Polynomials All Around Us

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IFIP WG 2.2 _ Aachen

In this talk ...

- Examples of Polynomials in computer science

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- Examples of Polynomials in computer science
- The geometry behind our algebraic methods

Symmetric VAS Reachability Problem

Example. Symmetric VAS = $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right\} \subseteq \mathbb{Z}^2$

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Polynomial Encoding: $(a, b \geq 0)$

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$x^a y^b - 1$	$1 - x^a y^b$	$x^a - y^b$	$y^b - x^a$

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Ring	Ideal	generators	members
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Hilbert basis thm

algebraic set

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what is $\overline{\mathbb{Q}}$? (field of) algebraic numbers

root of polynomials with rational coefficients : $\sqrt{2} \in \overline{\mathbb{Q}}$, $\pi \notin \overline{\mathbb{Q}}$

Equivalent Problems

Ideal membership Problem

Input: generator polynomials $f_1, \ldots, f_k$ and a target polynomial g.

Question: $g \in \langle f_1, \ldots, f_k \rangle$?

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same complexity

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$$\boxed{C_1 \wedge C_2 \wedge C_3 = T \text{ iff } \exists\, (a_1, a_2, a_3) \text{ s.t. } \begin{cases} f_i(a_1, a_2, a_3) = 0 \\ g_i(a_1, a_2, a_3) = 0 \end{cases}}$$

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$$\bar{J} := \langle f_i, g_i, i = 1,2,3 \rangle$$

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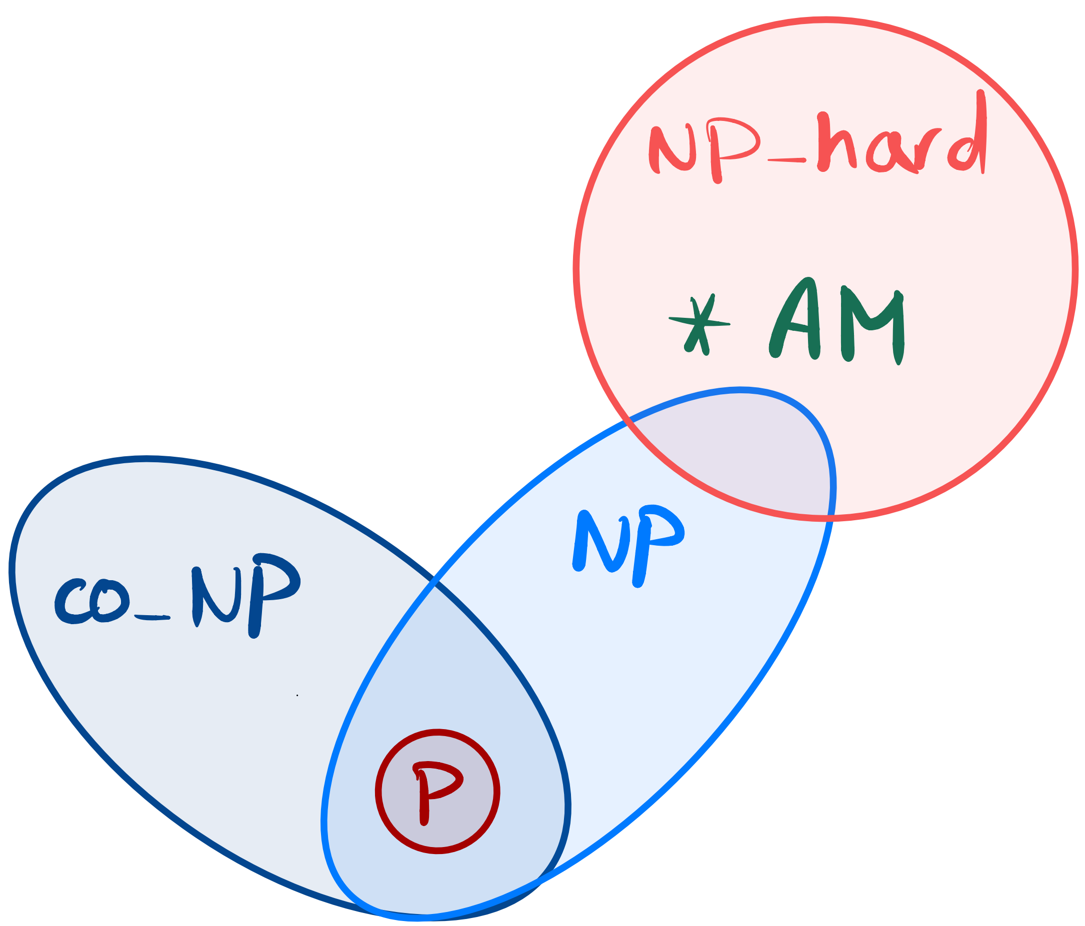

