Geometry of Interaction and Principal Types

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IFIP WG 2.2 Bologna 6-8/9/7E7, (9+8)x7x(6x5-4x3-2+1)

Mathematics and Politics: the issue of **inequity**

In 2022, the Region FVG has had a significant increase in fiscal revenues. Last December we had over 750 million Euros more than expected. How should we use them?

Wait a minute! How much of this comes from VAT, which is an indirect tax, and hence it has an inversely proportional impact on household incomes, therefore unconstitutional, *strictu sensu*?
Actually, how much of this comes just from inflation?
I asked the question and computed the answer: 150 million Euros!
MORAL: the cost of bread increases because of inflation, the government gets richer?
How should the Region use this extra profit?

A little history and my IFIP WG 2.2. CV

- “Formal Language description Languages” 1966
  - Corrado Böhm, CUCH Machine,
  - LCF, ISWIM, OWHY Dana Scott,
  - Peter Landin, The Next 700 Programming Languages
  - ...
- first invitation - **Palo Alto 1989**
- member - **Hamilton (CA) 1993**
- local organizer IFIP WG 2.2- **Udine 1999** and **Udine 2006**

A question at the Semantics of Data Types Symposium, Sophia-Antipolis, June 27-29, 1984.

- I asked D.Scott, G.Plotkin, etc. :"**Are any two λ-terms equated in all Scott Domain Models β-convertible**?"
- The *Completeness* issue: let $\mathcal{C}$ be a class of models of $\mathcal{T}$; and let $\mathcal{F}$ be a set of sentences, $\mathcal{C}$ is $\mathcal{F}$-complete if

$$\forall \phi \in \mathcal{F}.(\forall \mathcal{M} \in \mathcal{C}.\mathcal{M} \models \phi) \implies (\forall \mathcal{M} \models \mathcal{T}.\mathcal{M} \models \phi)$$

- The reverse issue is $\mathcal{F}$-consistency of $\mathcal{C}$.
- Inconsistency has been gradually settled in the negative: F.Honsell, S.Ronchi (quantifier-free 1992), P.Selinger (2003), F.H. G.Plotkin ($\Pi_2(Eq)$, 2009), A.Salibra, A.Carraro (equational 2013, using unsolvable terms);
- $\Pi_1(POS)$-completeness holds (F.H.-G.P. 2009) $\Sigma_1(Eq)$-completeness fails (P.Selinger 2001);
- **Equational completeness, original question, is still open!** (YES, for ω_1-continuous functions in NJC1992 P.Di Gianantonio-F.H.-G.P. .)

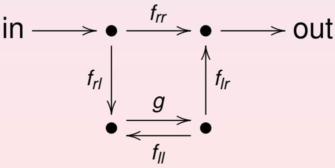
Girard's "Geometry of Interaction" Semantics

- Linear Logic: make multiple occurrences of variables explicit;
- decompose application $[\![MN]\!] = [\![M]\!] \bullet_{linear} !([\![N]\!])$;
- introduce two abstractions $\lambda x.x$ and $\lambda !x.xx$ and pattern matching reduction,
  i.e. $(\lambda !x.M)N$ is stuck but $(\lambda !x.M)!N \to M[N/x]$

Many *game (interaction) models* were developed since the early '90's.
Equational Completeness is even more problematic, in TLCA 1999 P.D.G.- G.Franco - F.H. essentially only one theory is modeled.

Girard's GoI - Abramsky's version in MSCS (2002)

- derived from approach using *traced monoidal categories*;
- T_Σ, the language of *moves*, is defined by the signature $\Sigma_0 = \{\epsilon\} \cup Vars$, $\Sigma_1 = \{\mathrm{l},\mathrm{r}\}$, $\Sigma_2 = \{\langle\ ,\ \rangle\}$;
  $r(t)$ are *output words*, terms $l(t)$ are *input words* ;
- $\mathcal{I}$ is the set of *strategies i.e. partial involutions* over T_Σ: partial functions $f : T_\Sigma \rightharpoonup T_\Sigma$ such that
  $$f(t) = t' \Leftrightarrow f(t') = t;$$
- the operation of *replication* is defined by
  $!f = \{(\langle t, t_1\rangle, \langle t, t_2\rangle) \mid t \in T_\Sigma \wedge (t_1, t_2) \in f\}$;
- the notion of *linear application* is defined by
  $f \cdot g = f_{rr} \cup (f_{rl}; g; (f_{ll}; g)^*; f_{lr})$, where
  $f_{ij} = \{(t_1, t_2) \mid (i(t_1), j(t_2)) \in f\}$, for $i, j \in \{r, l\}$



