A Complexity Dichotomy for Semilinear Target Sets in Automata with One Counter

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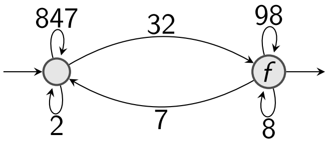
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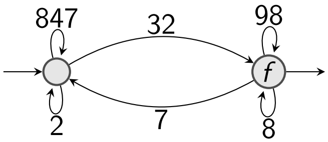

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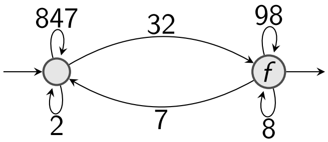
Given Automaton, number $t \in \mathbb{N}$

Question Can we reach (f, t)?



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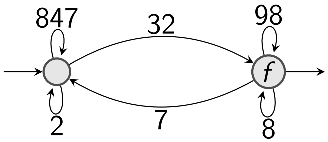

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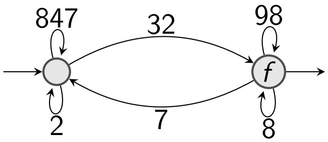

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General toolbox beyond coverability?

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- $\mathbb{Z}$-VASS (counters can go negative): modified density notion

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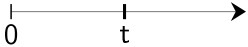

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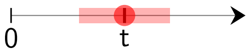

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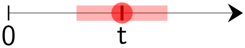

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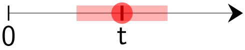

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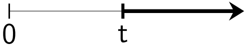

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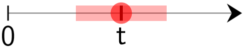
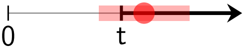


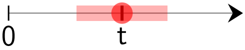


$D(S) = 0$

$S_t = \{t\}$ $\qquad$ $S_t = [t, \infty)$

$D(S) = 0$ $\qquad$ $D(S) = \frac{1}{2}$

$S_t = \{t\}$



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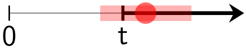
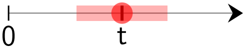
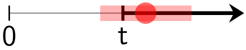


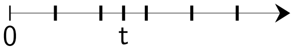
$D(S) = \frac{1}{2}$

Similarly: $S_{r,s,t} = [t, \infty) \backslash \{r, s\}$

$S_t = \{t\}$ $\quad$ $S_t = [t, \infty)$ $\quad$ $S_t = \{2t + 1\} \cup 2\mathbb{N}$







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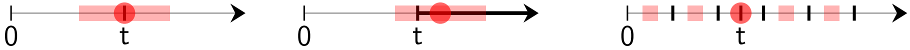

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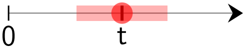

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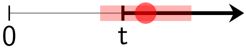

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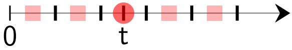

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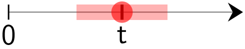

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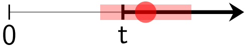

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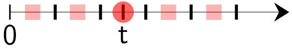

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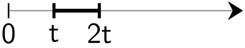

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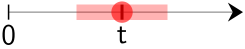

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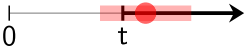

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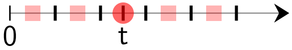

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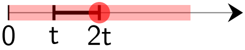

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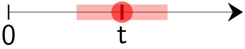

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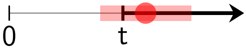

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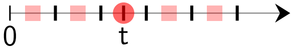

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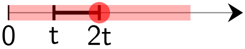
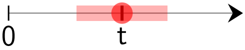


$D(S) = \frac{1}{4}$

$S_t = \{t\}$



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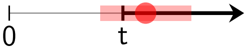

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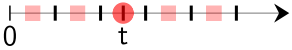

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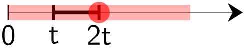

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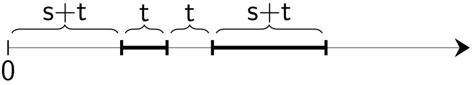

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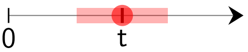

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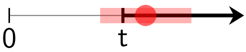

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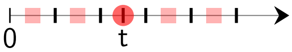

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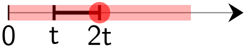