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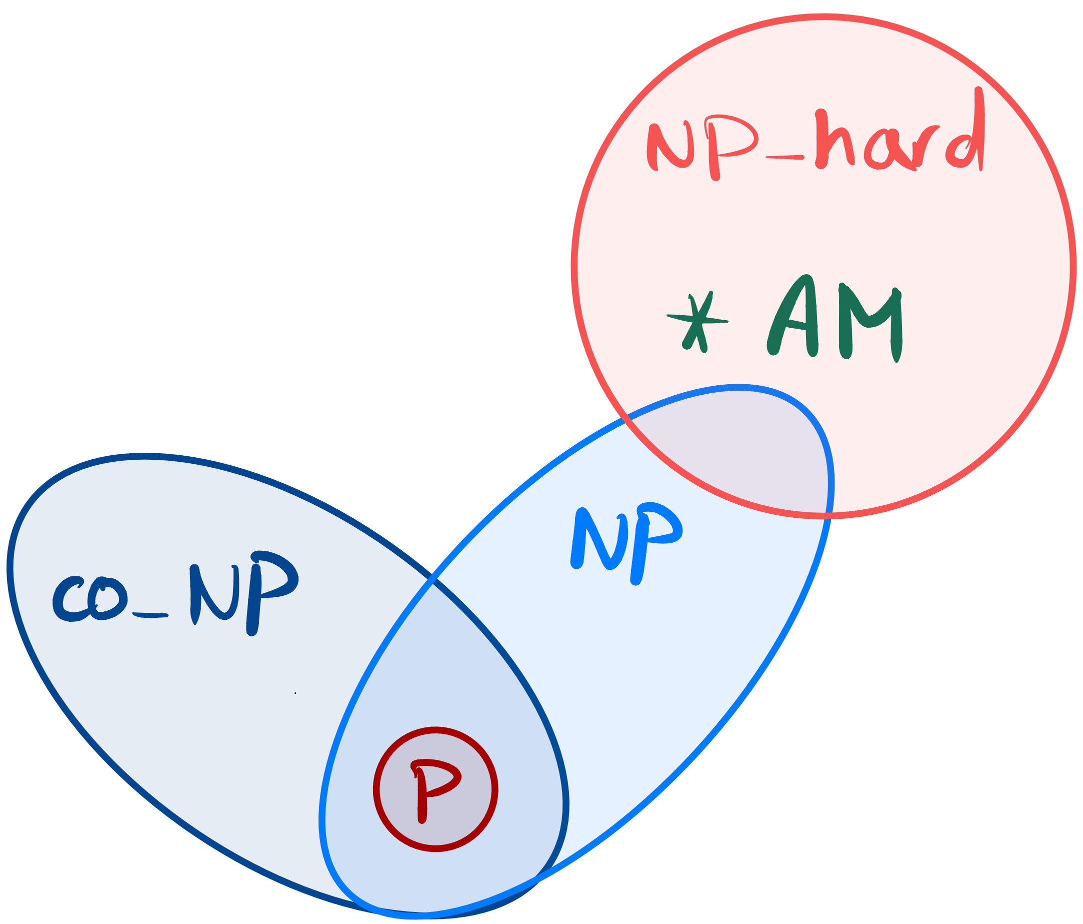

Strong Nullstellensatz:

