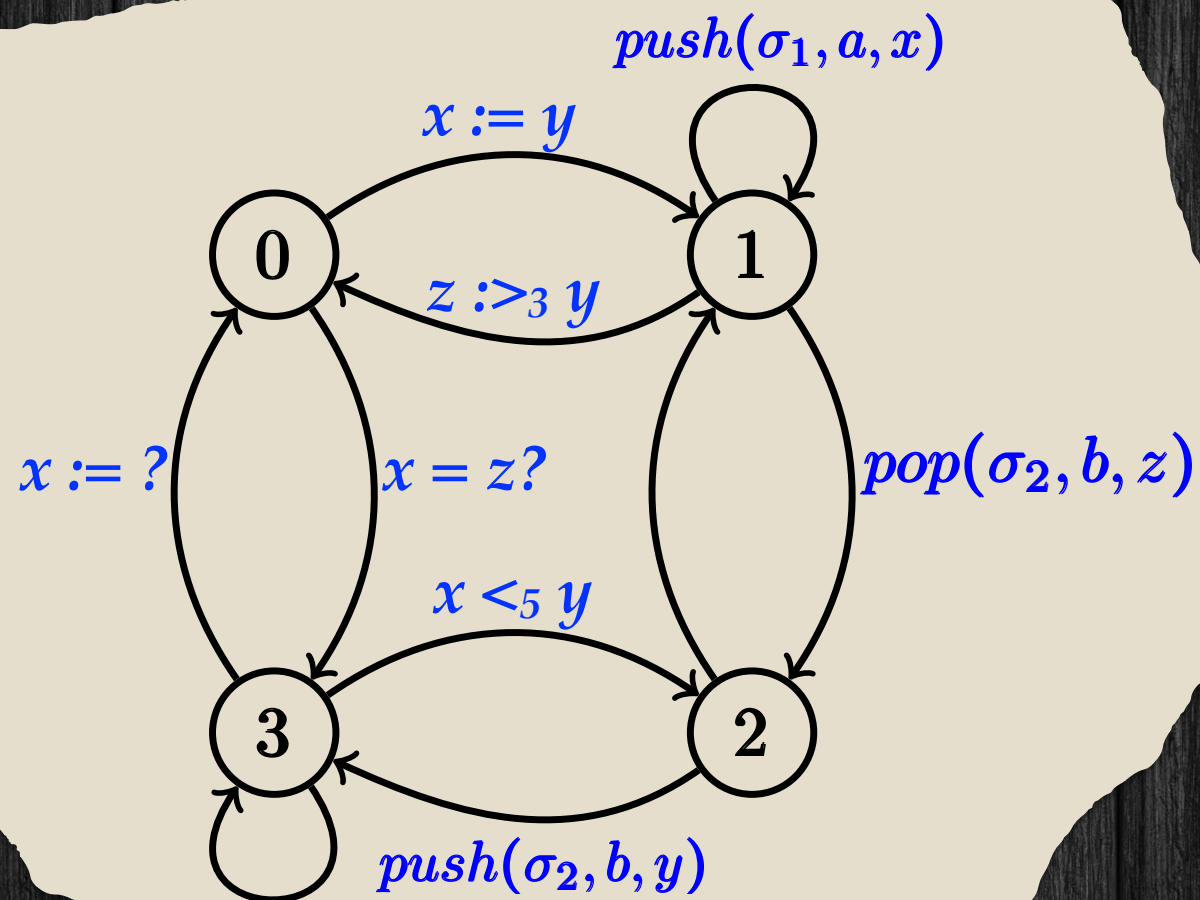
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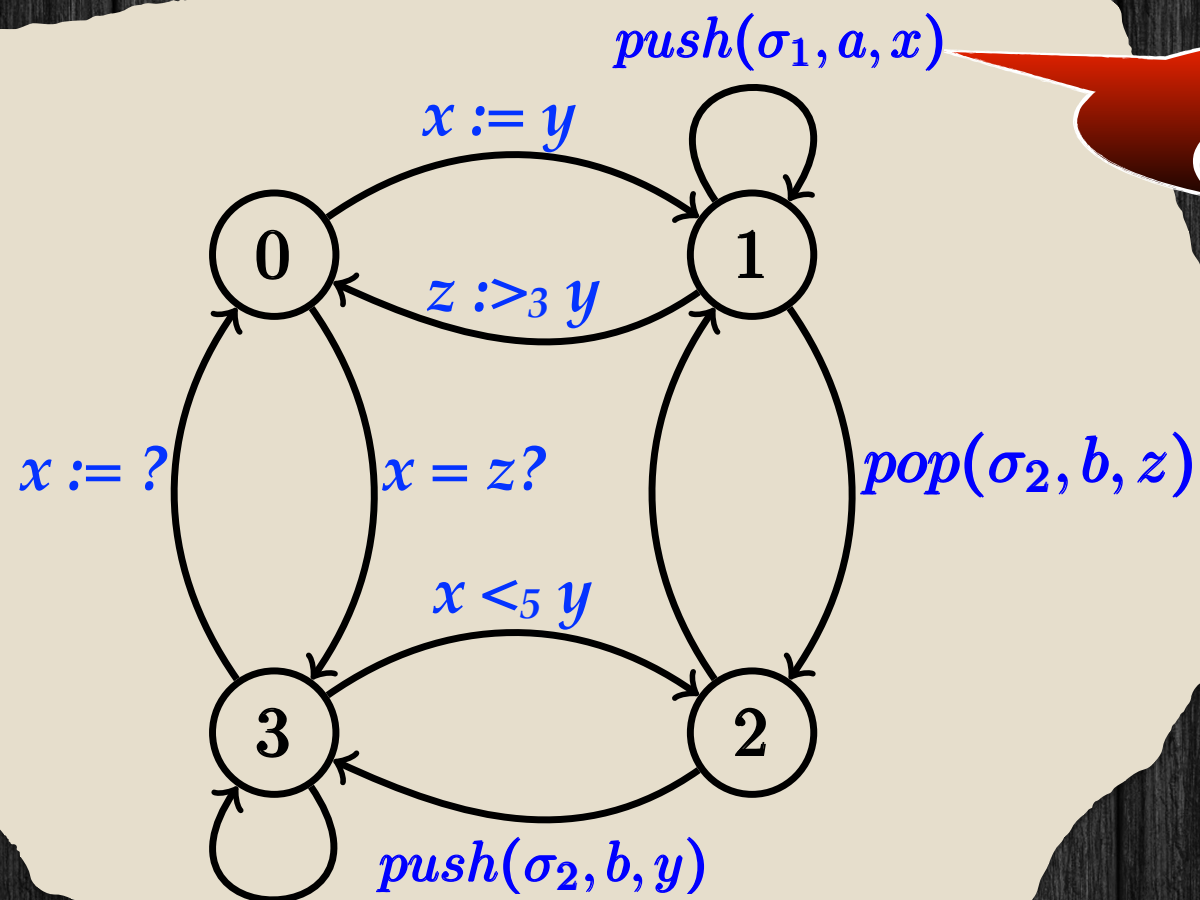
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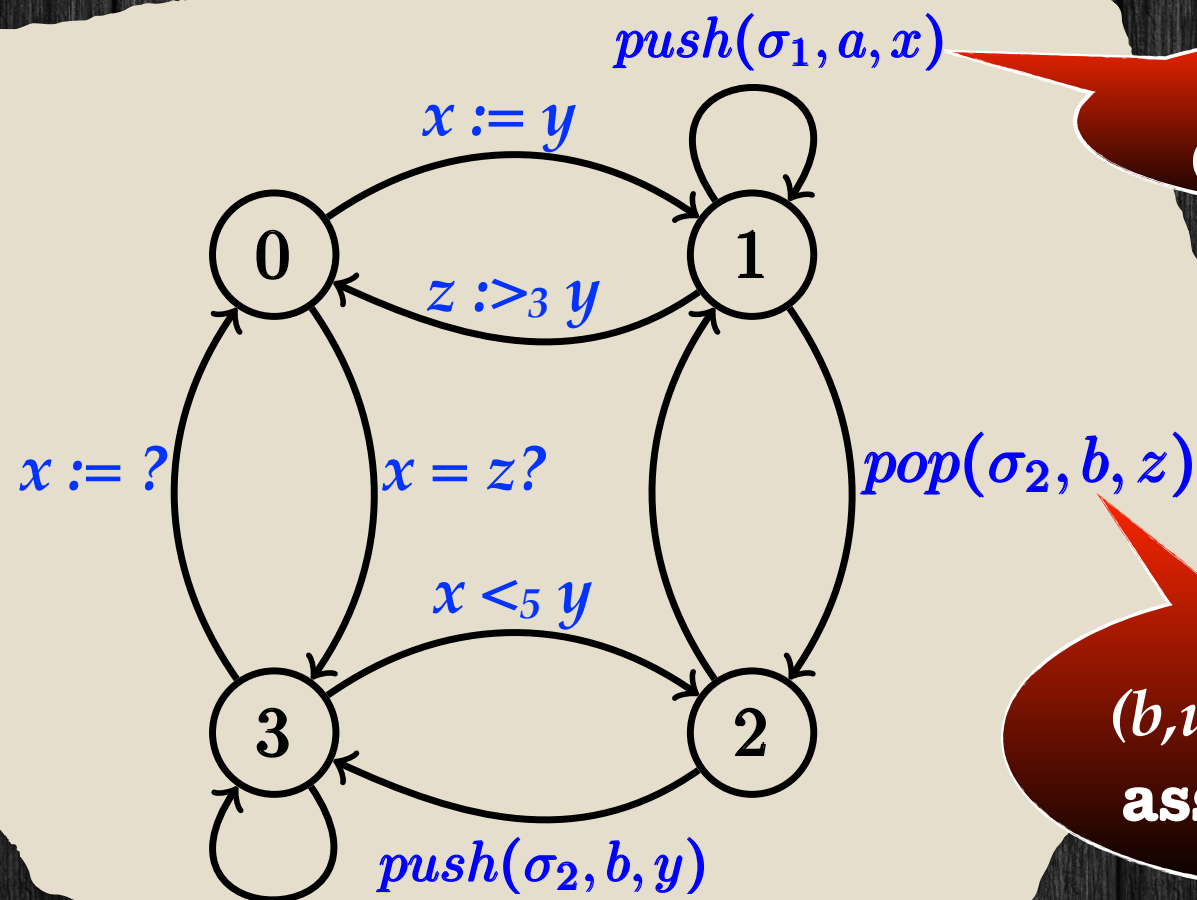
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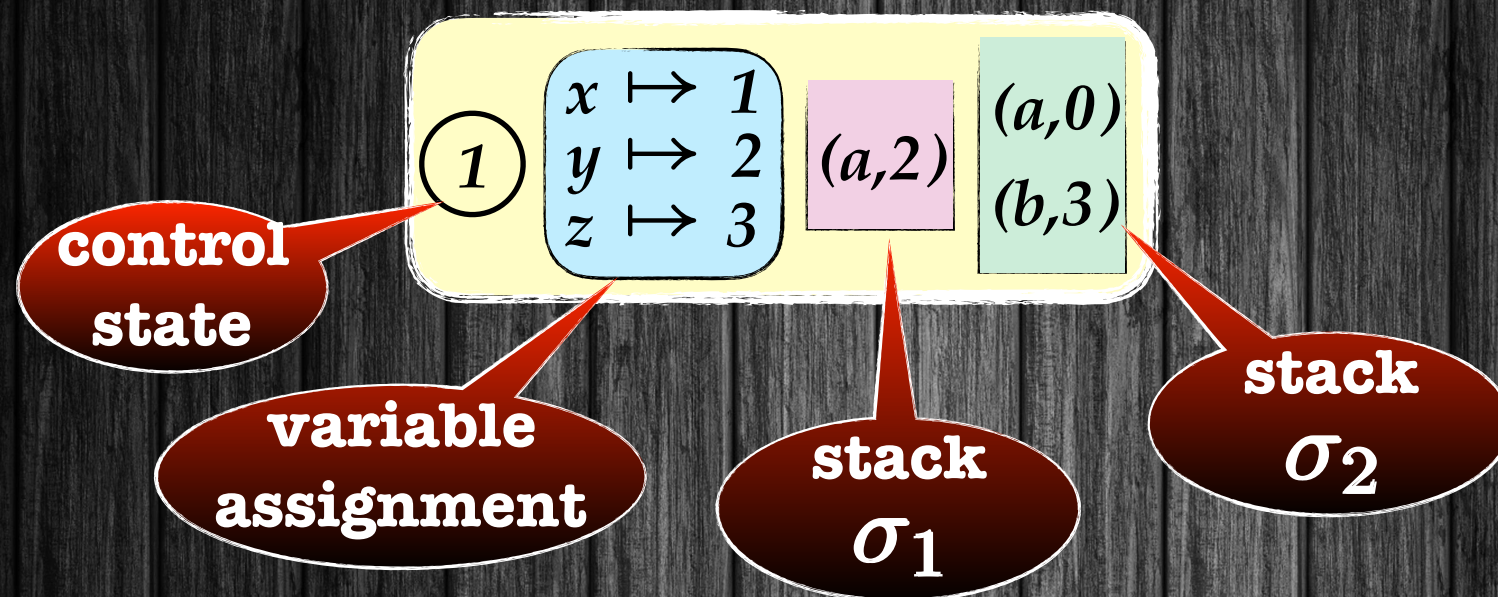
σ_1, σ_2



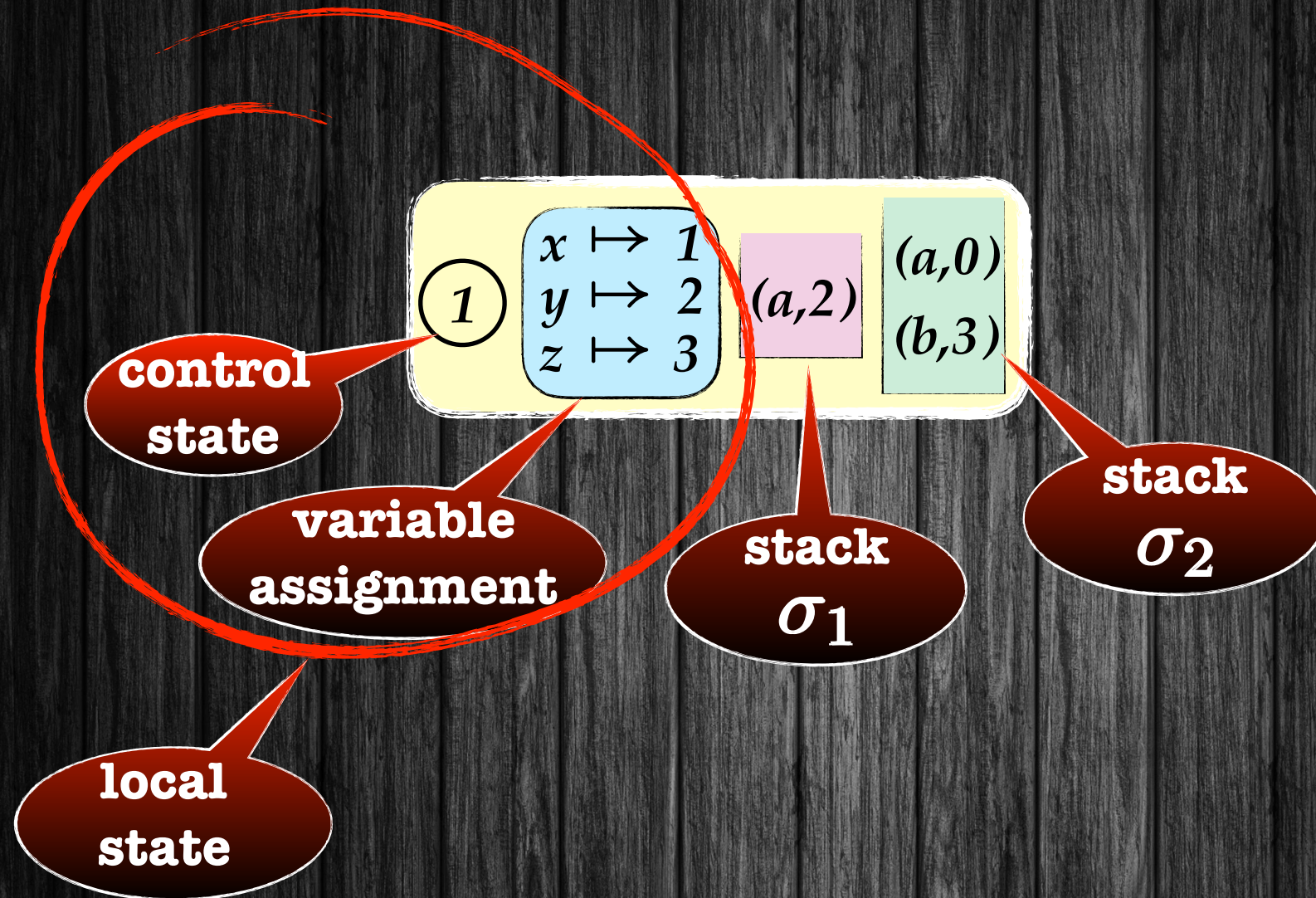
“push
($a, v(x)$) to σ_1 ”

“pop
(b, u) from σ_2 ,
assign u to z ”

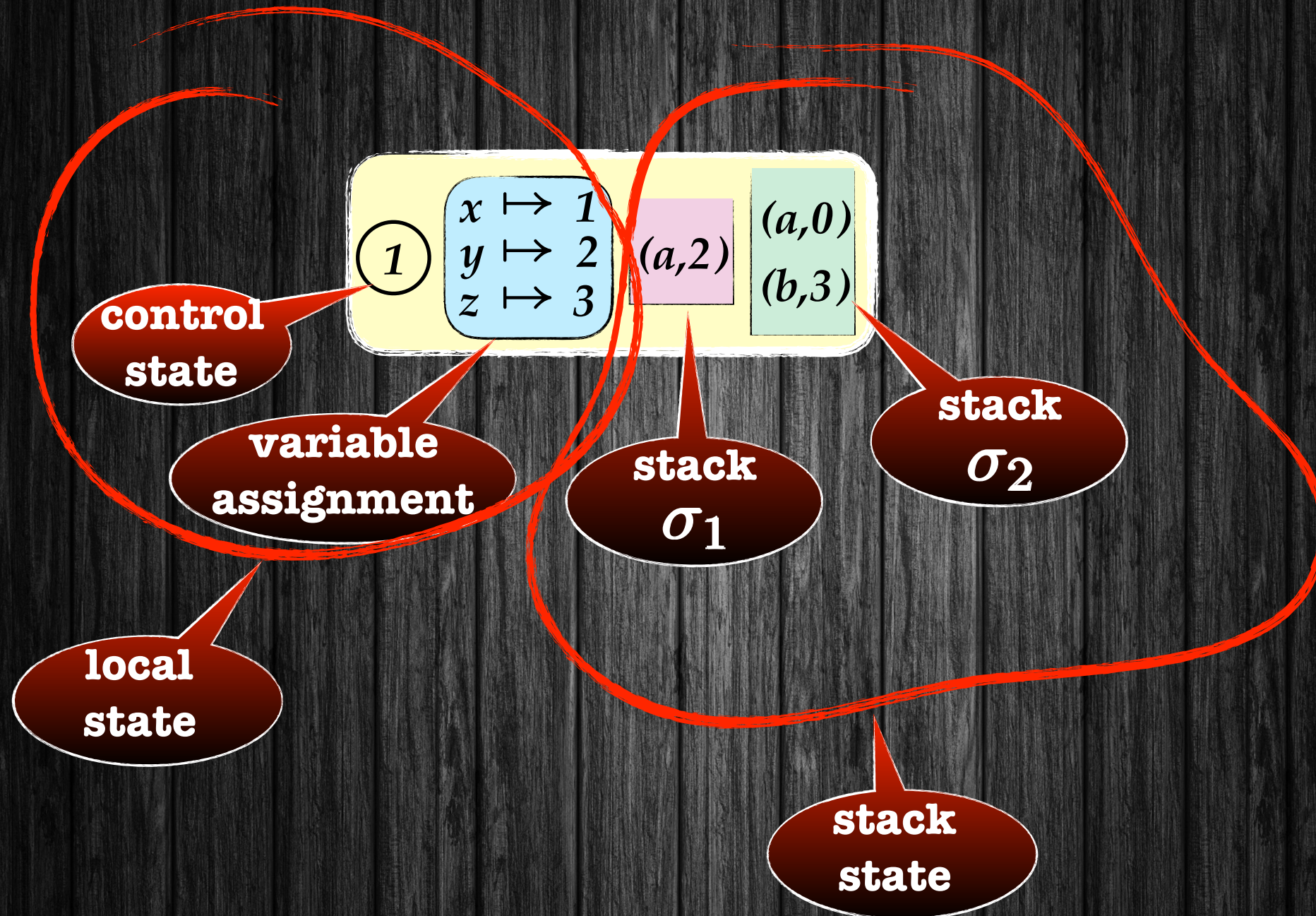
Configurations



Configurations



Configurations



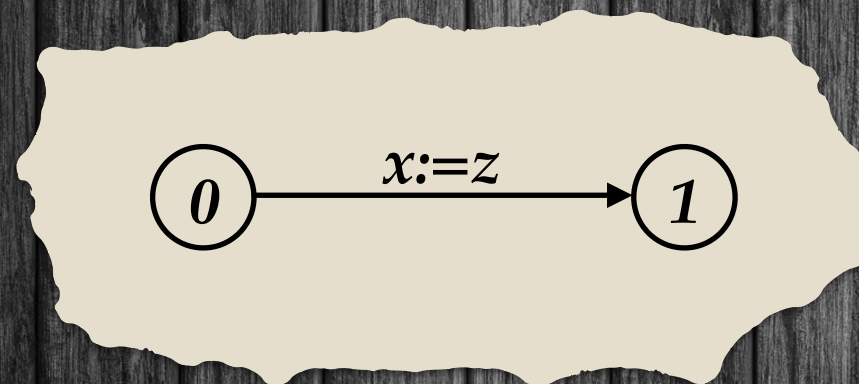
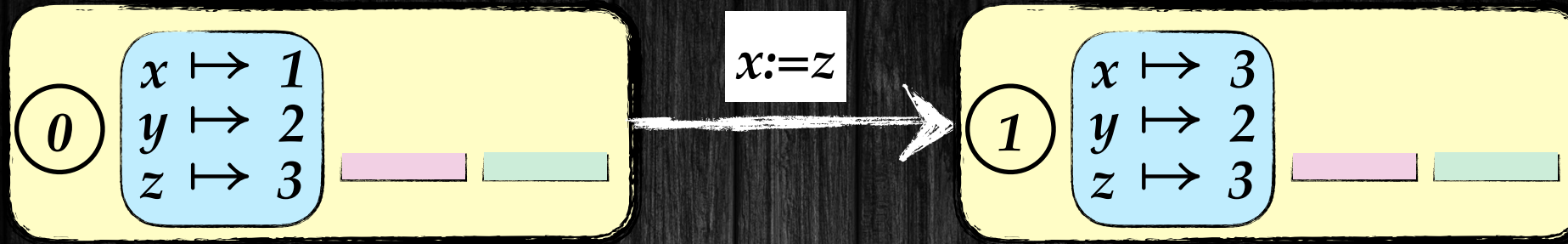
Runs

0

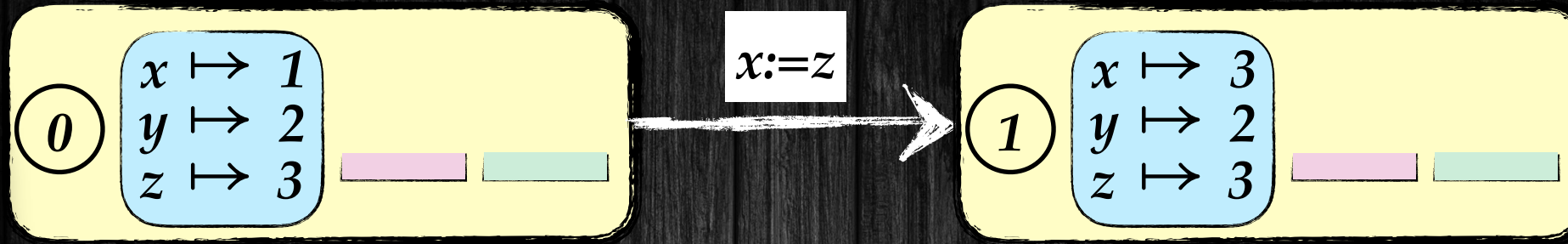
$x \mapsto 1$
 $y \mapsto 2$
 $z \mapsto 3$



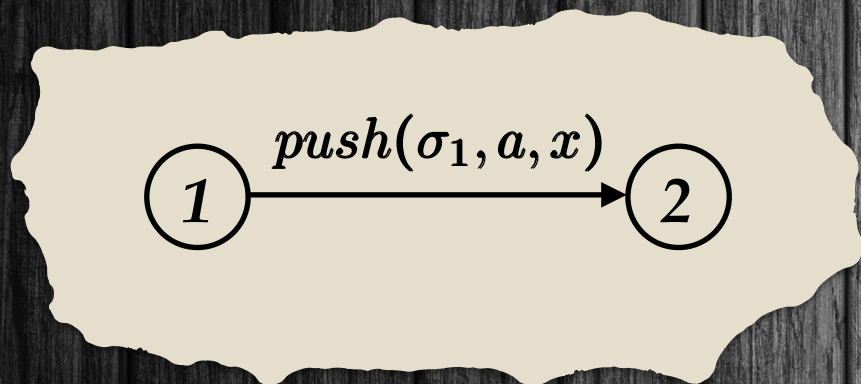
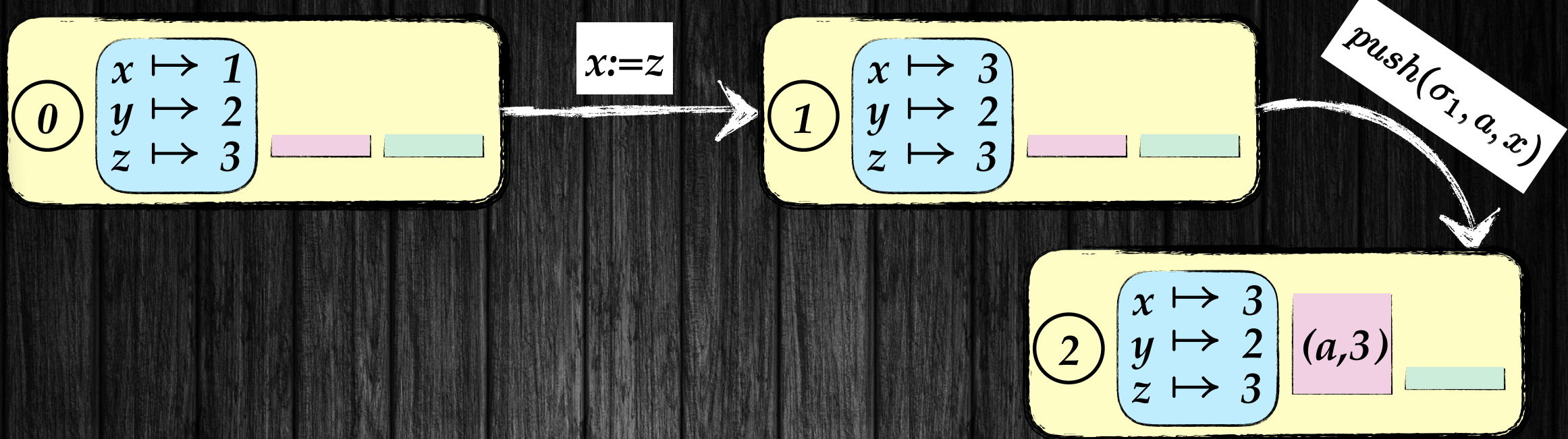
Runs



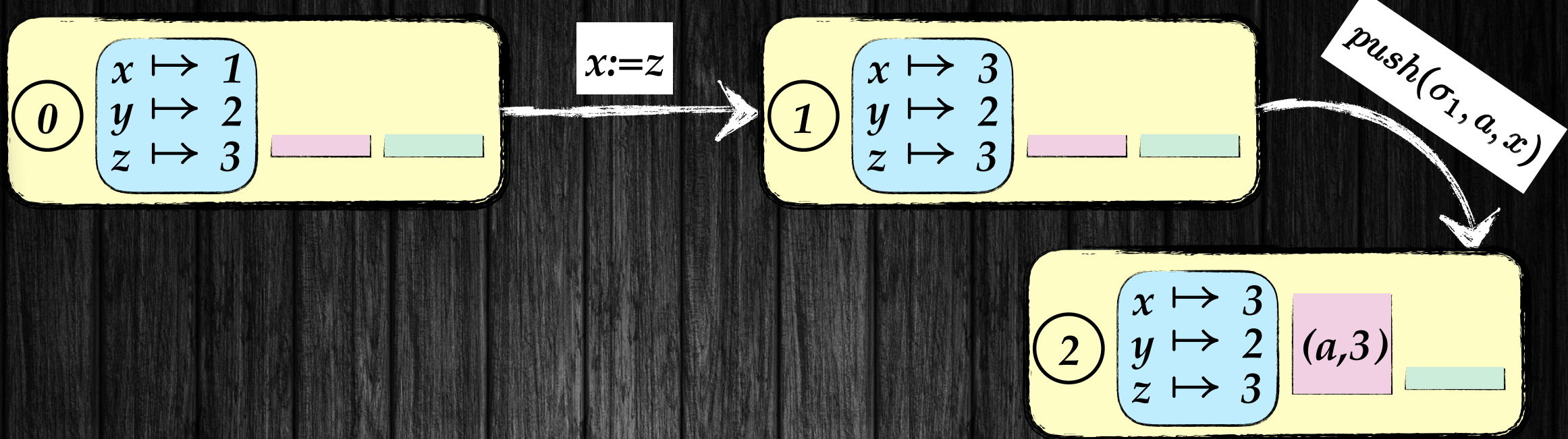
Runs



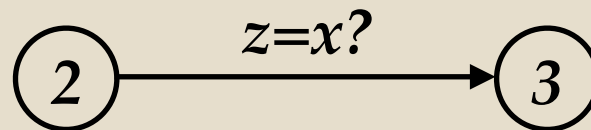
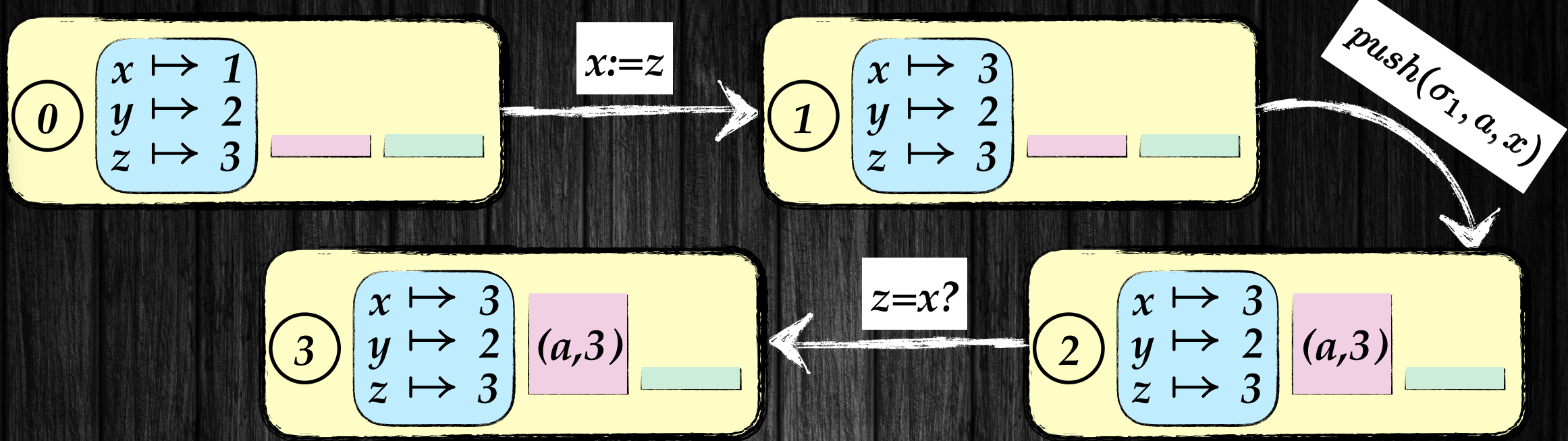
Runs