Models: Affine, Light, Elementary, Linear λ-Calculi via Combinators

- **affine** λ^A: no !-abstraction no multiple occurrences of vars; the calculus terminates in linear time and models Grzegorczyk $\mathcal{E}^1$.

$$(\mathbf{B})_\lambda = \lambda xyz.x(yz) \quad (\mathbf{C})_\lambda = \lambda xyz.(xz)y \quad (\mathbf{I})_\lambda = \lambda x.x \quad (\mathbf{K})_\lambda = \lambda xy.x$$

- **light** λ^L … terminates in polynomial time and models $\mathcal{E}^2$;
- **elementary** λ^E: variables !-abstracted only if they occur in the scope of a single !; terminates in elementary time and models $\mathcal{E}^3$

$$(\mathbf{W})_\lambda = \lambda x!y.x!y!y \qquad (\mathbf{F})_\lambda = \lambda!x!y.!(xy)$$

- **full** $\lambda^!$: no restrictions on !-abstractions

$$(\mathbf{D})_\lambda = \lambda!x.x \qquad (\delta)_\lambda = \lambda!x.!!x$$

The interpretation of Combinators in $\mathcal{I}$

$$
\begin{array}{lcl}
[\![\mathbf{B}]\!] & = & \{r^3x \leftrightarrow lrx\ ,\ l^2x \leftrightarrow rlrx\ ,\ rl^2x \leftrightarrow r^2lx\} \\
[\![\mathbf{I}]\!] & = & \{lx \leftrightarrow rx\} \\
[\![\mathbf{C}]\!] & = & \{l^2x \leftrightarrow r^2lx\ ,\ lrlx \leftrightarrow rlx\ ,\ lr^2x \leftrightarrow r^3x\} \\
[\![\mathbf{K}]\!] & = & \{lx \leftrightarrow r^2x\} \\
[\![\mathbf{F}]\!] & = & \{l\langle i, rx\rangle \leftrightarrow r^2\langle i, x\rangle\ ,\ l\langle i, lx\rangle \leftrightarrow rl\langle i, x\rangle\} \\
[\![\mathbf{W}]\!] & = & \{r^2x \leftrightarrow lr^2x\ ,\ l^2\langle i, x\rangle \leftrightarrow rl\langle li, x\rangle\ ,\ lrl\langle i, x\rangle \leftrightarrow rl\langle ri, x\rangle\} \\
[\![\delta]\!] & = & \{l\langle\langle i, j\rangle, x\rangle \leftrightarrow\ r\langle i, \langle j, x\rangle\rangle\} \\
[\![\mathbf{D}]\!] & = & \{l\langle \epsilon, x\rangle \leftrightarrow rx\}
\end{array}
$$

where $\{u_i[x] \leftrightarrow v_i[x] \mid i \in I\}$ denotes the partial involution $\{(u_i[t], v_i[t]) \mid i \in I, t \in T_\Sigma\}$

What is this all about? What are strategies? What are moves? What is G.o.I.? (λ^A-case)

- understanding this is essential for studying the fine structure of models
- **strategies** are **principal types**,
- **moves** are **occurrences of variables** in principal types;
- *Types* T_Σ are binary trees whose leaves are variables $\alpha, \beta, \ldots \in$ *TVar*, and nodes are denoted by $\multimap$, *i.e.*

$$(T_\Sigma \ni)\ \sigma, \tau ::= \ \alpha \mid \beta \mid \ \ldots \ \mid \sigma \multimap \tau \ .$$

- A type σ is *binary* if each variable in σ occurs at most twice.
- *Occurrences* of variables in types are denoted by:

$$(O_\Sigma \ni)\ u[\alpha] ::= [\alpha] \mid lu[\alpha] \mid ru[\alpha]\ ,$$

  - $[\alpha]$ denotes the occurrence of the variable α in the type α,
  - if $u[\alpha]$ denotes an occurrence of α in σ_1 (σ_2), then $lu[\alpha]$ ($ru[\alpha]$) denotes the corresponding occurrence of α in $\sigma_1 \multimap \sigma_2$.

