Bidimensional fixpoint operators

IFIP 2023

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Fixpoints

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- For a program M, a fixpoint is an input x such that there is a calculation sequence between $M(x)$ and x
  - $\mathbf{Y}$ fixpoint combinator in untyped λ-calculus:

$$\mathbf{Y}M =_\beta M(\mathbf{Y}M)$$

  - fixpoint operator in PCF

$$\frac{\Gamma, A \vdash M : A}{\Gamma \vdash \mathbf{Y}M : A}\ \text{FIX}$$

Categorical fixpoint operators

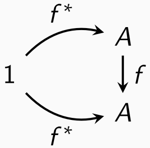
Definition

Let $\mathbb{C}$ be a category with a terminal object 1, a **fixed-point operator** on $\mathbb{C}$ is family of functions

$$(-)^*_A : \mathbb{C}(A, A) \to \mathbb{C}(1, A)$$
$$f \mapsto f^*$$

indexed by the objects A of $\mathbb{C}$ verifying that for all morphisms $f : A \to A$,

$$f \circ f^* = f^*$$



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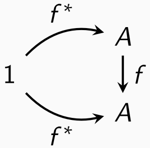

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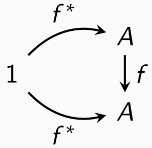

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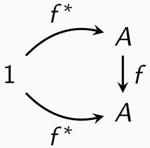
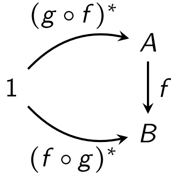


Example: for a domain D and a Scott-continuous map $f : D \to D$,

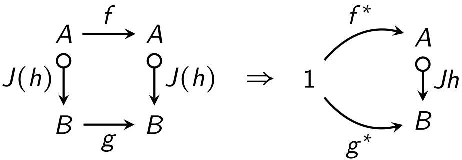
$$f(\bigvee_{n \in \omega} f^n(\bot)) = \bigvee_{n \in \omega} f^n(\bot)$$

Dinatural fixpoint operators

- A fixpoint operator $(-)^*$ on $\mathbb{D}$ is **dinatural** if for every $f : A \to B$ and $g : B \to A$ in $\mathbb{D}$,



- For $J : \mathbb{C} \to \mathbb{D}$ an identity-on-objects functor preserving terminal objects, a fixpoint operator $(-)^*$ on $\mathbb{D}$ is said to be **uniform with respect to J** if:



- $+$ naturality and diagonal axioms for parameterized fixpoints...

Replace equalities by arrows

- In λ-calculus, reductions are directed:

$$\mathbf{Y}M \quad \longrightarrow \quad M(\mathbf{Y}M)$$

- Initial algebra and final coalgebra semantics

$$F(X) \xrightarrow{\cong} X \qquad\qquad X \xrightarrow{\cong} F(X)$$

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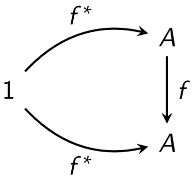
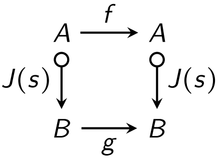
Use 2-morphisms in a 2-category/bicategory to model fixpoint reductions

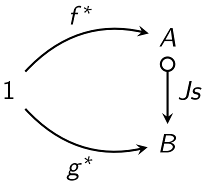
What is a 2-dimensional fixpoint operator?

for $f : A \to A$



for a square





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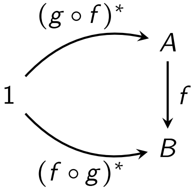
and $g : B \to A$



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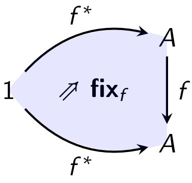

$f^* \overset{\text{fix}}{\Longrightarrow} f \circ f^*$

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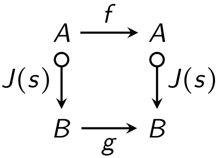

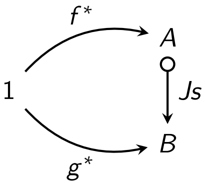

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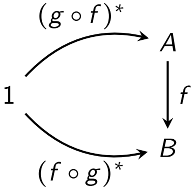

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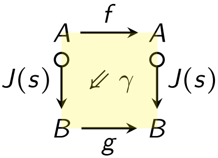

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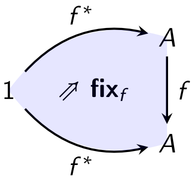

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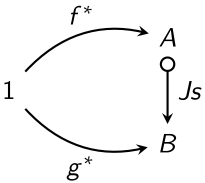

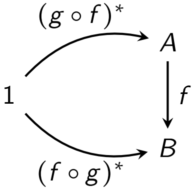

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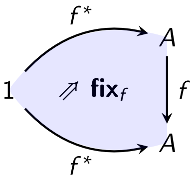

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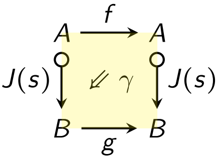