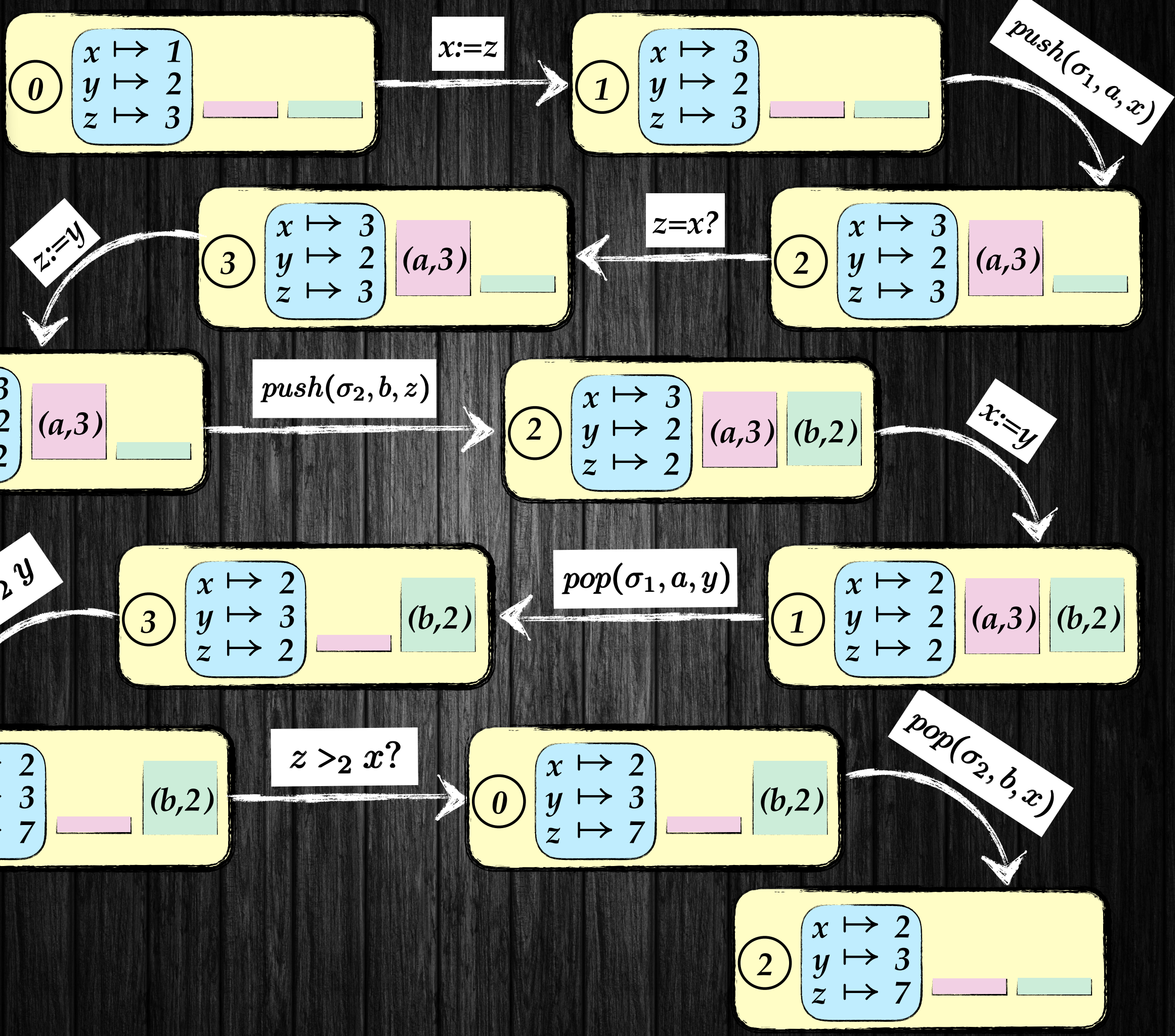
Runs



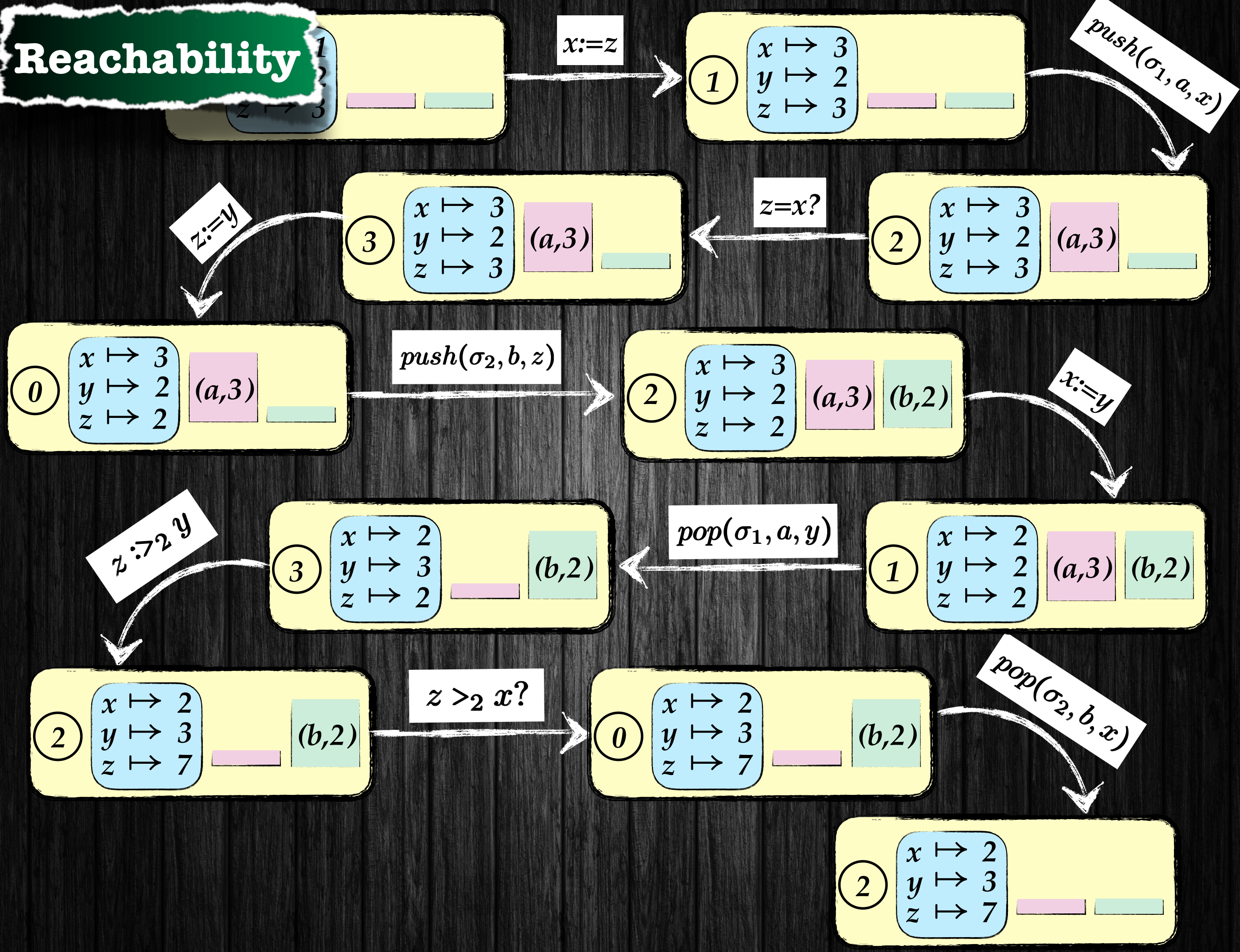
Runs



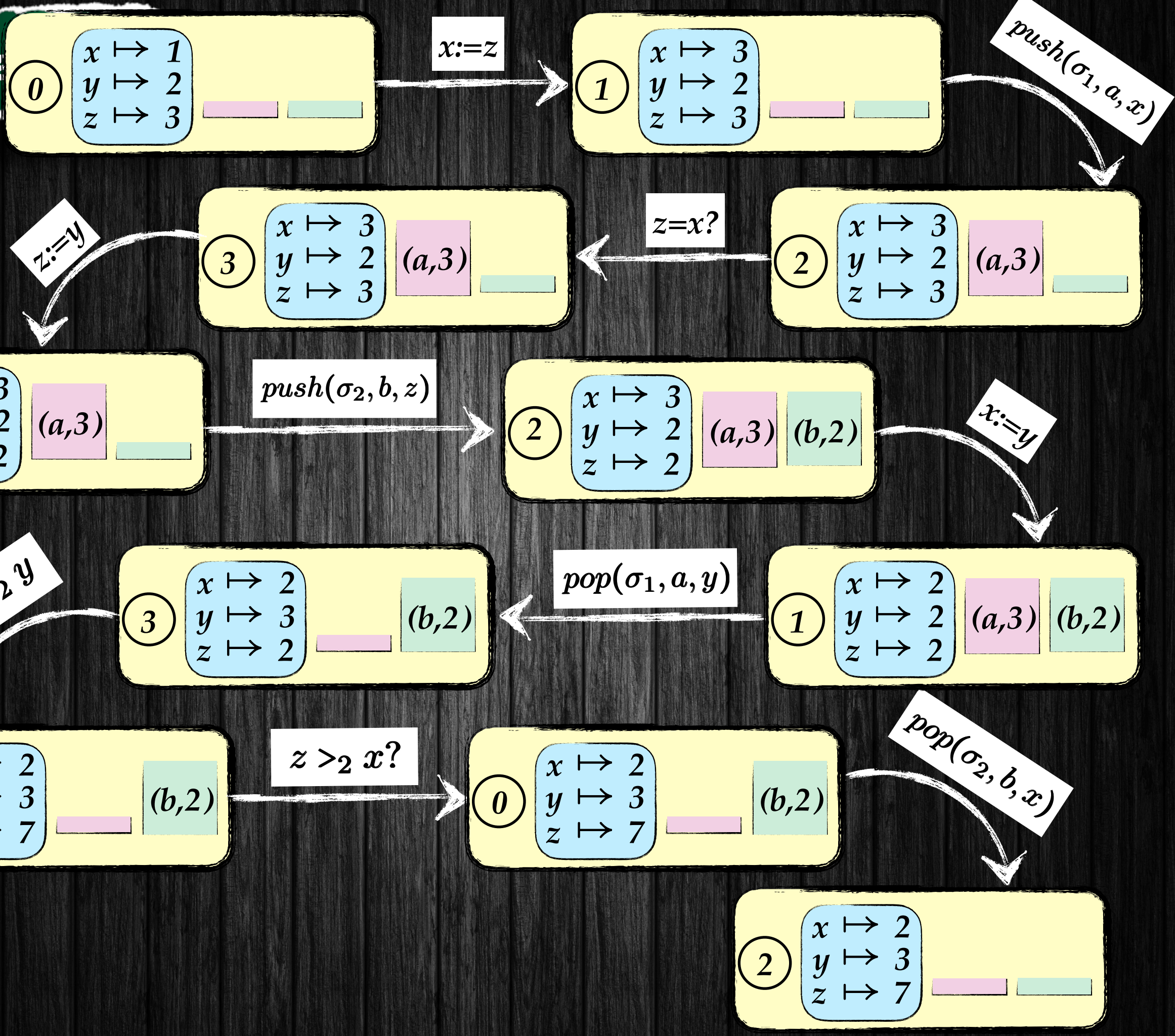
Runs



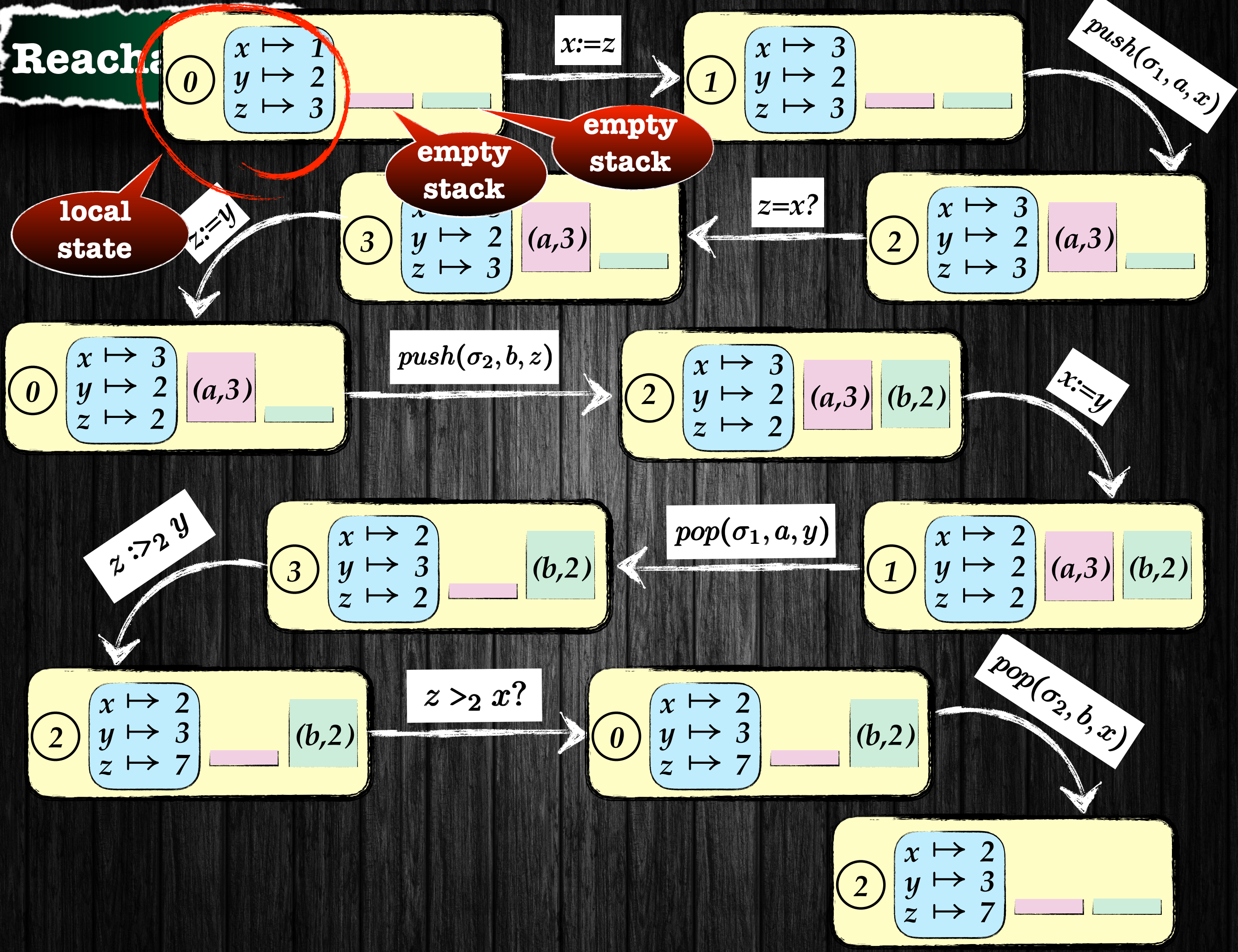
Reachability



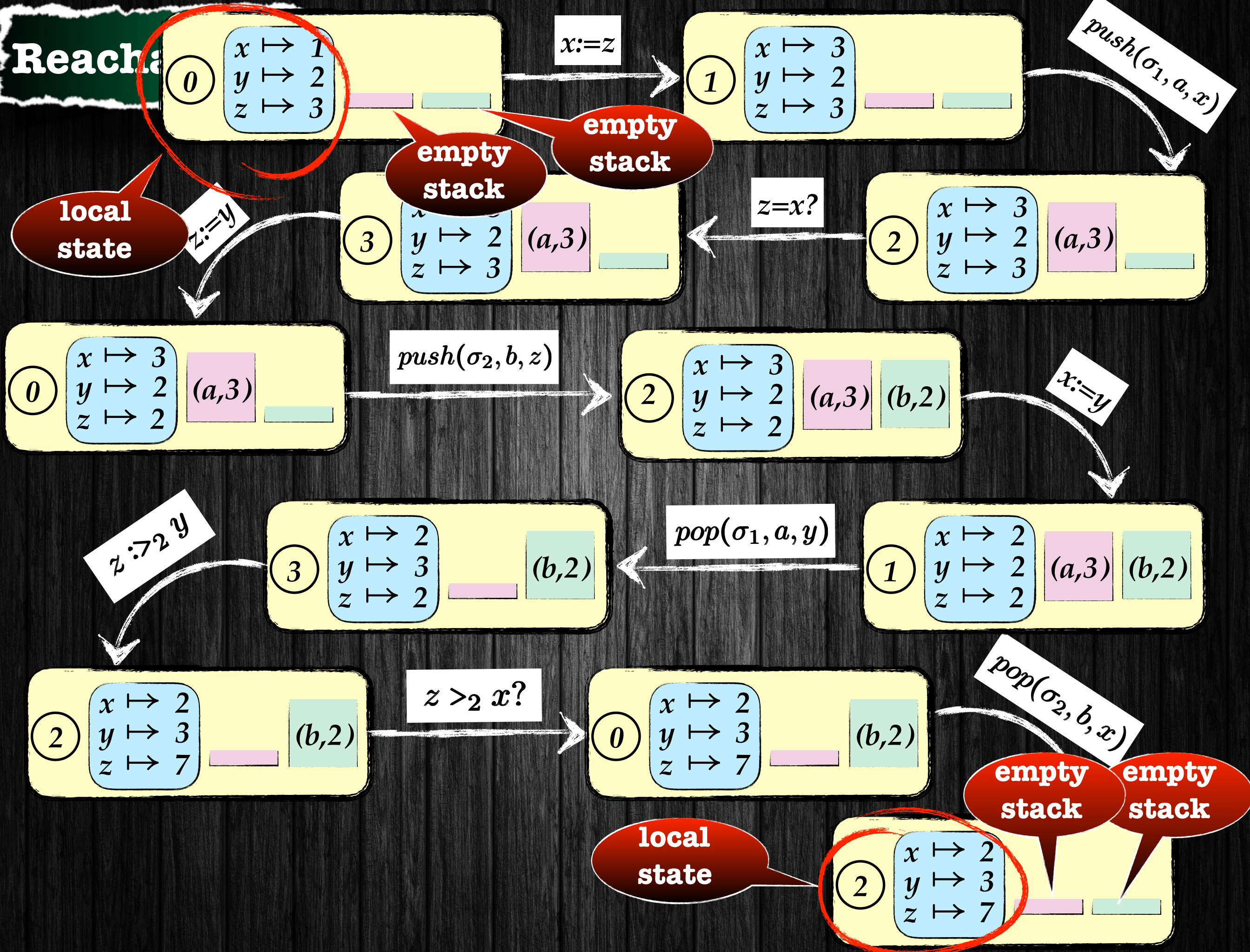
Reaches



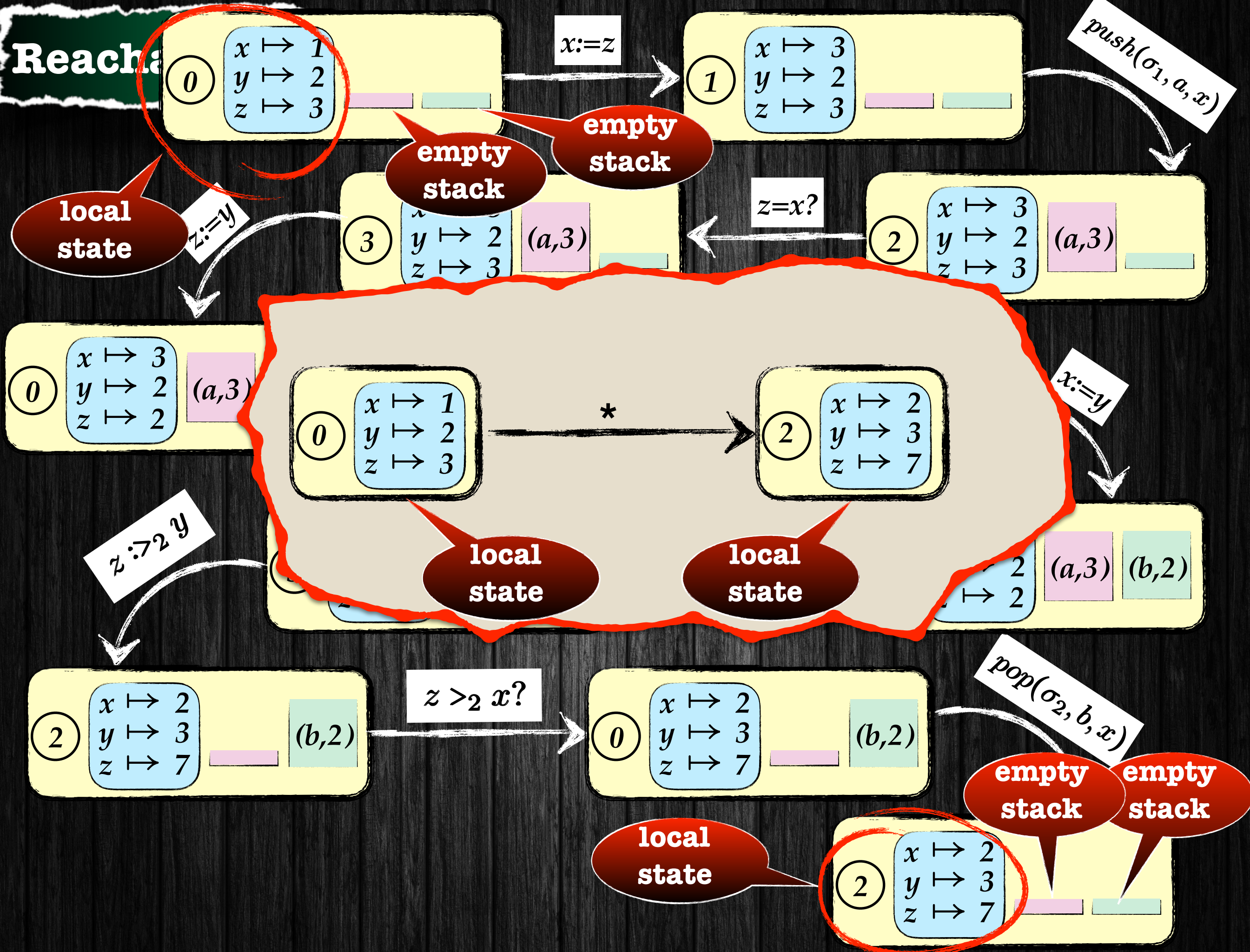
Reachable



Reached

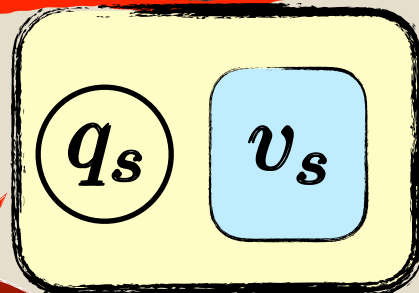


Reachable



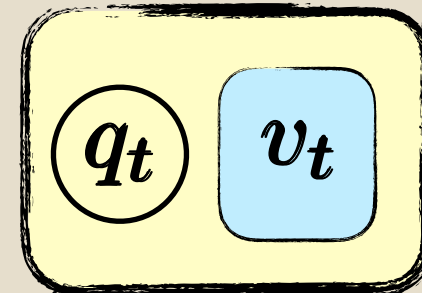
Reachability Problem

Instance:



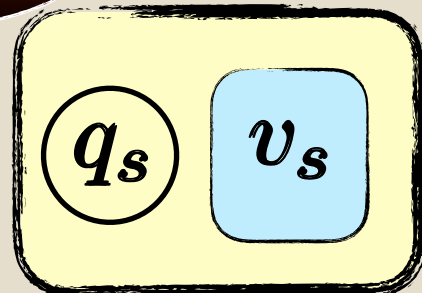
local
state

and

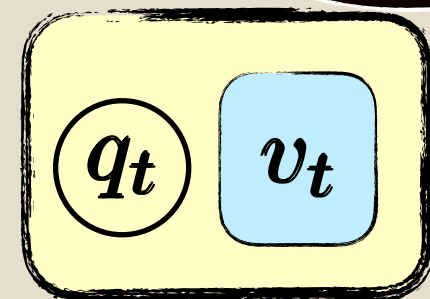


local
state

Question:

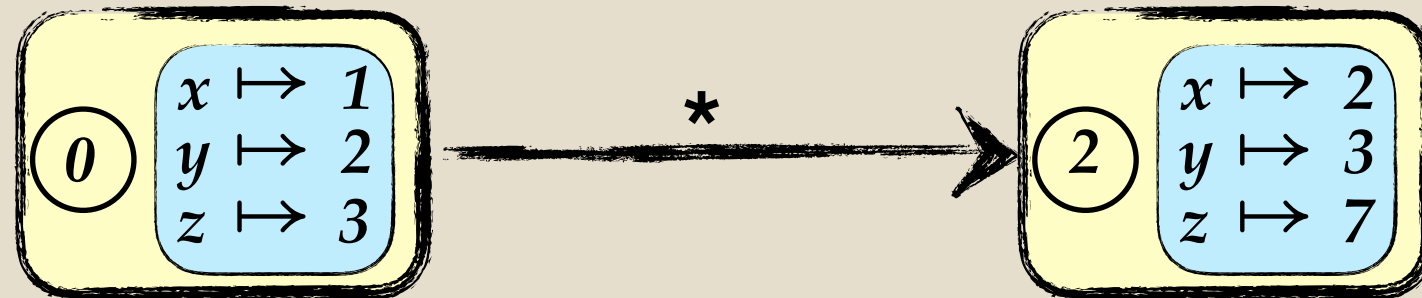


*

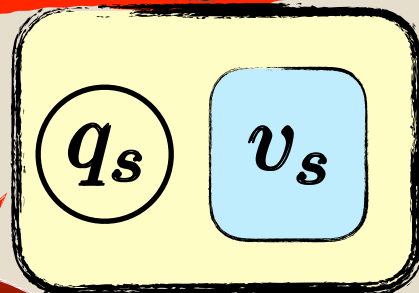


?

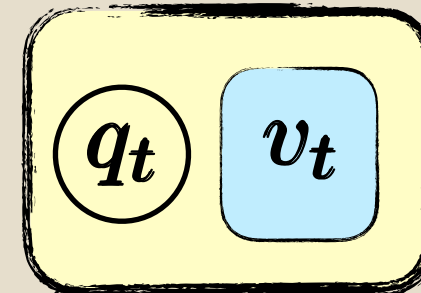
Reachability Problem



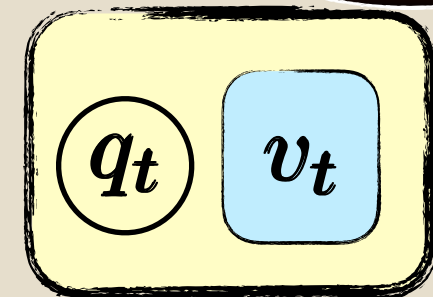
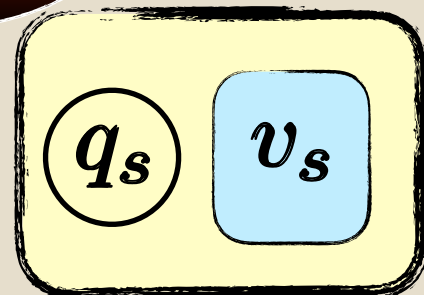
Instance:



and



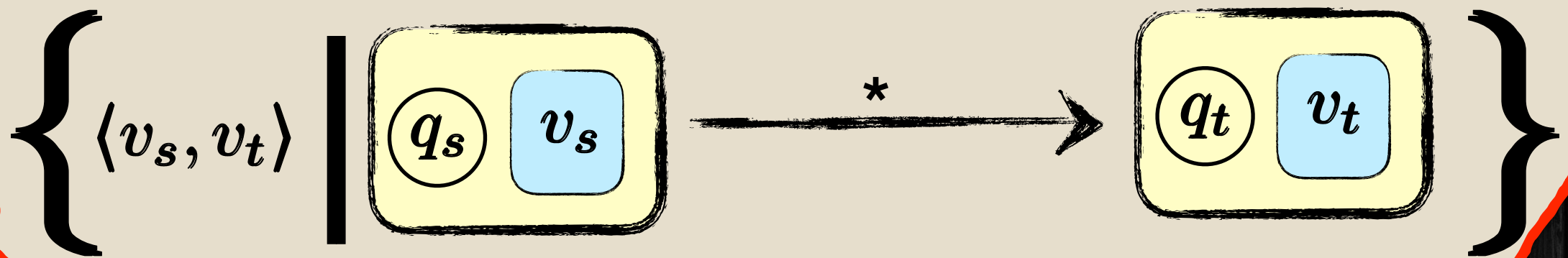
Question:



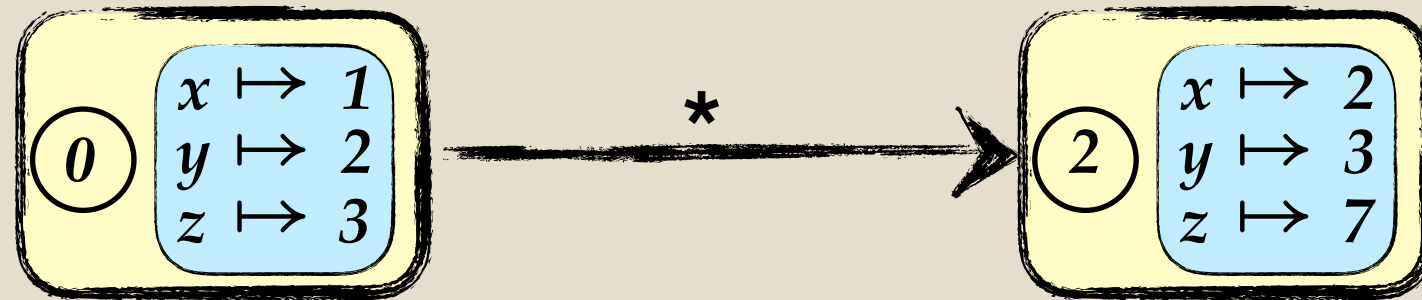
?

Reachability Relations

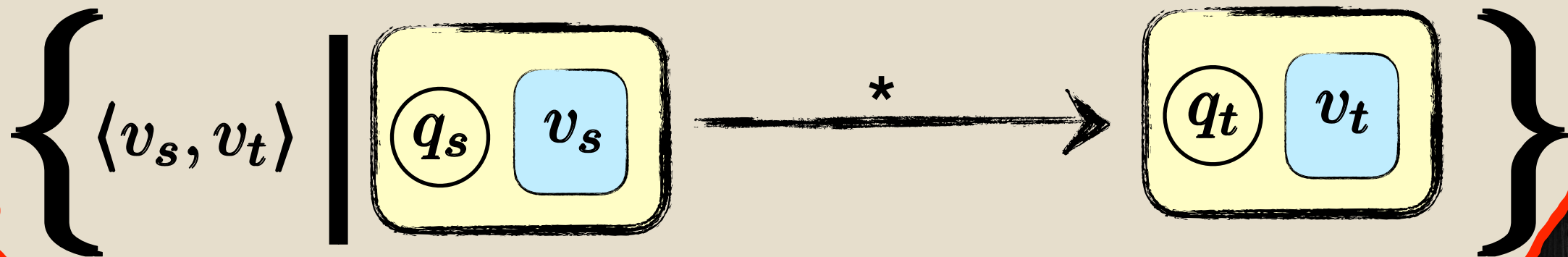
$[[q_s, q_t]] =$



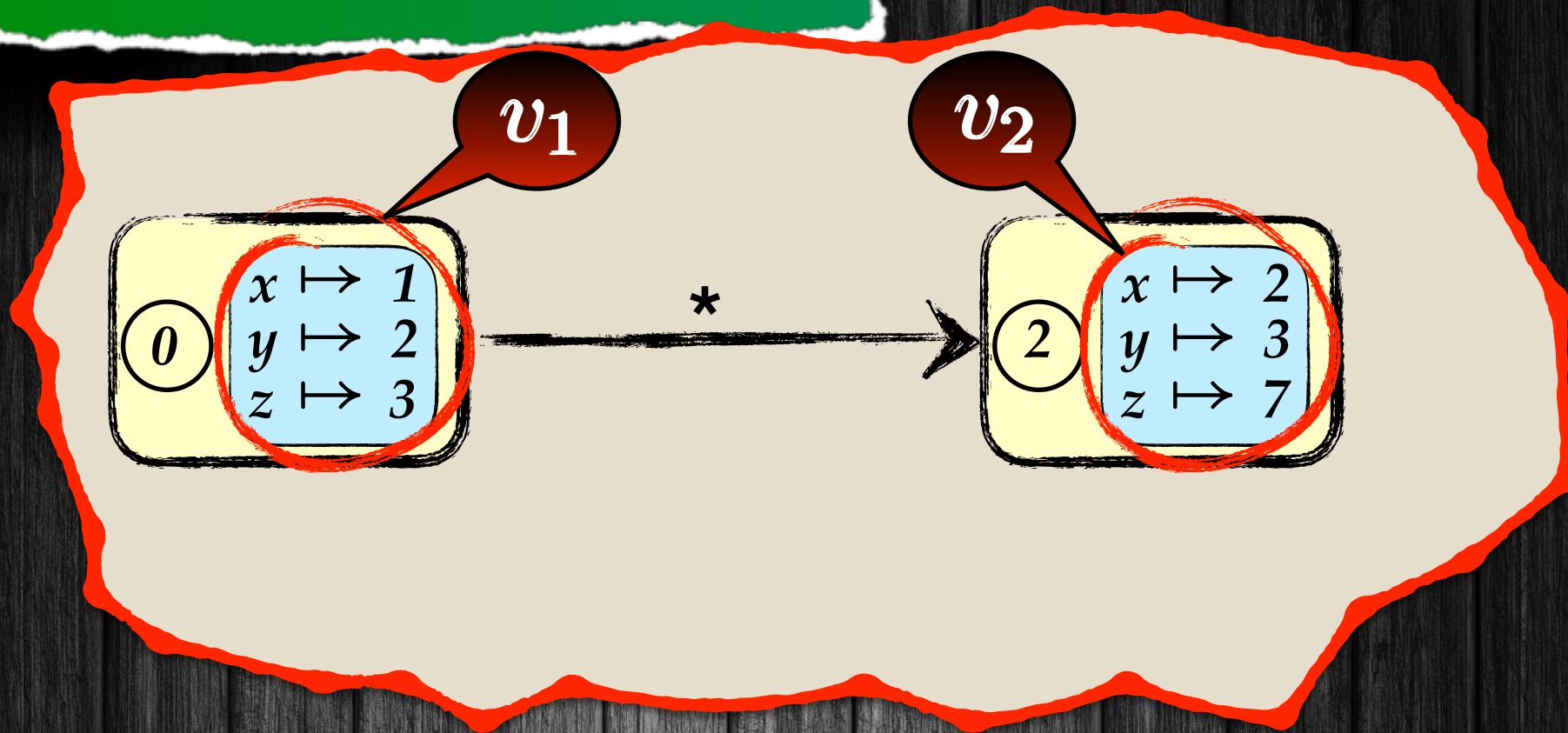
Reachability Relations



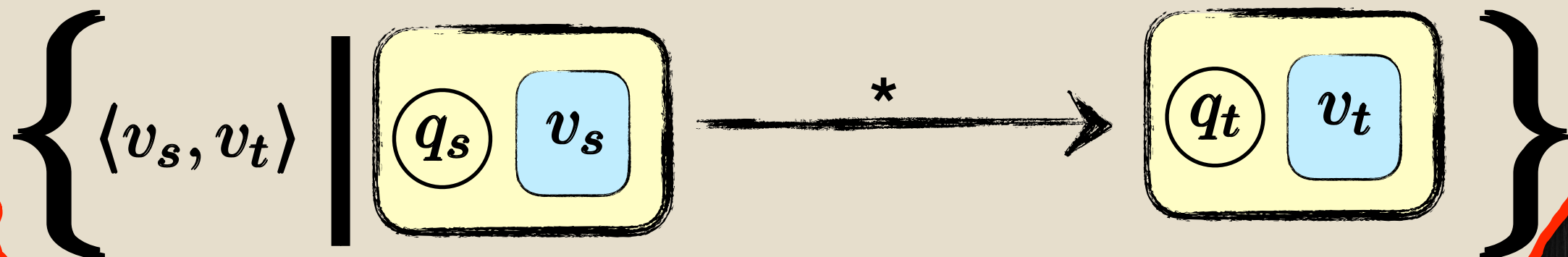
$\llbracket q_s, q_t \rrbracket =$



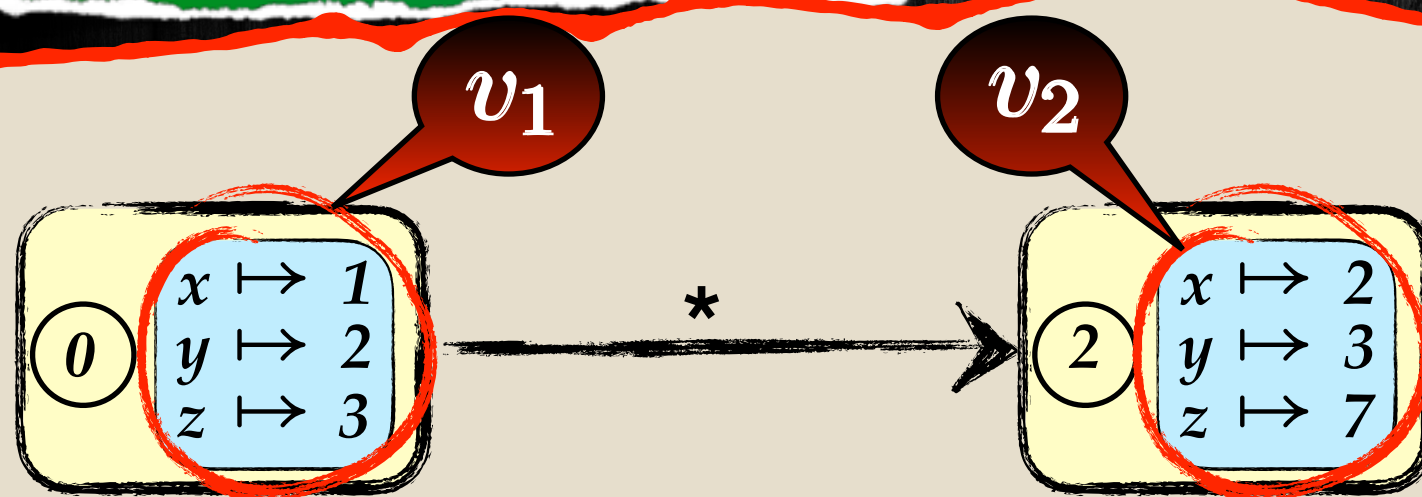
Reachability Relations



$\llbracket q_s, q_t \rrbracket =$

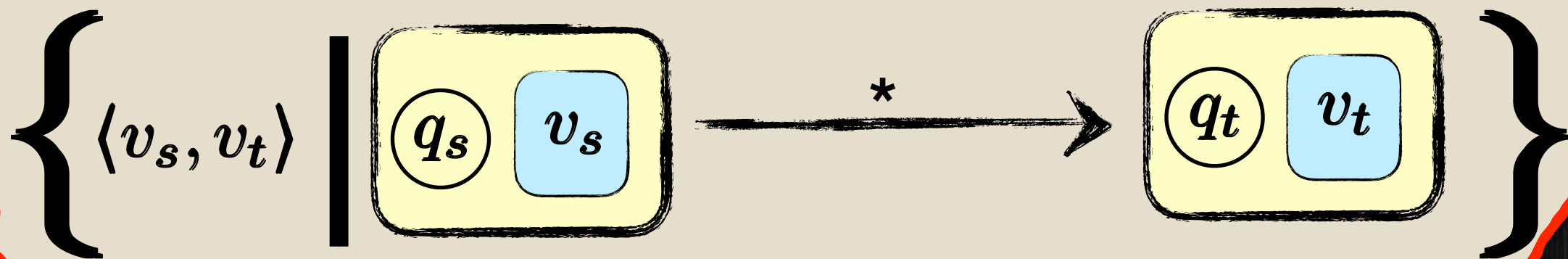


Reachability Relations



$$\langle v_1, v_2 \rangle \in \llbracket 0, 2 \rrbracket$$

$$\llbracket q_s, q_t \rrbracket =$$



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- **Syntax**
- **Saturation**
- **Quantifier elimination**
- **Entailment**

Gap-Order Constraints

finite set of
variables

x, y, z, u

Gap-Order Constraints

finite set of
variables

x, y, z, u

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 u$$

Gap-Order Constraints

finite set of
variables

x, y, z, u

conjunction of
constraints

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 u$$

ϕ

Gap-Order Constraints

finite set of
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conjunction of
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equality

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ϕ

Gap-Order Constraints

finite set of
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gap

$$y <_2 z$$

$$z <_3 u$$

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ϕ

Gap-Order Constraints

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conjunction of
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$$x = y$$

gap

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 u$$

ϕ

$$v_1 : \begin{array}{l} x \mapsto 2 \\ y \mapsto 2 \\ z \mapsto 6 \\ u \mapsto 10 \end{array}$$

$$v_1 \in \llbracket \phi \rrbracket$$

$$v_2 : \begin{array}{l} x \mapsto 1 \\ y \mapsto 2 \\ z \mapsto 6 \\ u \mapsto 10 \end{array}$$

$$v_2 \notin \llbracket \phi \rrbracket$$

$$v_3 : \begin{array}{l} x \mapsto 2 \\ y \mapsto 2 \\ z \mapsto 3 \\ u \mapsto 10 \end{array}$$

$$v_3 \notin \llbracket \phi \rrbracket$$

Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 u$$



Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 u$$

$$x <_2 z$$

Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

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Gap-Order Constraints Saturation

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Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 u$$

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$$y <_6 u$$

$$x <_6 u$$

Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

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$$x <_2 u$$

$$x <_2 z$$

$$y <_6 u$$

$$x <_6 u$$

Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

~~$$x <_2 u$$~~

$$x <_2 z$$

$$y <_6 u$$

$$x <_6 u$$

Gap-Order Constraints Saturation

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

~~$$x <_2 u$$~~

$$x <_2 z$$

$$y <_6 u$$

$$x <_6 u$$

Gap-Order Constraints Saturation

$x = y$
 $y <_2 z$
 $z <_3 u$
 ~~$x <_2 u$~~

+

$x <_2 z$
 $y <_6 u$
 $x <_6 u$



$x = y$
 $y <_2 z$
 $z <_3 u$
 $x <_2 z$
 $y <_6 u$
 $x <_6 u$

saturated

Gap-Order Quantifier Elimination

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 z$$

$$y <_6 u$$

$$x <_6 u$$

saturated

Gap-Order Quantifier Elimination

$\exists y.$

$$x = y$$

$$y <_2 z$$

$$z <_3 u$$

$$x <_2 z$$

$$y <_6 u$$

$$x <_6 u$$

saturated

Gap-Order Quantifier Elimination

$\exists y.$

~~$x =_1 y$~~

~~$y =_2 z$~~

$z <_3 u$

$x <_2 z$

~~$y <_6 u$~~

$x <_6 u$

saturated

Gap-Order Quantifier Elimination

$\exists y.$

~~$x <_1 y$~~

~~$y <_2 z$~~

$z <_3 u$

$x <_2 z$

~~$y <_6 u$~~

$x <_6 u$

$=$

$z <_3 u$

$x <_2 z$

$x <_6 u$

saturated

Gap-Order Constraints Entailment

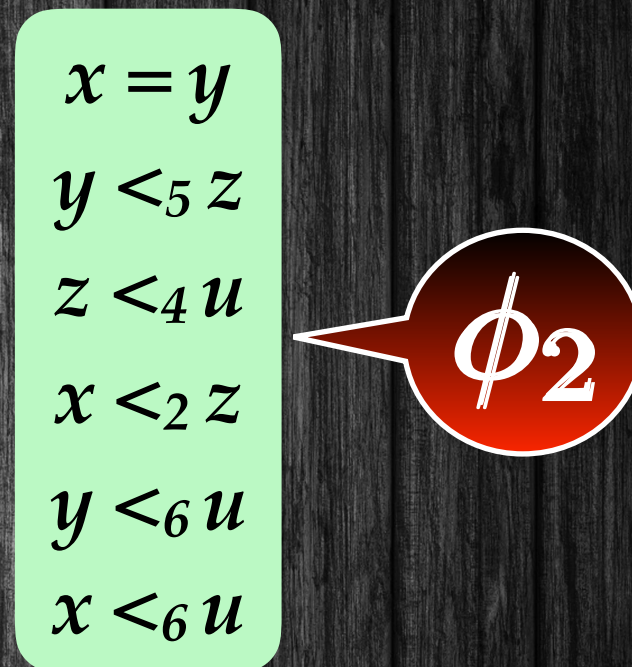
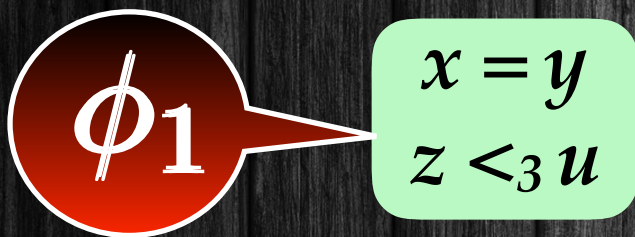
ϕ_1

$x = y$
 $z <_3 u$

$x = y$
 $y <_5 z$
 $z <_4 u$
 $x <_2 z$
 $y <_6 u$
 $x <_6 u$

ϕ_2

Gap-Order Constraints Entailment



$$\phi_1 \sqsubseteq \phi_2$$

Gap-Order Constraints Entailment

ϕ_1

$x = y$
 $z <_3 u$

$x = y$

$y <_5 z$

$z <_4 u$

$x <_2 z$

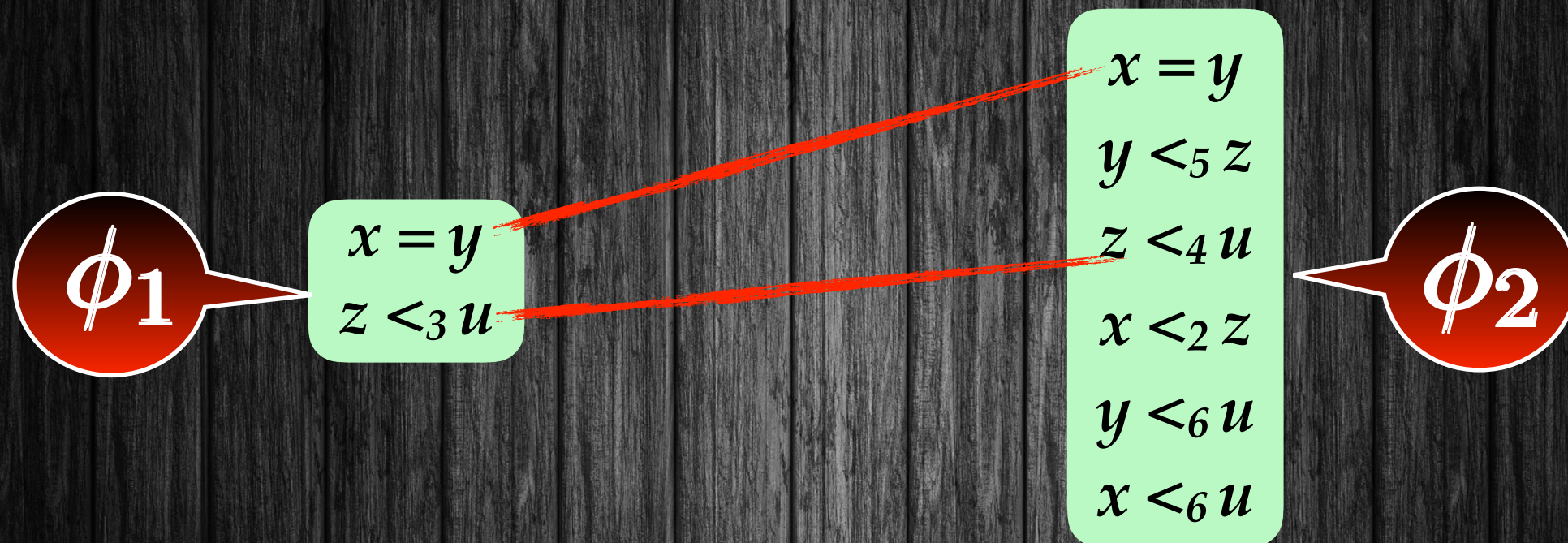
$y <_6 u$

$x <_6 u$

ϕ_2

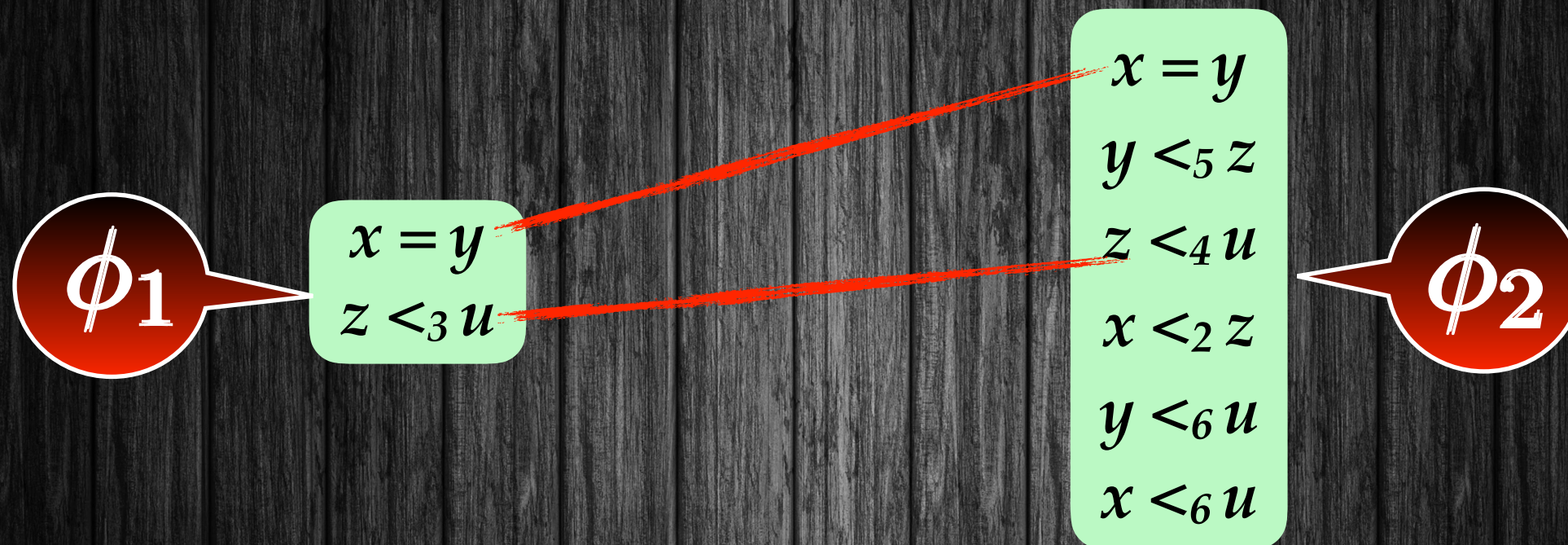
$\phi_1 \sqsubseteq \phi_2$

Gap-Order Constraints Entailment



$$\phi_1 \sqsubseteq \phi_2$$

Gap-Order Constraints Entailment



$$\phi_1 \sqsubseteq \phi_2$$

$$[\phi_2] \subseteq [\phi_1]$$

Outline

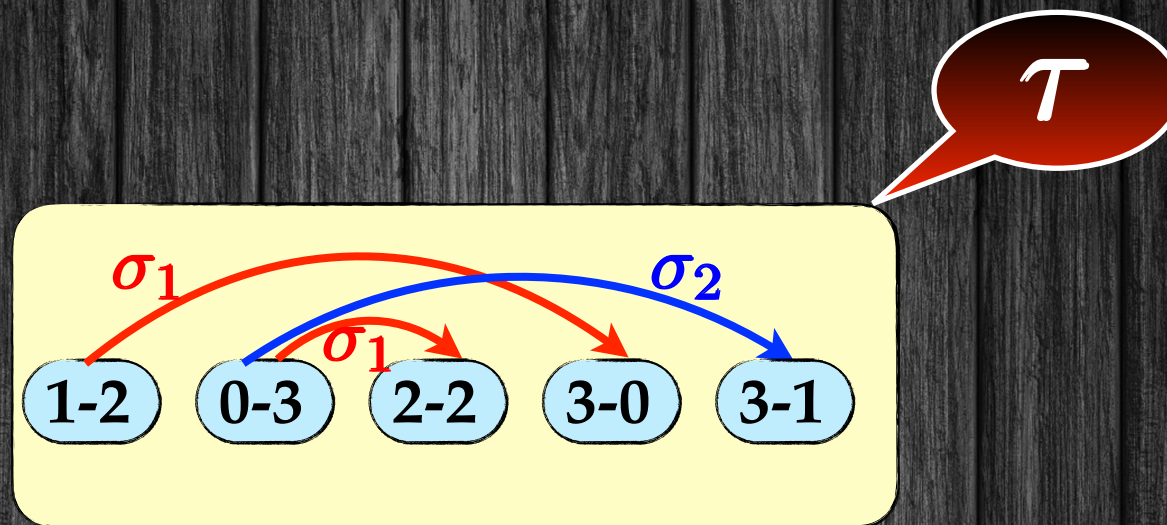
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- **Applications**

Outline

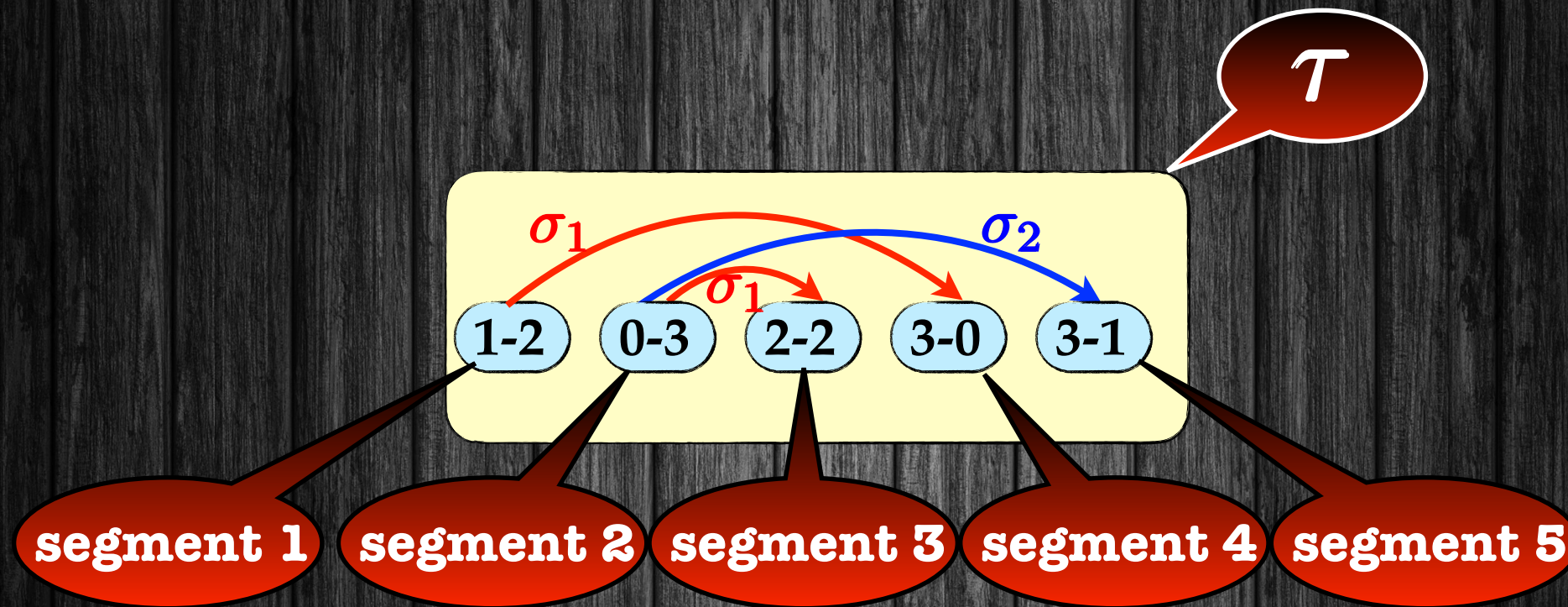
- **Background**
- **Model**
- **Gap-Order Constraints**
- **Signatures**
- **Reachability Algorithm**
- **Applications**