Correspondence between types and partial involutions

- A type τ gives rise to a set of variable occurrences

$$\mathcal{O}(\tau) = \{u[\alpha] \mid u[\alpha] \text{ is an occurrence of } \alpha \text{ in } \tau\}.$$

- A *binary* type τ gives rise to a partial involution on O_Σ

$$\mathcal{R}(\tau) = \{\langle u[\alpha], v[\alpha]\rangle \mid u[\alpha], v[\alpha] \text{ are different occurrences of } \alpha \text{ in } \tau\}$$

- Vice versa, from a set of variable occurrences, such that no path is the initial prefix of any other path of a different occurrence, we can build the tree of a type, by tagging possible missing leaves with fresh variables in $Z = \{\zeta_1, \ldots, \zeta_i, \ldots\}$.
-

$$\mathcal{T}_Z(S) = \begin{cases} \zeta & \text{if } \mathcal{S} = \emptyset \\ \alpha & \text{if } \mathcal{S} = \{[\alpha]\} \\ \mathcal{T}_Z(\{u \mid lu \in \mathcal{S}\}) \multimap \mathcal{T}_Z(\{u \mid ru \in \mathcal{S}\}) & \text{otherwise,} \end{cases}$$

- For all type σ, we have $\mathcal{T}_Z(\mathcal{O}(\sigma)) = \sigma$;

Two Perspectives on Unification - the top-down perspective of types

The perspectve of Types

A unification algorithm *à la* Martelli Montanari: it unifies simultaneously sets of pairs of types.

Let E be a set of pairs of types:

$$
\begin{array}{lcl}
\mathcal{U}(\{\langle \sigma_1 \multimap \sigma_2, \tau_1 \multimap \tau_2 \rangle\} \cup E) & \rightarrow & \mathcal{U}(\{\langle \sigma_1, \tau_1 \rangle, \langle \sigma_2, \tau_2 \rangle\} \cup E) \\
\mathcal{U}(\{\langle \alpha, \alpha \rangle\} \cup E) & \rightarrow & E \\
\mathcal{U}(\{\langle \sigma_1 \multimap \sigma_2, \alpha \rangle\} \cup E) & \rightarrow & \mathcal{U}(\{\langle \alpha, \sigma_1 \multimap \sigma_2 \rangle\} \cup E) \\
\mathcal{U}(\{\langle \alpha, \sigma \rangle\} \cup E) & \rightarrow & \mathcal{U}(\{\langle \alpha, \sigma \rangle\} \cup E[\sigma/\alpha]), \text{ if } \alpha \notin \sigma \wedge \alpha \in Var(E) \\
\mathcal{U}(\{\langle \alpha, \sigma \rangle\} \cup E) & \rightarrow & \text{fail, if } \alpha \in \sigma \ \wedge \ \alpha \neq \sigma
\end{array}
$$

The Bottom-up Perspective of Type-Variable Occurrences

Definition (Occurrence Unifiers)

Let σ, τ be types.
(i) Two occurrences $u[\alpha] \in \sigma$ and $v[\beta] \in \tau$ are *unifiable* if u is a prefix of v, *i.e.* there exists w such that $uw = v$, or vice versa.
(ii) If two occurrences $u[\alpha] \in \sigma$ and $v[\beta] \in \tau$ are *unifiable*, their *occurrence unifier* (occ-unifier) is the most general unifier of $\mathcal{T}_Z(\{u[\alpha]\})$ and $\mathcal{T}_Z(\{v[\beta]\})$.

The Bottom-up Perspective of Type Variable Occurrences

GoI application gives rise to a *variable-occurrence oriented* alternate characterization of unification. (Case of binary types where each variable occurs exactly twice).

Definition (GoI-unification)

Let $\sigma, \tau \in T_\Sigma$ be types. The types σ and τ *GoI-unify* if
(i) for every $\langle u[\alpha], v[\alpha]\rangle \in \mathcal{R}(\sigma)$ there exists $\langle u'[\gamma], v'[\gamma]\rangle \in \mathcal{R}(\tau)\hat{;}(\mathcal{R}(\sigma)\hat{;}\mathcal{R}(\tau))^*$, such that $uw = u'$ and $vw = v'$, and
(ii) for every $(u[\alpha], v[\alpha]) \in \mathcal{R}(\tau)$ there exists $\langle u'[\gamma], v'[\gamma]\rangle \in \mathcal{R}(\sigma)\hat{;}(\mathcal{R}(\tau)\hat{;}\mathcal{R}(\sigma))^*$, such that $uw = u'$ and $vw = v'$.
I.e.:

$$\mathcal{R}(\tau)\widehat{\subseteq}\mathcal{R}(\sigma)\hat{;}(\mathcal{R}(\tau)\hat{;}\mathcal{R}(\sigma))^* \quad \text{and} \quad \mathcal{R}(\sigma)\widehat{\subseteq}\mathcal{R}(\tau)\hat{;}(\mathcal{R}(\sigma)\hat{;}\mathcal{R}(\tau))^*$$

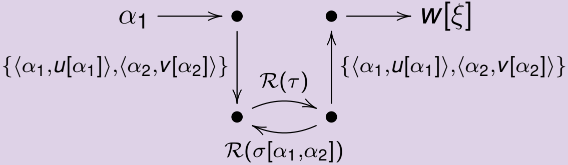
where $\widehat{\subseteq}$ denotes “inclusion up-to substitution”.