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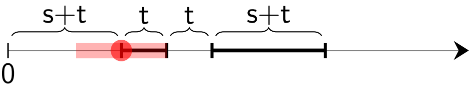

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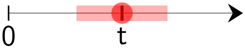

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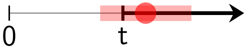

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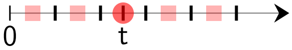

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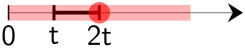

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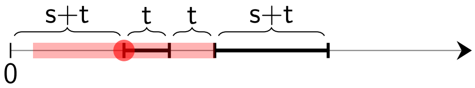

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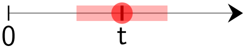

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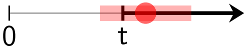

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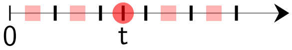

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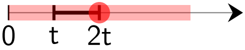

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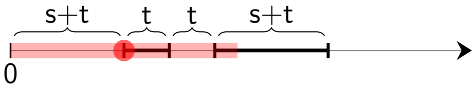

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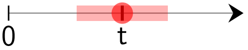

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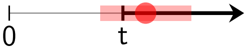

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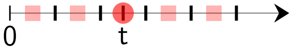

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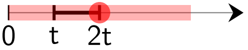

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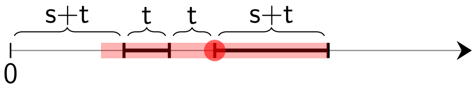

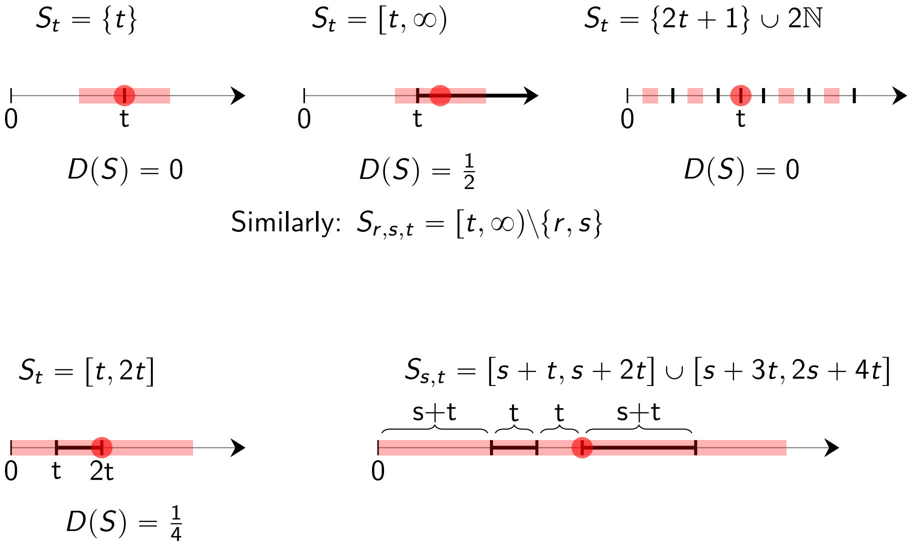
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t
s+t
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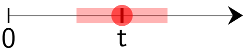

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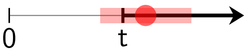

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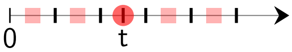

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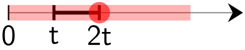

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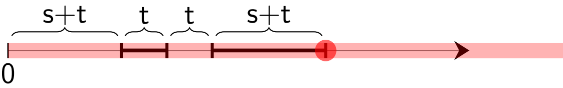

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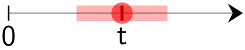

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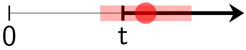

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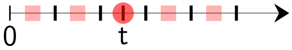

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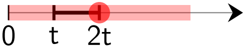

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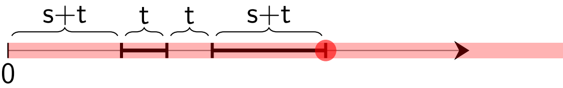

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How to solve cases with $D(S) > 0$?