- **Syntax**
- **Entailment**
- **Atomic signatures**
 - **Internal transitions**
 - **Stack transitions**
- **Composition**
 - **Shuffling**
 - **Contraction**
- **Monotonicity**

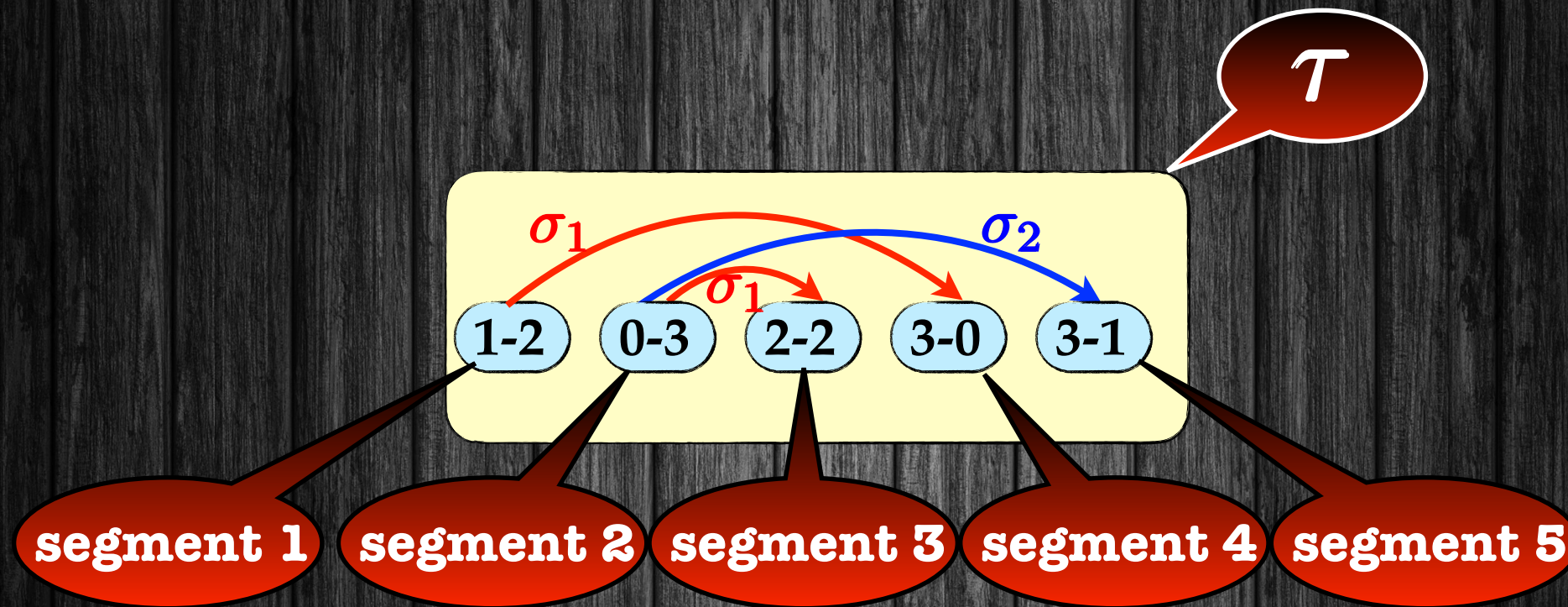
Signatures



Signatures



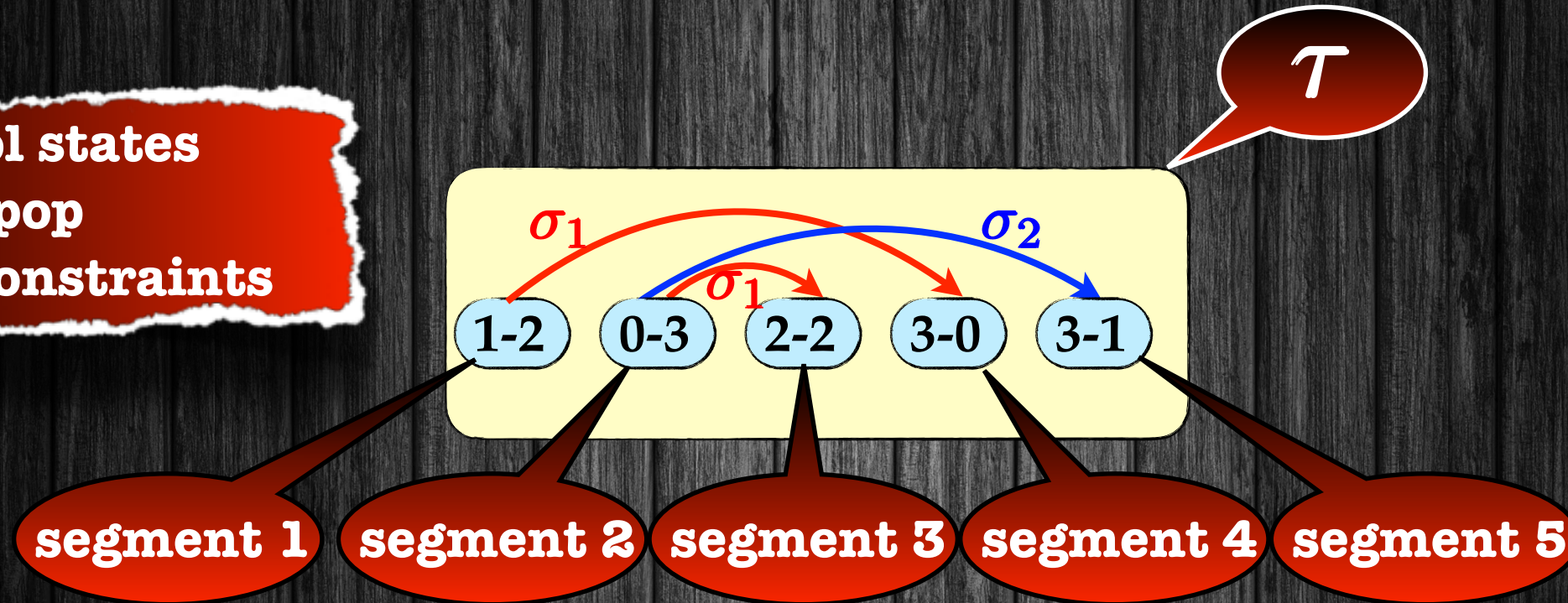
Signatures



- segment: "part of run"
- signature: parts of runs with "holes" in between

Signatures

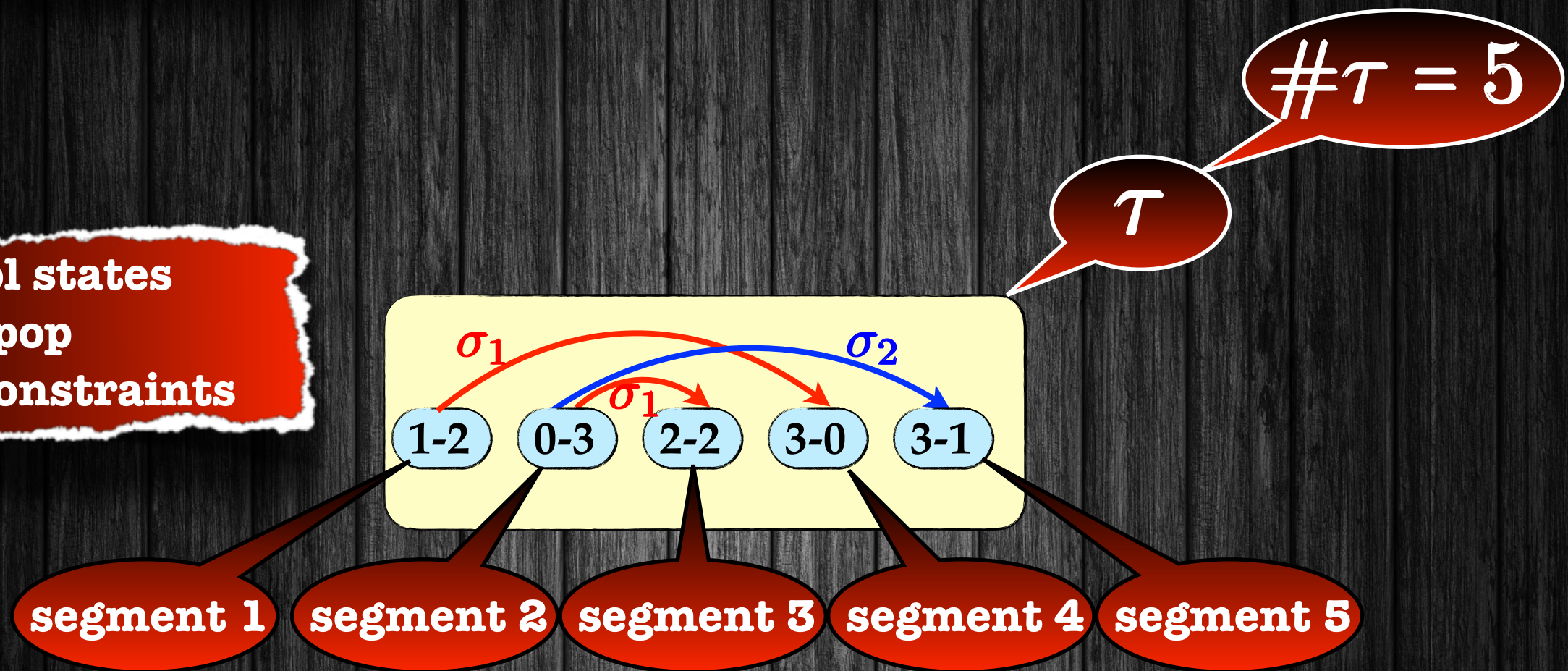
- control states
- push/pop
- data constraints



- segment: “part of run”
- signature: parts of runs with "holes" in between

Signatures

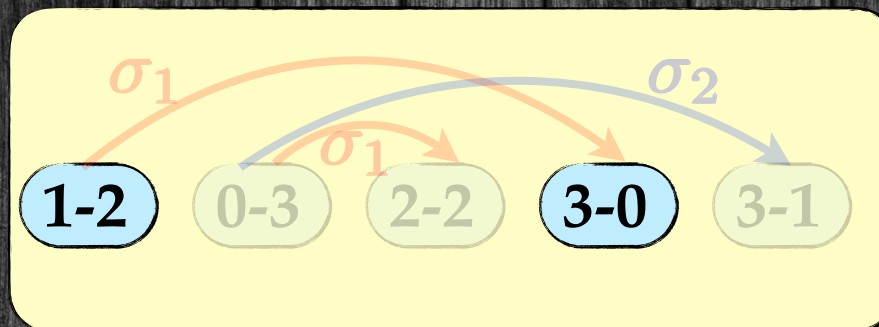
- control states
- push/pop
- data constraints



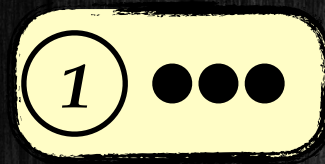
- segment: "part of run"
- signature: parts of runs with "holes" in between

Signatures

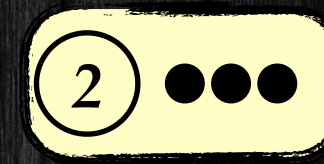
- control states ✓
- push/pop
- data constraints



Signatures



$push(\sigma_1, a, x)$



- control states ✓
- push/pop
- data constraints

1-2

0-3

2-2

3-0

3-1

σ_1

σ_1

σ_2

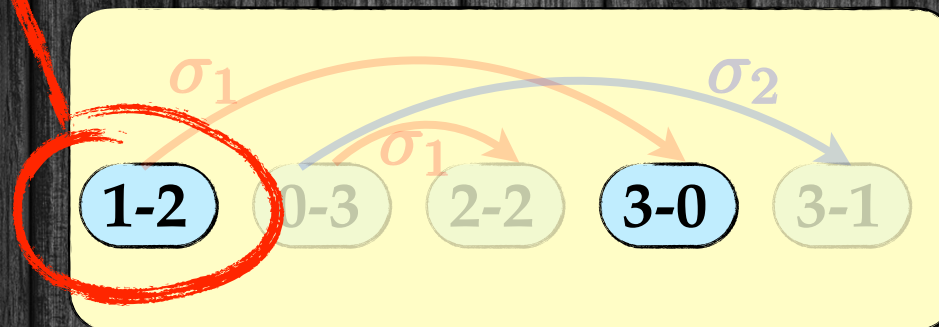
Signatures



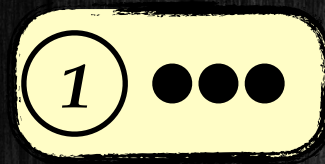
$push(\sigma_1, a, x)$



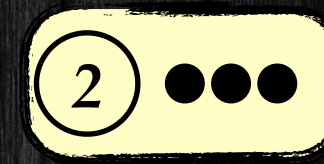
- control states ✓
- push/pop
- data constraints



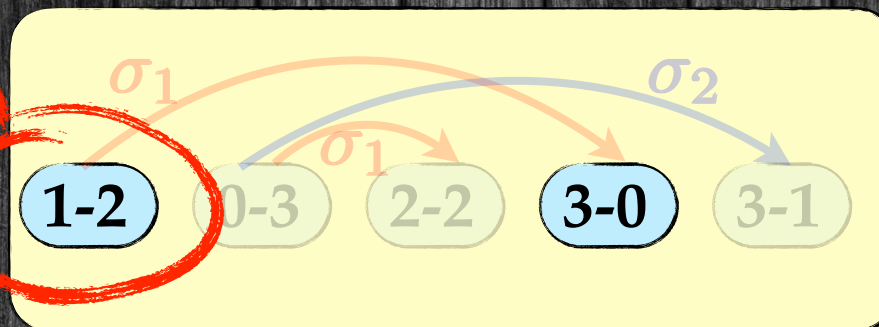
Signatures



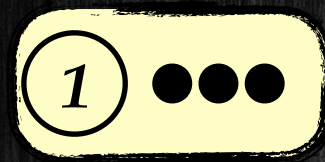
$push(\sigma_1, a, x)$



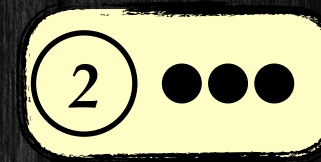
- control states ✓
- push/pop
- data constraints



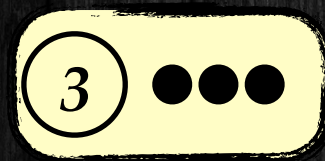
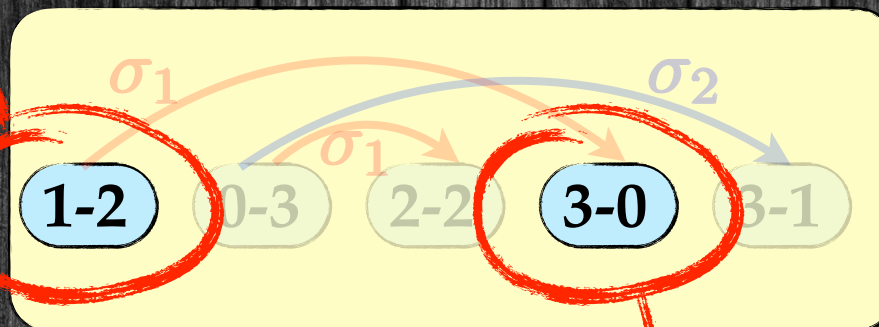
Signatures



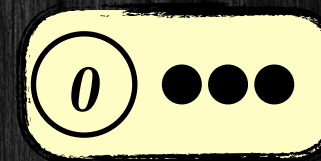
$push(\sigma_1, a, x)$



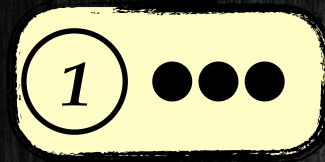
- control states ✓
- push/pop
- data constraints



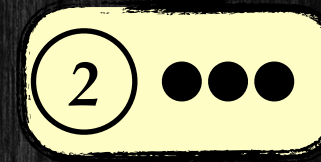
$pop(\sigma_1, a, y)$



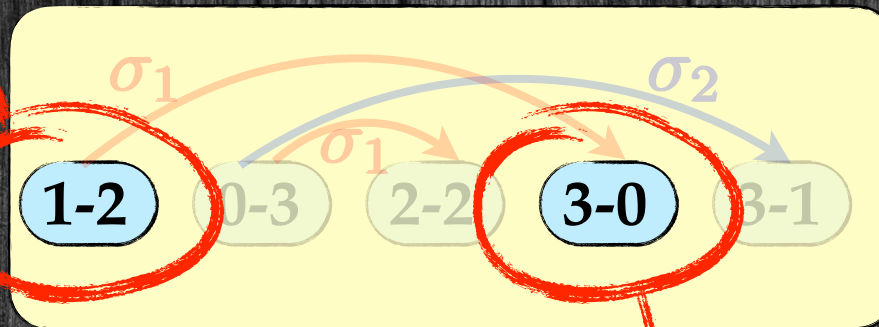
Signatures



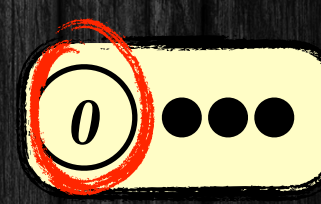
$push(\sigma_1, a, x)$



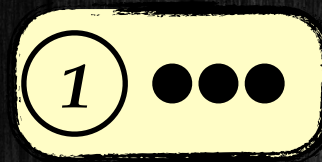
- control states ✓
- push/pop
- data constraints



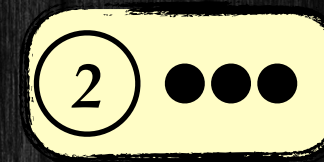
$pop(\sigma_1, a, y)$



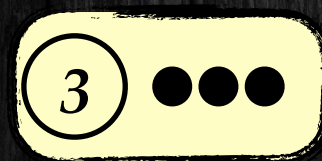
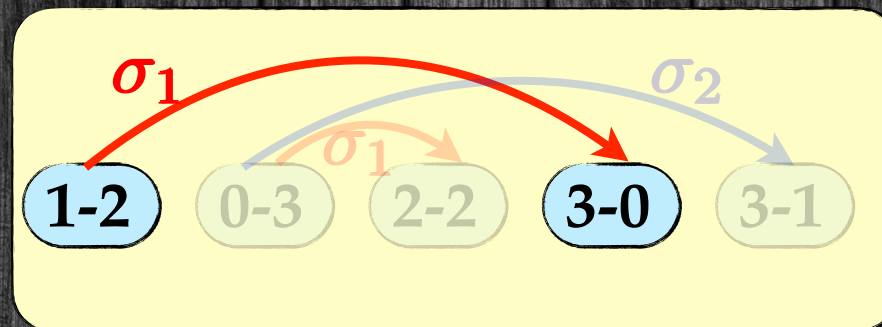
Signatures



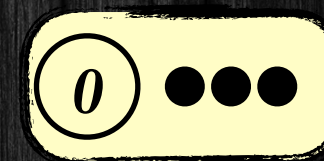
$push(\sigma_1, a, x)$



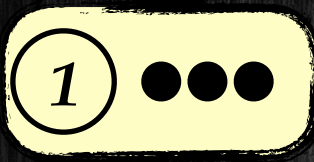
- control states
- push/pop ✓
- data constraints



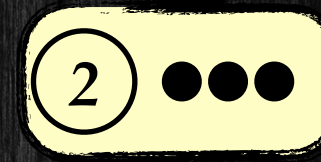
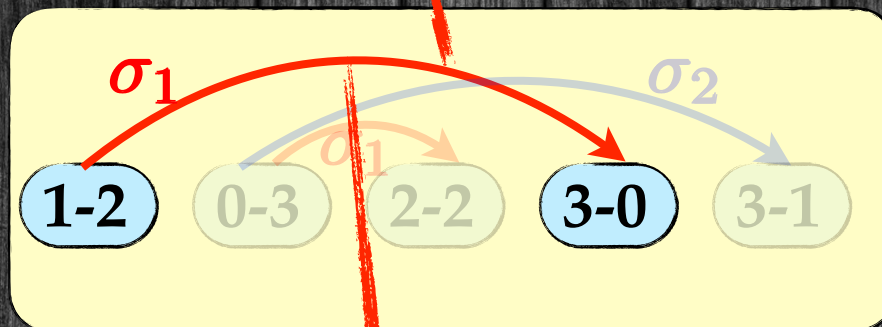
$pop(\sigma_1, a, y)$



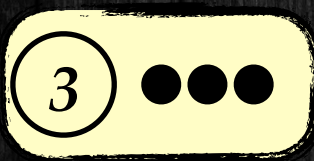
Signatures



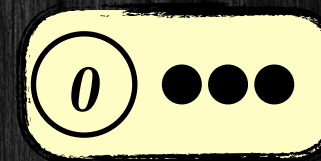
- control states
- push/pop ✓
- data constraints



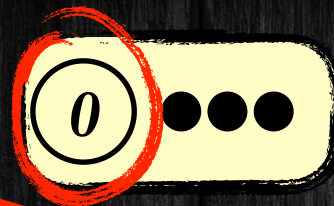
$push(\sigma_1, a, x)$



$pop(\sigma_1, a, y)$



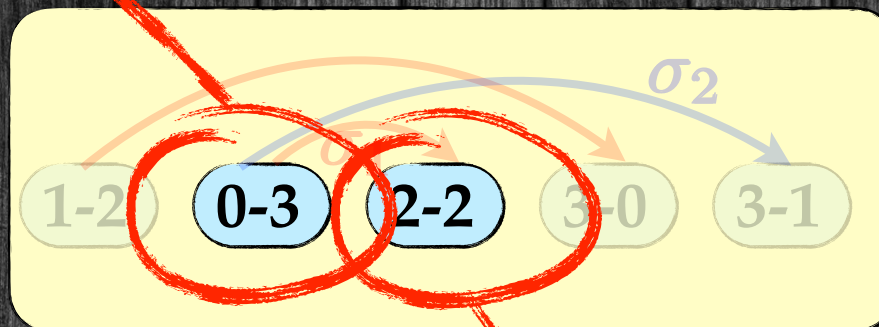
Signatures



$push(\sigma_1, a, x)$



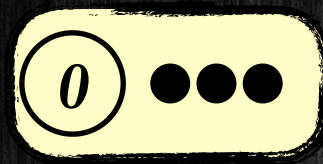
- control states ✓
- push/pop
- data constraints



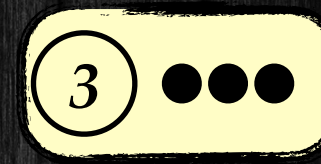
$pop(\sigma_1, a, y)$



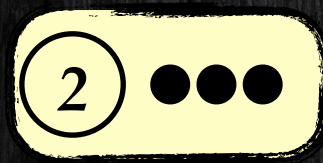
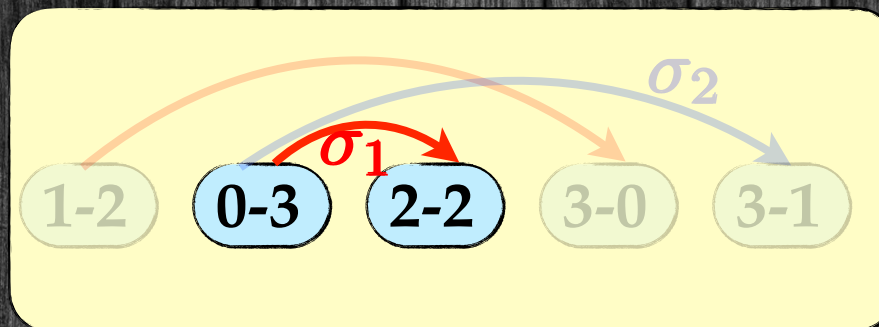
Signatures



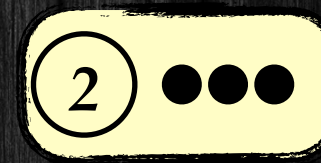
$push(\sigma_1, a, x)$



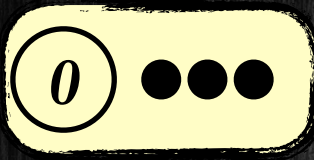
- control states
- push/pop ✓
- data constraints



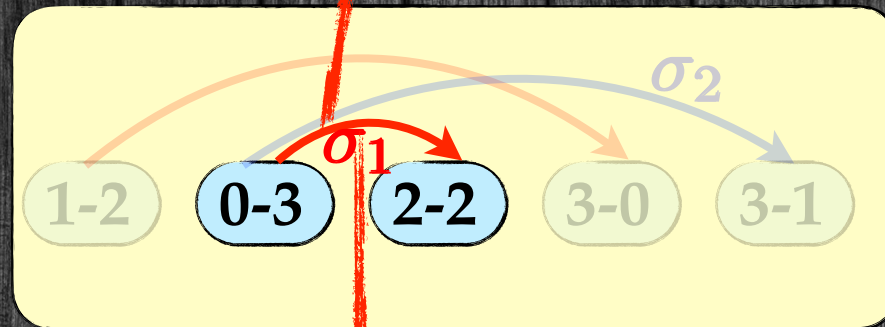
$pop(\sigma_1, a, y)$



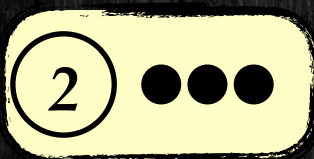
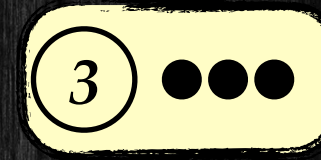
Signatures



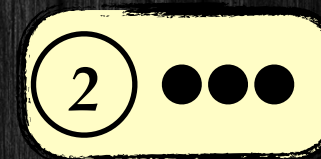
- control states
- push/pop ✓
- data constraints



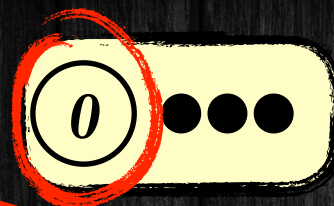
$push(\sigma_1, a, x)$



$pop(\sigma_1, a, y)$



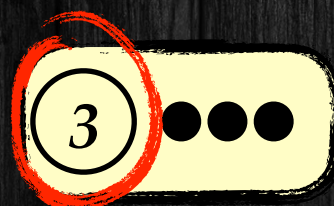
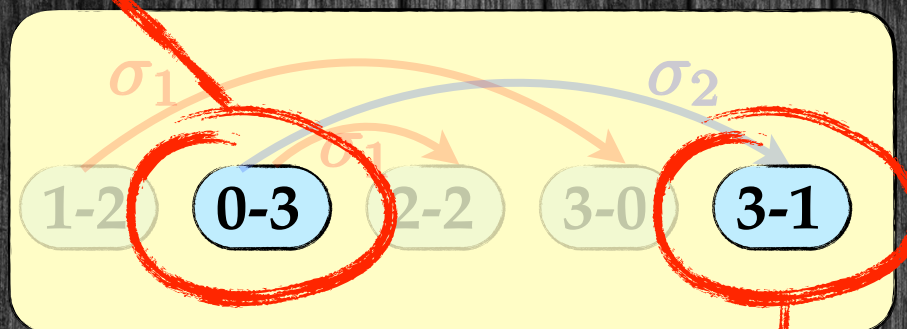
Signatures



$push(\sigma_2, b, z)$



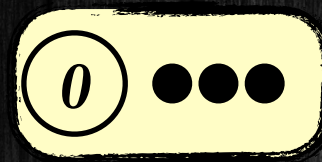
- control states ✓
- push/pop
- data constraints



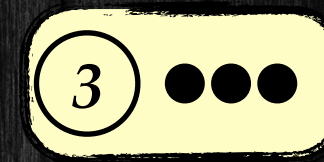
$pop(\sigma_2, b, x)$



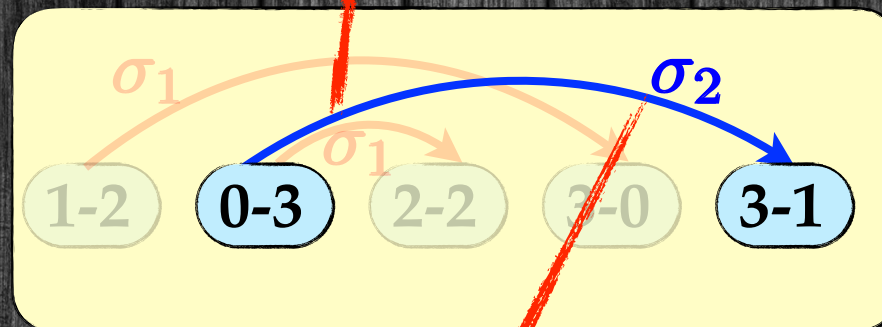
Signatures



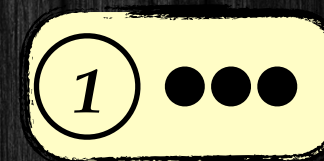
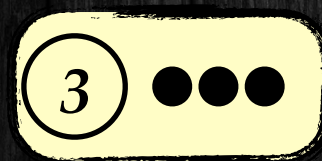
$push(\sigma_2, b, z)$



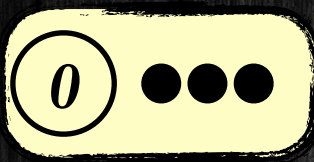
- control states
- push/pop ✓
- data constraints



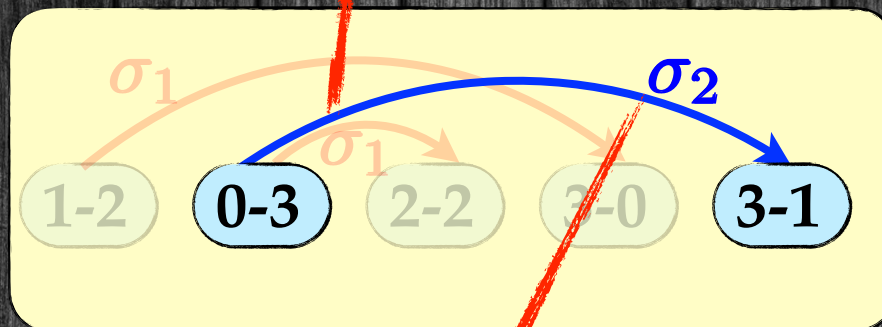
$pop(\sigma_2, b, x)$



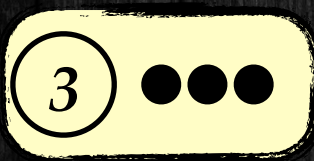
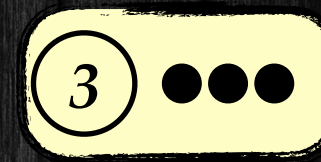
Signatures



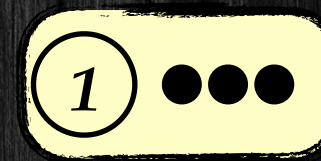
- control states
- push/pop ✓
- data constraints



$push(\sigma_2, b, z)$



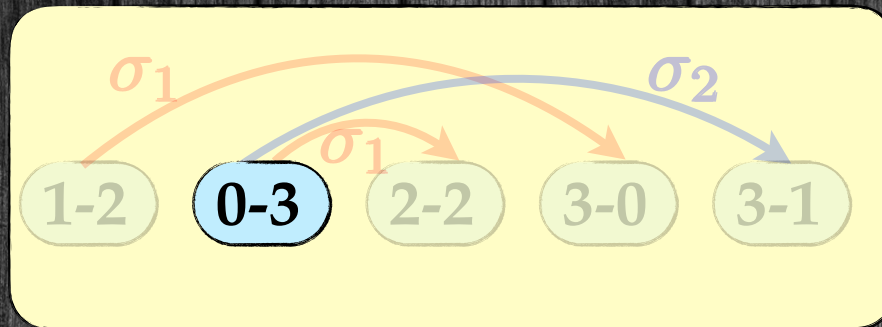
$pop(\sigma_2, b, x)$



Signatures

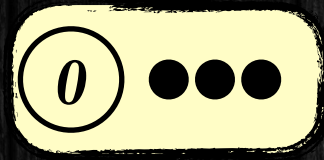


- control states
- push/pop ✓
- data constraints



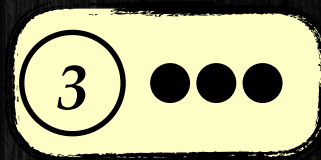
“push/pop operations within same segment not shown”

Signatures

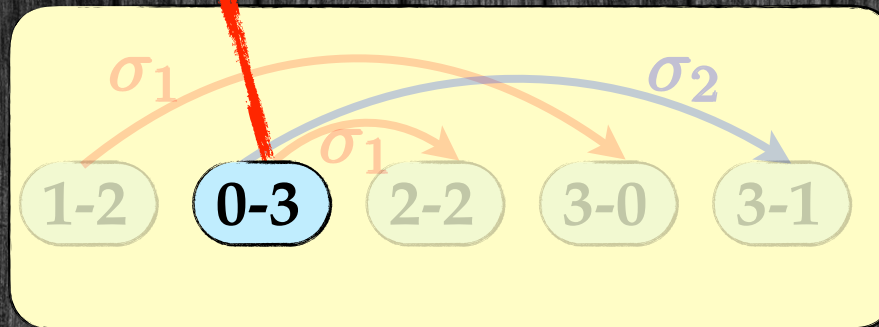


$push(\sigma_2, b, z)$

$pop(\sigma_2, b, x)$



- control states
- push/pop ✓
- data constraints



“push/pop operations within same segment not shown”

Signatures

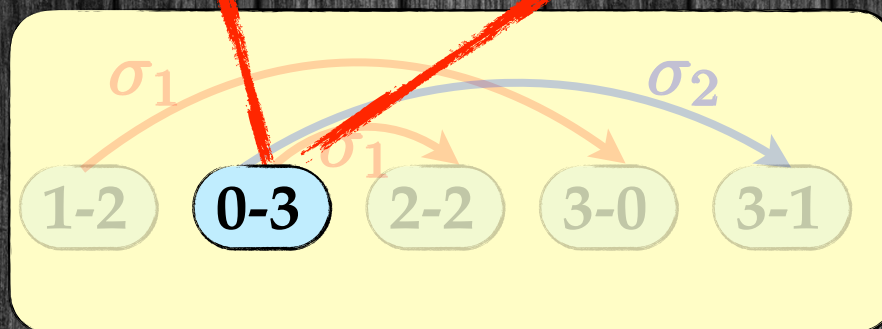
0 ...

$push(\sigma_2, b, z)$

$pop(\sigma_2, b, x)$

3 ...

- control states
- push/pop ✓
- data constraints



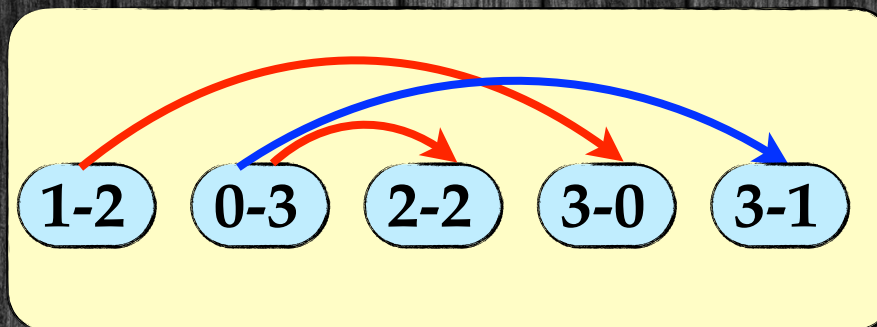
“push/pop operations within same segment not shown”

Signatures

finite set of
variables

x, y, z

- control states
- push/pop
- data constraints ✓

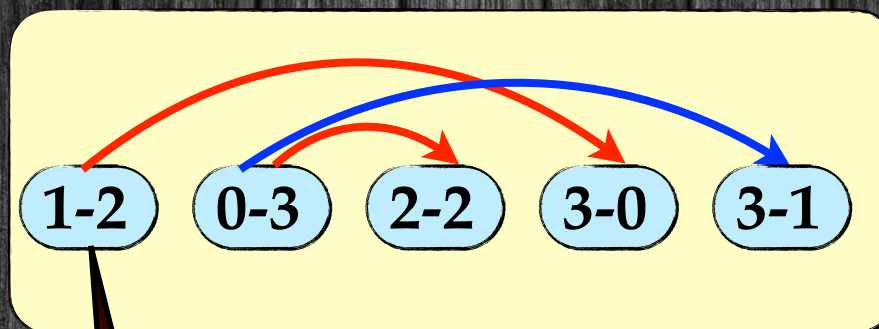


Signatures

finite set of
variables

x, y, z

- control states
- push/pop
- data constraints ✓



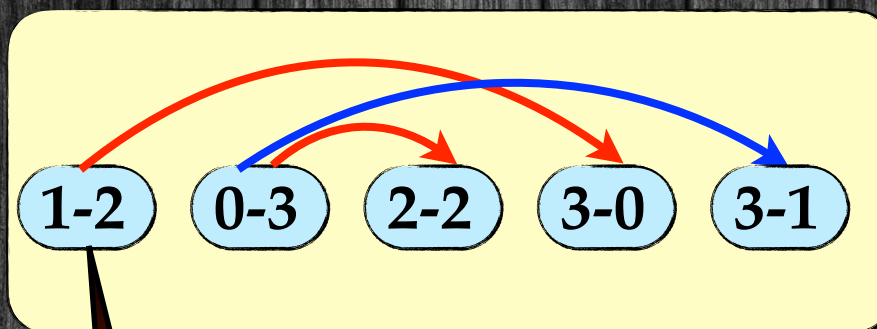
segment 1

Signatures

finite set of
variables

x, y, z

- control states
- push/pop
- data constraints ✓



segment 1

$x_s^1, x_t^1,$
 $y_s^1, y_t^1,$
 z_s^1, z_t^1

Signatures

finite set of
variables

x, y, z

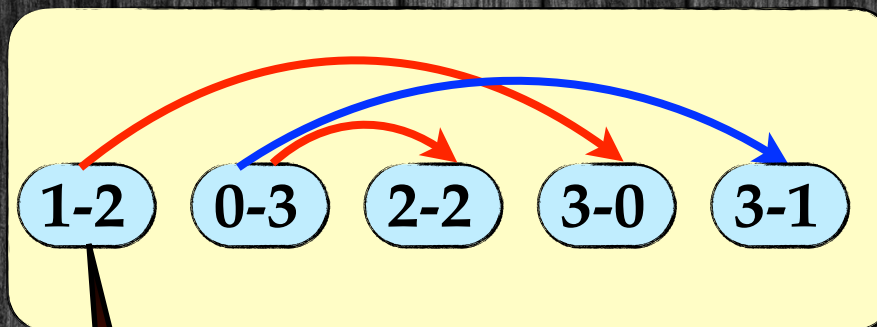
- control states
- push/pop
- data constraints ✓

value of x at
the start of
segment 1

$x_s^1, x_t^1,$
 $y_s^1, y_t^1,$
 z_s^1, z_t^1

segment 1

value of x at
the end of
segment 1

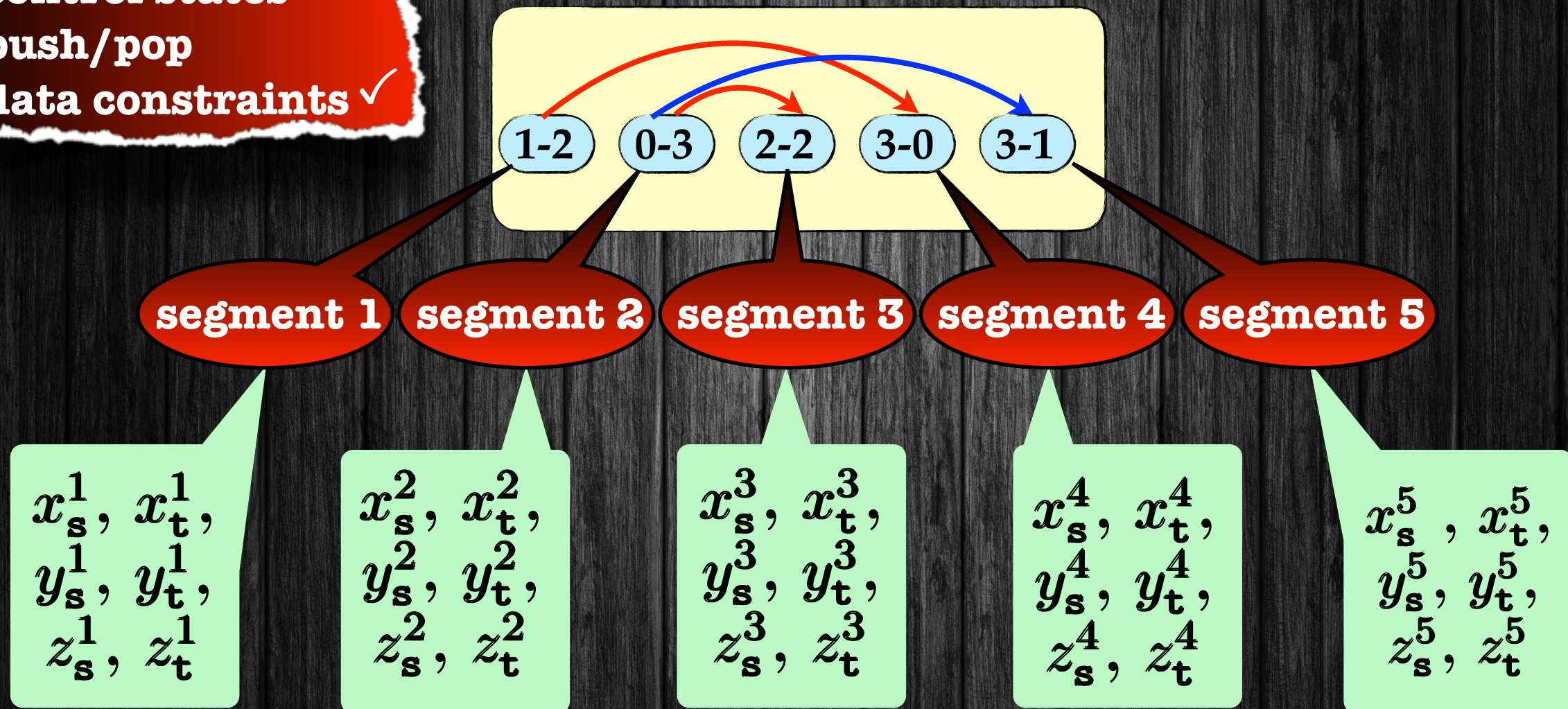


Signatures

finite set of
variables

x, y, z

- control states
- push/pop
- data constraints ✓



Signatures

finite set of
variables

x, y, z

- control states
- push/pop
- data constraints ✓

gap-order
constraint

$$\begin{array}{l} x_t^1 <_3 y_s^2 \\ z_t^3 = x_s^4 \\ y_t^5 <_4 z_s^2 \\ y_t^2 <_2 x_s^1 \\ x_t^1 = y_s^4 \\ z_t^2 <_2 x_s^5 \end{array}$$

segment 1

segment 2

segment 3

segment 4

segment 5

$$\begin{array}{l} x_s^1, x_t^1, \\ y_s^1, y_t^1, \\ z_s^1, z_t^1 \end{array}$$

$$\begin{array}{l} x_s^2, x_t^2, \\ y_s^2, y_t^2, \\ z_s^2, z_t^2 \end{array}$$

$$\begin{array}{l} x_s^3, x_t^3, \\ y_s^3, y_t^3, \\ z_s^3, z_t^3 \end{array}$$

$$\begin{array}{l} x_s^4, x_t^4, \\ y_s^4, y_t^4, \\ z_s^4, z_t^4 \end{array}$$

$$\begin{array}{l} x_s^5, x_t^5, \\ y_s^5, y_t^5, \\ z_s^5, z_t^5 \end{array}$$

Signatures

finite set of
variables

x, y, z

- control states
- push/pop
- data constraints ✓

gap-order
constraint

$$\begin{array}{l} x_t^1 <_3 y_s^2 \\ z_t^3 = x_s^4 \\ y_t^5 <_4 z_s^2 \\ y_t^2 <_2 x_s^1 \\ x_t^1 = y_s^4 \\ \textcircled{z_t^2 <_2 x_s^5} \end{array}$$

segment 1

segment 2

segment 3

segment 4

segment 5

$x_s^1, x_t^1,$
 $y_s^1, y_t^1,$
 z_s^1, z_t^1

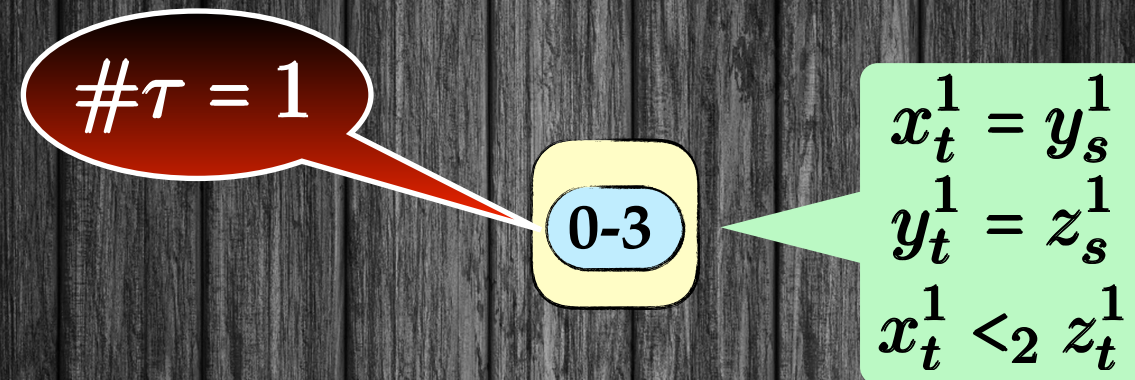
$x_s^2, x_t^2,$
 $y_s^2, y_t^2,$
 z_s^2, z_t^2