Proposition

Let $\sigma, \tau \in T_\Sigma$ be binary types where each variable occurs exactly twice. Then σ, τ unify if and only if σ, τ GoI unify.

Proof.

Let $\sigma[\alpha, \alpha]$ be σ where we have highlighted the two occurrences of a variable α, $u[\alpha]$, $v[\alpha]$, and consider the new type $\sigma[\alpha_1, \alpha_2] \multimap \alpha_1 \multimap \alpha_2$. Now compute $\mathcal{R}(\sigma[\alpha_1, \alpha_2] \multimap \alpha_1 \multimap \alpha_2) \cdot \mathcal{R}(\tau)$. Then σ and τ unify with unifier U if and only if, by Propositions below, for some fresh variable ξ,



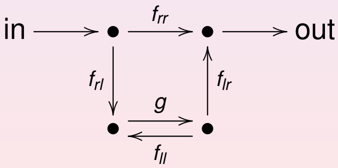
If S is the collection of all such possible outcomes we have $U(\alpha) = \mathcal{T}_Z(S)$, for a suitable set Z of fresh variables. □

GoI using binary types - http://158.110.146.197:31780/automata/

(i) $\mathcal{I}$ is the set of *partial involutions* induced by binary types, *i.e.* $\mathcal{I} = \{\mathcal{R}(\tau) \mid \tau \in T_\Sigma \ \wedge \ \tau \text{ binary}\}$. (ii) The notion of *linear application* is defined, for $f, g \in \mathcal{I}$, by

$$f \cdot g = \ f_{rr} \cup (f_{rl} \,\hat{;}\, g \,\hat{;}\, (f_{ll} \,\hat{;}\, g)^* \,\hat{;}\, f_{lr}) \ ,$$

where $f_{ij} \ = \ \{\langle u, v\rangle \mid \langle i(u), j(v)\rangle \in f\}$, for $i, j \in \{r, l\}$. Variables in different pairs of $f \cdot g$ to be disjoint.
(iii) $\mathcal{O}(f \cdot g) = \{u \mid \exists v. \langle u, v\rangle \in f \cdot g\}$.



Principal Type Assignment System $\Vdash$ for λ^A

$$\frac{}{x : \alpha \Vdash_A x : \alpha} \quad \text{(var)}$$

$$\frac{\Gamma, x : \sigma \Vdash_A M : \tau}{\Gamma \Vdash_A \lambda x.M : \sigma \multimap \tau} \quad \text{(abs)} \qquad \frac{\Gamma \Vdash_A M : \sigma \quad \alpha \text{ fresh}}{\Gamma \Vdash_A \lambda x.M : \alpha \multimap \sigma} \quad (\text{abs}_\emptyset)$$

$$\frac{\begin{array}{cc} \Gamma \Vdash_A M : \sigma & \Delta \Vdash_A N : \tau \\ (dom(\Gamma) \cap dom(\Delta)) = \emptyset & (TVar(\Gamma) \cap TVar(\Delta)) = \emptyset \\ (TVar(\sigma) \cap TVar(\tau)) = \emptyset & U' = \mathcal{U}(\sigma, \alpha \multimap \beta) \\ U = \mathcal{U}(U'(\alpha), \tau) & \alpha, \beta \textit{ fresh} \end{array}}{U \circ U'(\Gamma, \Delta) \Vdash_A MN : U \circ U'(\beta)} \quad \text{(app)}$$

Principal Types for Affine Combinators

$$
\begin{aligned}
[\![\mathbf{I}]\!] &= \alpha \multimap \alpha \\
[\![\mathbf{B}]\!] &= (\alpha \multimap \beta) \multimap (\gamma \multimap \alpha) \multimap \gamma \multimap \beta \\
[\![\mathbf{C}]\!] &= (\alpha \multimap \beta \multimap \gamma) \multimap \beta \multimap \alpha \multimap \gamma \\
[\![\mathbf{K}]\!] &= \alpha \multimap \beta \multimap \alpha
\end{aligned}
$$