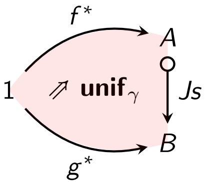

$$g^* \overset{\textbf{unif}}{\Longrightarrow} Js \circ f^*$$

for $f : A \to B$
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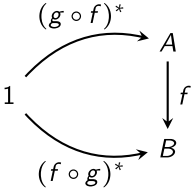

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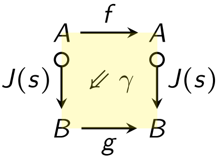

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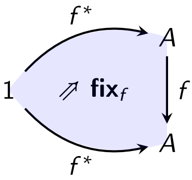

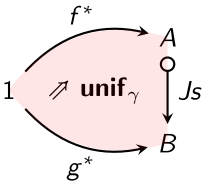

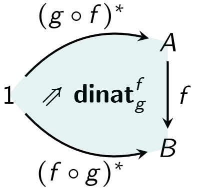

$f^* \xRightarrow{\textbf{fix}} f \circ f^*$

$g^* \xRightarrow{\textbf{unif}} Js \circ f^*$

$(fg)^* \xRightarrow{\textbf{dinat}} f \circ (gf)^*$

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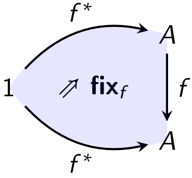

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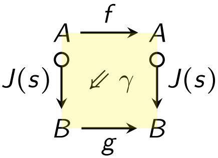

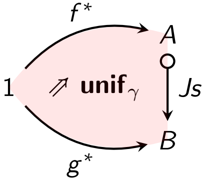

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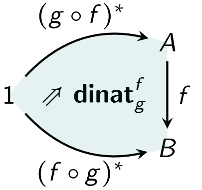

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What axioms and coherences do they satisfy?

General strategy

- **step 1** Restrict to fixpoints that are obtained via a universal property (Plotkin-Simpson theorem constructing a unique fixpoint operator when certain conditions hold)
- **step 2** Categorify the Plotkin-Simpson theorem to extract the coherence equations needed to obtain the 2-categorical universal property
- **step 3** Verify that the axiomatization holds for general fixpoint operators

Induction $\Leftrightarrow$ coinduction

A bifree algebra for an endofunctor $T : \mathbb{C} \to \mathbb{C}$ consists of an initial algebra $(X, a : T(X) \to X)$ such that its inverse a^{-1} is a final coalgebra.

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Theorem (Plotkin-Simpson, 2000)

Let $\mathbb{C}$ be a category equipped with a comonad (T, δ, ε) and a terminal object.

- *If the endofunctor T has a bifree algebra, then the co-Kleisli $\mathbb{C}_T$ has a unique fixpoint operator uniform wrt $\mathbb{C} \xrightarrow{\textit{free}} \mathbb{C}_T$.*
- *If $\mathbb{C}$ is cartesian and the endofunctor $T \circ T$ has a bifree algebra, then the co-Kleisli $\mathbb{C}_T$ has a unique dinatural fixpoint operator uniform wrt $\mathbb{C} \xrightarrow{\textit{free}} \mathbb{C}_T$.*

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Examples: the category **Cppo** in domain theory and the relational model in linear logic

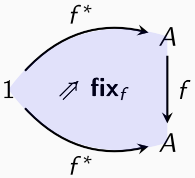
2-categorical fixpoint operator

Definition

Let $\mathscr{D}$ be a 2-category with a terminal object 1. A *pseudo fixpoint operator on* $\mathscr{D}$ consists of a family of functors indexed by the objects A of $\mathscr{D}$:

$$(-)^*_A : \mathscr{D}(A, A) \to \mathscr{D}(1, A)$$

together with a family of natural isomorphisms **fix** with components

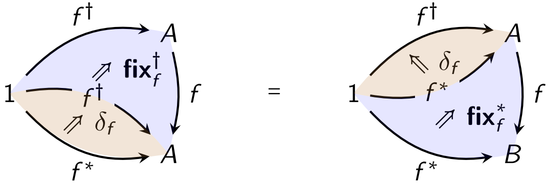


for a 1-cell $f : A \to A$ in $\mathscr{D}$.

Fixpoint operators form a category

Define the category $\mathbf{Fix}(\mathscr{D})$

- objects: pseudo-fixpoint operators on $\mathscr{D}$
- morphisms: a morphism $((-)^*, \mathbf{fix}^*) \to ((-)^\dagger, \mathbf{fix}^\dagger)$ is a family of natural transformations with components $\delta_f : f^* \Rightarrow f^\dagger$ for $f : A \to A$ such that:



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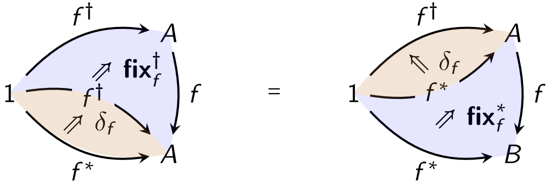

least fixpoint $\rightsquigarrow$ initial object in $\mathbf{Fix}(\mathscr{D})$

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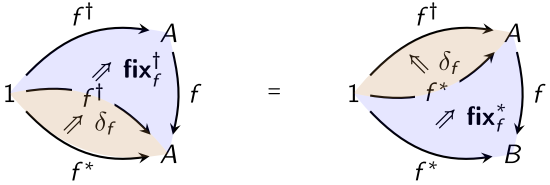

least fixpoint	$\rightsquigarrow$	initial object in $\mathbf{Fix}(\mathscr{D})$
greatest fixpoint	$\rightsquigarrow$	terminal object in $\mathbf{Fix}(\mathscr{D})$