A system has no solution iff $1 \in \sqrt{\langle f_1, \dots, f_k \rangle}$

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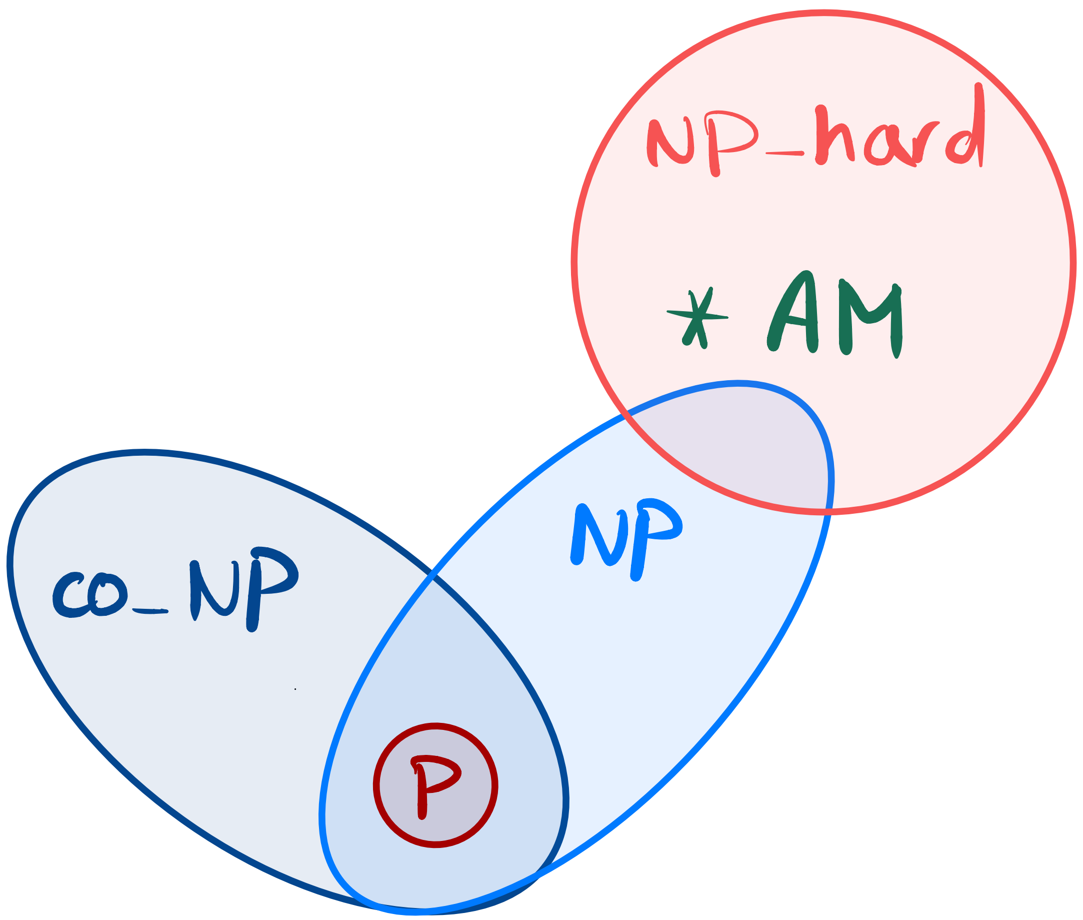

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what is radical ideal ?

Eugene Wigner

(Nobel Prize in Physics 1963)

Reprinted from *Communications in Pure and Applied Mathematics*, Vol. 13, No. I (February 1960). New York: John Wiley & Sons, Inc.

THE UNREASONABLE EFFECTIVENSS OF MATHEMATICS IN THE NATURAL SCIENCES

Eugene Wigner

Mathematics, rightly viewed, possesses not only truth, but supreme beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

- BERTRAND RUSSELL, Study of Mathematics

There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is pi." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

Naturally, we are inclined to smile about the simplicity of the classmate's approach. Nevertheless, when I heard this story, I had to admit to an eerie feeling because, surely, the reaction of the classmate betrayed only plain common sense. I was even more confused when, not many days later, someone came to me and expressed his bewilderment [*The remark to be quoted was made by F. Werner when he was a student in Princeton.*] with the fact that we make a rather narrow selection