Resolution of principal types is G.o.I. application

Proposition

Let $f, g \in \mathcal{I}$. *Then* $\langle u, v\rangle \in f \cdot g$ *if and only if there exists an even sequence,* $\langle u_1[\alpha_1], u'_1[\alpha_1]\rangle, \ldots, \langle u_{n+1}[\alpha_{n+1}], u'_{n+1}[\alpha_{n+1}]\rangle$*:*

- *either* $n = 0$ *and* $\langle u_1[\alpha_1], u'_1[\alpha_1]\rangle \in f_{rr}$ *or* $n > 0$, $\langle u_1[\alpha_1], u'_1[\alpha_1]\rangle \in f_{rl}$, $\langle u_{n+1}[\alpha_{n+1}], u'_{n+1}[\alpha_{n+1}]\rangle \in f_{lr}$, $\langle u_i[\alpha_i], u'_i[\alpha_i]\rangle \in g$, *for* $i < n$, i *even, and* $\langle u_i[\alpha_i], u'_i[\alpha_i]\rangle \in f_{ll}$, *for* $1 < i < n+1$, i *odd;*
- *the set of types* $\Pi = \{\langle \mathcal{T}_Z(u'_i[\alpha_i]), \mathcal{T}_Z(u_{i+1}[\alpha_{i+1}])\rangle \mid 1 \leq i \leq n\}$ *(where* Z*-variables used in different types are different) is unifiable*

The sequence $\langle u_1[\alpha_1], u'_1[\alpha_1]\rangle, \ldots, \langle u_{n+1}[\alpha_{n+1}], u'_{n+1}[\alpha_{n+1}]\rangle$ *is called a trajectory and* $\langle u, v\rangle$ *its output.*

Resolution of principal types is G.o.I. application

Proposition

Let $\sigma_1 \multimap \sigma_2$ and τ be binary types, let Θ be the set of type-variables in $TVar(\sigma_1 \multimap \sigma_2, \tau)$ which are not involved in any trajectory of $\mathcal{R}(\sigma_1 \multimap \sigma_2) \cdot \mathcal{R}(\tau)$, and let $U_{\pi_1}, \ldots, U_{\pi_n}$ be the unifiers arising from all trajectories $\pi_1, \ldots, \pi_n$ of

$$\mathcal{R}(\sigma_1 \multimap \sigma_2) \cdot \mathcal{R}(\tau).$$

If σ_1 and τ are unifiable with m.g.u. $\overline{U}$, then:

1. $U_{\pi_i} \leq \overline{U}$ *for all* i;
2. $\bigoplus_i U_{\pi_i} = \overline{U}_{\restriction (TVar \setminus \Theta)}$;
3. $\mathcal{T}_Z(\mathcal{O}(\mathcal{R}(\sigma_1 \multimap \sigma_2) \cdot \mathcal{R}(\tau))) = \overline{U}(\sigma_2)$.

G.o.I. application is Resolution of (ancestral) types

Proposition

Let $\sigma_1 \multimap \sigma_2$ and τ be binary types such that σ_1 and τ are unifiable with m.g.u. $\overline{U}$, then

$$\mathcal{R}(\sigma_1 \multimap \sigma_2) \cdot \mathcal{R}(\tau) = \mathcal{R}(\overline{U}(\sigma_2)) \ .$$

Theorem (A.C. - F.H.- M.L. - I.S. 2018)

*For any closed term M of affine combinatory logic, $\{$**I**, **B**, **C**, **K** $\}$, we have: $[\![M]\!]^{\mathcal{I}} = \mathcal{R}(\sigma)$, where σ is the principal type of $(M)_\lambda$.*

The above theorem provides a partial answer to a problem raised by Abramsky in *Structural Approach to Reversible Computation (2011)*

Ancestral types

No full resolution can apply to the following two types

$\tau \equiv ((\alpha \multimap \beta) \multimap (\gamma \multimap (\gamma \multimap \delta) \multimap \delta)) \multimap \alpha \multimap \beta$

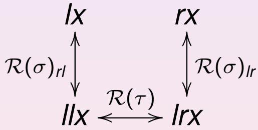
$\sigma \equiv (\alpha \multimap \alpha) \multimap (\gamma \multimap \gamma)$

but,

$\mathcal{R}(\sigma) = \{\mathit{rlx} \leftrightarrow \mathit{lllx}, \mathit{rrx} \leftrightarrow \mathit{llrx}, \mathit{lrlx} \leftrightarrow \mathit{lrrlx}, \mathit{lrrlrx} \leftrightarrow \mathit{lrrrx}\}$