Step I: Make automaton acyclic

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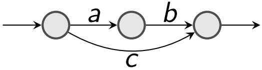

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matrix over $\mathbb{B}[X]$

$=$ polynomials in X over Boolean semiring

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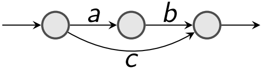

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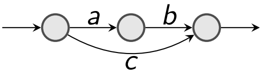

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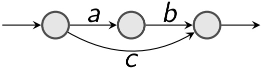

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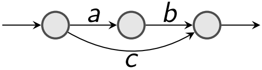

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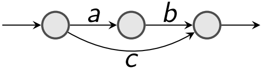

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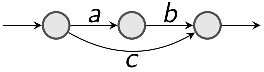

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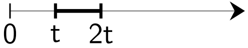
Equivalently: impose $X^i + X^j = X^{\max(i,j)}$


0
t
2t



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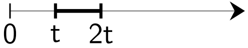

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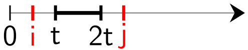

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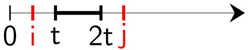

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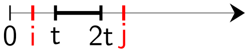

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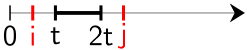

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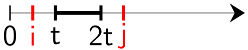

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3 terms that cannot be merged $\rightsquigarrow$ 2 gaps of size $> t$ $\rightsquigarrow$ some term $> 2t$

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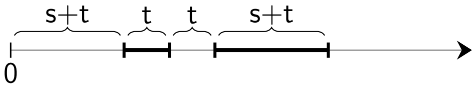

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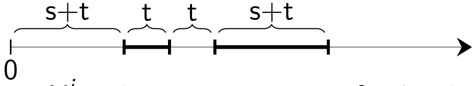

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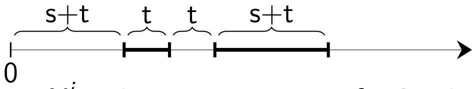

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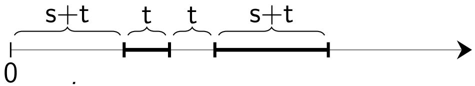

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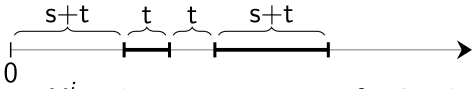

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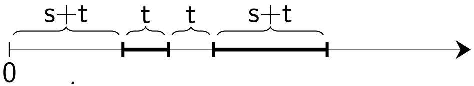

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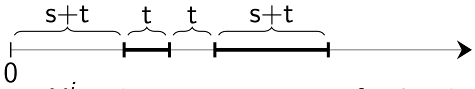

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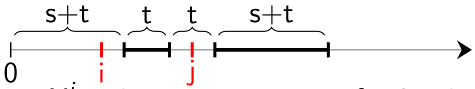

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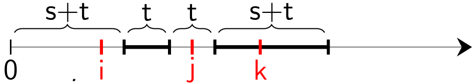

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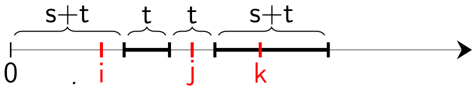

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Lemma

Three equations $\rightsquigarrow$ bounded number of $X^{[i,j]}$ terms

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  In particular: Improved NC^2 upper bound (by Almagor, Cohen, Pérez, Shirmohammadi, and Worrell in 2020) to AC^1

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- $\mathbb{Z}$-VASS (counters can go negative): modified density notion
- VASS (must stay non-negative): "uniformly quasi-upward closed"
  In particular: Improved NC^2 upper bound (by Almagor, Cohen, Pérez, Shirmohammadi, and Worrell in 2020) to AC^1

Toolbox: Weighted automata over suitable semirings

This is always possible:

Whenever $D(S) > 0$, we can find equation system

To get AC^1 upper bound:

- Compute A^n by repeated squaring $\rightsquigarrow$ logarithmic number of steps
- Show that matrix product can be computed in AC^0

Theorem (Shakiba, Sinclair-Banks, Z. 2025)

Let $S \subseteq \mathbb{N}^p \times \mathbb{N}$ be Presburger-definable.

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Open: Multiple counters, e.g. $S = \{(x, y) \in \mathbb{N}^2 \mid x \leqslant y \leqslant 2x\}$