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Algebra	Geometry
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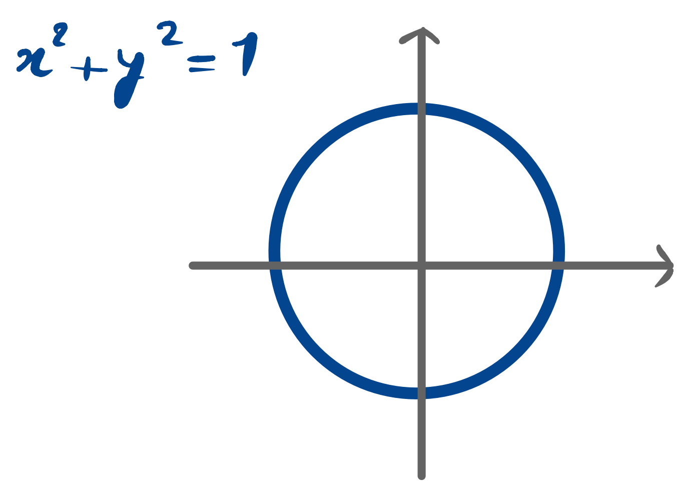
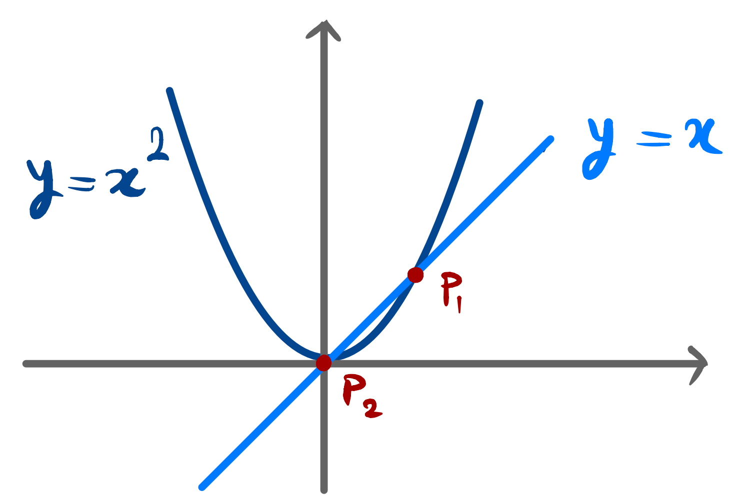


$f(x,y) = x^2+y^2-1$



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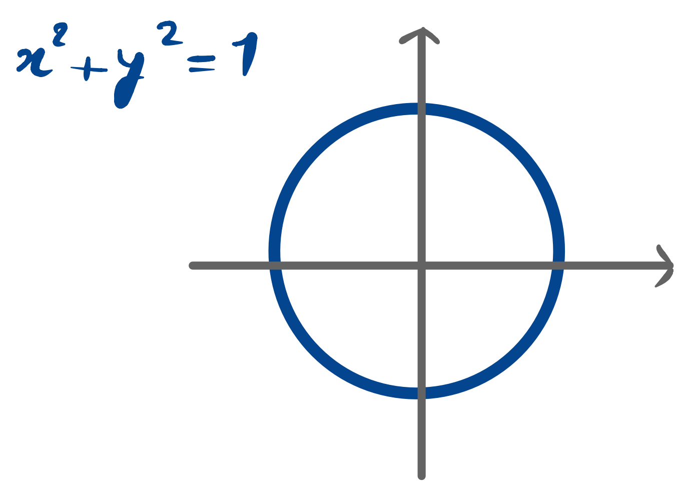

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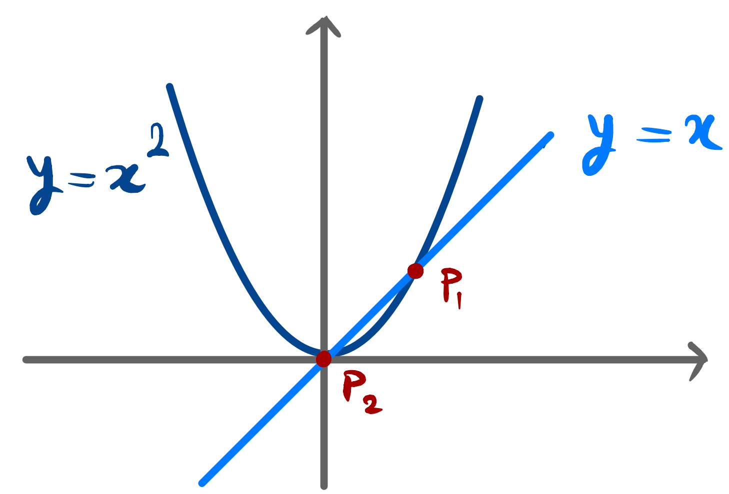

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$J := \langle y-x^2, y-x \rangle$