$\mathcal{R}(\tau) = \{\mathit{llx} \leftrightarrow \mathit{lrx}, \mathit{rlx} \leftrightarrow \mathit{rrx}\}$

$\mathcal{R}(\sigma) \cdot \mathcal{R}(\tau) = \{\mathit{lx} \leftrightarrow \mathit{rx}\}$



Exact resolution can be obtained considering *ancestral types*:

$\sigma' \equiv ((\alpha \multimap \beta) \multimap \gamma) \multimap \alpha \multimap \beta$

$\tau' \equiv (\alpha \multimap \alpha) \multimap \gamma$

$\mathcal{R}(\sigma') = \{\mathit{rlx} \leftrightarrow \mathit{lllx}, \mathit{rrx} \leftrightarrow \mathit{llrx}\}$

$\mathcal{R}(\tau') = \{\mathit{llx} \leftrightarrow \mathit{lrx}\}$

Beyond λ^A: Coppo -Dezani Intersection Types with Modalities

- Finitary logical description of λ-models: M. Coppo, M. Dezani-Ciancaglini, F. Honsell, G. Longo. Extended type structures and filter lambda models. **Logic Colloquium '82**, North Holland, Amsterdam, (1984).
- P. Di Gianantonio, F. Honsell, M. Lenisa. A type assignment system for game semantics. *Theor. Comput. Sci.* **398** (1-3), (2008).
- A. Ciaffaglione, P. Di Gianantonio, F. Honsell, M. Lenisa, I. Scagnetto, λ!-calculus, Intersection Types, and Involutions. *FSCD 2019*, LIPIcs **131**, (2019).
- Antonio Bucciarelli, Delia Kesner, Simona Ronchi Della Rocca - Inhabitation for Non-idempotent Intersection Types, *Logical Methods in Computer Science*, **14**(3), (2018).
- Many papers by Delia Kesner *et al*.

The language of types and indices

The language of types, *Type*$^!$, is a two sorted language given by the following grammars

$$(\mathit{Type}^! \ni)\ \sigma, \tau \ ::=\ \omega \mid \alpha \mid \ldots \mid \sigma \multimap \tau \mid \widehat{\sigma}$$
$$(\widehat{\mathit{Type}^!} \ni)\ \widehat{\sigma}, \widehat{\tau} \ ::= !_u\sigma \mid \widehat{\sigma} \wedge \widehat{\tau}$$
$$(\mathit{Index}^! \ni)\ u, v ::= \epsilon \mid i \mid j \mid \ldots \mid \langle u, v \rangle.$$

- i, j, ... are index variables.
- The syntactic category $\widehat{\sigma}$-types isolates types whose main constructor is ! or $\wedge$.
- The equivalence relation on types $\sim$ is induced by $\omega \sim \sigma$, for any type σ which contains only the constant ω and no type variables.

Principal Types for non-linear Combinators

$$
\begin{aligned}
[\![\mathbf{F}]\!] &= \;!_i(\alpha \multimap \beta) \multimap !_i\alpha \multimap !_i\beta \\
[\![\mathbf{W}]\!] &= (!_i\alpha \multimap !_j\beta \multimap \gamma) \multimap (!_i\alpha \wedge !_j\beta) \multimap \gamma \\
[\![\mathbf{D}]\!] &= \;!_\epsilon\alpha \multimap \alpha \\
[\![\delta]\!] &= \;!_{\langle j,i\rangle}\alpha \multimap !_i!_j\alpha
\end{aligned}
$$

General Principal Type Assignment System $\Vdash$

$$\frac{}{x : \alpha \Vdash x : \alpha} \quad \text{(var)}$$

$$\frac{\Gamma, x : \sigma \Vdash M : \tau}{\Gamma \Vdash \lambda x.M : \sigma \multimap \tau} \quad \text{(abs)} \qquad \frac{\Gamma \Vdash M : \sigma \quad \alpha \text{ fresh}}{\Gamma \Vdash \lambda x.M : \alpha \multimap \sigma} \quad \text{(abs}_\emptyset\text{)}$$

$$\frac{\Gamma \Vdash M : \sigma}{\widehat{!}_i(\Gamma) \Vdash !M : !_i(\sigma)} \quad \text{(!-Intro)} \qquad \frac{\Gamma \Vdash !M : !_\epsilon \sigma \quad !(\Gamma)}{\Gamma \Vdash M : \sigma} \quad \text{(!-Elim)}$$