$x_s^3, x_t^3,$
 $y_s^3, y_t^3,$
 z_s^3, z_t^3

$x_s^4, x_t^4,$
 $y_s^4, y_t^4,$
 z_s^4, z_t^4

“the gap between
the value of x at
the start of
segment 5
and the value
of z at the end
of segment 2 is
more than 2”

Signatures

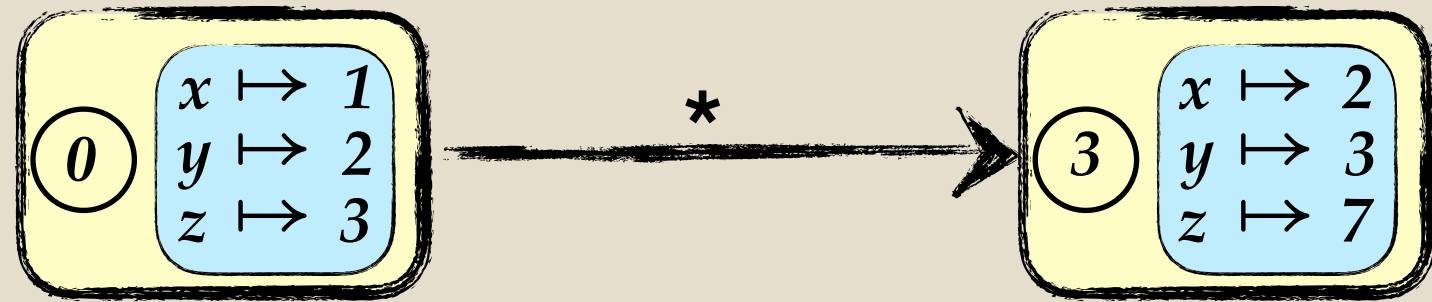


“push/pop operations within same segment not shown”

“single-node signatures characterize full runs”

“single-node signatures characterise reachability relations”

Signatures



$$\# \tau = 1$$

0-3

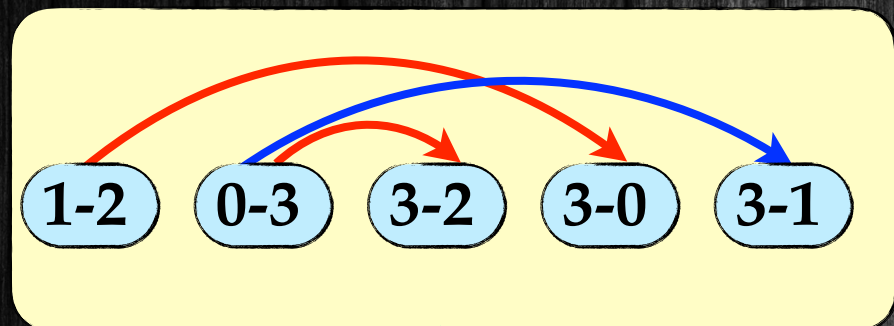
$$\begin{aligned} x_t^1 &= y_s^1 \\ y_t^1 &= z_s^1 \\ x_t^1 &<_2 z_t^1 \end{aligned}$$

“push/pop operations within same segment not shown”

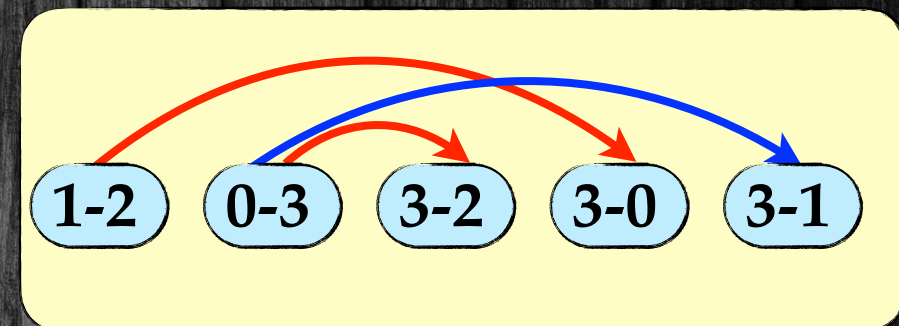
“single-node signatures characterize full runs”

“single-node signatures characterise reachability relations”

Signatures Entailment



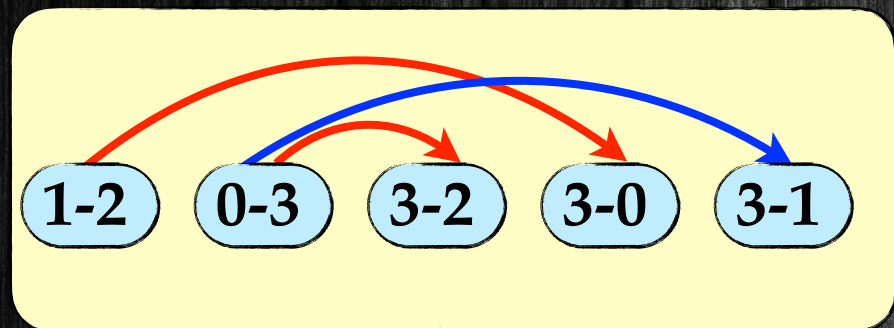
$\sqsubseteq$



$z_t^5 <_2 z_s^2$
 $y_t^5 <_3 x_s^4$

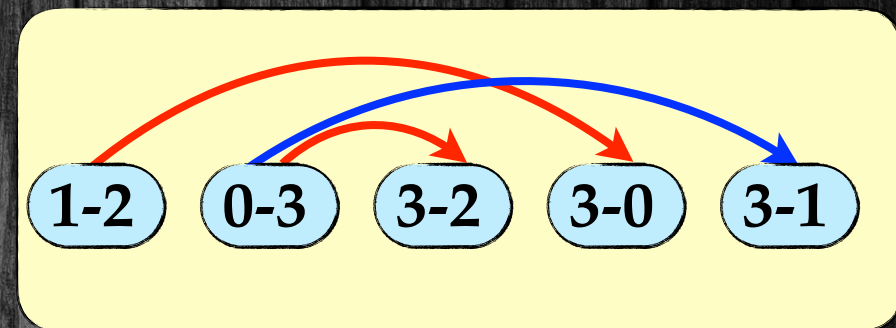
$x_s^1 <_3 y_t^2$
 $z_t^5 <_4 z_s^2$
 $y_s^3 <_2 z_s^4$
 $y_t^5 <_5 x_s^4$

Signatures Entailment



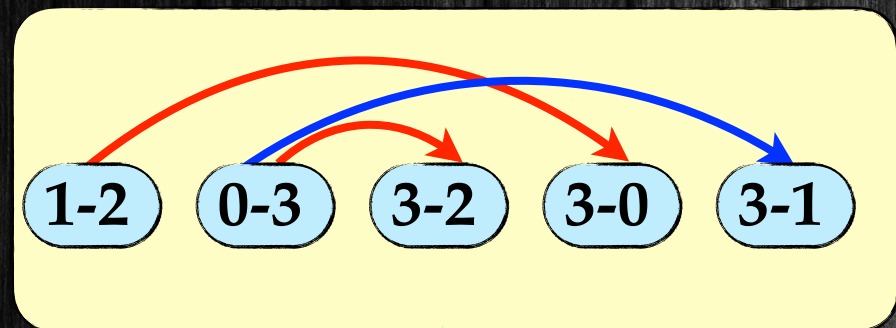
$z_t^5 <_2 z_s^2$
 $y_t^5 <_3 x_s^4$

$\sqsubseteq$

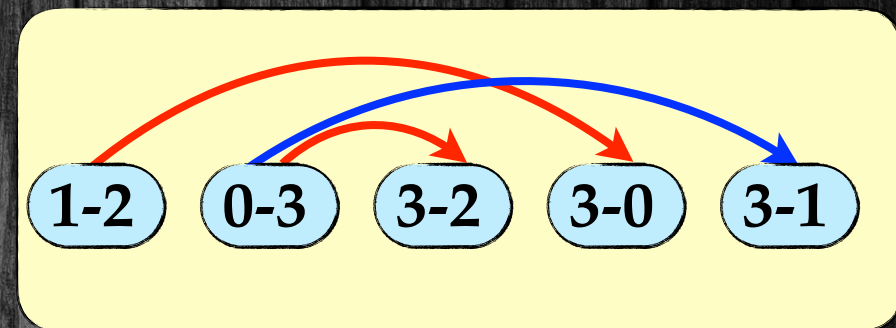


$x_s^1 <_3 y_t^2$
 $z_t^5 <_4 z_s^2$
 $y_s^3 <_2 z_s^4$
 $y_t^5 <_5 x_s^4$

Signatures Entailment



$z_t^5 <_2 z_s^2$
 $y_t^5 <_3 x_s^4$



$x_s^1 <_3 y_t^2$
 $z_t^5 <_4 z_s^2$
 $y_s^3 <_2 z_s^4$
 $y_t^5 <_5 x_s^4$



Signatures

Atomic Signatures

Signatures

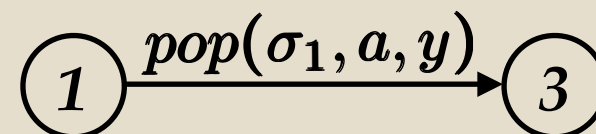
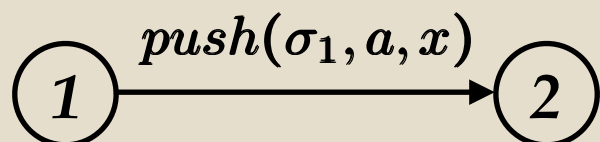
Atomic Signatures

Stack Transitions

Signatures

Atomic Signatures

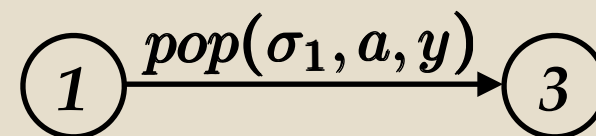
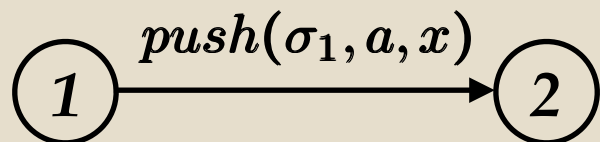
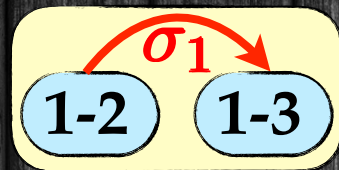
Stack Transitions



Signatures

Atomic Signatures

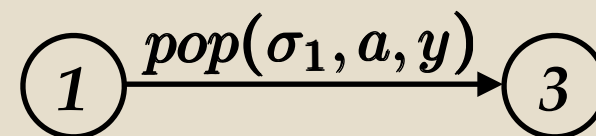
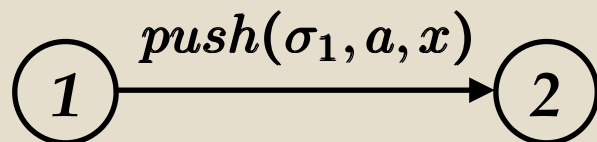
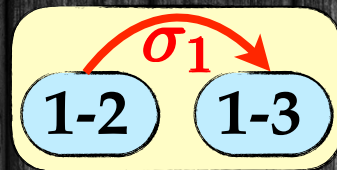
Stack Transitions



Signatures

Atomic Signatures

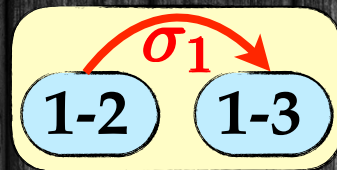
Stack Transitions



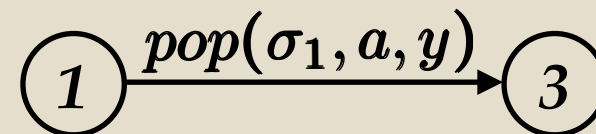
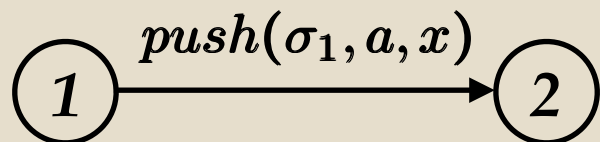
Signatures

Atomic Signatures

Stack Transitions



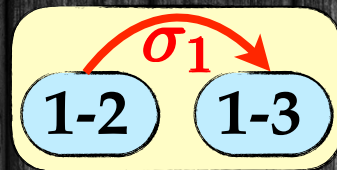
$$\begin{aligned}x_t^1 &= x_s^1 \\y_t^1 &= y_s^1 \\z_t^1 &= z_s^1\end{aligned}$$



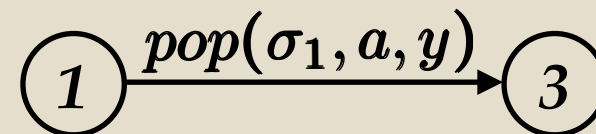
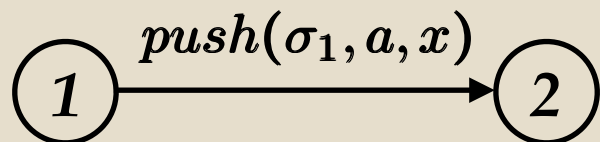
Signatures

Atomic Signatures

Stack Transitions



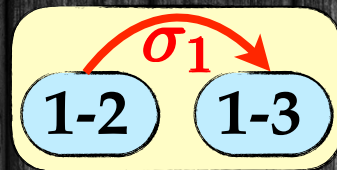
$$\begin{aligned}x_t^1 &= x_s^1 \\ y_t^1 &= y_s^1 \\ z_t^1 &= z_s^1\end{aligned}$$



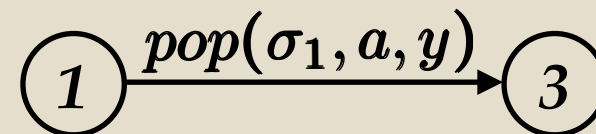
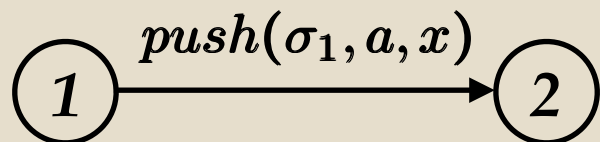
Signatures

Atomic Signatures

Stack Transitions



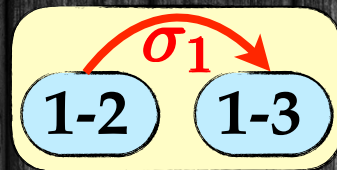
$$\begin{array}{ll} x_t^1 = x_s^1 & x_t^2 = x_s^2 \\ y_t^1 = y_s^1 & y_t^2 = x_s^1 \\ z_t^1 = z_s^1 & z_t^2 = z_s^2 \end{array}$$



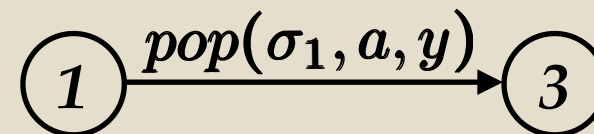
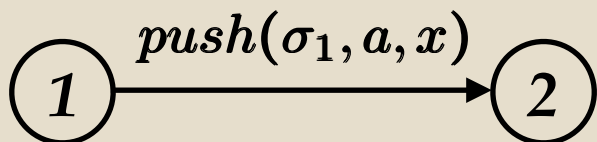
Signatures

Atomic Signatures

Stack Transitions



$$\begin{array}{ll} x_t^1 = x_s^1 & x_t^2 = x_s^2 \\ y_t^1 = y_s^1 & y_t^2 = x_s^1 \\ z_t^1 = z_s^1 & z_t^2 = z_s^2 \end{array}$$



Signatures

Atomic Signatures

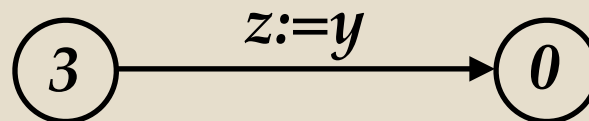
Internal Transitions

$$\begin{array}{ll} x_1^1 < x_3^2 & x_1^3 = x_2^4 \\ y_1^5 = x_2^6 & y_1^2 = x_2^1 \\ x_1^1 = y_1^5 & x_1^2 = x_2^1 \end{array}$$

Signatures

Atomic Signatures

Internal Transitions



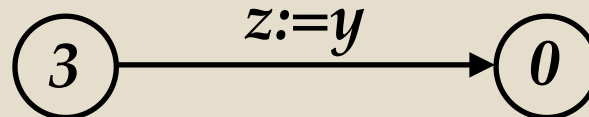
$$\begin{array}{ll} x_1^1 <_3 y_1^2 & z_1^3 = x_1^4 \\ y_1^5 = x_1^6 & y_1^2 = x_1^1 \\ x_1^1 = y_1^5 & z_1^2 = x_1^5 \end{array}$$

Signatures

Atomic Signatures

Internal Transitions

3-0



$$\begin{array}{ll} x_1^1 <_3 y_1^2 & z_1^3 = x_1^4 \\ y_1^5 = x_1^6 & y_1^2 = x_1^1 \\ x_1^1 = y_1^5 & z_1^2 = x_1^3 \end{array}$$

Signatures

Atomic Signatures

Internal Transitions

3-0

$$\begin{aligned}x_t^1 &= x_s^1 \\ y_t^1 &= y_s^1 \\ z_t^1 &= y_s^1\end{aligned}$$

3

$z := y$

0

$$\begin{aligned}x_t^1 &<_3 x_t^2 & z_t^3 &= x_s^4 \\ y_t^5 &= x_t^6 & y_t^2 &= x_s^1 \\ x_t^1 &= y_t^3 & x_t^2 &= x_s^5\end{aligned}$$

Signatures

Atomic Signatures

Internal Transitions

3-0

$$x_t^1 = x_s^1$$

$$y_t^1 = y_s^1$$

$$z_t^1 = y_s^1$$

3

$z := y$

0

$$\begin{array}{ll} x_t^1 <_3 x_s^2 & z_t^3 = x_s^4 \\ y_t^5 = x_s^6 & y_t^2 = x_s^1 \\ x_t^1 = y_s^3 & x_t^2 = x_s^5 \end{array}$$

Signatures

Atomic Signatures

Internal Transitions

3-0

$$x_t^1 = x_s^1$$

$$y_t^1 = y_s^1$$

$$z_t^1 = y_s^1$$

3

$z := y$

0

$$\begin{array}{ll} x_t^1 <_3 x_t^2 & z_t^3 = x_s^4 \\ y_t^5 = x_t^6 & y_t^2 = x_t^1 \\ x_t^1 = y_t^5 & z_t^2 = x_t^5 \end{array}$$