Algebraic set: vanishing set of J

A Bridge between Algebra and Geometry

- J an ideal of $\overline{\mathbb{Q}}[X]$ $\quad (X = (x_1, \dots, x_n))$

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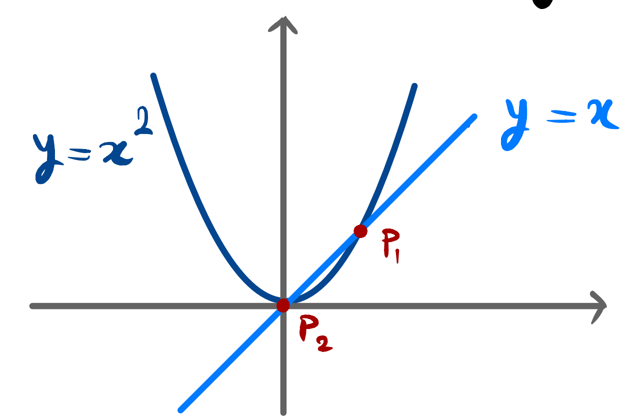

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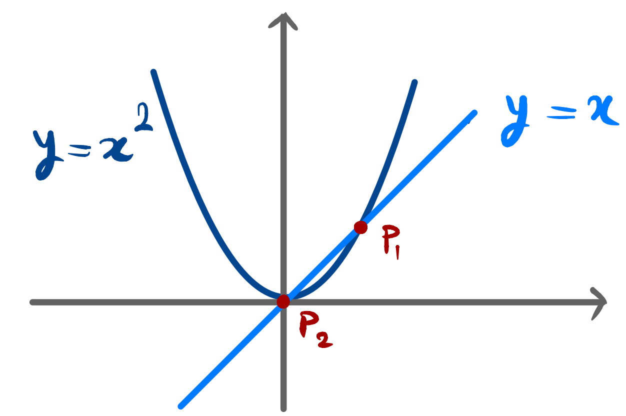

▷ S set of points in $\overline{\mathbb{Q}}^n$.

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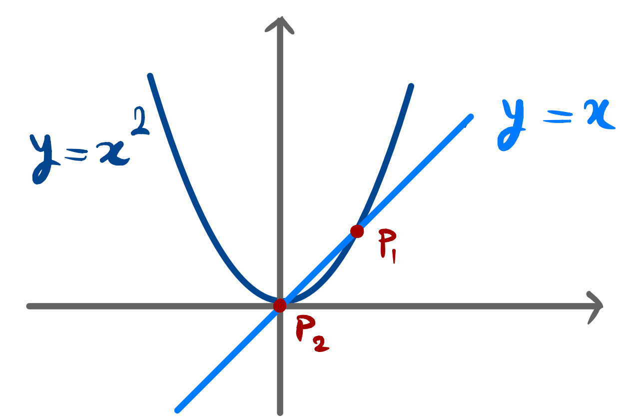

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$V(\langle y \rangle) = (\text{all points where } y \text{ is } 0) = x\text{-axis}$

fill the hole!

Noetherian Property

A ring is **Noetherian** if it has no infinitely long, strictly ascending chains of ideals. $I_1 \subsetneq I_2 \subsetneq I_3 \subseteq \cdots$ $\exists N$ s.t. $I_N = I_{N+1} = I_{N+2} = \cdots$

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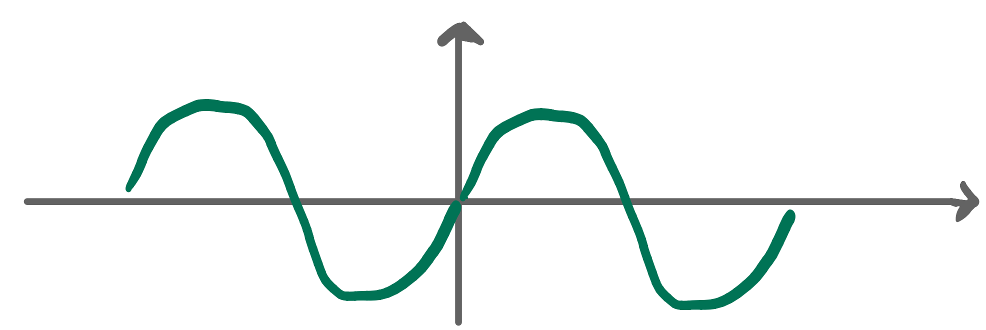

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Thank you!