$$\frac{\Gamma, !x : \widehat{\sigma}, !y : \widehat{\tau} \Vdash M[x, y] : \rho}{\Gamma, !x : \widehat{\sigma} \wedge \widehat{\tau} \Vdash M[x, x] : \rho} \quad (\wedge\text{-Intro-L})$$

$$\frac{\begin{array}{cc} \Gamma \Vdash M : \sigma & \Delta \Vdash N : \tau \\ (dom(\Gamma) \cap dom(\Delta)) = \emptyset & (TVar(\Gamma) \cap TVar(\Delta)) = \emptyset \\ (TVar(\sigma) \cap TVar(\tau)) = \emptyset & U' = \mathcal{M}(\langle \ldots, \sigma, \alpha \multimap \beta) \\ U = \mathcal{M}(\ldots, U'(\alpha), \tau) & \alpha, \beta \textit{ fresh} \end{array}}{U \circ U'(\Gamma, \Delta) \Vdash MN : U \circ U'(\beta)} \quad \text{(app)}$$

Notation

- $!(\Gamma)$ means that all variables in Γ are banged (Γ is possibly empty);
- $\widehat{!}_u(\Gamma, x:\tau) = \widehat{!}_u(\Gamma), !x :!_u\tau$ and
  $\widehat{!}_u(\Gamma, !x:\widehat{\tau}) = \widehat{!}_u(\Gamma), !x : \widehat{!}_u(\widehat{\tau})$, where
  $$\widehat{!}_u(\widehat{\tau}) = \begin{cases} !_{\langle u,v\rangle}\tau & \text{if } \widehat{\tau} = !_v\tau \\ \widehat{!}_u(\widehat{\tau}_1) \wedge \widehat{!}_u(\widehat{\tau}_2) & \text{if } \widehat{\tau} = \widehat{\tau}_1 \wedge \widehat{\tau}_2 \end{cases}$$

Principal Type for $2 \equiv \lambda !x.\lambda !y.x!(x!y)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{z : \delta \Vdash z : \delta}{!z :!_{\epsilon}\delta \Vdash !z :!_{\epsilon}\delta}
}{!z :!_{\epsilon}\delta \Vdash z : \delta}
\qquad
\dfrac{
\dfrac{
w : \beta \Vdash w : \beta
\qquad
\dfrac{y : \alpha \Vdash y : \alpha}{!y :!_i\alpha \Vdash !y :!_i\alpha}
}{w :!_i\alpha \multimap \gamma,\ !y :!_i\alpha \Vdash w!y : \gamma}
}{!w :!_j(!_i\alpha \multimap \gamma),\ !y :!_{\langle j,i\rangle}\alpha \Vdash !(w!y) :!_j\gamma}
}{!z :!_{\epsilon}(!_j\gamma \multimap \delta),\ w :!_j(!_i\alpha \multimap \gamma), !y :!_{\langle j,i\rangle}\alpha \Vdash z!(w!y) : \delta}
}{!x :!_k(!_j\gamma \multimap \delta) \wedge !_j(!_i\alpha \multimap \gamma),\ !y :!_{\langle j,i\rangle}\alpha \Vdash x!(x!y) : \delta}
}{\langle\rangle \Vdash \lambda !x.\lambda !y.x!(x!y) :!_k(!_j\gamma \multimap \delta) \wedge !_j(!_i\alpha \multimap \gamma) \multimap (!_{\langle j,i\rangle}\alpha \multimap \delta)}
$$

Principal Type for $2!2 = 4$

$$\dfrac{\dfrac{\cdots}{\langle\rangle \Vdash 2 :!_k(!_j\gamma \multimap \delta)\wedge!_j(!_i\alpha \multimap \gamma) \multimap (!_{\langle j,i\rangle}\alpha \multimap \delta)} \qquad \dfrac{\dfrac{\cdots}{\langle\rangle \Vdash 2 :!_k(!_j\gamma \multimap \delta)\wedge!_j(!_i\alpha \multimap \gamma) \multimap (!_{\langle j,i\rangle}\alpha \multimap \delta)}}{\langle\rangle \Vdash !2 :!_m(!_k(!_j\gamma \multimap \delta)\wedge!_j(!_i\alpha \multimap \gamma) \multimap (!_{\langle j,i\rangle}\alpha \multimap \delta))}}{\langle\rangle \Vdash 2!2 :??}$$