Signatures

Combining Signatures

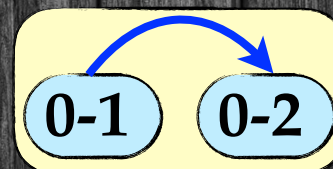
Signatures

Combining Signatures

Shuffling

3-0

$$\begin{aligned} y_t^1 &<_4 z_s^1 \\ y_t^1 &= y_s^1 \end{aligned}$$



$$\begin{aligned} x_t^1 &<_3 y_s^2 \\ z_t^2 &= x_s^1 \end{aligned}$$

Signatures

Combining Signatures

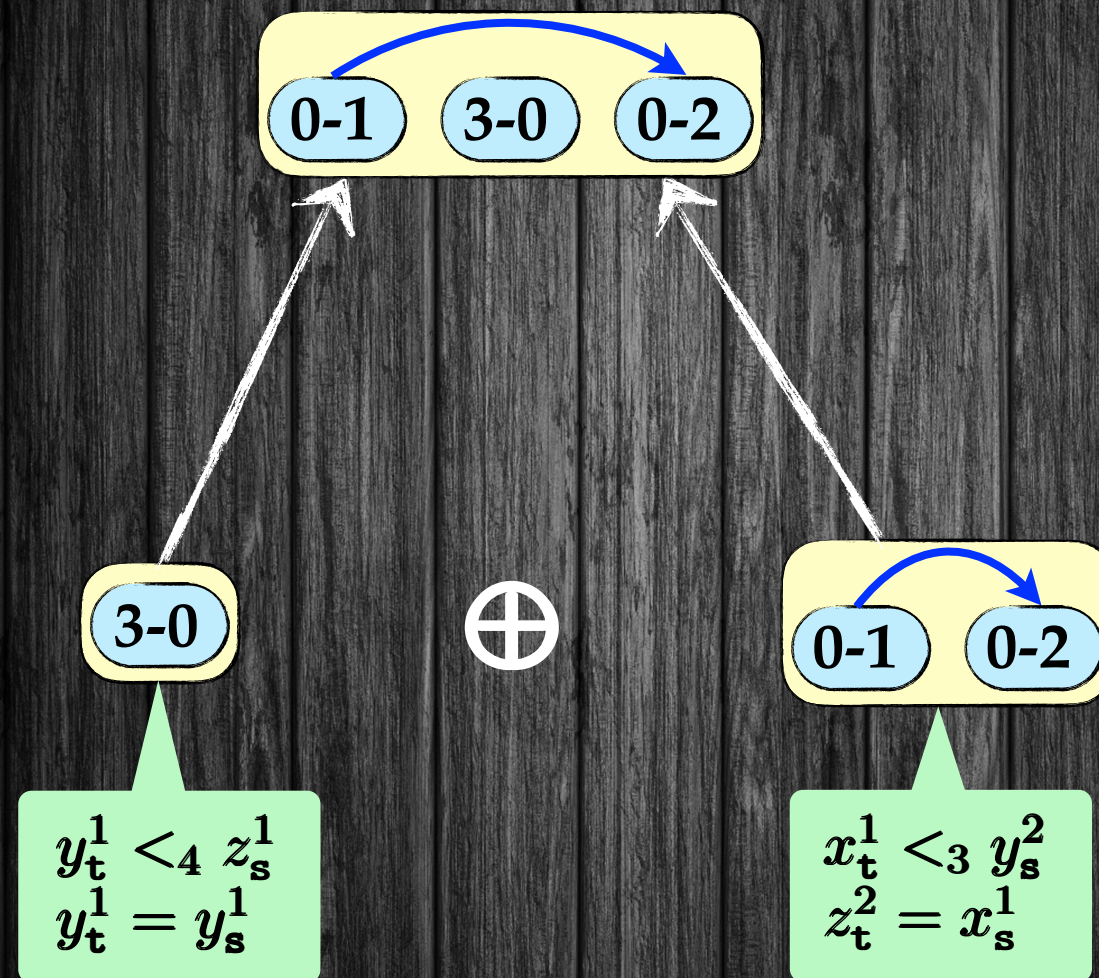
Shuffling



Signatures

Combining Signatures

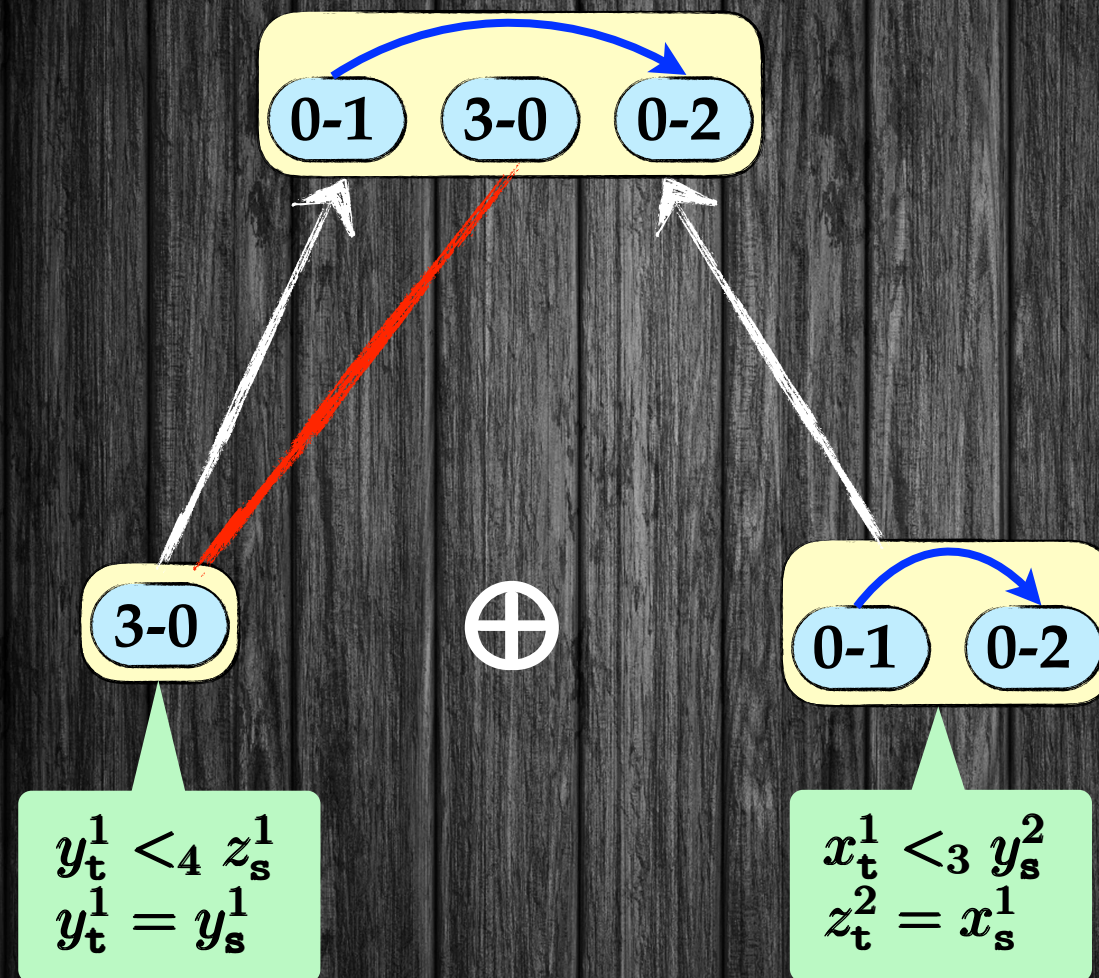
Shuffling



Signatures

Combining Signatures

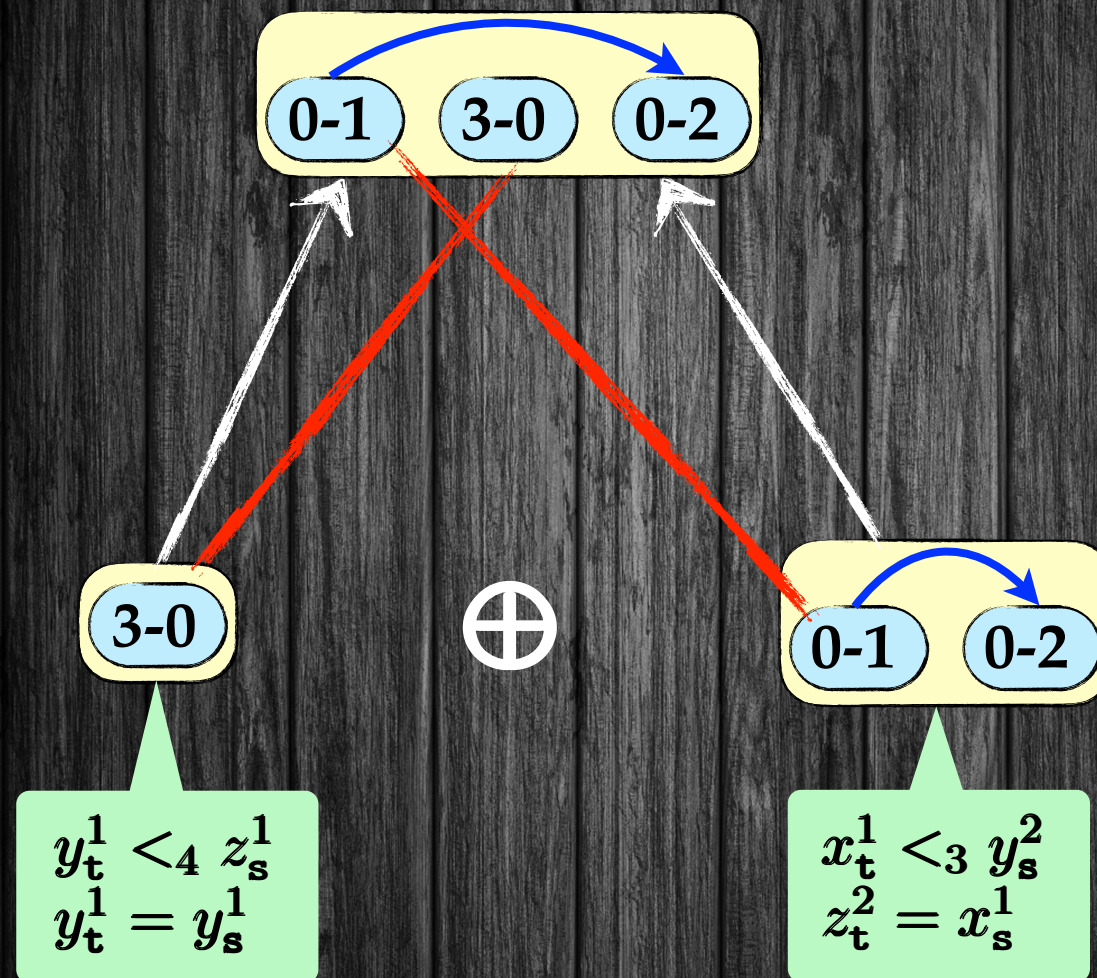
Shuffling



Signatures

Combining Signatures

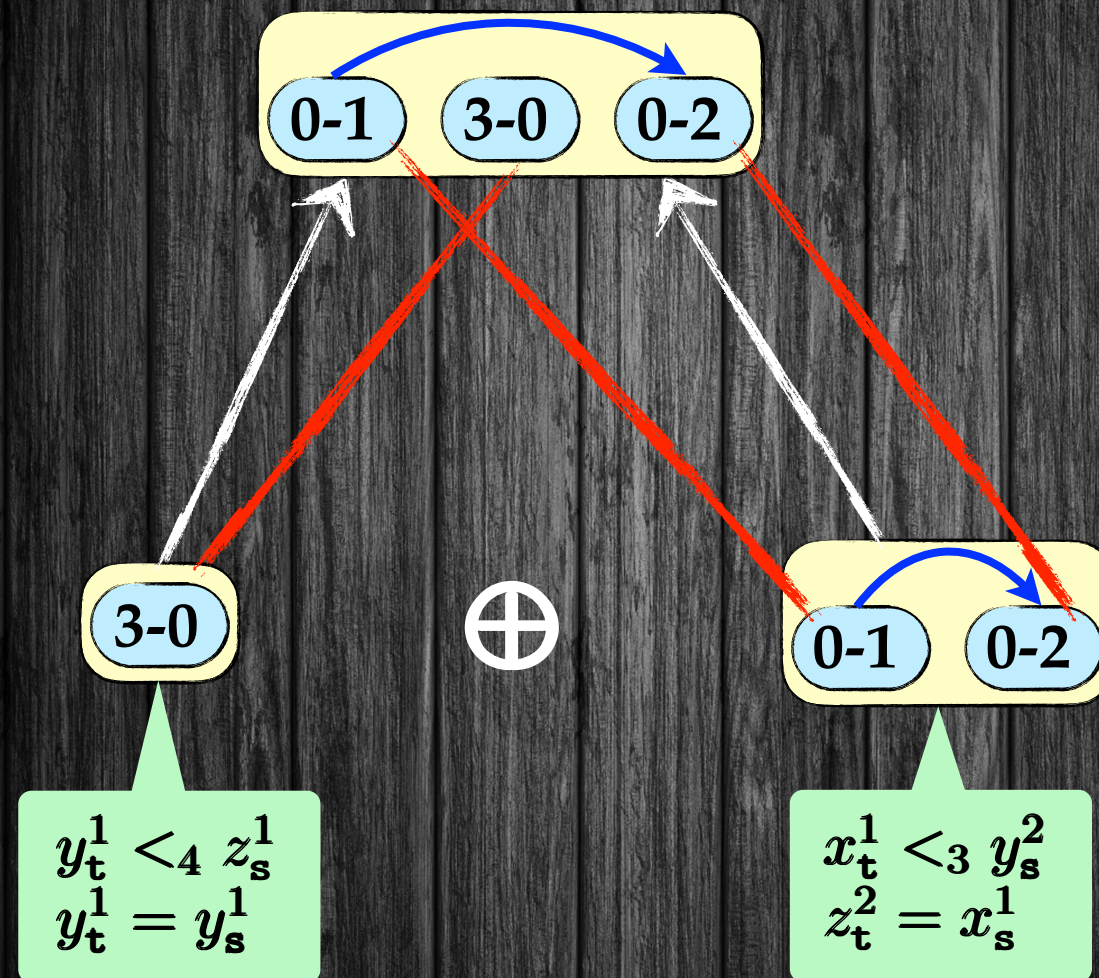
Shuffling



Signatures

Combining Signatures

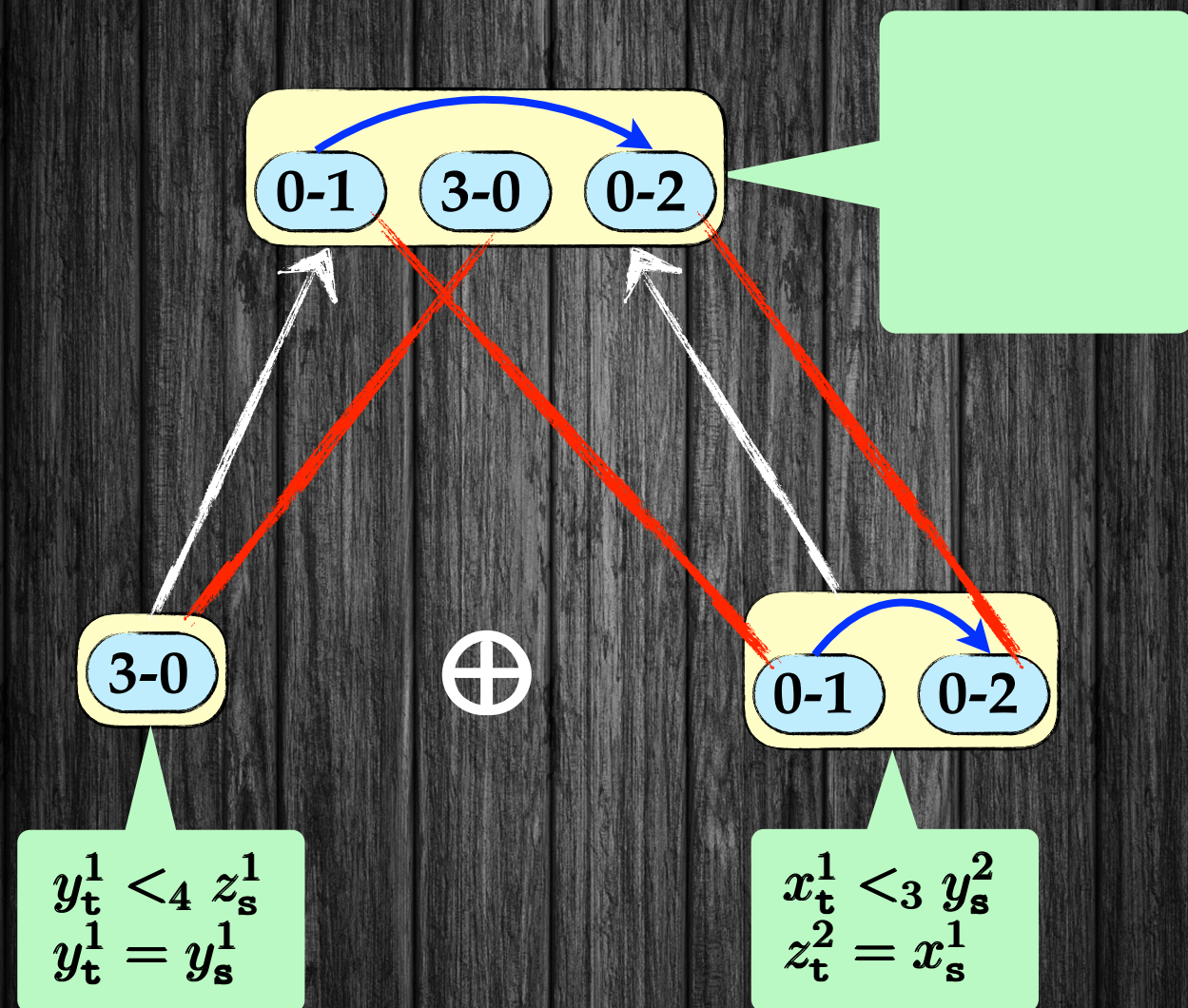
Shuffling



Signatures

Combining Signatures

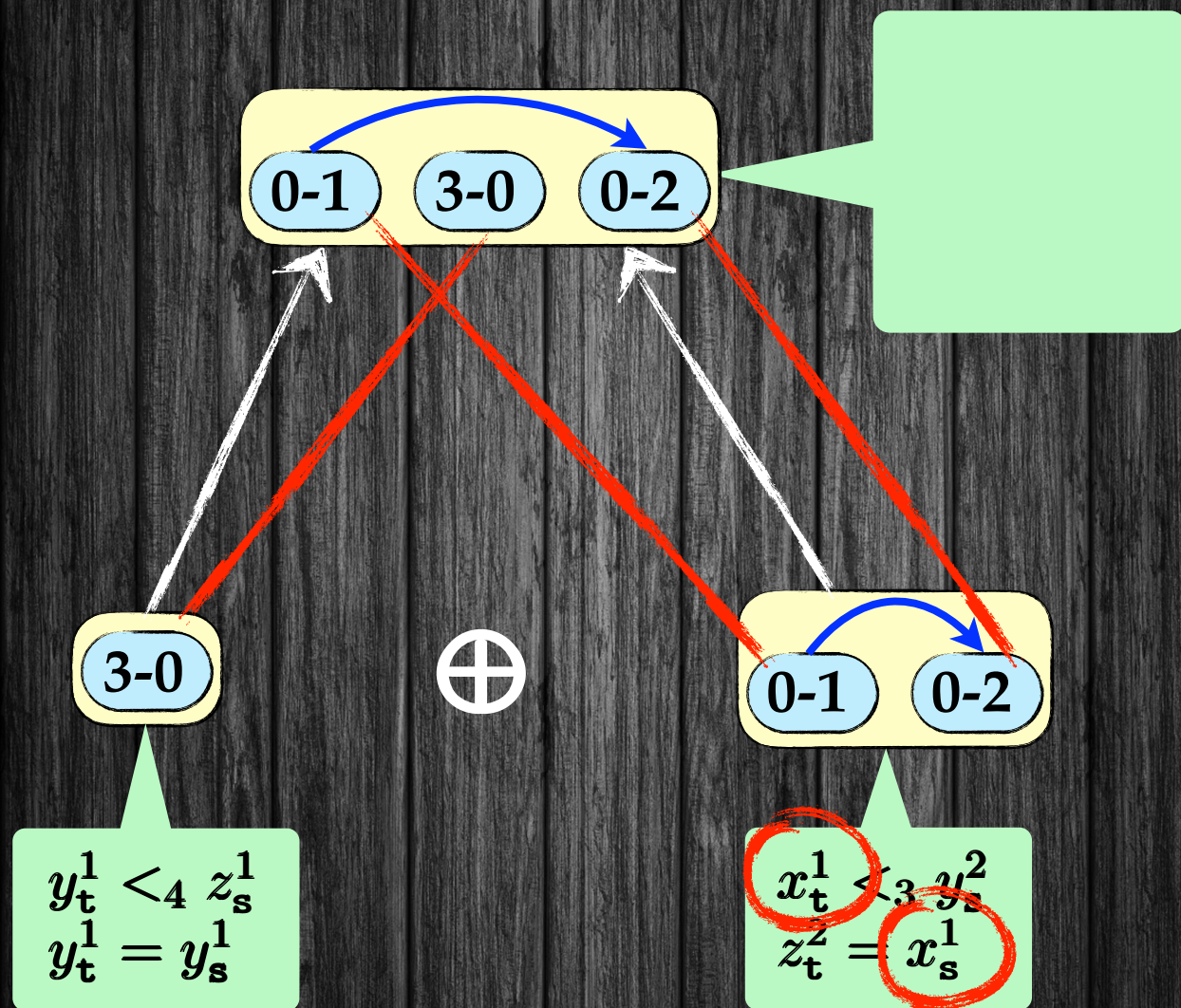
Shuffling



Signatures

Combining Signatures

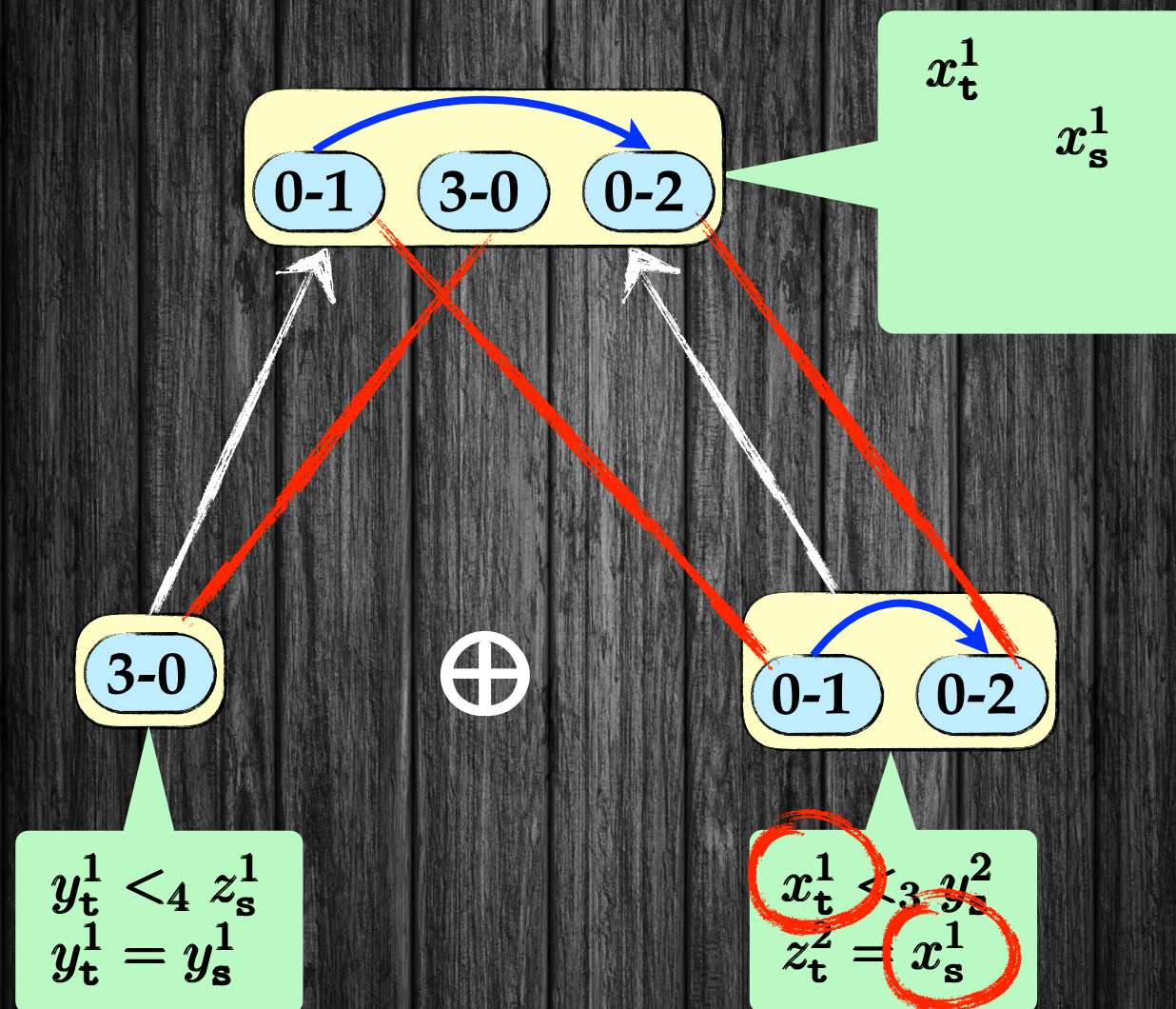
Shuffling



Signatures

Combining Signatures

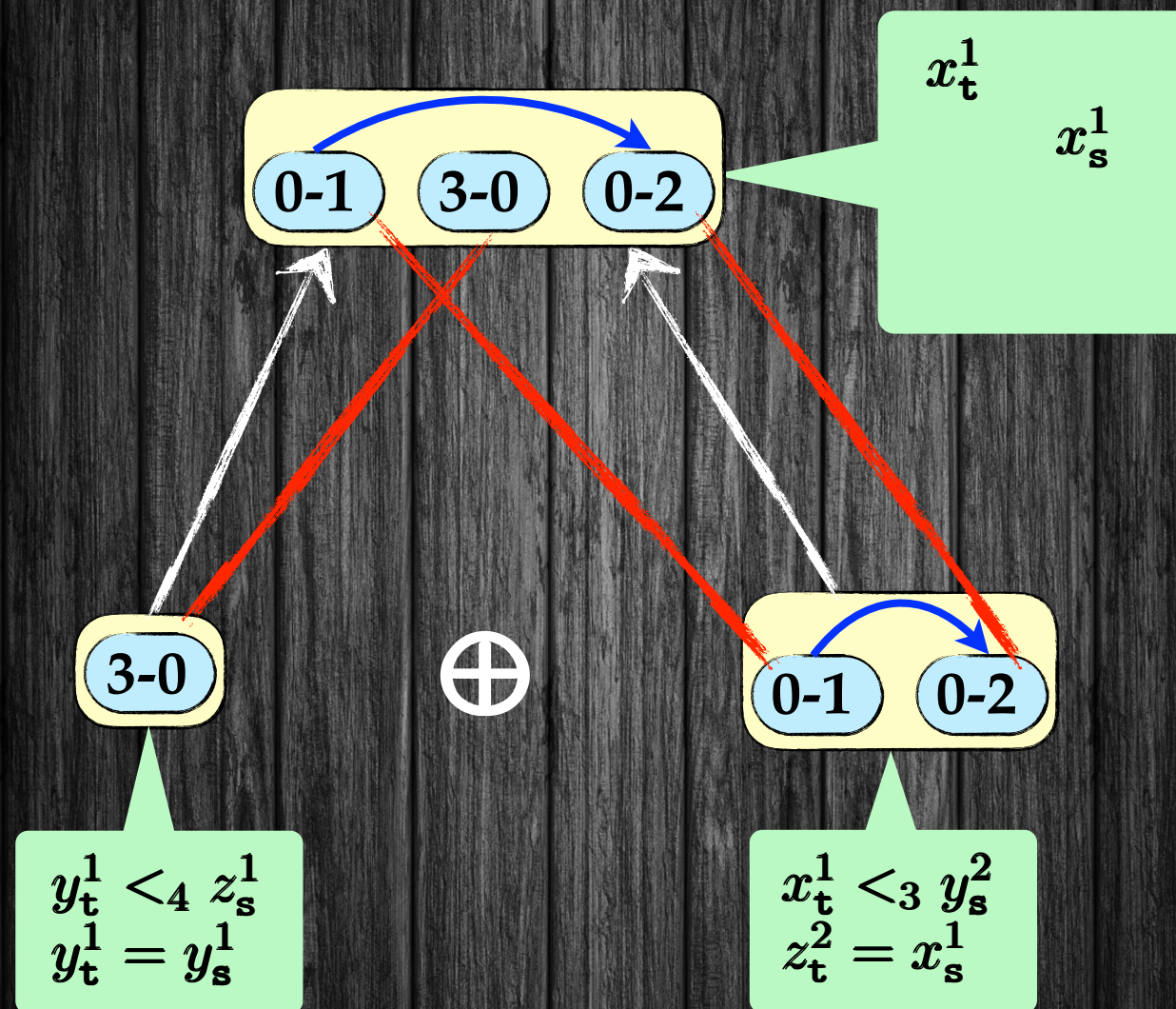
Shuffling



Signatures

Combining Signatures

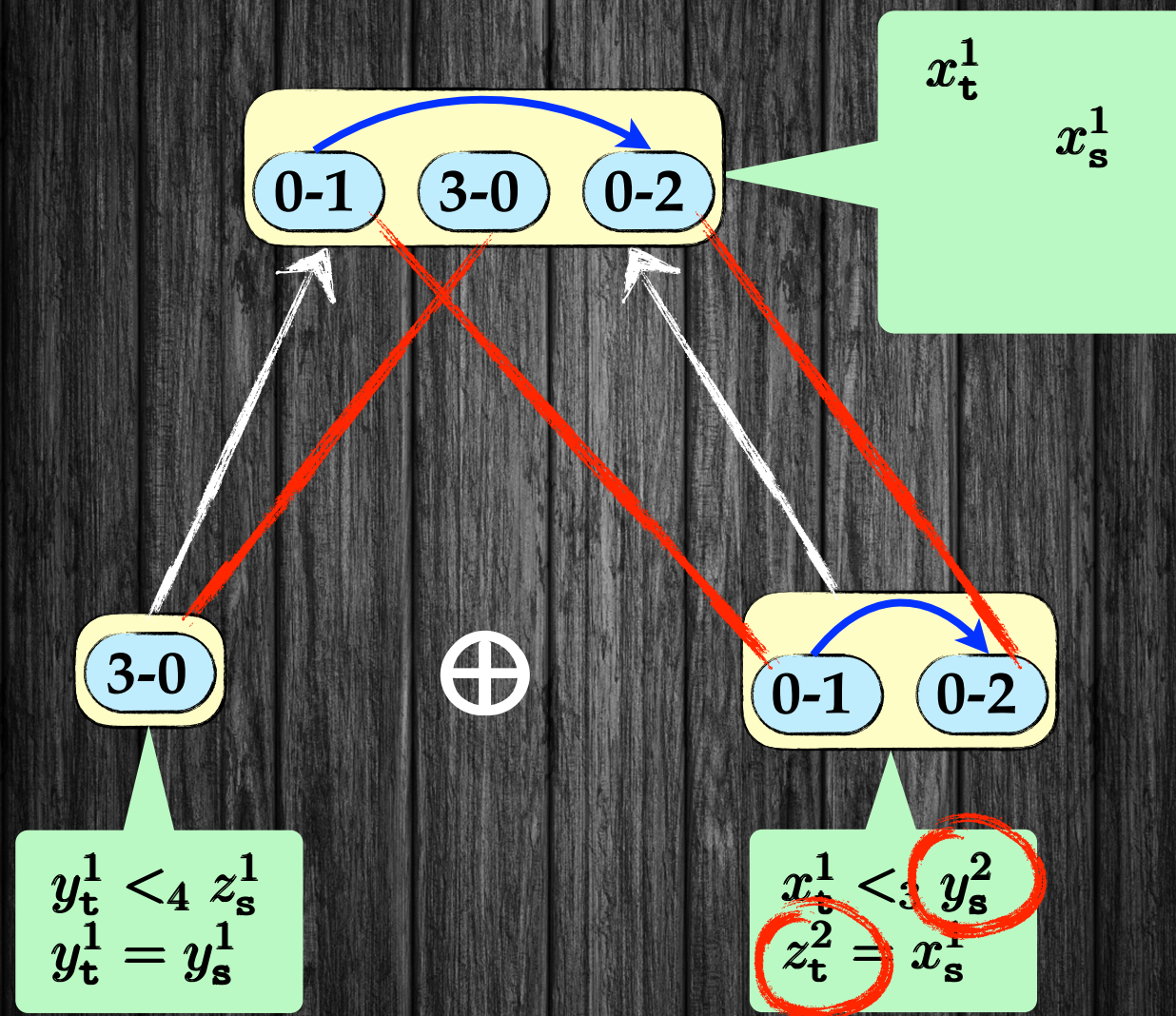
Shuffling



Signatures

Combining Signatures

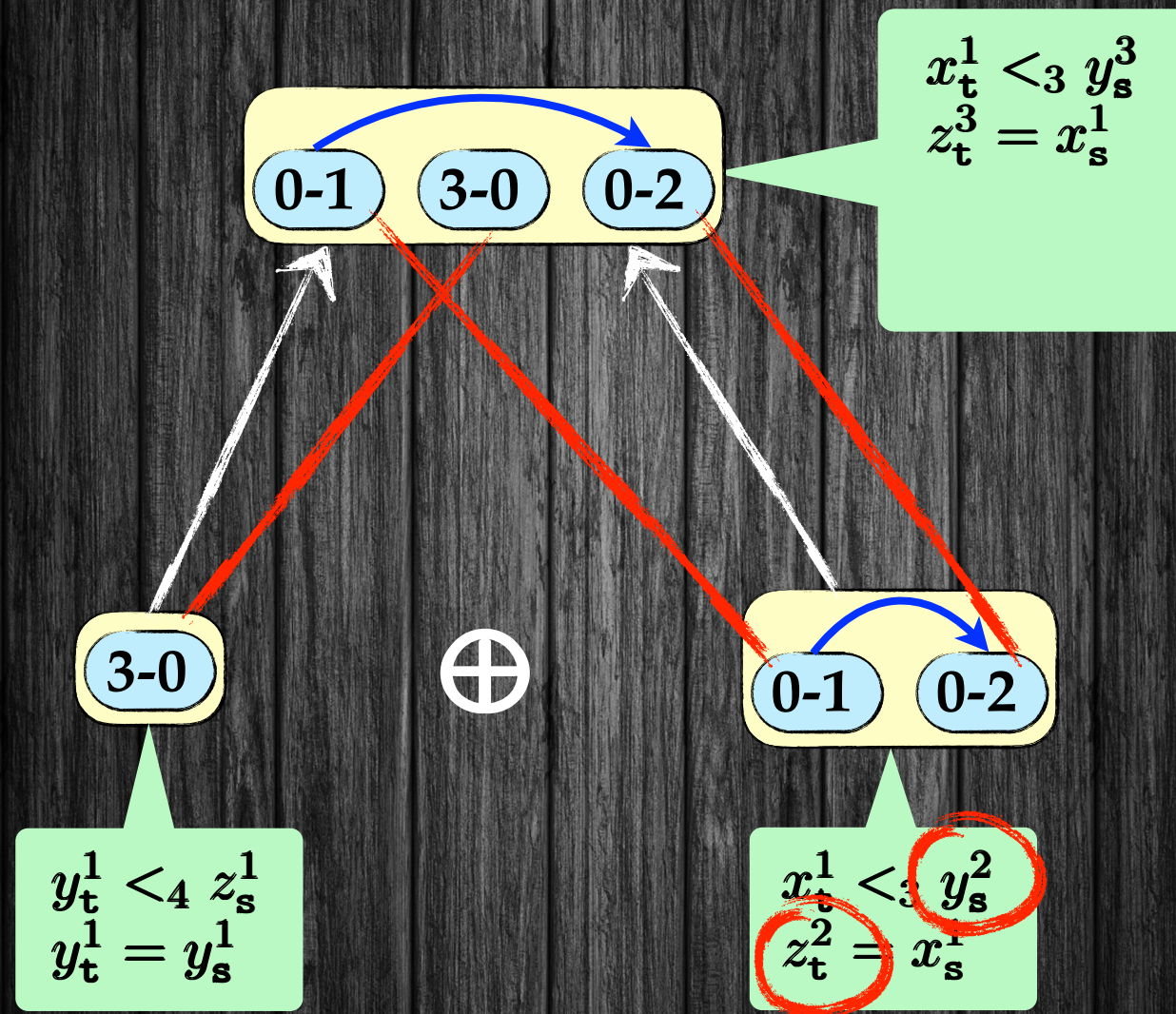
Shuffling



Signatures

Combining Signatures

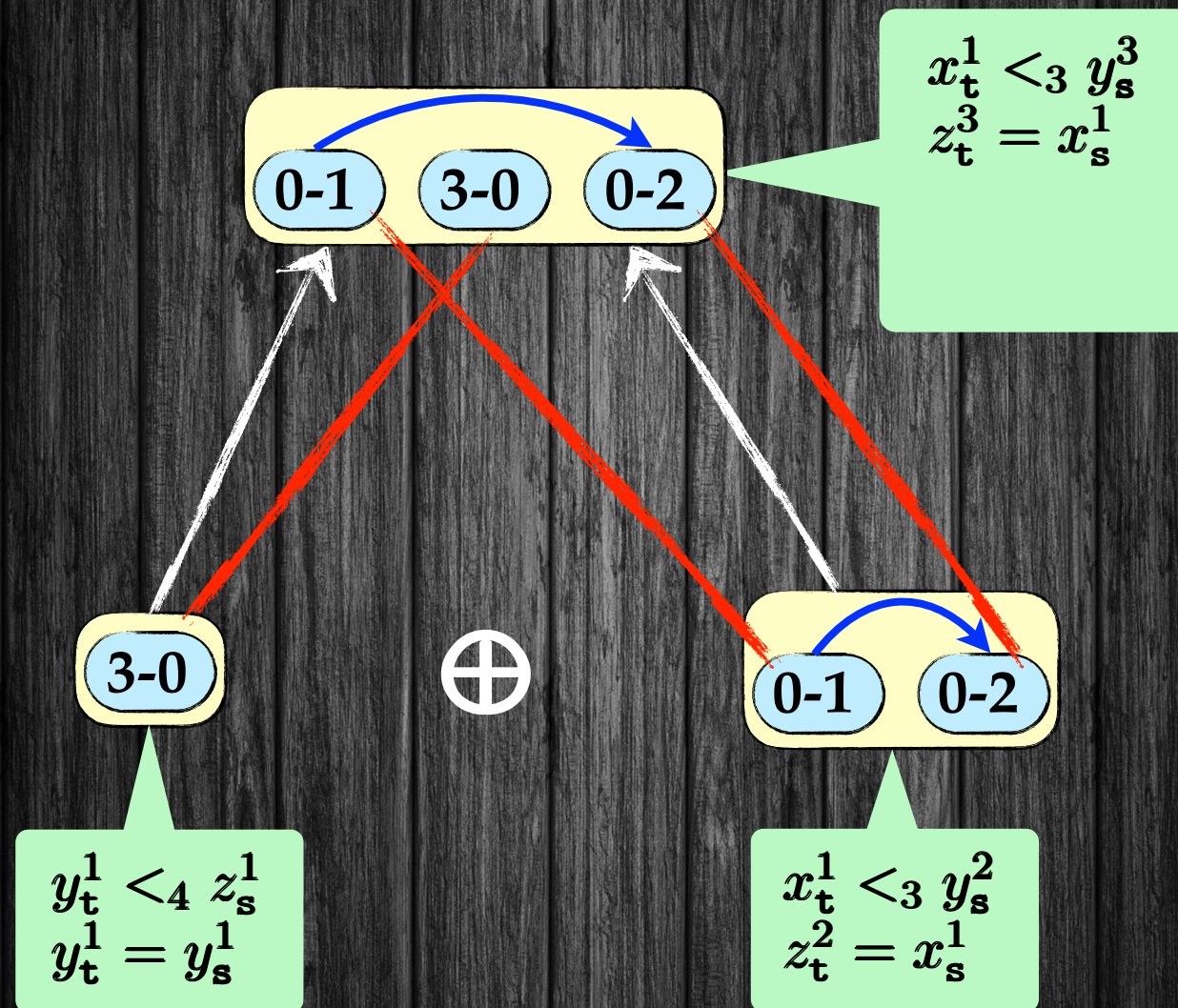
Shuffling



Signatures

Combining Signatures

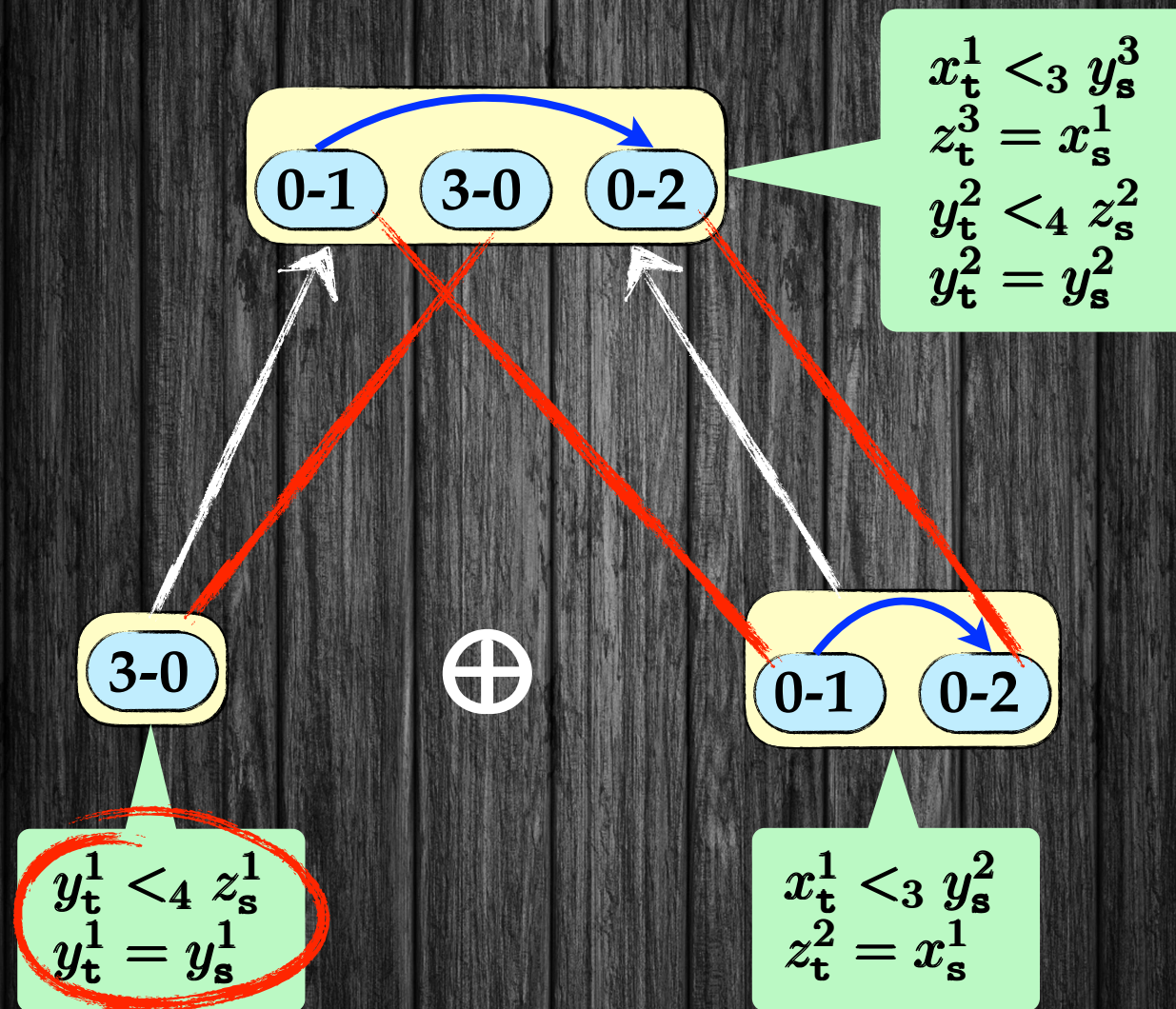
Shuffling



Signatures

Combining Signatures

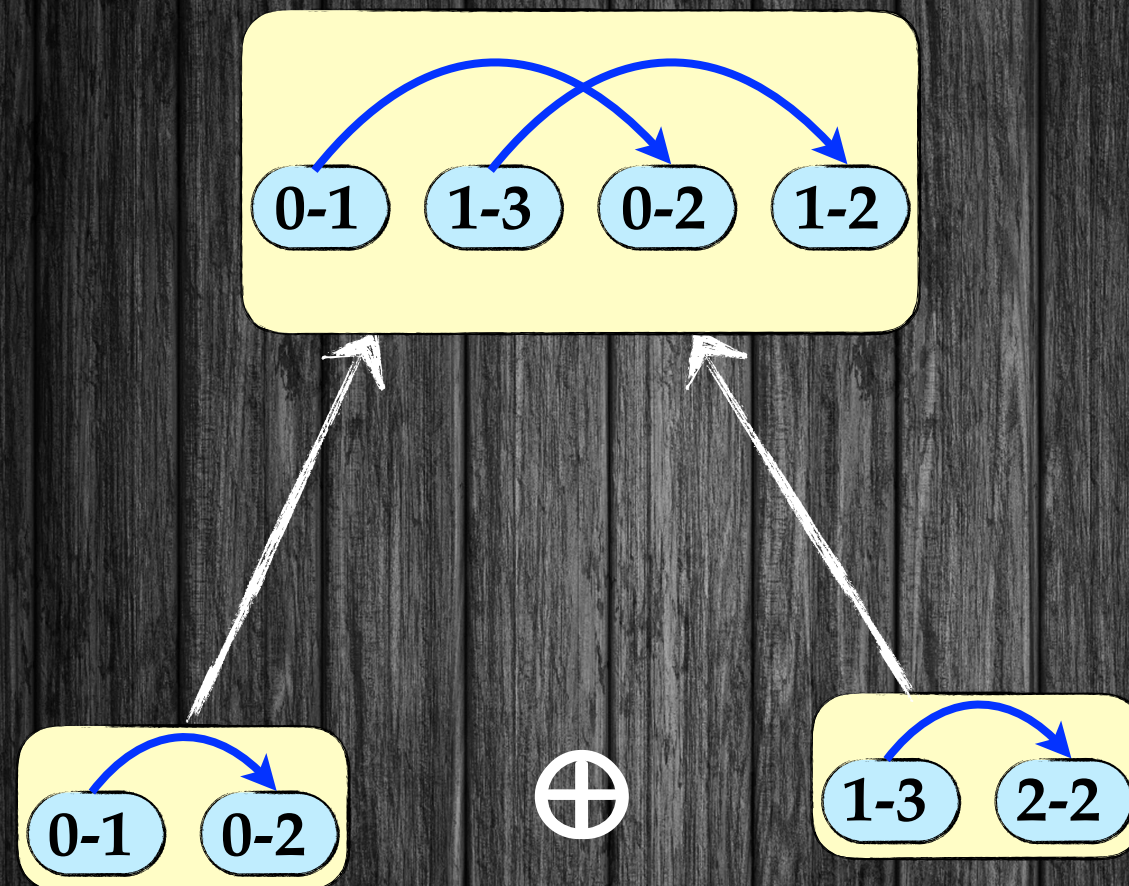
Shuffling



Signatures

Combining Signatures

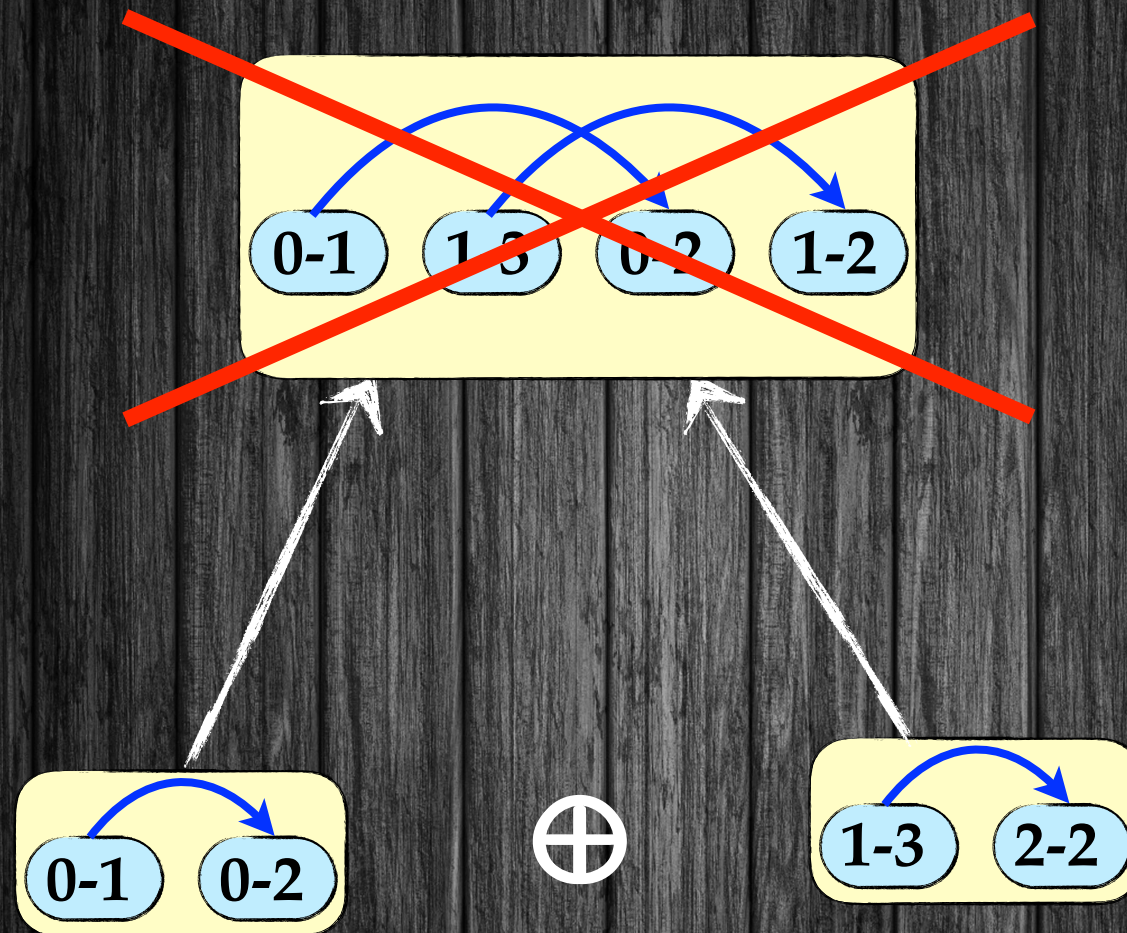
Shuffling



Signatures

Combining Signatu

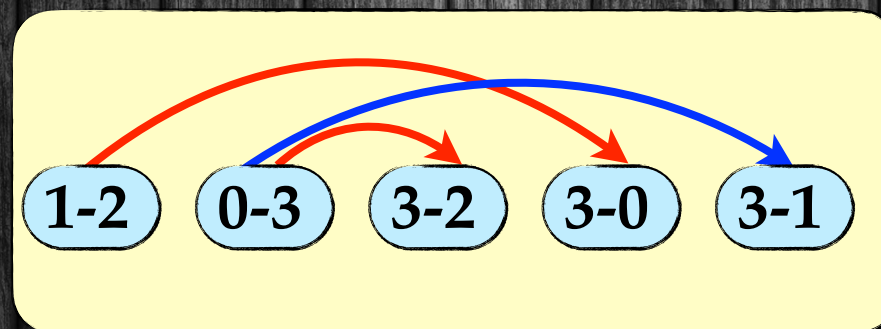
Shuffling



Signatures

Combining Signatu

Contraction



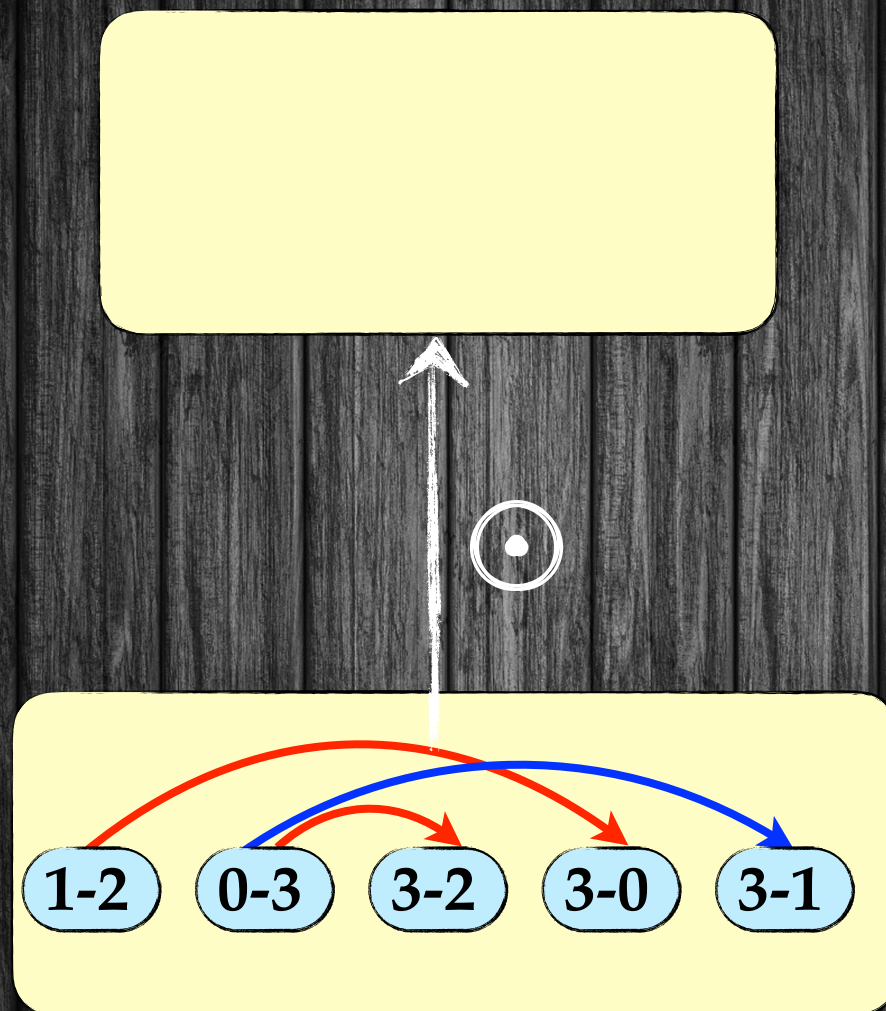
$x_s^1 <_3 y_t^2$
 $z_t^5 <_4 z_s^2$
 $y_s^3 <_2 z_s^4$
 $y_t^5 <_5 x_s^4$

Signatures

Combining Signatu

Contraction

- control states ✓
- push/pop
- data constraints



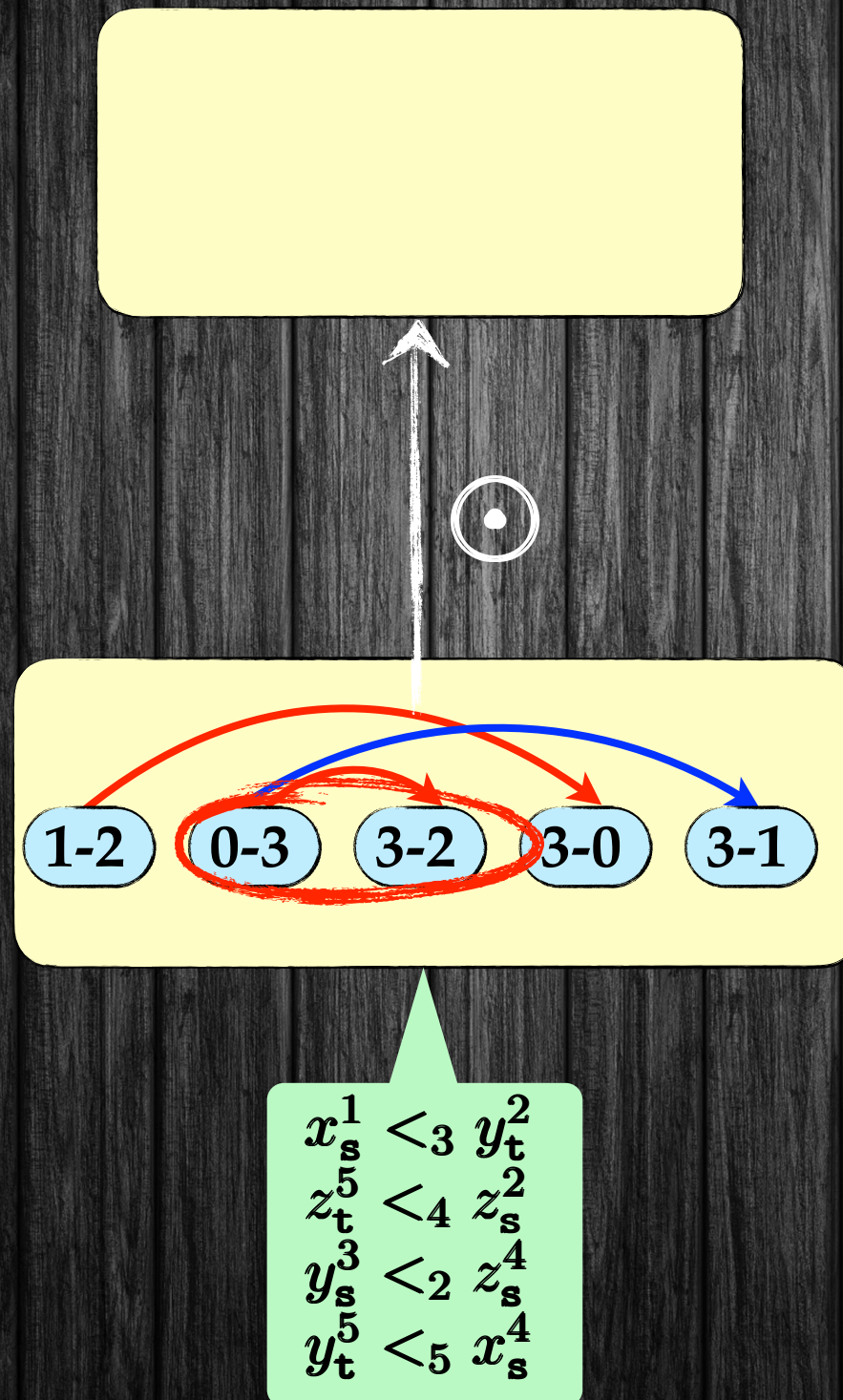
$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

Combining Signatures

Contraction

- **control states** ✓
- **push/pop**
- **data constraints**

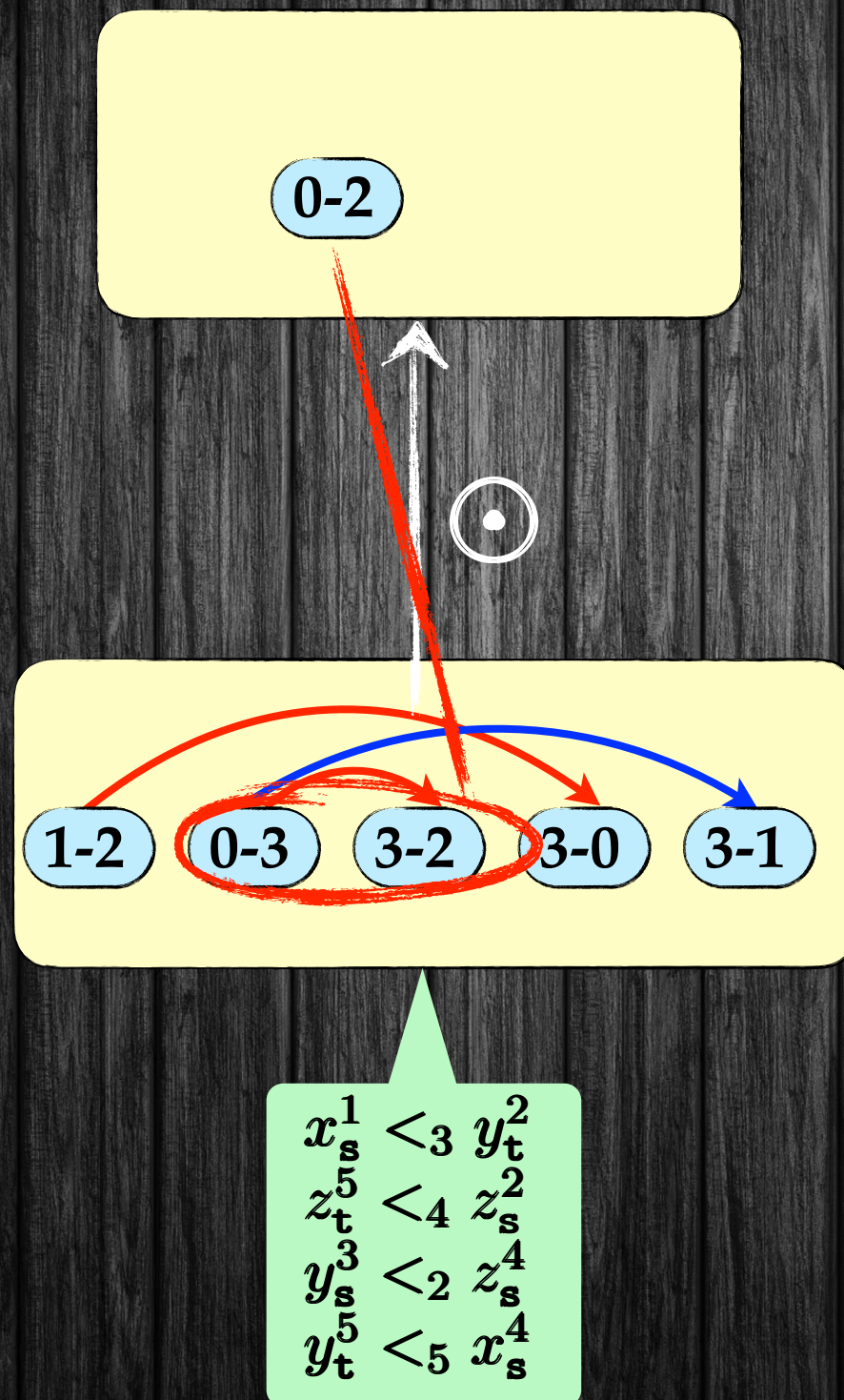


Signatures

Combining Signatu

Contraction

- control states ✓
- push/pop
- data constraints

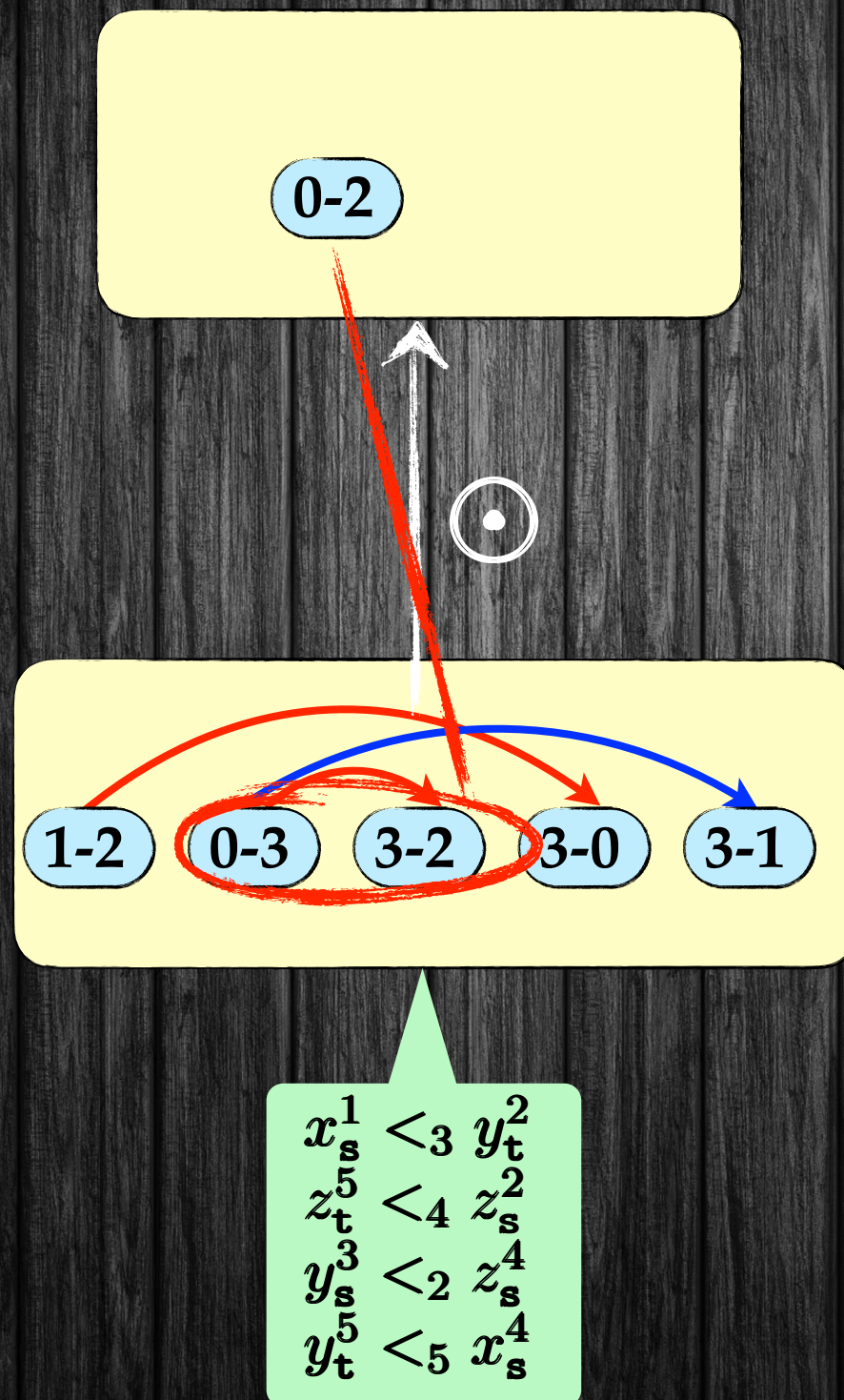


Signatures

Combining Signatu

Contraction

- control states ✓
- push/pop
- data constraints

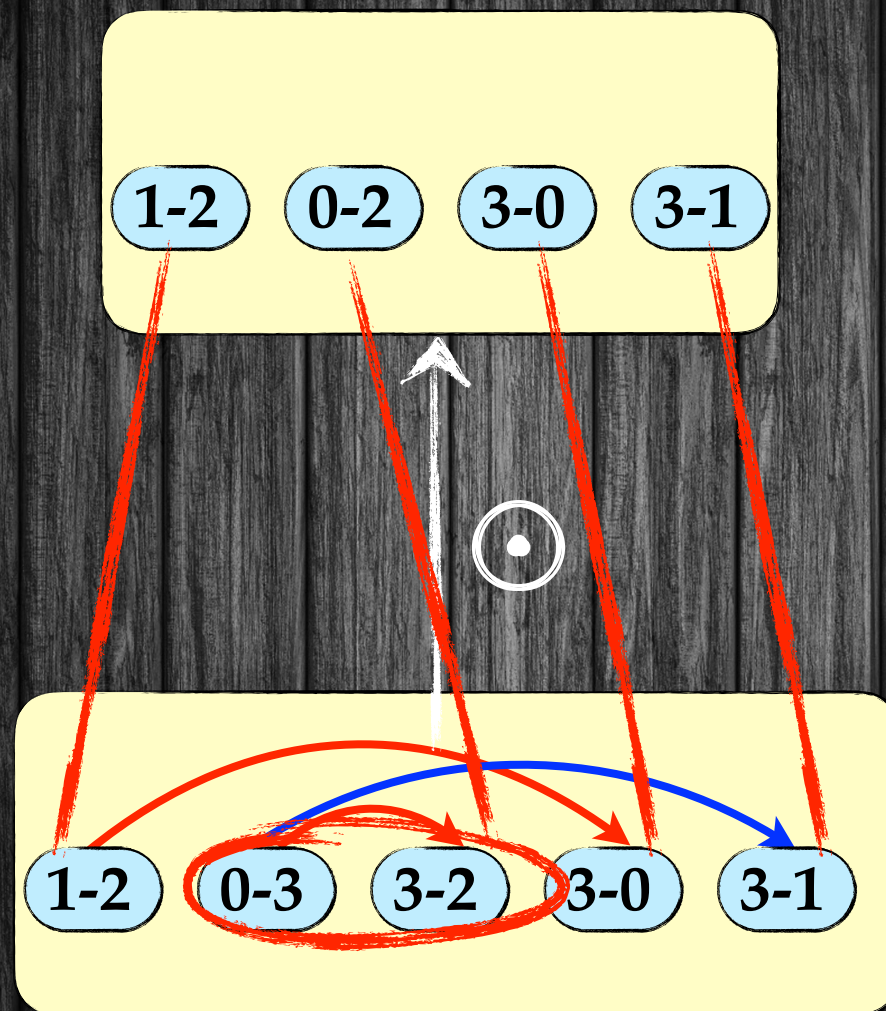


Signatures

Combining Signatu

Contraction

- control states ✓
- push/pop
- data constraints



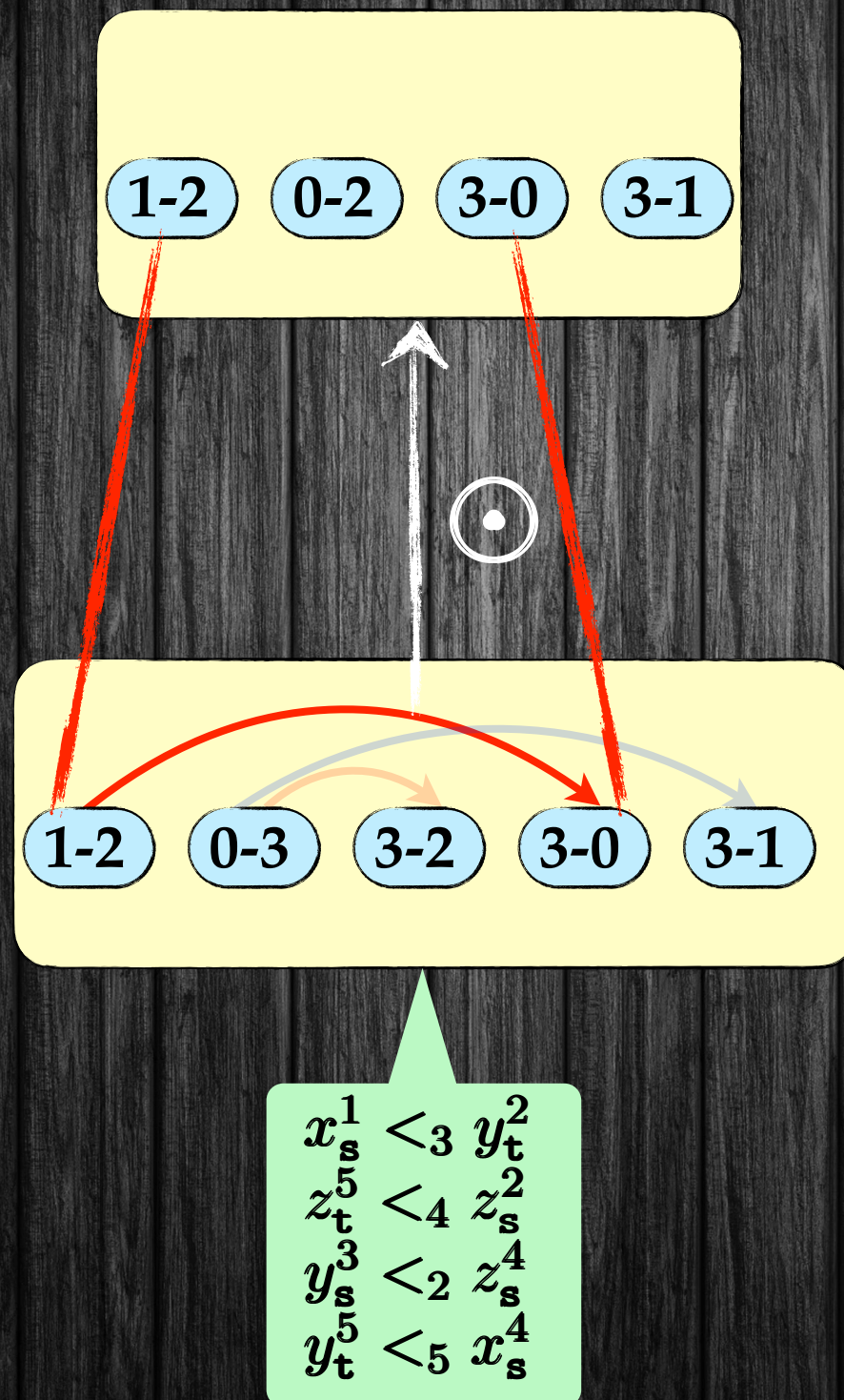
$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures

Combining Signatures

Contraction

- control states
- push/pop ✓
- data constraints

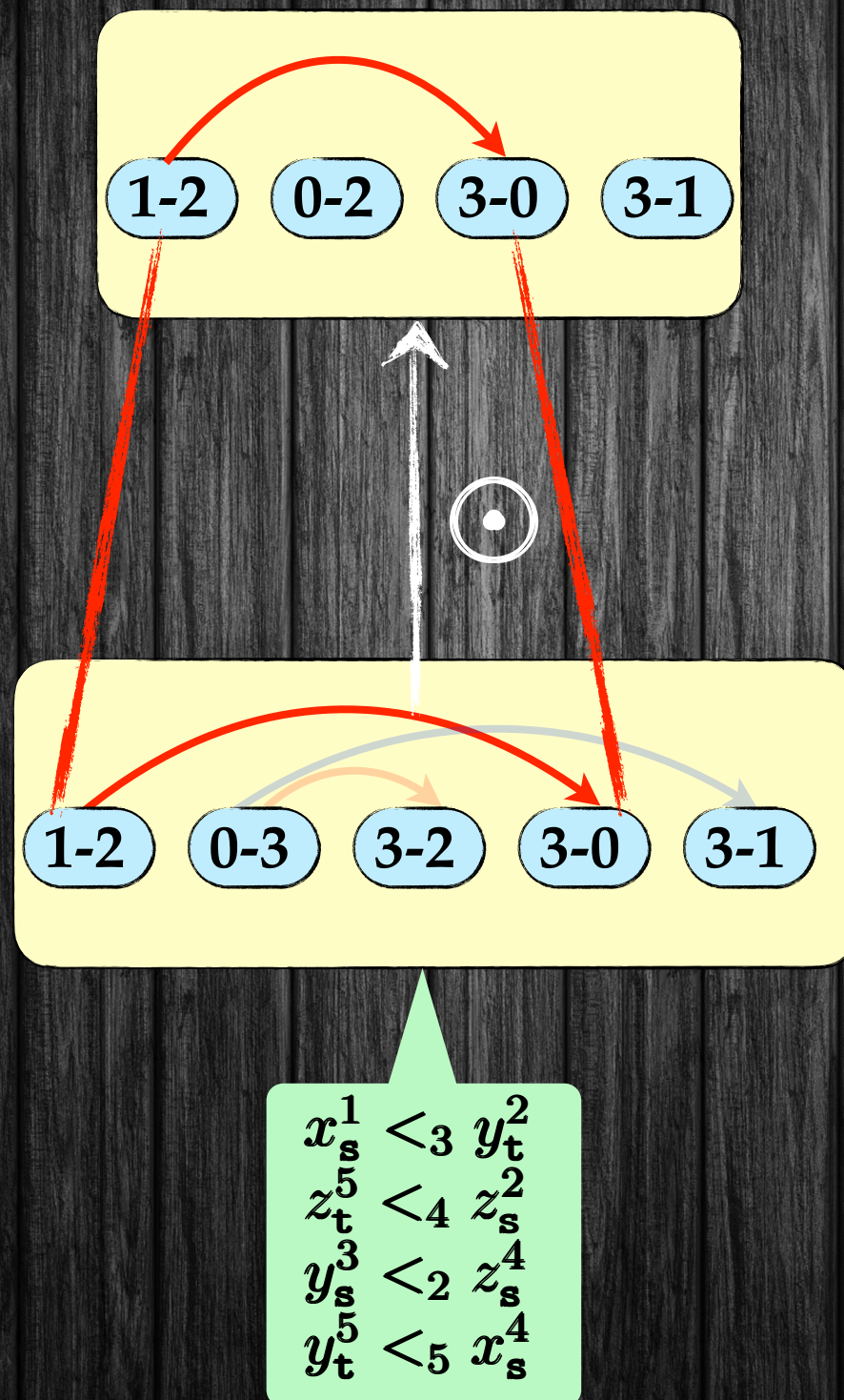


Signatures

Combining Signatures

Contraction

- control states
- push/pop ✓
- data constraints

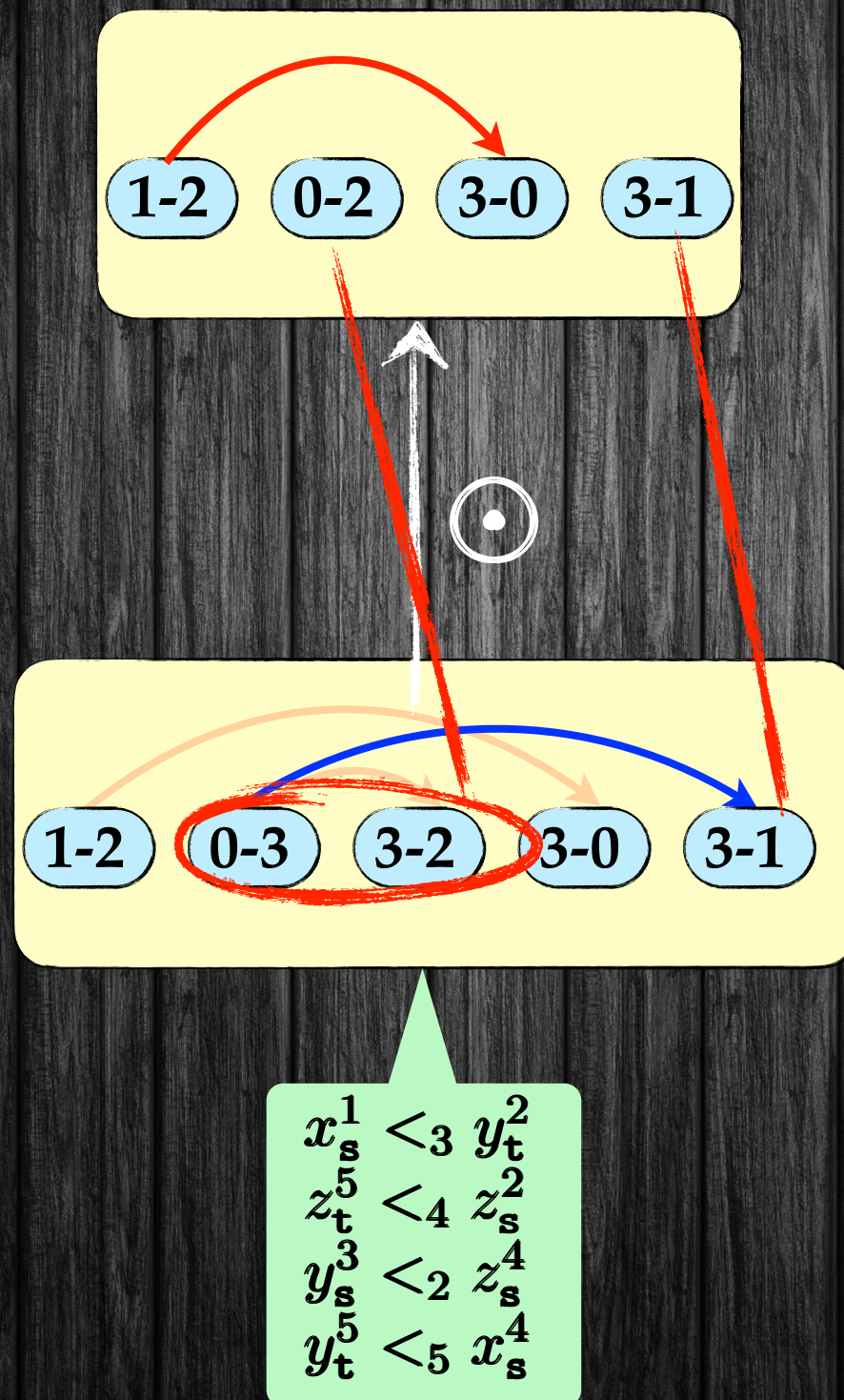


Signatures

Combining Signatu

Contraction

- control states
- push/pop ✓
- data constraints

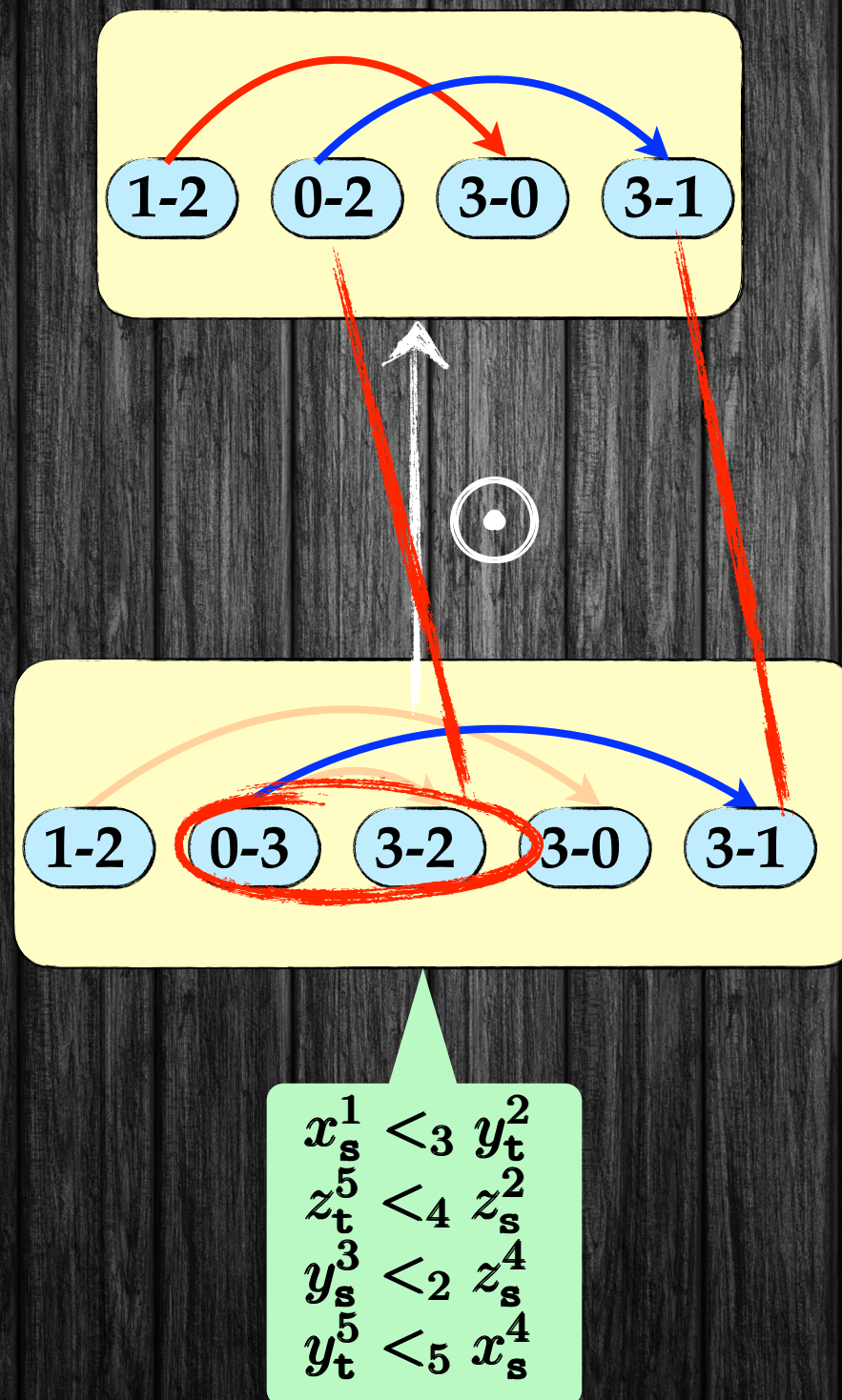


Signatures

Combining Signatu

Contraction

- control states
- push/pop ✓
- data constraints

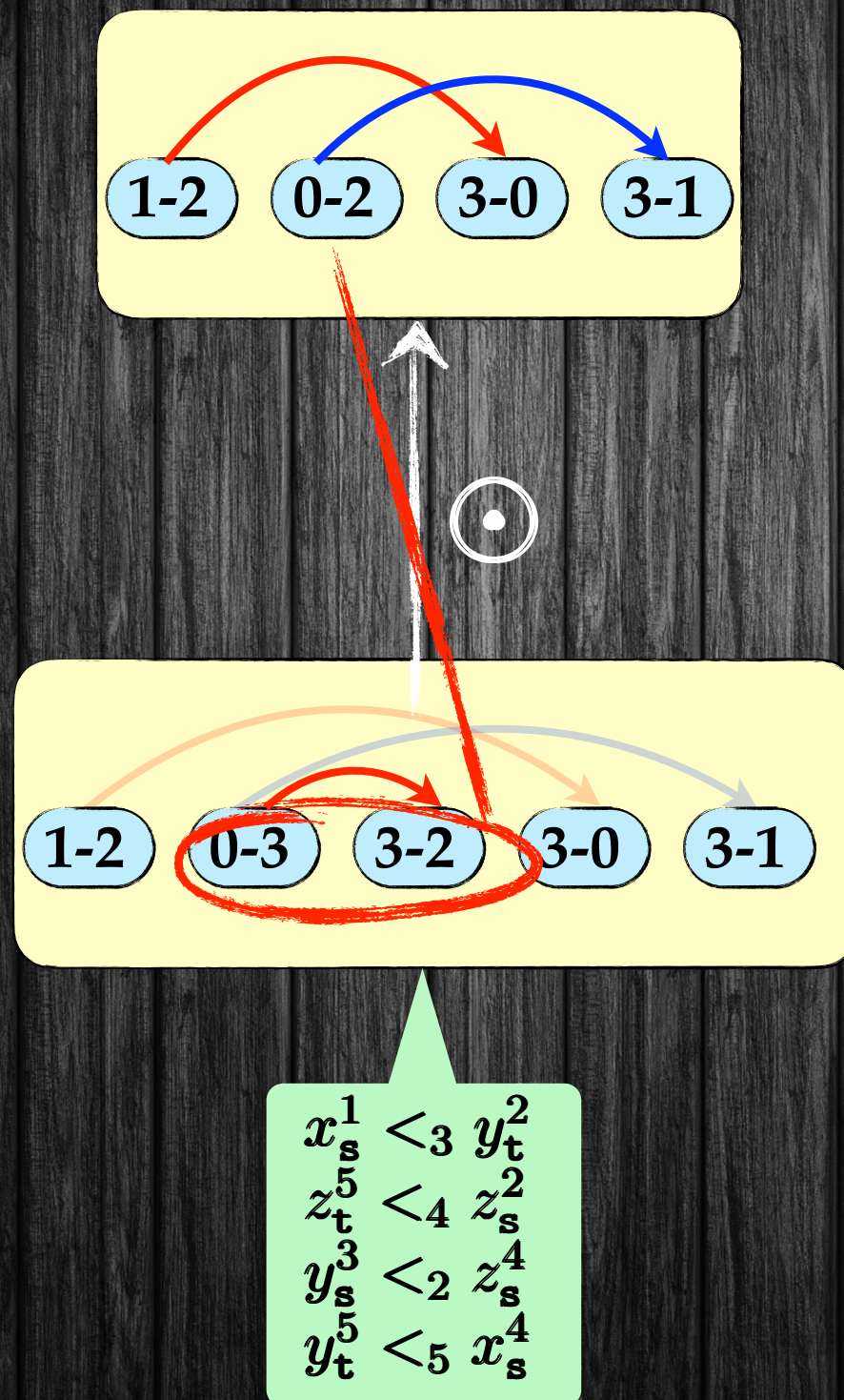


Signatures

Combining Signatu

Contraction

- control states
- push/pop ✓
- data constraints

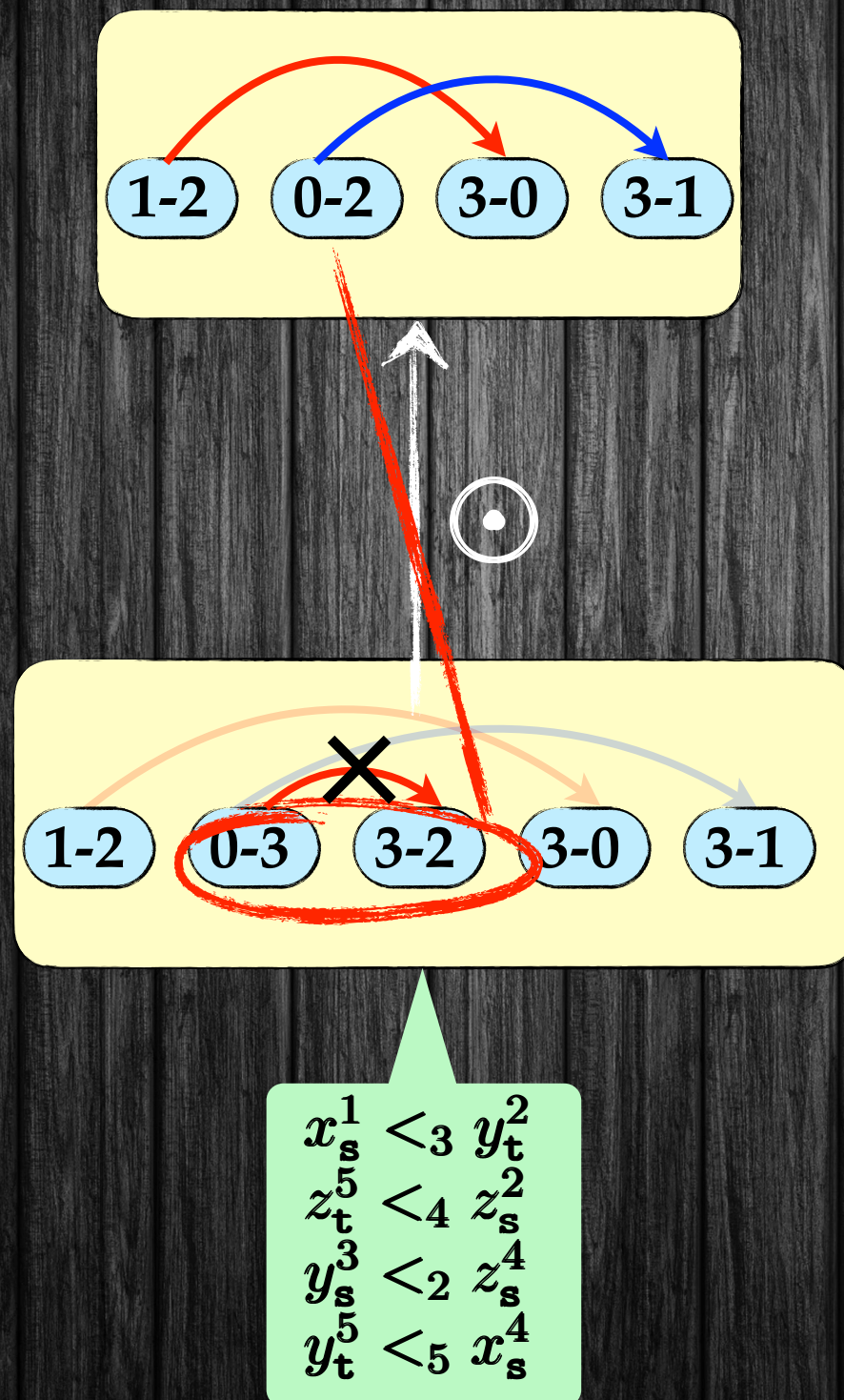


Signatures

Combining Signatu

Contraction

- control states
- push/pop ✓
- data constraints

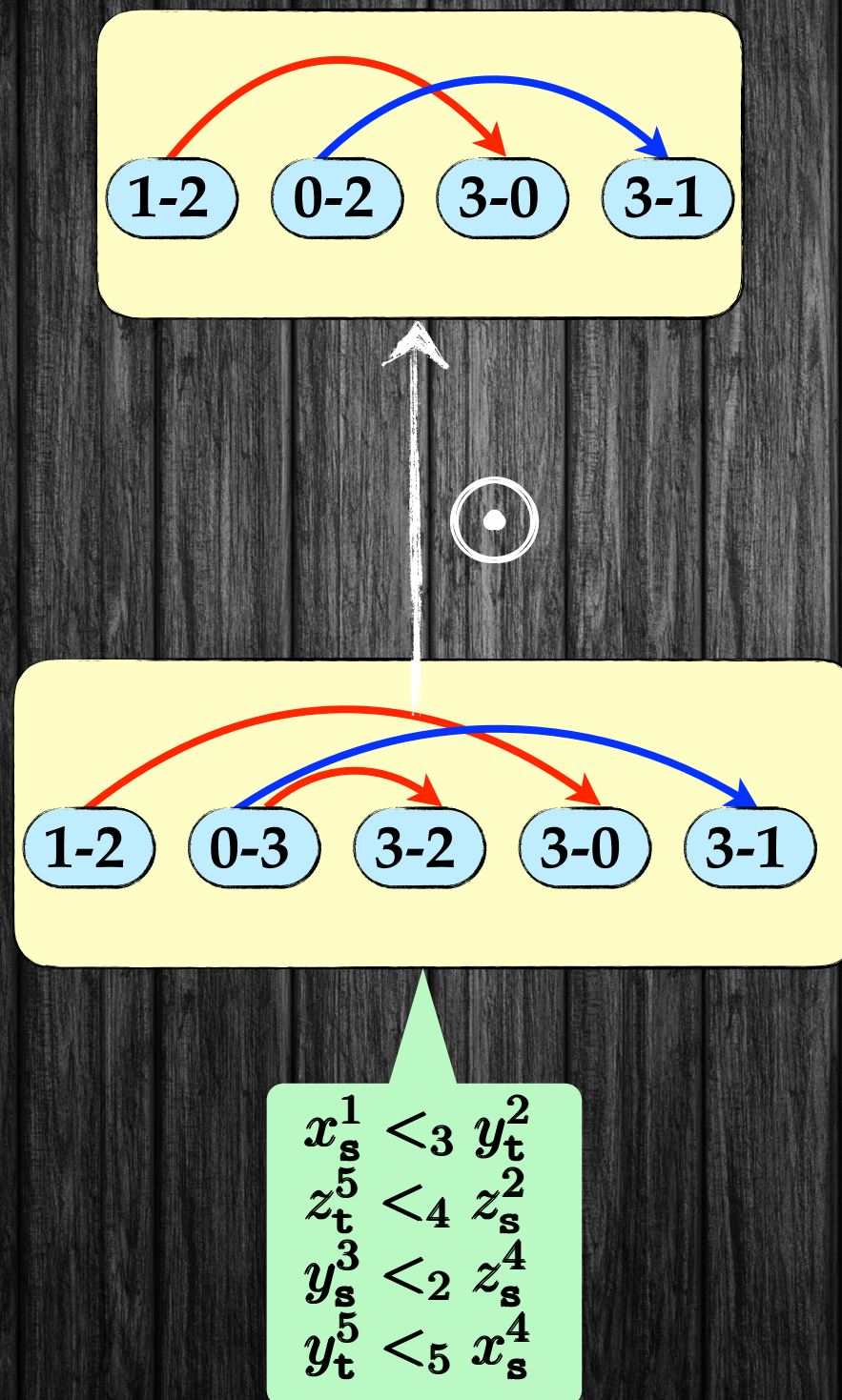


Signatures

Combining Signatu

Contraction

- control states
- push/pop
- data constraints ✓

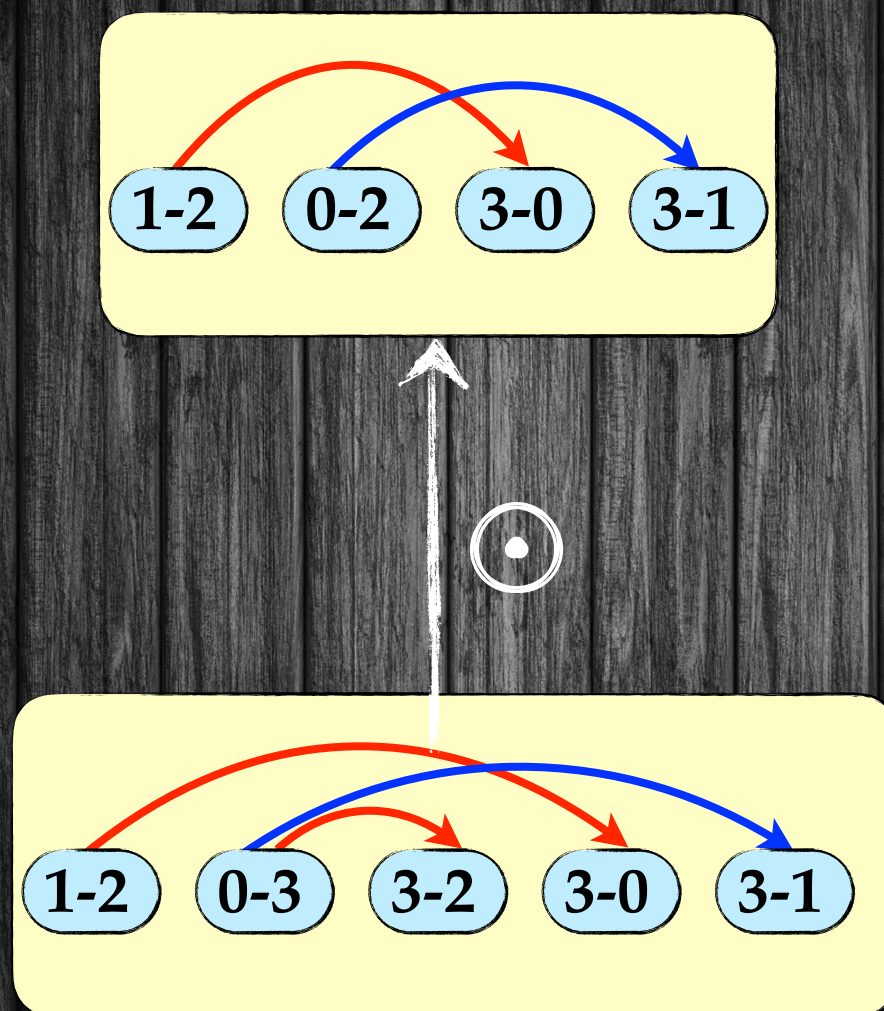


Signatures

Combining Signatures

Contraction

- control states
- push/pop
- data constraints ✓



1. equating variables
2. saturation
3. variable elimination
4. renaming

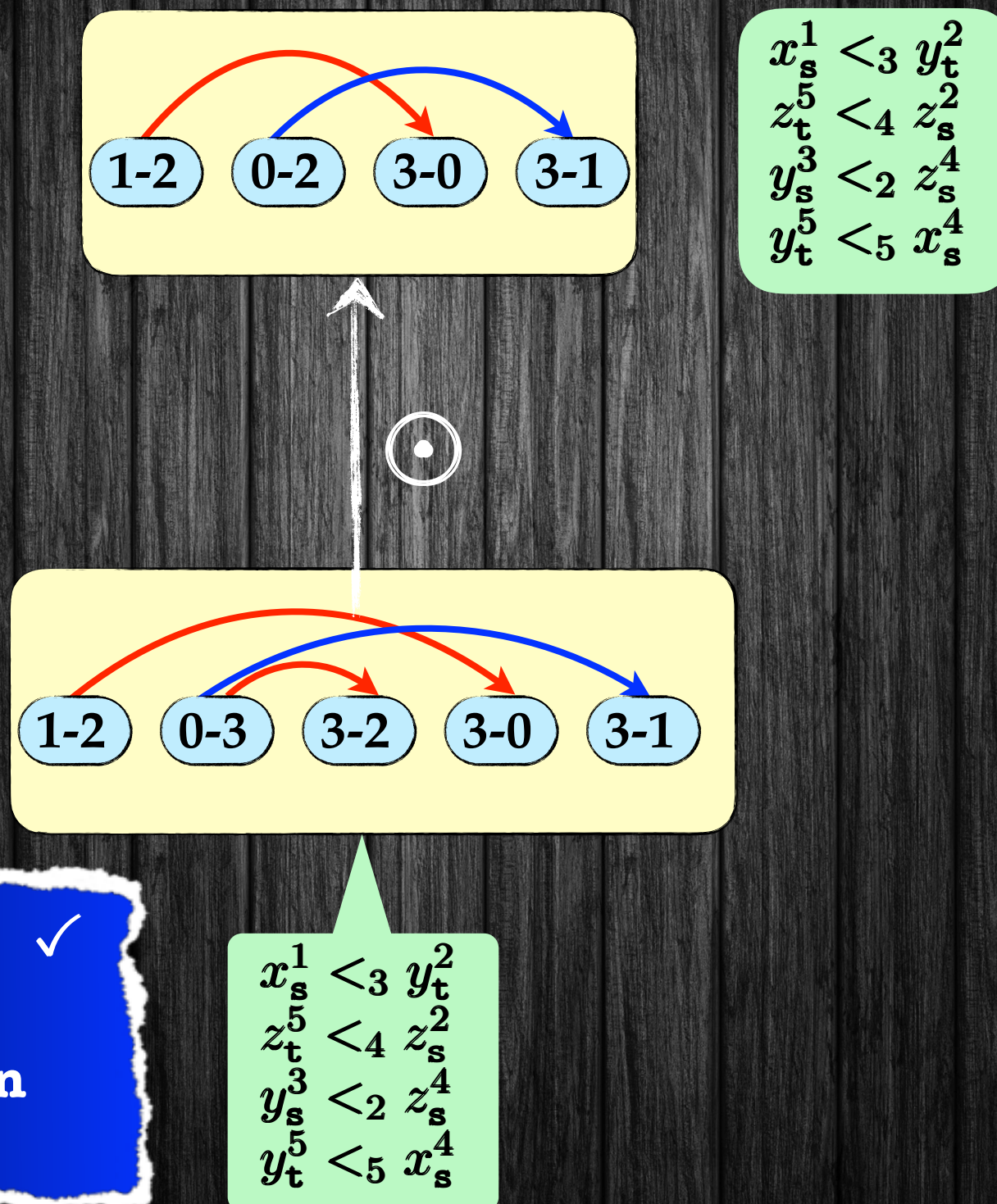
$$\begin{array}{lcl} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures

Combining Signatures

Contraction

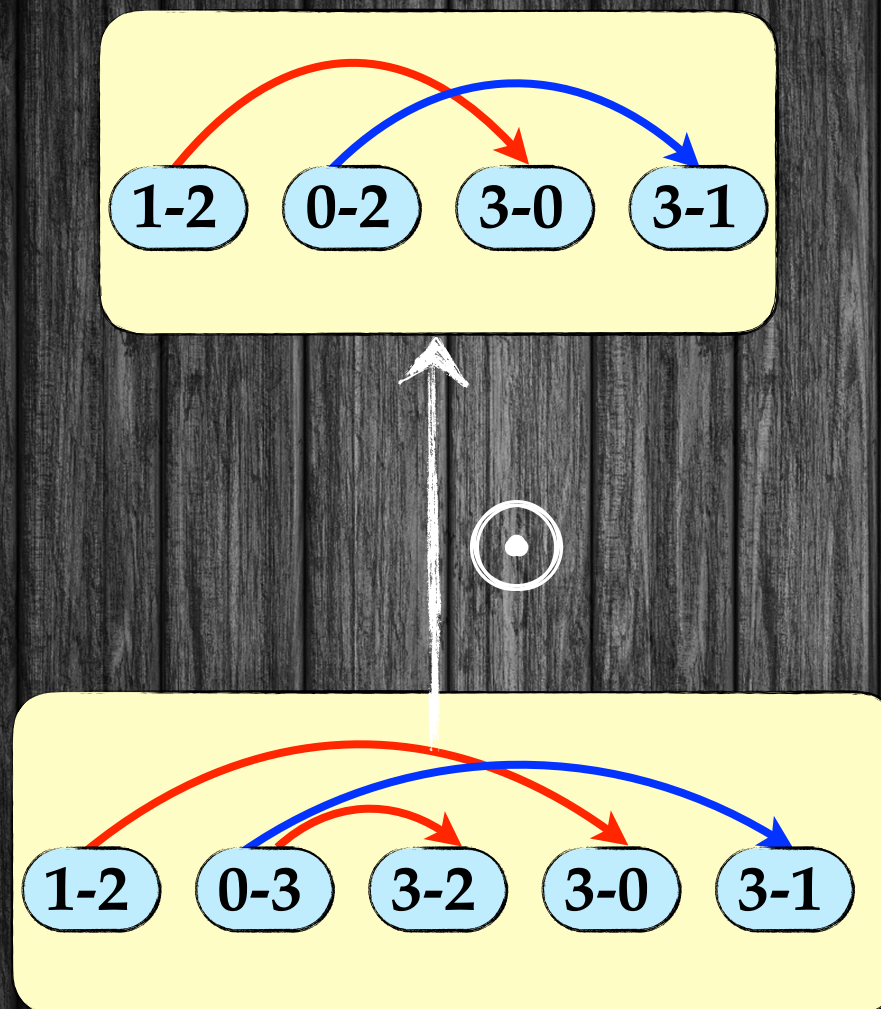
- control states
- push/pop
- data constraints ✓



1. equating variables ✓
2. saturation
3. variable elimination
4. renaming

Signatures Combining Signatu Contraction

- control states
- push/pop
- data constraints ✓



$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

+

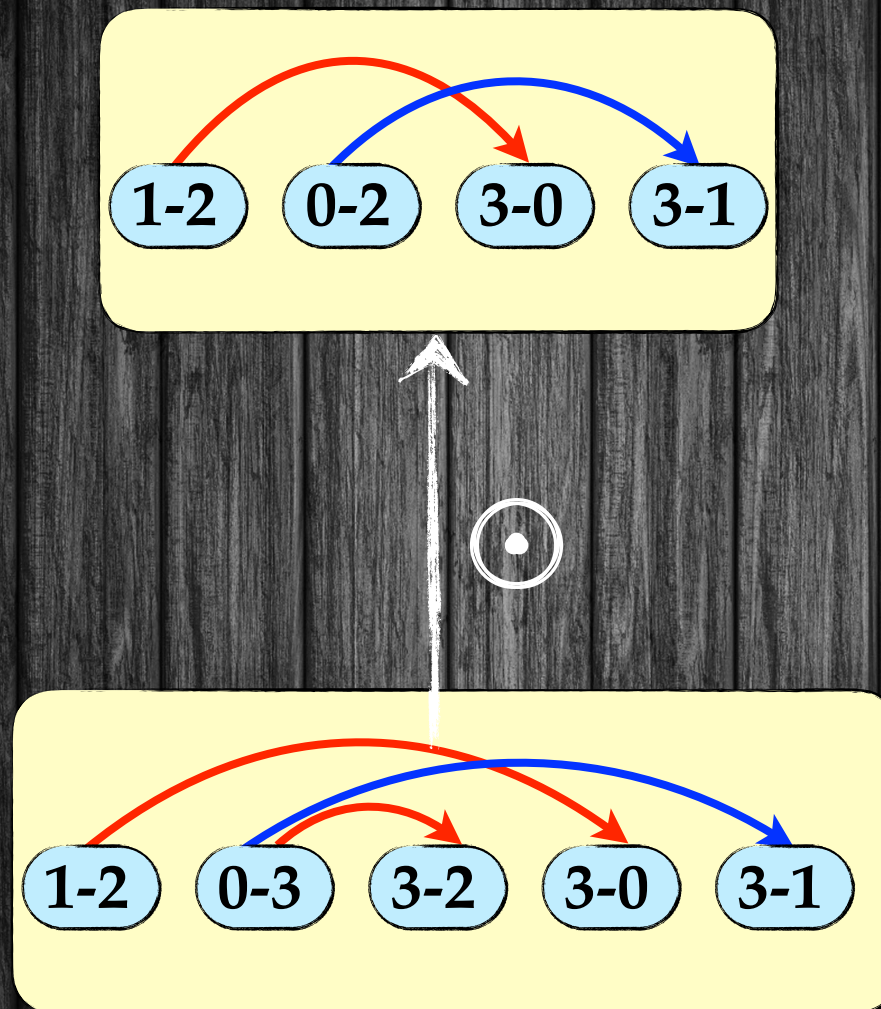
$$\begin{array}{l} x_s^3 = x_t^2 \\ y_s^3 = y_t^2 \\ z_s^3 = z_t^2 \end{array}$$

1. equating variables ✓
2. saturation
3. variable elimination
4. renaming

$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatu Contraction

- control states
- push/pop
- data constraints ✓



$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

+

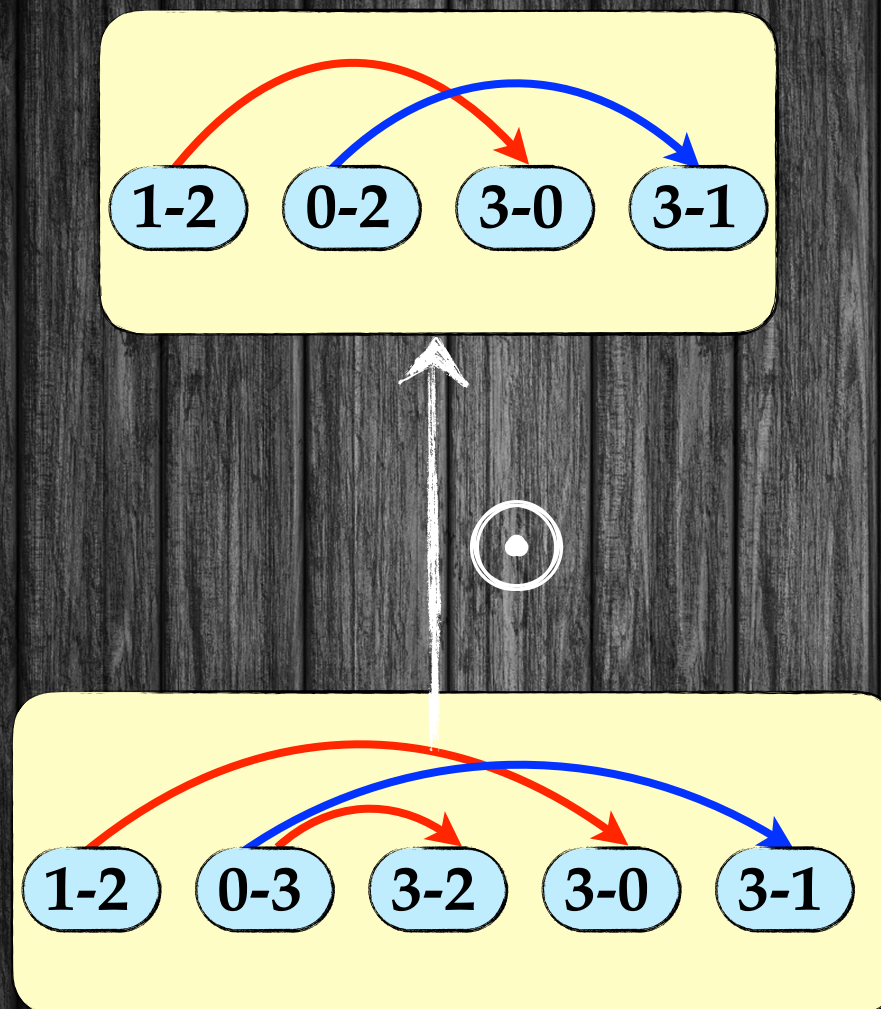
$$\begin{array}{l} x_s^3 = x_t^2 \\ y_s^3 = y_t^2 \\ z_s^3 = z_t^2 \end{array}$$