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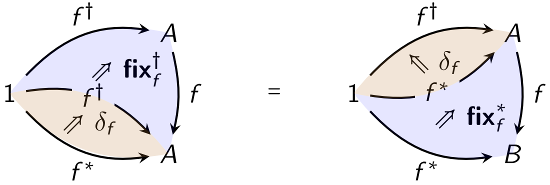
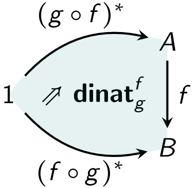


least fixpoint	$\rightsquigarrow$	initial object in $\mathbf{Fix}(\mathscr{D})$
greatest fixpoint	$\rightsquigarrow$	terminal object in $\mathbf{Fix}(\mathscr{D})$
unique fixpoint	$\rightsquigarrow$	$\mathbf{Fix}(\mathscr{D})$ is contractible

Dinatural fixpoints for 2-categories

A *pseudo dinatural fixpoint operator on* $\mathscr{D}$ consists a pseudo fixpoint operator $((-)^*, \mathbf{fix})$ on $\mathscr{D}$ together with 2-cells

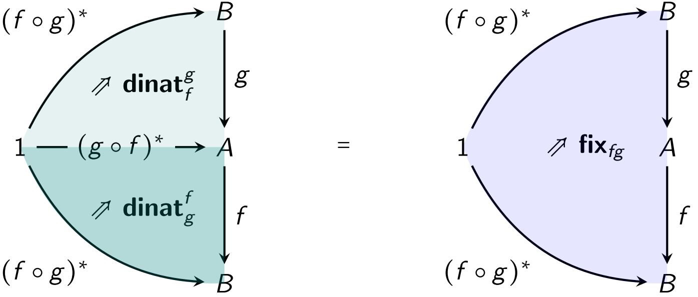


for all $f : A \to B$ and $g : B \to A$ satisfying the axioms of a pseudo dinatural transformation:

$$\mathbf{dinat} : \mathscr{D}(-,-) \Rightarrow \mathscr{D}(1,-) : \mathscr{D}^{op} \times \mathscr{D} \to \mathbf{CAT}$$

and ...

Coherence between **fix** and **dinat**



Characterizing fixpoint operators on 2-categories

Theorem

Let $\mathscr{C}$ be a 2-category equipped with a 2-comonad (T, δ, ε) and a terminal object. We denote by $\mathscr{D}$ the co-Kleisli 2-category $\mathscr{C}_T$ and by $J : \mathscr{C} \to \mathscr{D}$ the free functor.

- *If the endofunctor T has a pseudo-bifree algebra, the category of uniform pseudo-fixpoint operators $\mathbf{Fix}(\mathscr{D}, J)$ is contractible.*
- *If $\mathscr{C}$ is cartesian and the endofunctor $T \circ T$ has a pseudo-bifree algebra, then the category of dinatural uniform pseudo-fixpoint operators $\mathbf{DinFix}(\mathscr{D}, J)$ is contractible.*

Examples

- The 2-category $\mathbf{Cat}_{\omega,\perp}$

  objects: ω-complete categories with initial object
  1-cells: functors pres. colimits of ω-chains and initial objects
  2-cells: natural transformations

  with the lifting 2-comonad

$$(-)_\perp \circlearrowright \mathbf{Cat}_{\omega,\perp} \xrightarrow{\text{Free}} \mathbf{Kleisli}(\mathbf{Cat}_{\omega,\perp}) \simeq \mathbf{Cat}_\omega$$

- The bicategory **Prof**

  objects: small categories
  1-cells: profunctors
  2-cells: natural transformations

  with the free symmetric strict monoidal completion pseudo-comonad **Sym**

$$\mathbf{Sym} \circlearrowright \mathbf{Prof} \xrightarrow{\text{Free}} \mathbf{Prof}_{\mathbf{Sym}} \simeq \mathbf{Esp}$$

Polynomial functors

	induction	coinduction
	W-types	**M**-types
	well-founded trees	non-well-founded trees
$X \mapsto 1 + X$	$\mathbb{N}$	$\overline{\mathbb{N}}$
$X \mapsto 1 + X^2$	finite binary trees	all binary trees
$X \mapsto 1 + A \times X$	finite streams	all streams

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	pseudo-dinatural fixpoint uniform wrt *spans*	pseudo-dinatural fixpoint uniform wrt *monomials*

Future work

- Parameterized fixpoints operators
- Traced monoidal bicategories for cyclic λ-calculi
- Guarded recursion
- Develop the syntactic counterpart
- Coherence theorems

Thank you