Duplication at work

$$!_m\tau \equiv !_m(!_k(!_j\gamma \multimap \delta)\wedge !_j(!_i\alpha \multimap \gamma) \multimap (!_{\langle j,i\rangle}\alpha \multimap \delta)))$$
$$!'_k(!_{j'}\gamma' \multimap \delta')\wedge !_{j'}(!_{i'}\alpha' \multimap \gamma') \qquad \multimap \qquad (!_{\langle i',j'\rangle}\alpha' \multimap \delta')$$

duplicate $!_m$ *i.e.* $!_m\tau \mapsto !_{m_l}\tau_l \wedge !_{m_l}\tau_r$

$$\tau_l \equiv \quad !_{k_l}(!_{j_l}\gamma_l \multimap \delta_l)\wedge !_{j_l}(!_{i_l}\alpha_l \multimap \gamma_l) \quad \multimap \quad (!_{\langle i_l,j_l\rangle}\alpha_l \multimap \delta_l)$$
$$!_{j'}\gamma' \quad \multimap \quad \delta'$$

duplicate $!_{j'}$ *i.e.* $!_{j'}\gamma' \mapsto !_{j'_l}\gamma'_l \wedge !_{j'_r}\gamma'_r$

$$!_{k_l}(!_{j_l}\gamma_l \multimap \delta_l)\wedge !_{j_l}(!_{i_l}\alpha_l \multimap \gamma_l)$$
$$!_{j^l}\gamma^1_l \wedge !_{j^r}\gamma^1_r$$

but since on the right hand side we had

$$!_{m_r}\tau_r \equiv !_{m_r}(\quad !_{k_r}(!_{j_r}\gamma_r \multimap \delta_r)\wedge !_{j_r}(!_{i_r}\alpha_r \multimap \gamma_r) \quad \multimap \quad (!_{\langle i_r,j_r\rangle}\alpha_r \multimap \delta_r) \quad)$$
$$!_{j'}(\quad !_{i'}\alpha' \quad \multimap \quad \gamma' \quad)$$

and $!_{j'}$ has been duplicated we have, in fact,

$$!_{m_r}(\quad !_{k_r}(!_{j_r}\gamma_r \multimap \delta_r)\wedge !_{j_r}(!_{i_r}\alpha_r \multimap \gamma_r) \quad \multimap \quad (!_{\langle i_r,j_r\rangle}\alpha_r \multimap \delta_r) \quad)$$
$$!_{j'_l}(!_{i'_l}\alpha'_l \multimap \gamma'_l) \quad \wedge \quad !_{j'_r}(!_{i'_r}\alpha'_r \multimap \gamma'_r)$$

hence we have to duplicate also $!_{m_r}\tau_r$, namely we have to meta-unify

$$(!_{m_{rl}}(!_{k_{rl}}(!_{j_{rl}}\gamma_{rl} \multimap \delta_{rl})\wedge !_{j_{rl}}(!_{i_{rl}}\alpha_{rl} \multimap \gamma_{rl}) \multimap (!_{\langle i_{rl},j_{rl}\rangle}\alpha_{rl} \multimap \delta_{rl}))) \qquad \wedge$$
$$!_{j'_l}(!_{i'_l}\alpha'_l \multimap \gamma'_l) \qquad \wedge$$

$$(!_{m_{rr}}(!_{k_{rr}}(!_{j_{rr}}\gamma_{rr} \multimap \delta_{rr})\wedge !_{j_{rr}}(!_{i_{rr}}\alpha_{rr} \multimap \gamma_{rr}) \multimap (!_{\langle i_{rr},j_{rr}\rangle}\alpha_{rr} \multimap \delta_{rr})))$$
$$!_{j'_r}(!_{i'_r}\alpha'_r \multimap \gamma'_r)$$

...

Duplication at work

Duplications triggered by $!_{j'}$ are reflected also in

$$!_{\langle j', i' \rangle}\alpha' \multimap \delta'$$

which is meta-substituted into

$$(!_{\langle j'_l, i'_l \rangle}\alpha'_l \wedge !_{\langle j'_r, i'_r \rangle}\alpha'_r) \multimap \delta'$$

Metaunification (in progress)

Two issues

- duplication of !-indices trigger, even at-a-distance, duplication of subtypes which they encapsulate;
- the connection with the encapsulating !-indices has to be maintained:

we need a new meta-unification judgment, $\mathcal{M}$, which rewrites sets of *t*-ples consisting of appropriate parameters to take care of the above issues and two types, $\langle \ldots, \sigma, \tau \rangle$.