1. equating variables
2. saturation ✓
3. variable elimination
4. renaming

$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

- control states
- push/pop
- data constraints ✓



$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

+

$$\begin{array}{l} x_s^3 = x_t^2 \\ y_s^3 = y_t^2 \\ z_s^3 = z_t^2 \end{array}$$

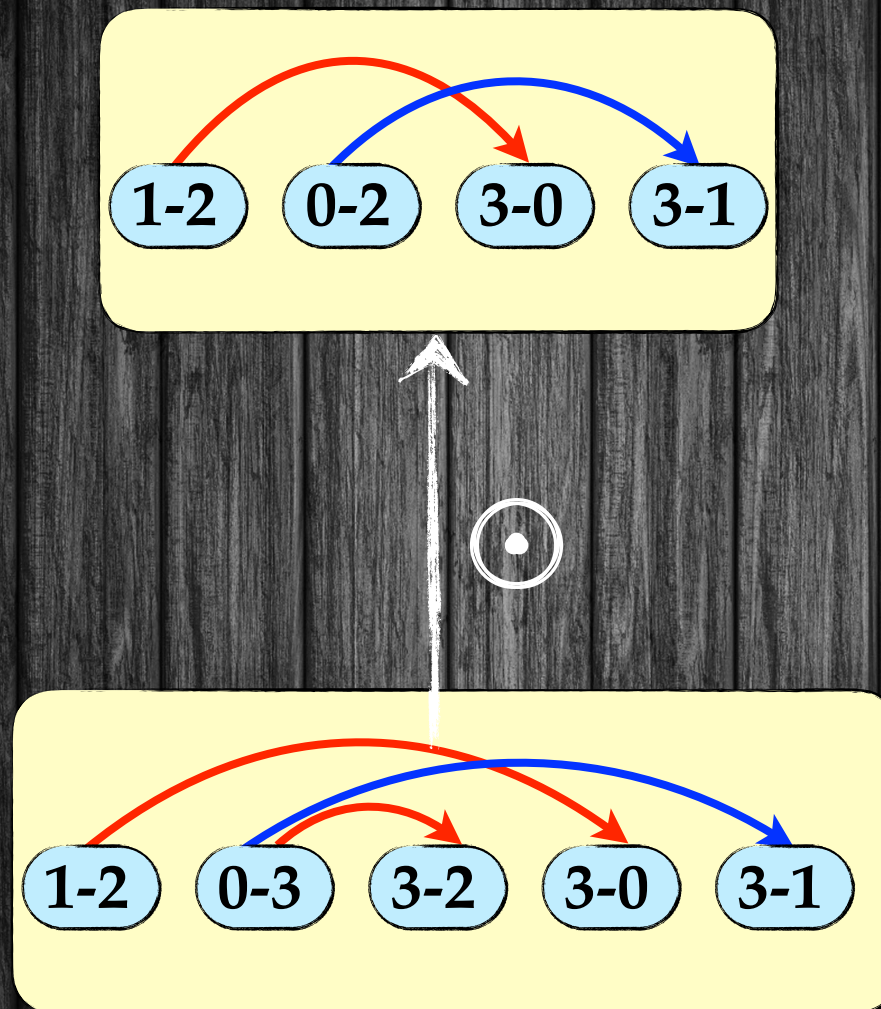
$$\begin{array}{l} x_s^1 <_3 y_s^3 \\ y_t^2 <_2 z_s^4 \\ x_s^1 <_6 z_s^4 \end{array}$$

1. equating variables
2. saturation
3. variable elimination ✓
4. renaming

$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

- control states
- push/pop
- data constraints ✓



$$\exists x_t^2 y_t^2 z_t^2 x_s^3 y_s^3 z_s^3.$$

$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

+

$$\begin{array}{l} x_s^3 = x_t^2 \\ y_s^3 = y_t^2 \\ z_s^3 = z_t^2 \end{array}$$

$$\begin{array}{l} x_s^1 <_3 y_s^3 \\ y_t^2 <_2 z_s^4 \\ x_s^1 <_6 z_s^4 \end{array}$$

1. equating variables
2. saturation
3. variable elimination ✓
4. renaming

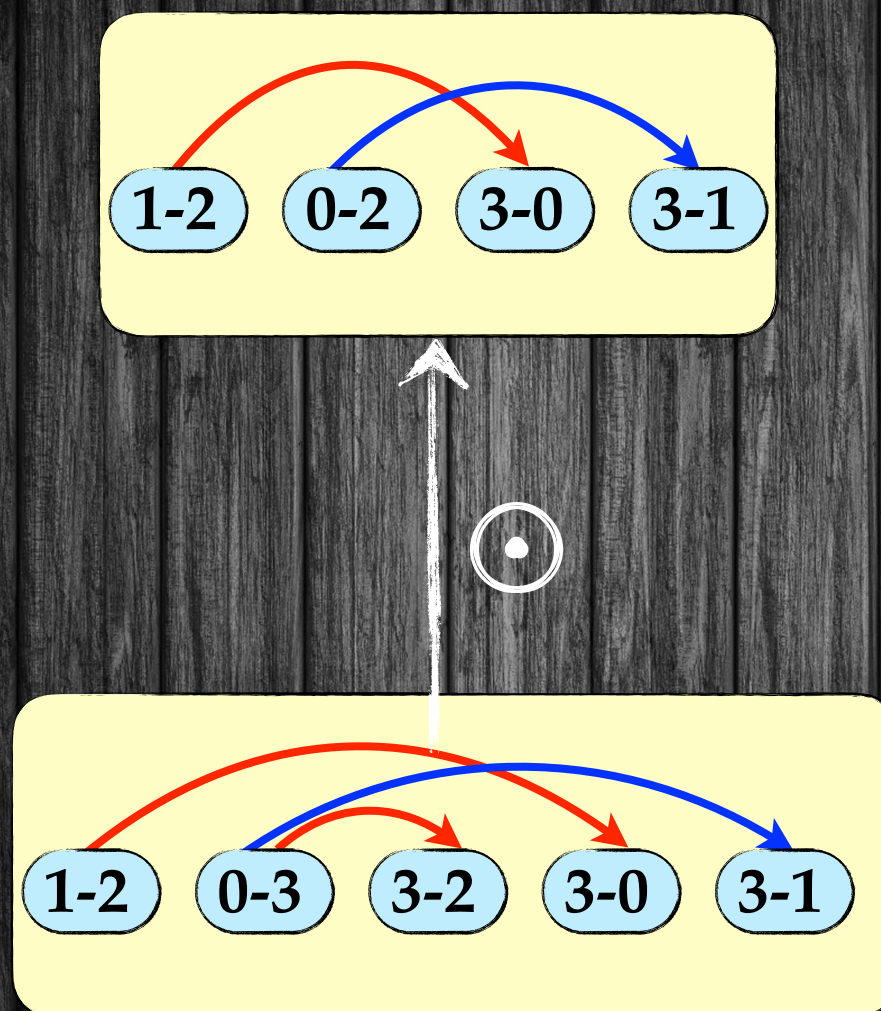
$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures

Combining Signatures

Contraction

- control states
- push/pop
- data constraints ✓



$$z_t^5 <_4 z_s^2$$

$$y_t^5 <_5 x_s^4$$

$$x_s^1 <_6 z_s^4$$

1. equating variables
2. saturation
3. variable elimination
4. renaming ✓

$$\begin{array}{l} x_s^1 <_3 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

- control states
- push/pop
- data constraints ✓

segment 1

segment 2

segment 3

segment 4

1-2

0-2

3-0

3-1

$$z_t^5 <_4 z_s^2$$

$$y_t^5 <_5 x_s^4$$

$$x_s^1 <_6 z_s^4$$

1-2

0-3

3-2

3-0

3-1

segment 1

segment 2

segment 3

segment 4

segment 5

1. equal
2. saturation
3. variable elimination
4. renaming ✓

$$\begin{array}{l} x_s^1 <_6 y_t^5 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

- control states
- push/pop
- data constraints ✓

segment 1

segment 2

segment 3

segment 4

1-2

0-2

3-0

3-1

$$z_t^5 <_4 z_s^2$$

$$y_t^5 <_5 x_s^4$$

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1-2

0-3

3-2

3-0

3-1

segment 1

segment 2

segment 3

segment 4

segment 5

1. equal
2. saturation
3. variable elimination
4. renaming ✓

$$\begin{array}{l} x_s^1 <_6 y_t^5 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatu Contraction

segment 1

segment 2

segment 3

segment 4

- control states
- push/pop
- data constraints ✓

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0-2

3-0

3-1

$$z_t^5 <_4 z_s^2$$

$$y_t^5 <_5 x_s^4$$

$$x_s^1 <_6 z_s^4$$

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3-1

segment 1

segment 2

segment 3

segment 4

segment 5

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2. saturation
3. variable elimination
4. renaming ✓

$$\begin{array}{l} x_s^1 <_6 y_t^5 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

- control states
- push/pop
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0-3

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3-1

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segment 5

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$$\begin{array}{l} x_s^5 <_5 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures Combining Signatures Contraction

- control states
- push/pop
- data constraints ✓

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segment 2

segment 3

segment 4

1-2

0-2

3-0

3-1

$$z_t^5 <_4 z_s^2$$

$$z_t^4 <_4 z_s^2$$

$$y_t^5 <_5 x_s^4$$

$$y_t^4 <_5 x_s^3$$

$$x_s^1 <_6 z_s^4$$

$$x_s^1 <_6 z_s^3$$

1-2

0-3

3-2

3-0

3-1

segment 1

segment 2

segment 3

segment 4

segment 5

1. equal
2. saturation
3. variable elimination
4. renaming ✓

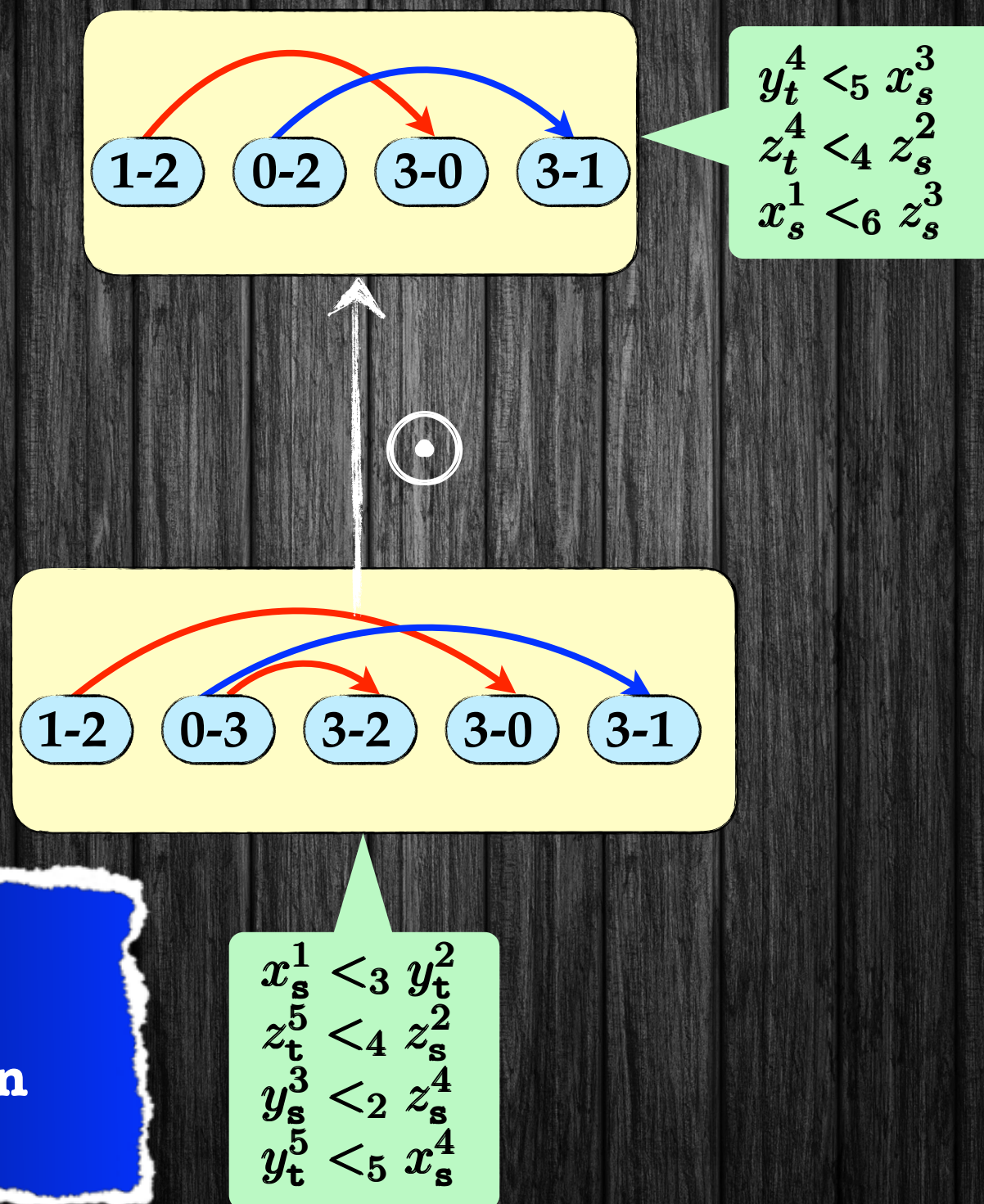
$$\begin{array}{l} x_s^5 <_5 y_t^2 \\ z_t^5 <_4 z_s^2 \\ y_s^3 <_2 z_s^4 \\ y_t^5 <_5 x_s^4 \end{array}$$

Signatures

Combining Signatures

Contraction

- control states
- push/pop
- data constraints ✓

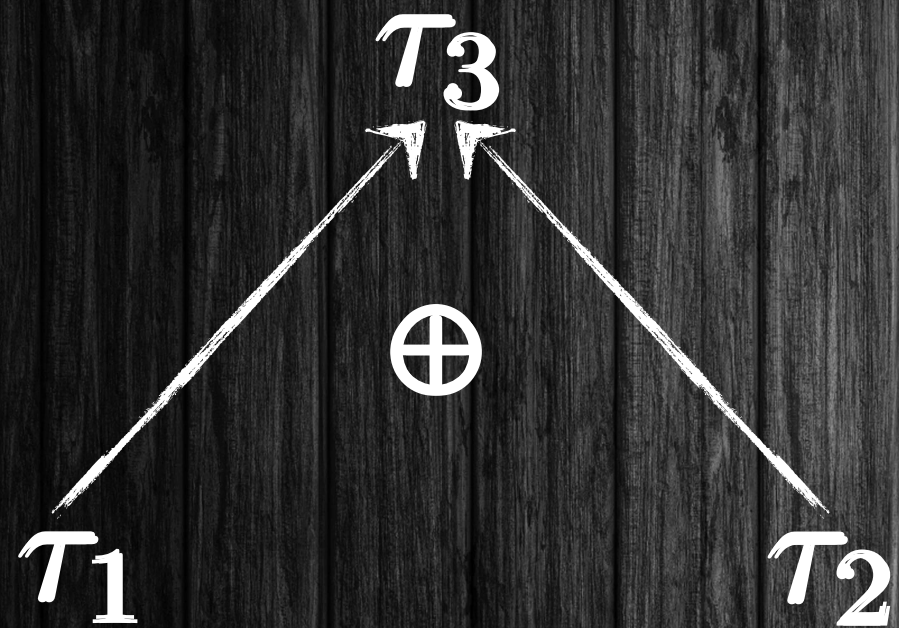


1. equating variables
2. saturation
3. variable elimination
4. renaming ✓

Signatures

Combining Signatu

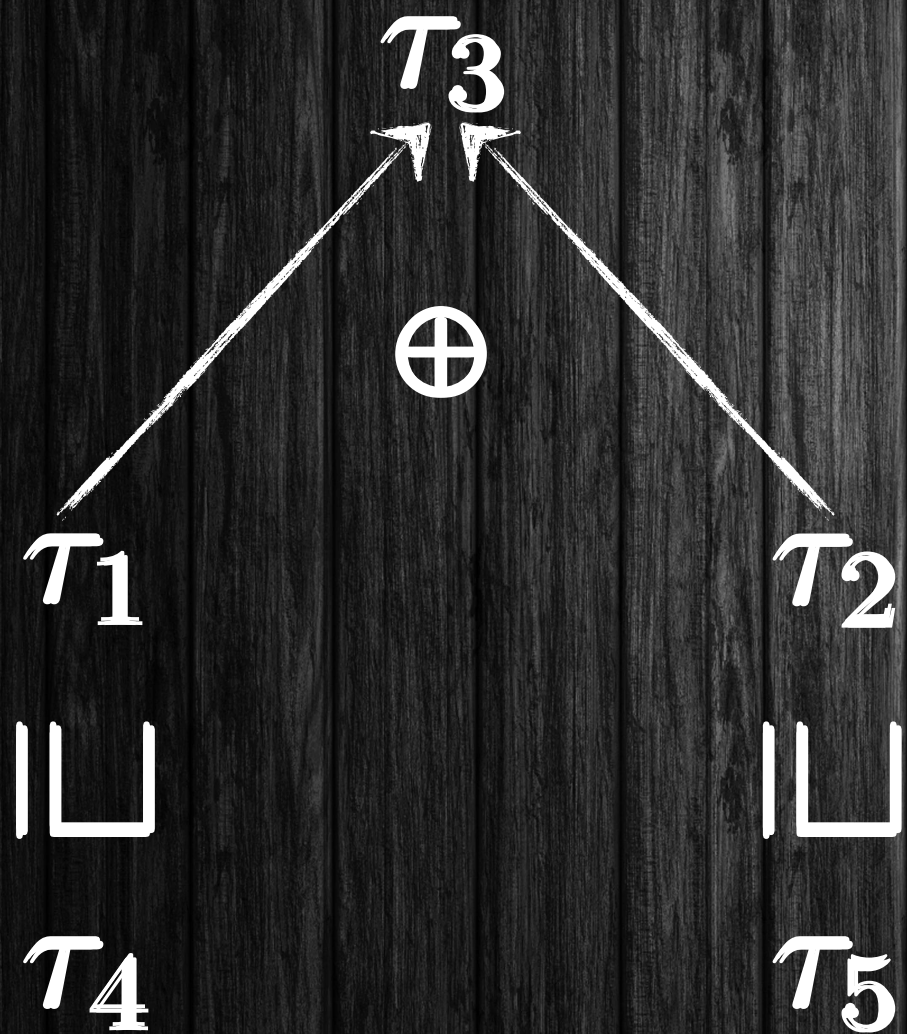
Monotonicity



Signatures

Combining Signatu

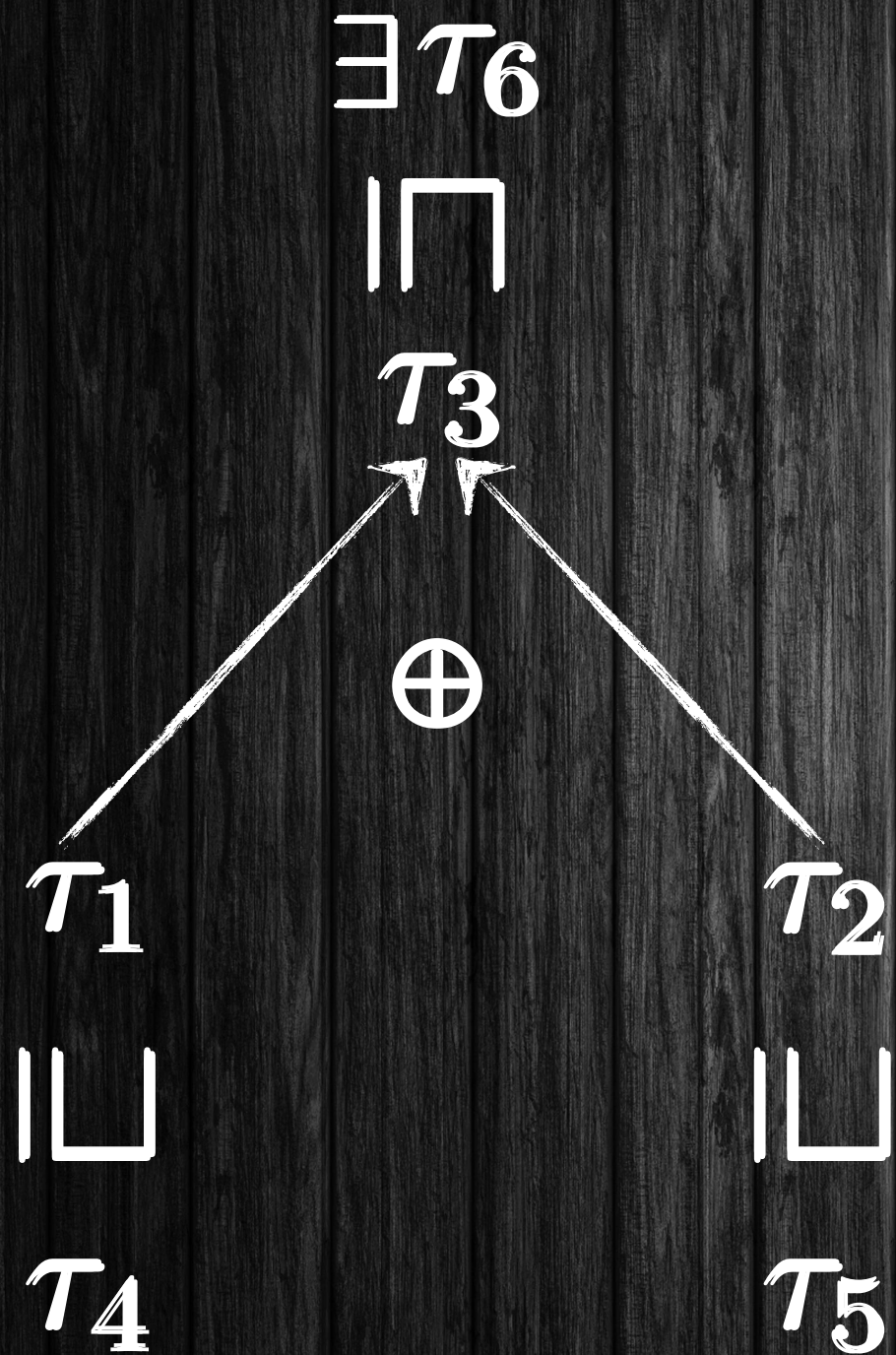
Monotonicity



Signatures

Combining Signatu

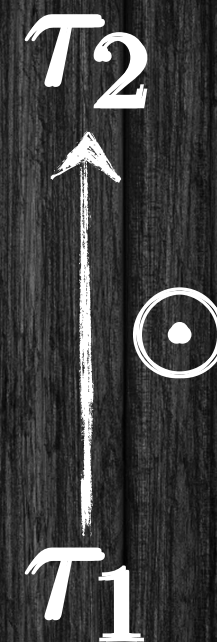
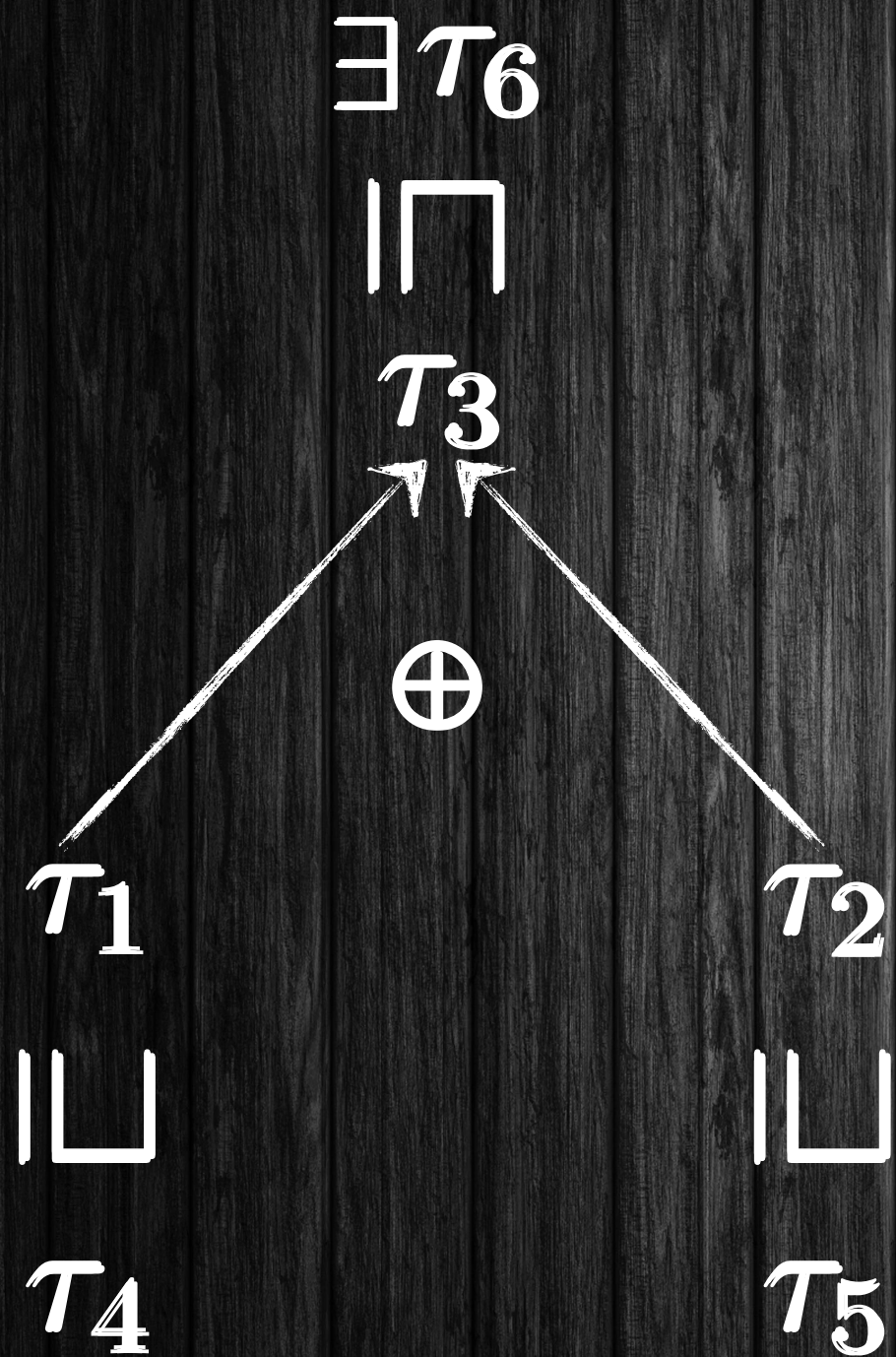
Monotonicity



Signatures

Combining Signatu

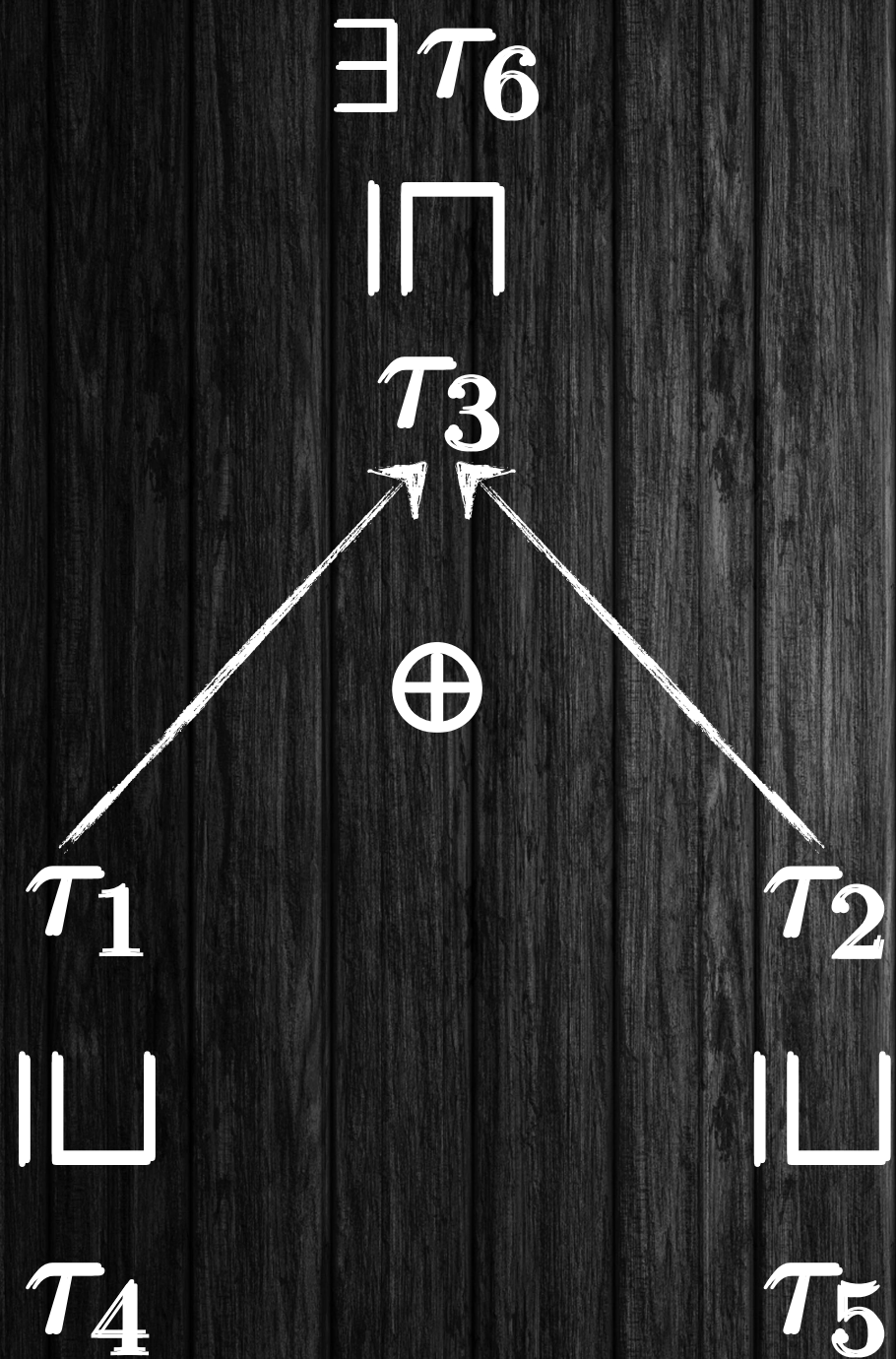
Monotonicity



Signatures

Combining Signatu

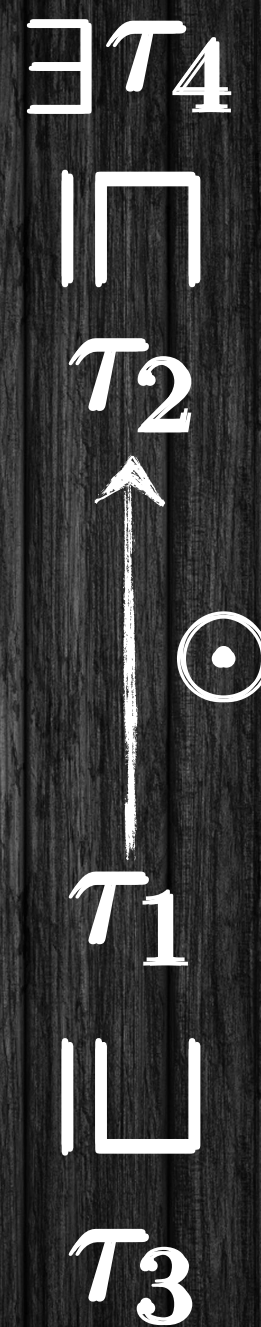
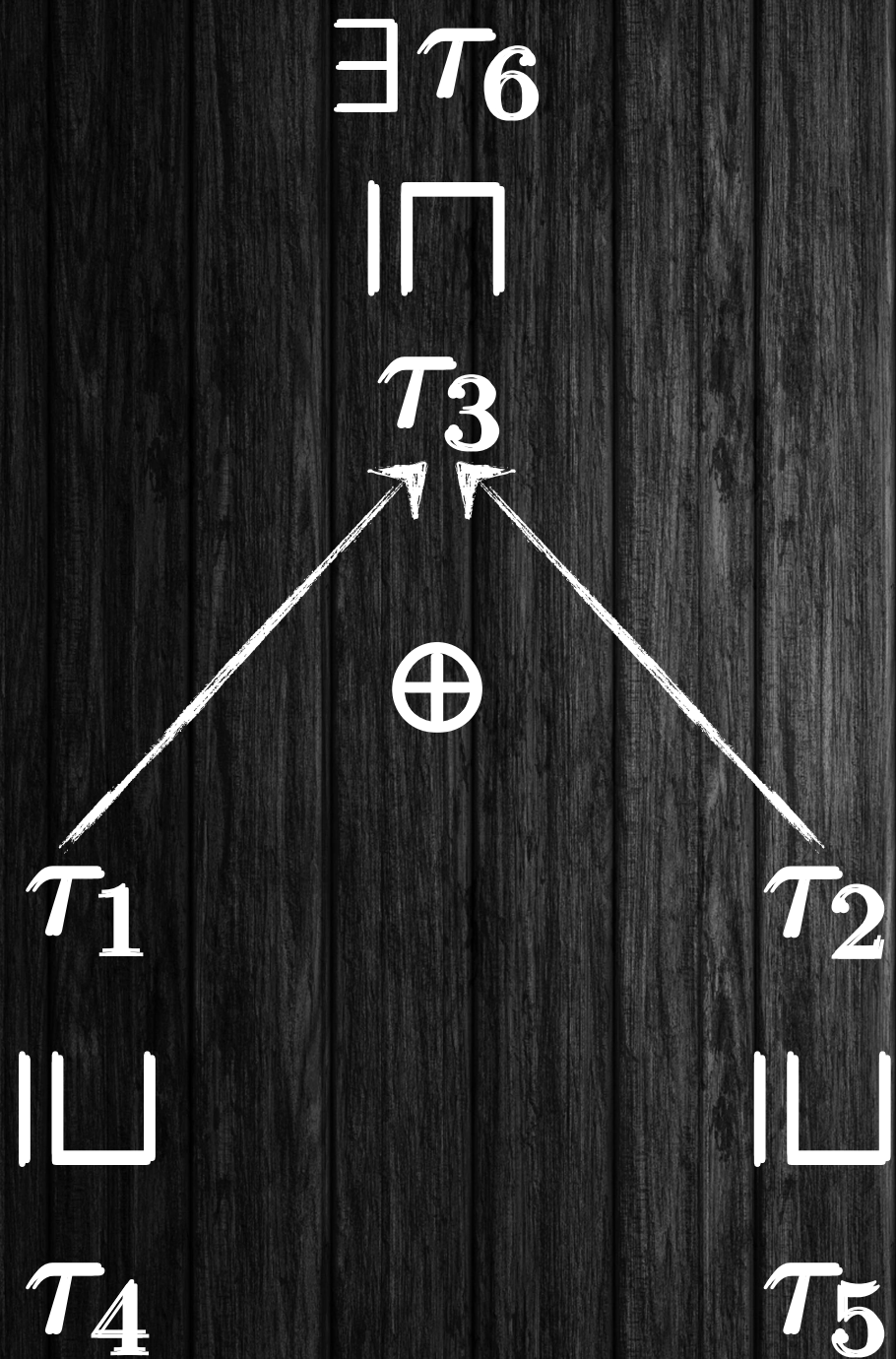
Monotonicity



Signatures

Combining Signatures

Monotonicity



Outline

- **Background**
- **Model**
- **Gap-Order Constraints**
- **Signatures**
- **Reachability Algorithm**
- **Applications**

Outline

- Background
- Model
- Gap-Order Constraints
- Signatures
- Reachability Algorithm
- Applications

- Derivation Trees
- Algorithm

Reachability Algorithm

Generate all $\tau : SW(\tau) \leq k$

Reachability Algorithm

Generate all τ : $\text{SW}(\tau) \leq k$

Reachability Algorithm

Generate all τ : $\text{SW}(\tau) \leq k$

Build signatures systematically
under the restriction $\#\tau \leq k$

Reachability Algorithm

Generate all τ : $\text{SW}(\tau) \leq k$

Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

Build all atomic signatures

Reachability Algorithm

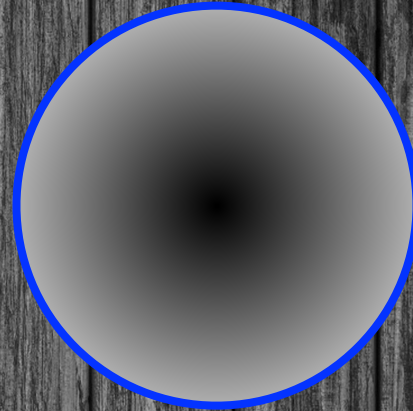
Generate all τ : $SW(\tau) \leq k$

Build all atomic signatures

Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

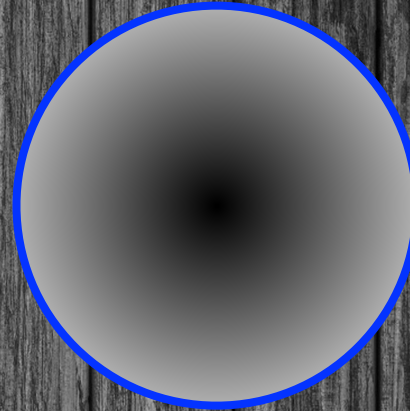
Build all atomic signatures



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

Build all atomic signatures

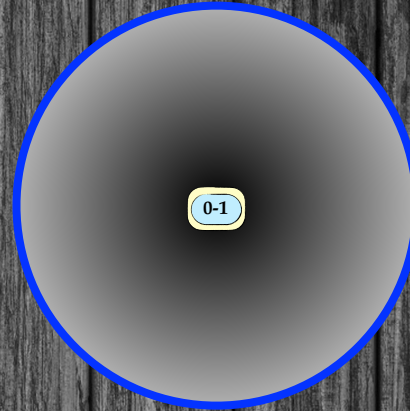


0-1

Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

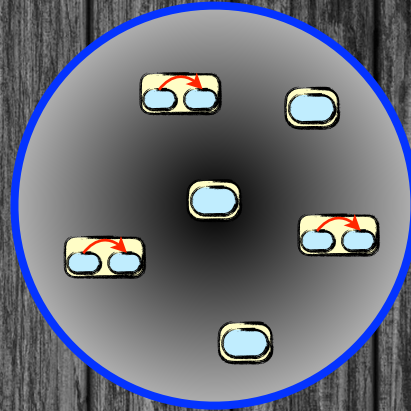
Build all atomic signatures



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

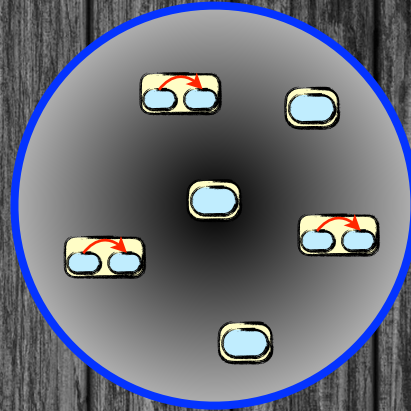
Build all atomic signatures



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

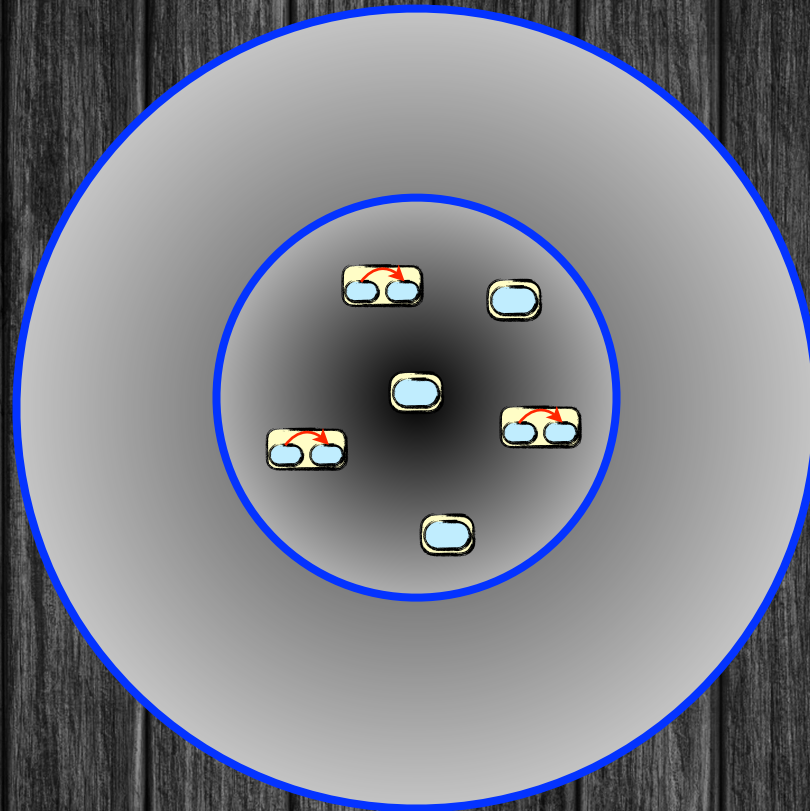
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

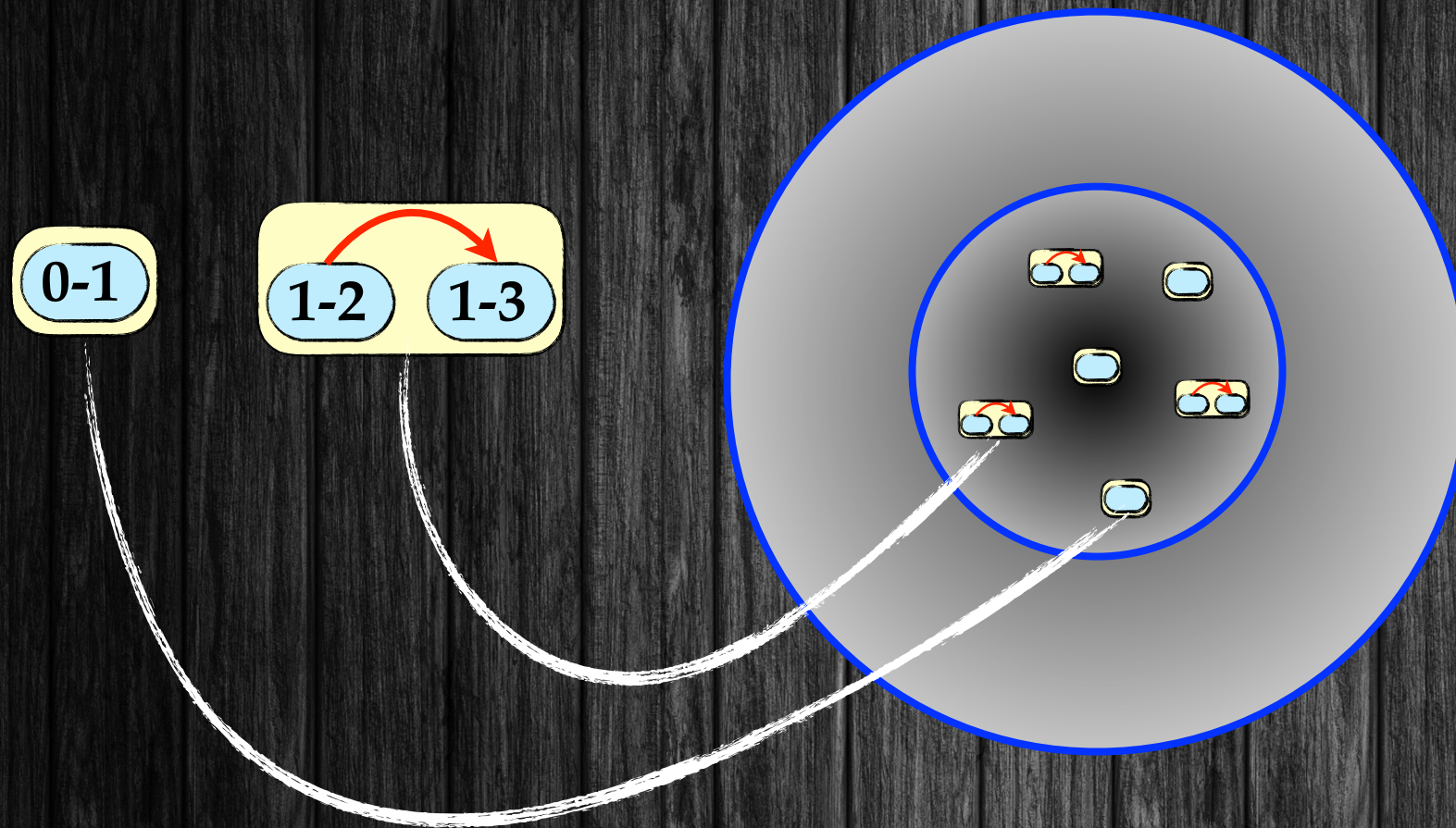
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

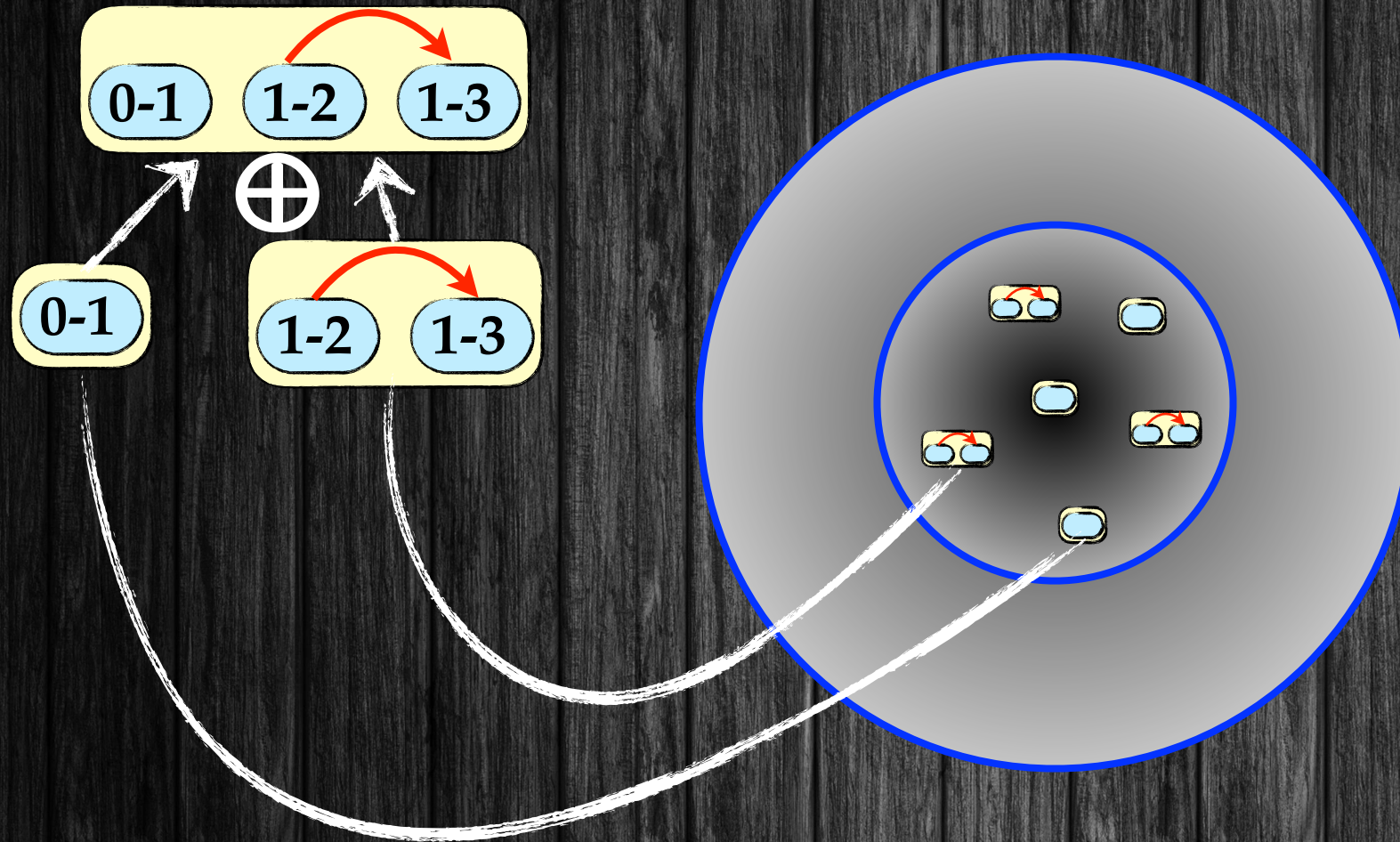
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $\text{SW}(\tau) \leq k$

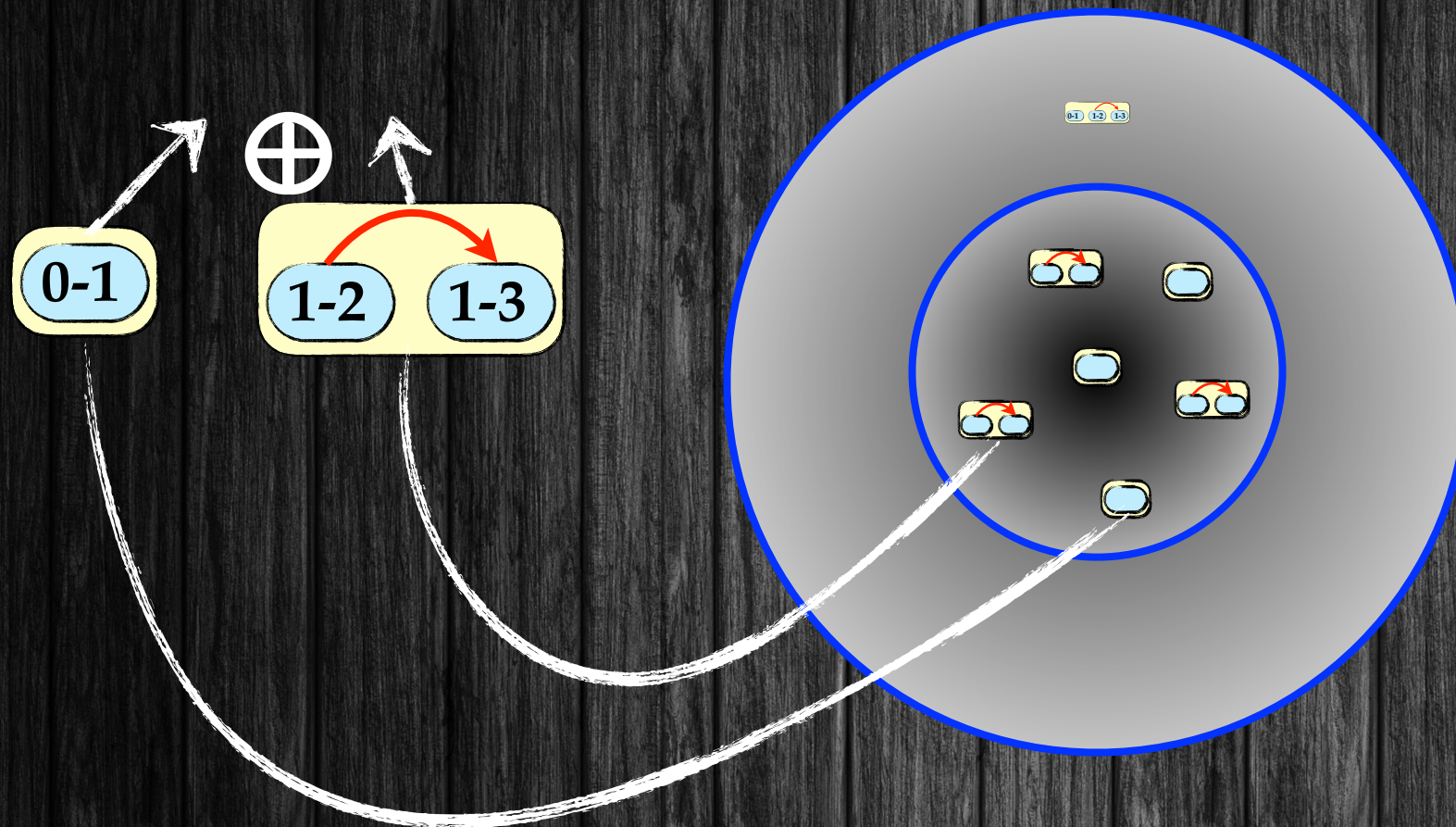
apply $\oplus$ **and** $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

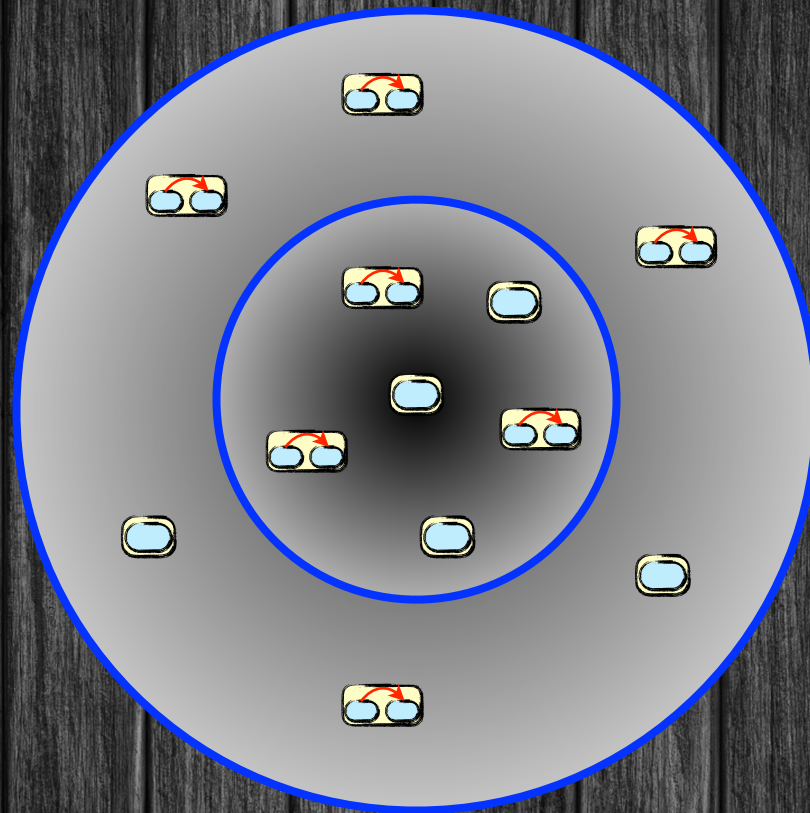
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

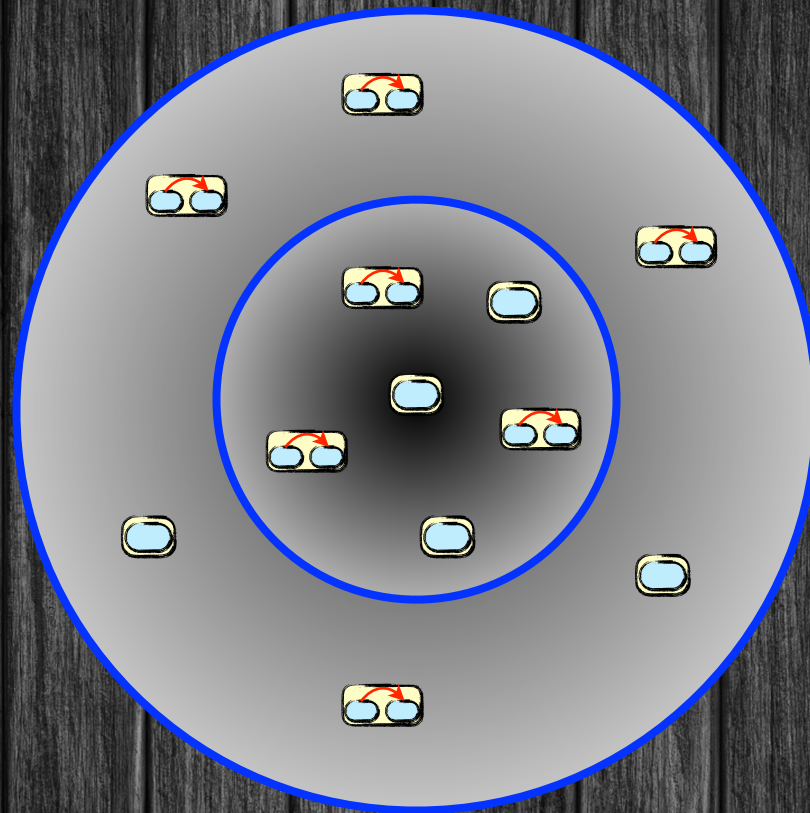
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$

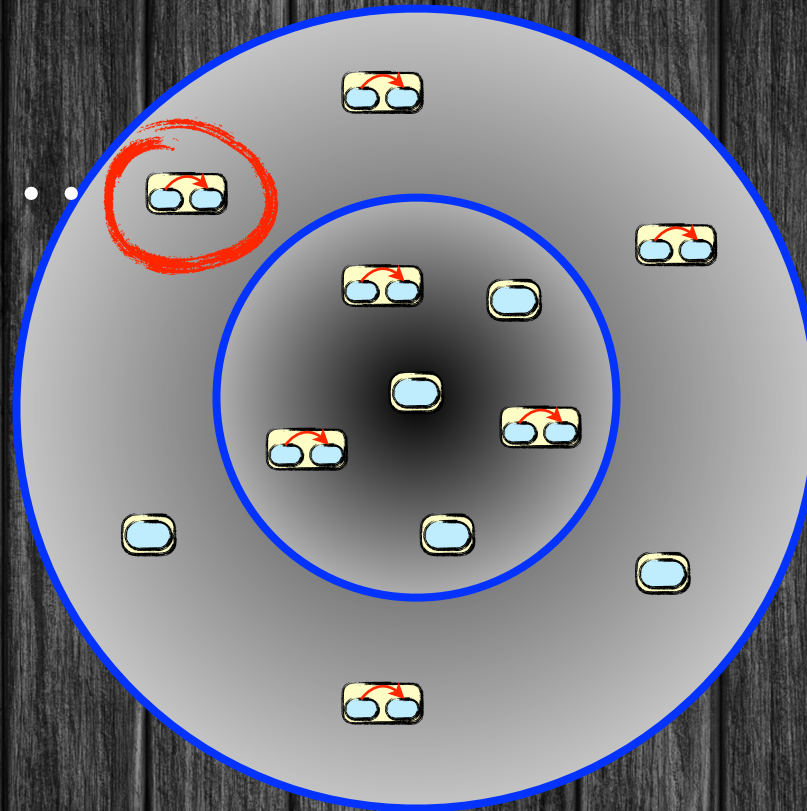


Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$

$SW(\text{👁️}) > k \dots\dots\dots$

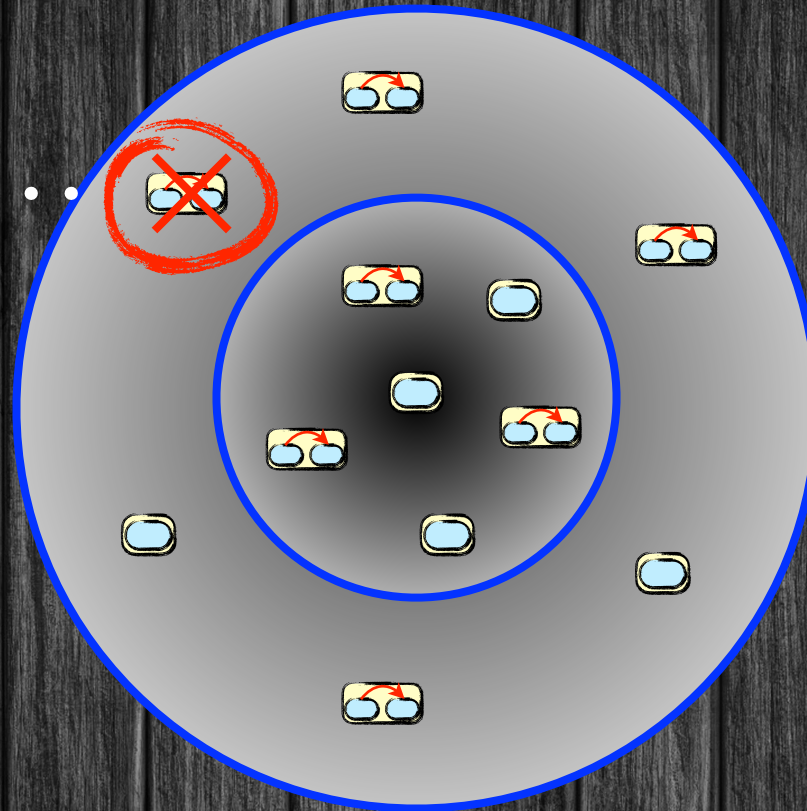


Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$

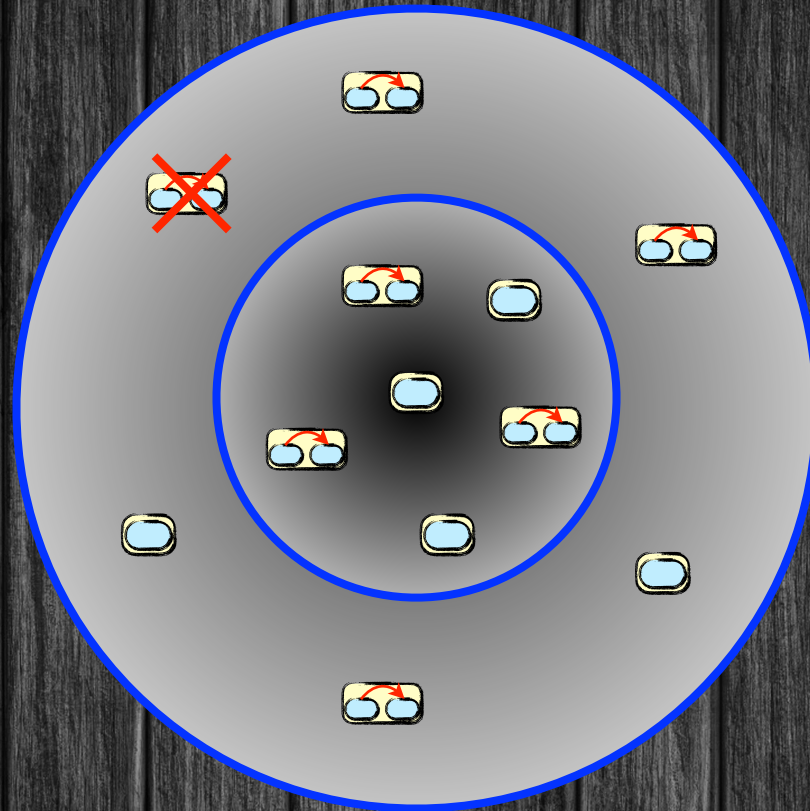
$SW(\text{👁️}) > k \dots\dots\dots$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

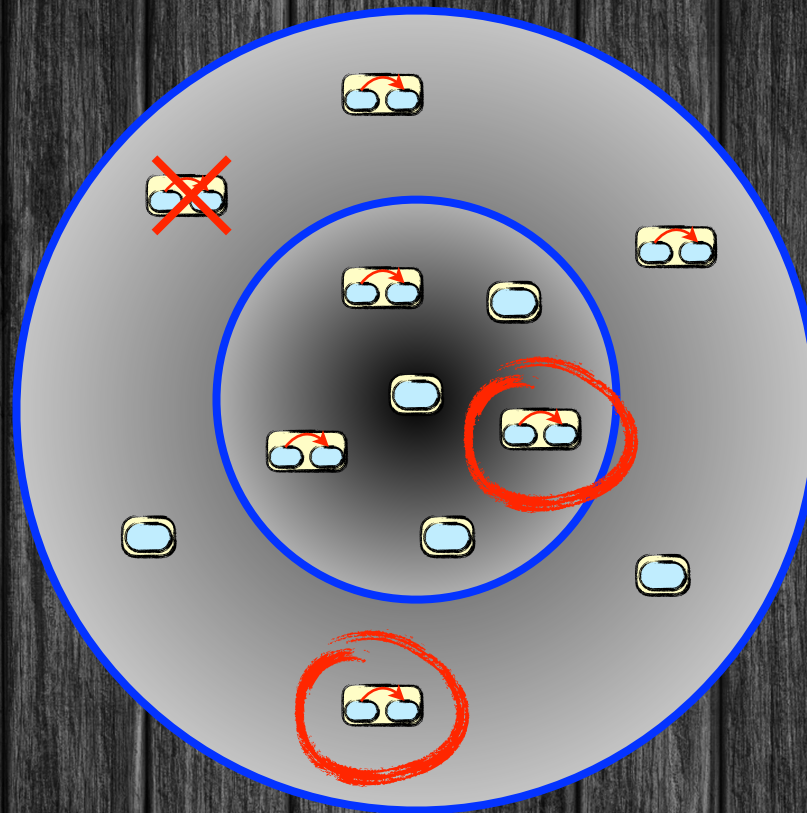
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

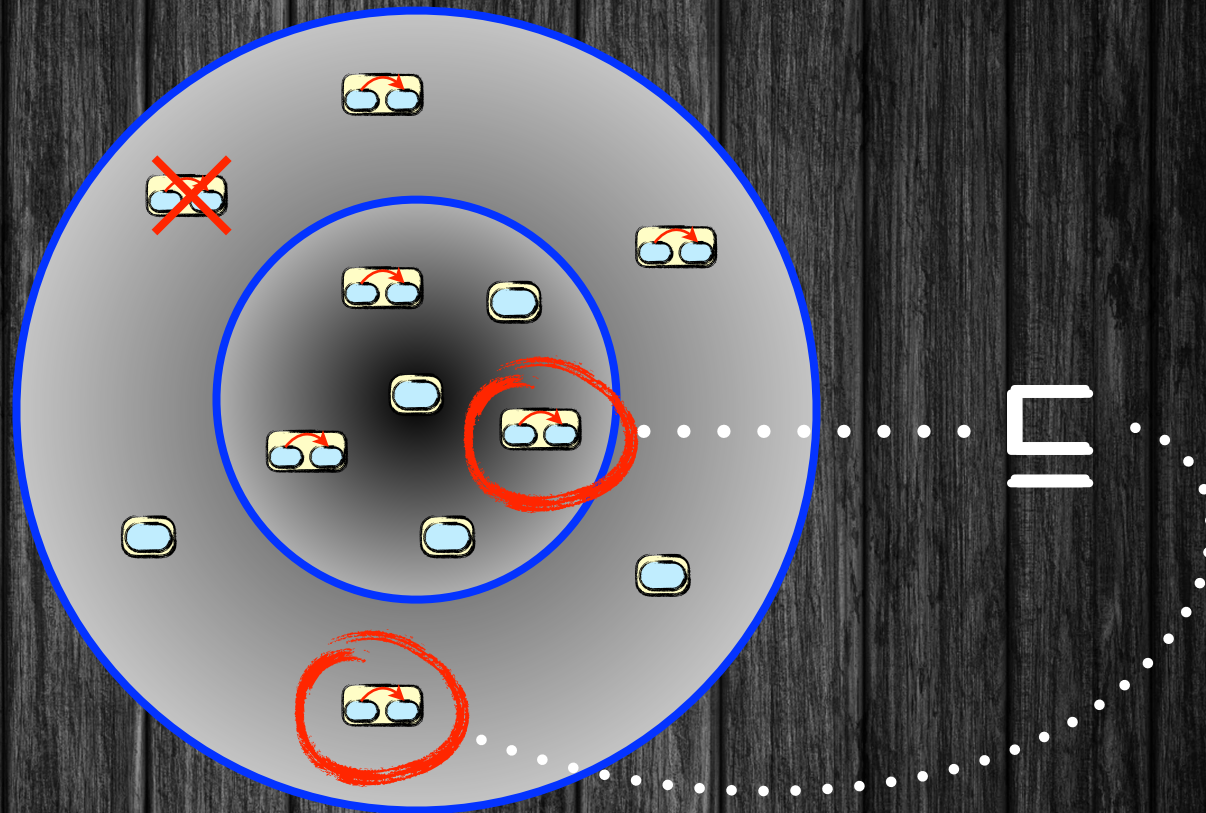
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

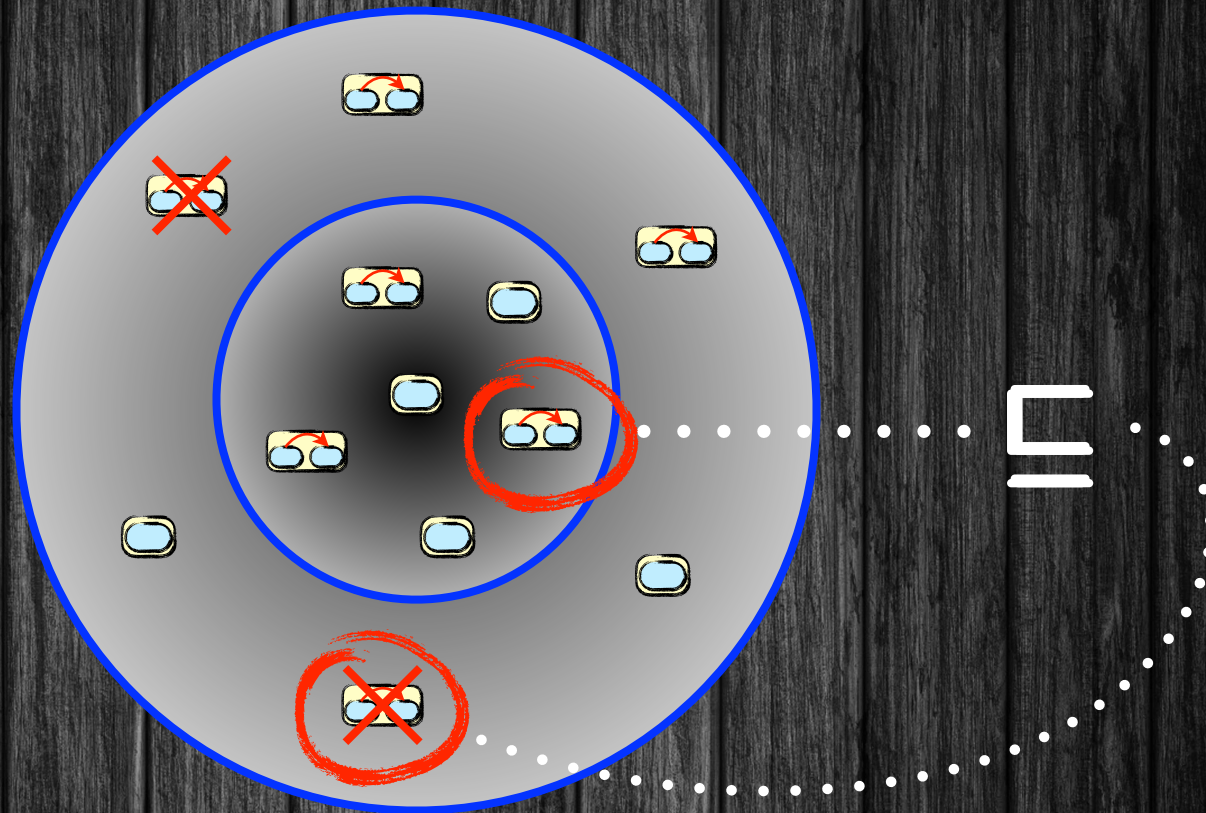
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

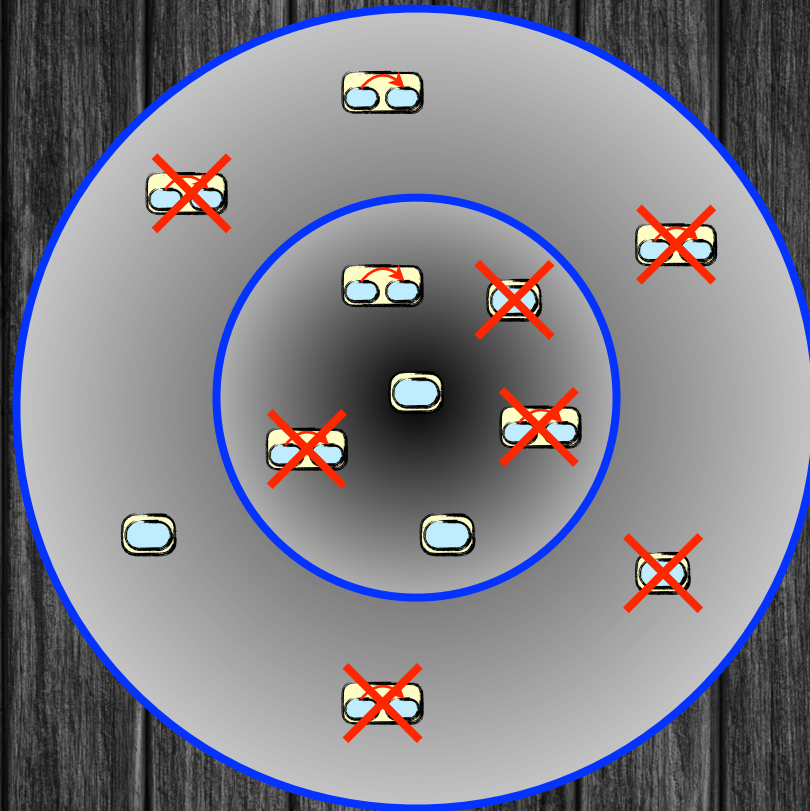
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

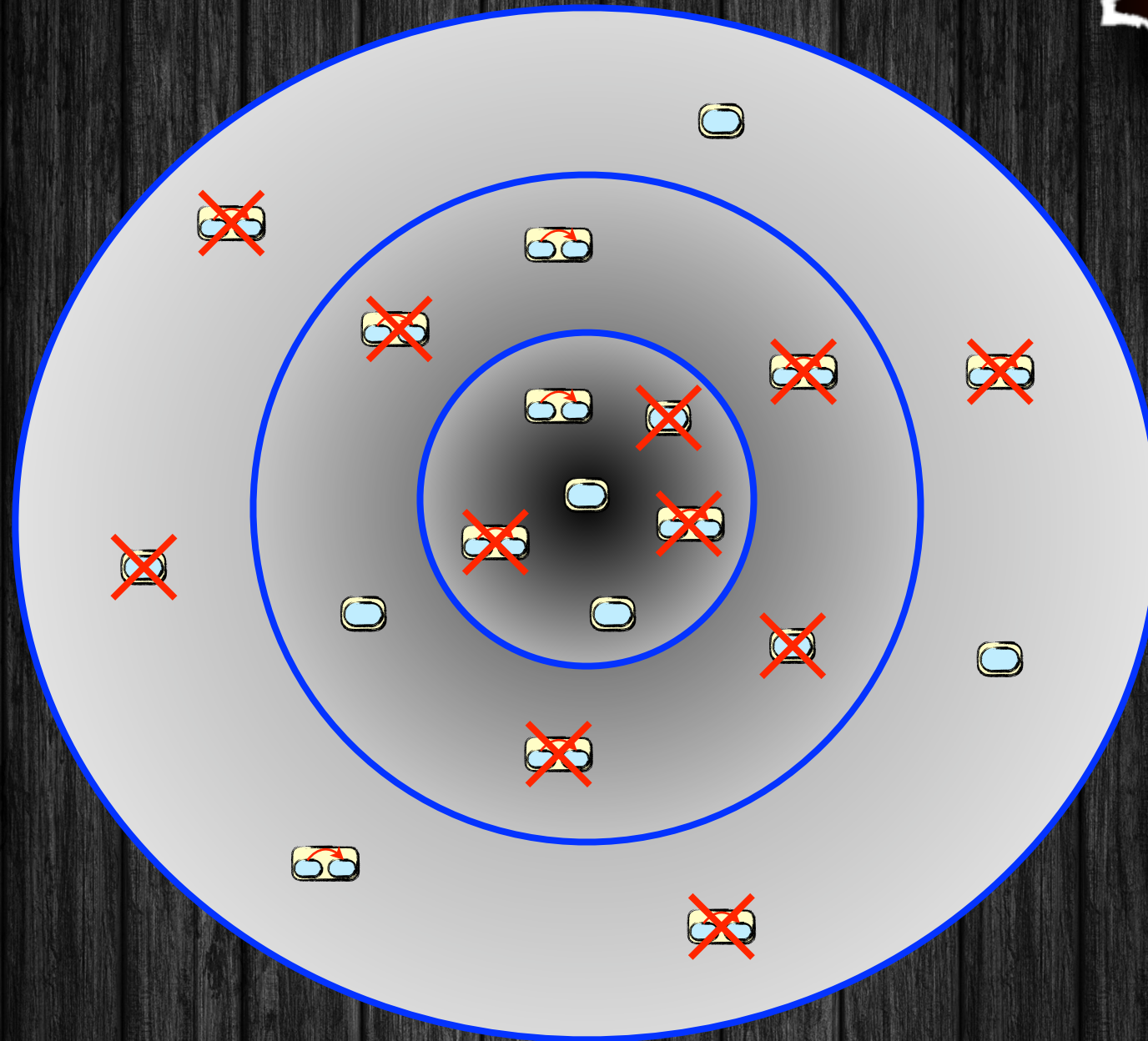
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

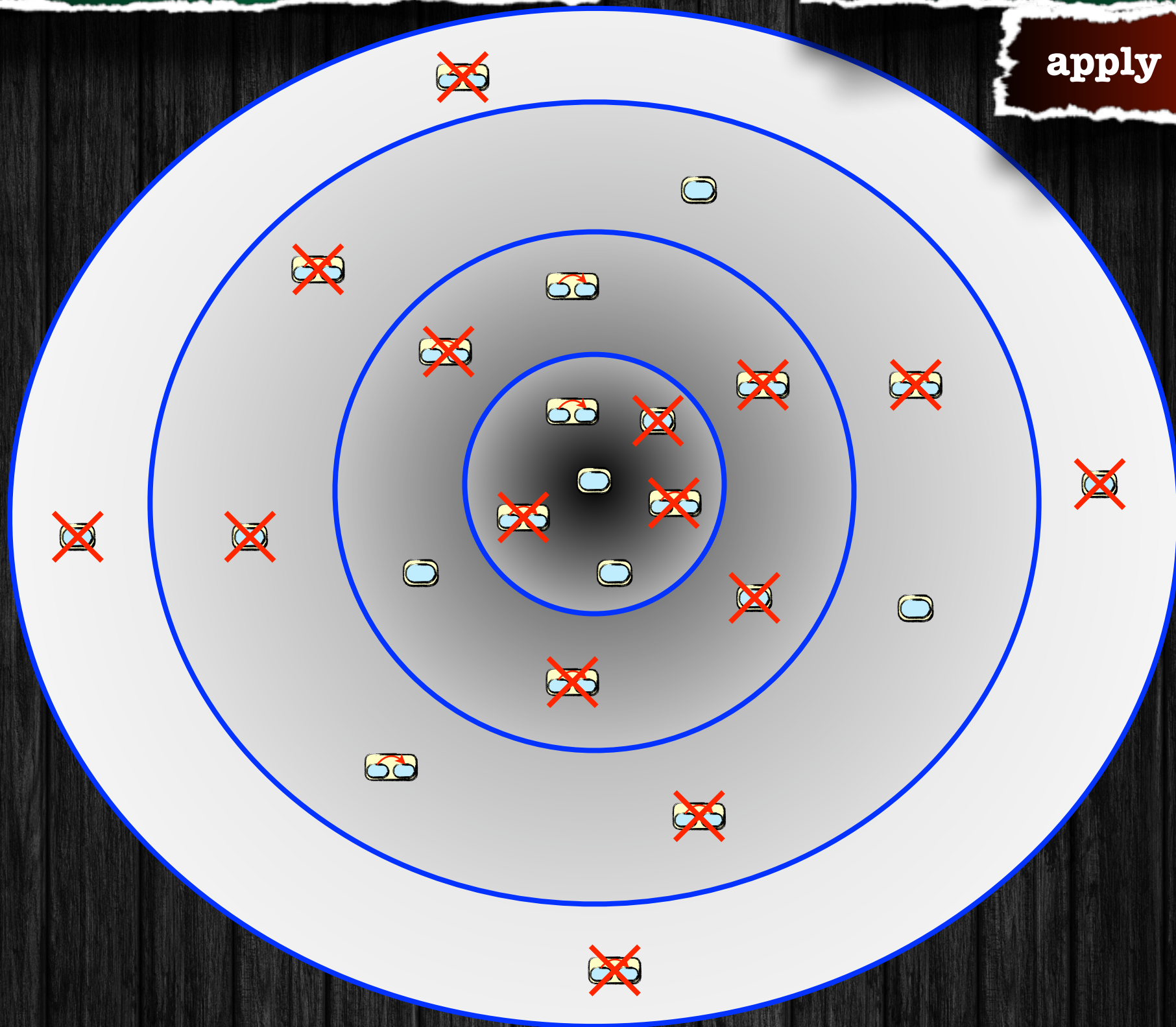
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$

Termination:

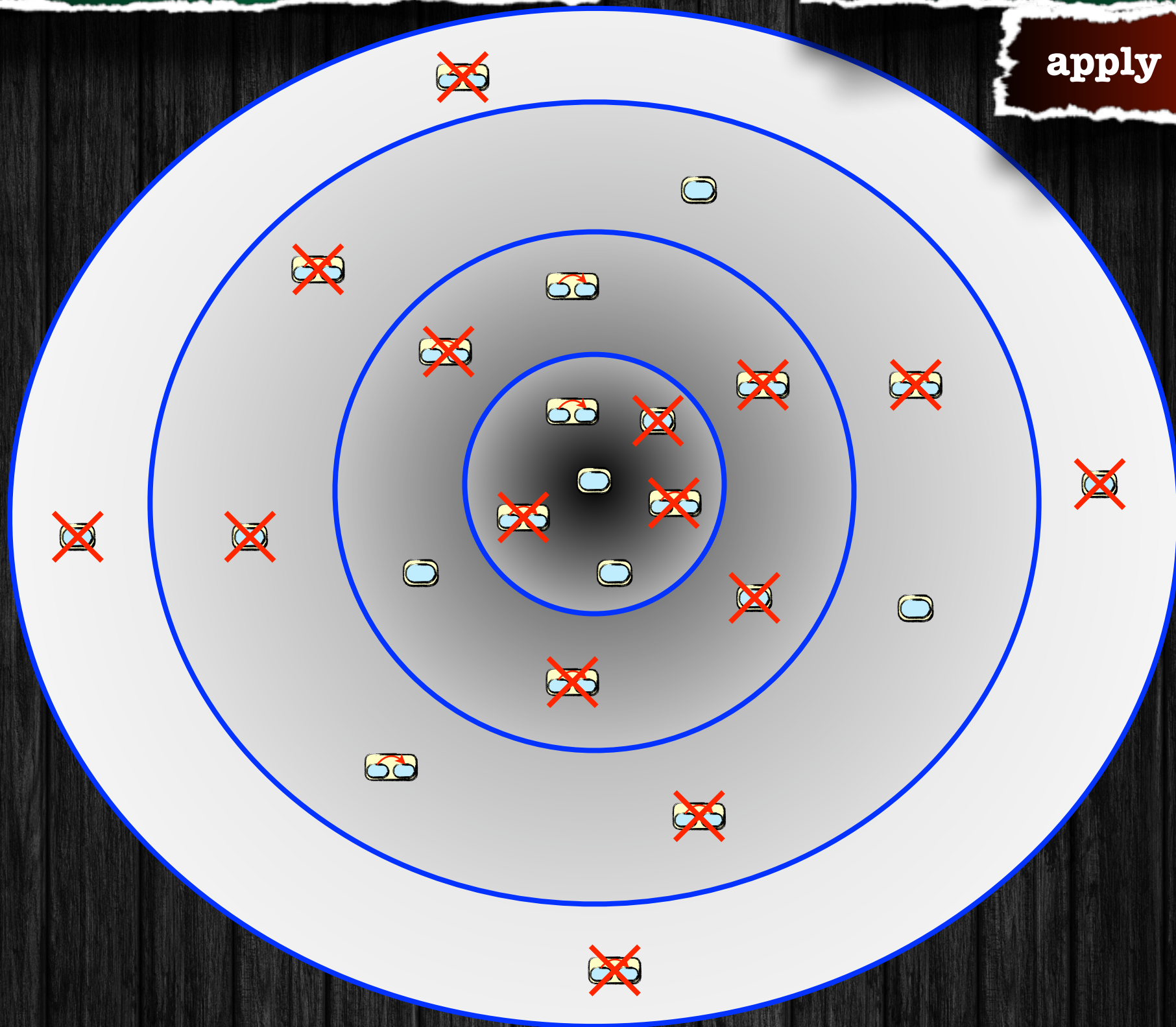
- finitely many signatures with $SW \leq k$
- $\sqsubseteq$ WQO (Dickson's lemma)



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

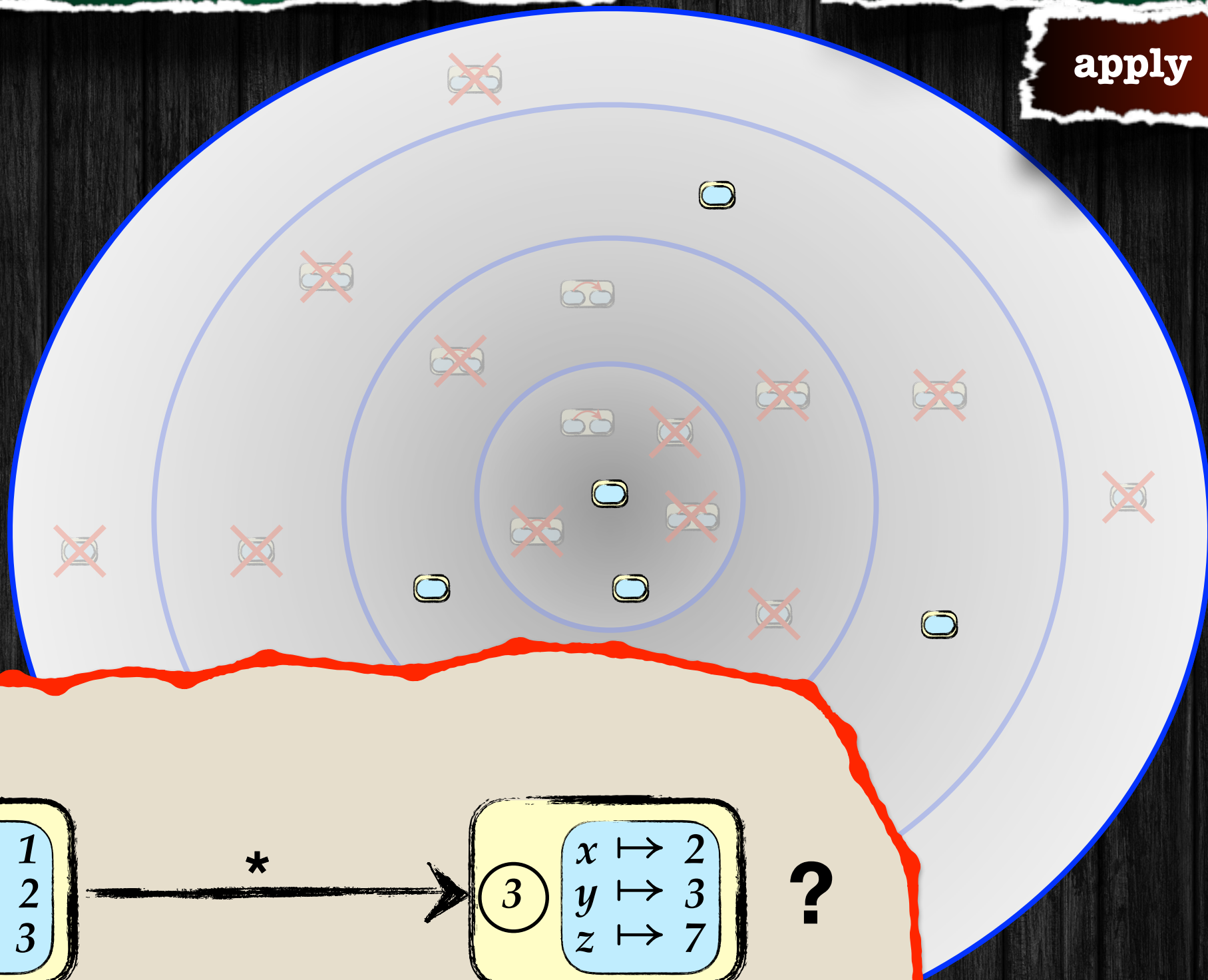
apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$



Reachability Algorithm

Generate all τ : $SW(\tau) \leq k$

apply $\oplus$ and $\odot$



Outline

- **Background**
- **Model**
- **Gap-Order Constraints**
- **Signatures**
- **Reachability Algorithm**
- **Applications**

Applications: Decidability

- Single-Stack $SW = 3$
- k -context bounded runs $SW = k + 2$
- k -scope bounded $SW = k + 2$
- k -phase bounded $SW = 2^k$
- ordered-stack runs $SW = 2^k$

decidable with gap-order
constrained data