Weighted Rewriting: Semiring Semantics for Abstract Reduction Systems

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RWTH Aachen University
26.09.2025

Provenance Analysis in Databases

$$\psi = Ra \wedge (Pab \vee Pbb)$$

R	
a	true
b	false

P		
a	a	true
b	a	false
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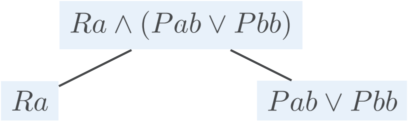

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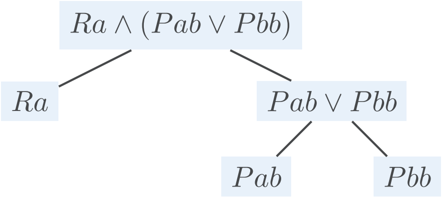

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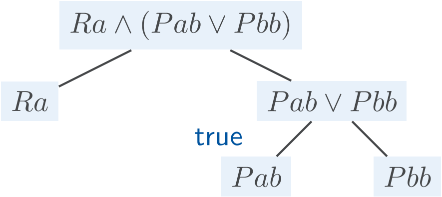

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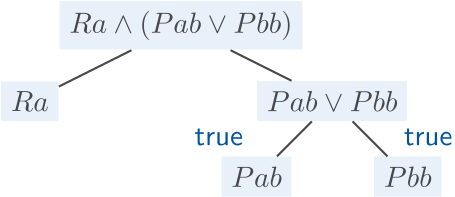

Provenance Analysis in Databases

$$\psi = Ra \land (Pab \lor Pbb)$$

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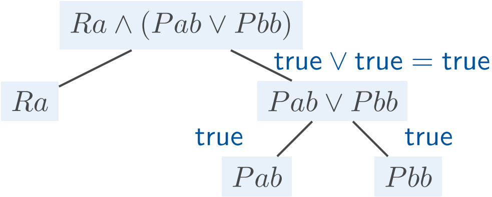

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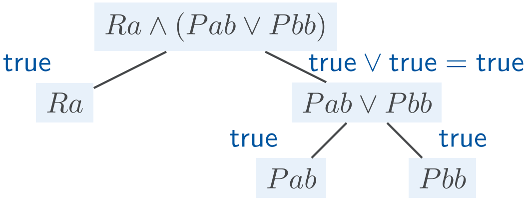

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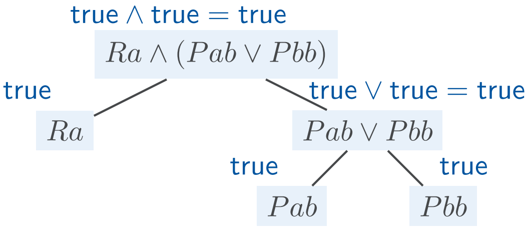

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$$\psi = Ra \wedge (Pab \vee Pbb)$$

R		cost
a	true	2
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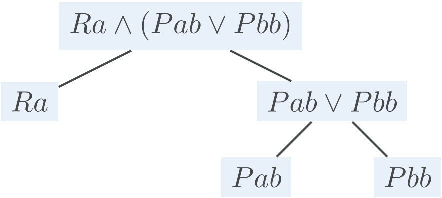
b	a	false	∞
a	b	true	7
b	b	true	10



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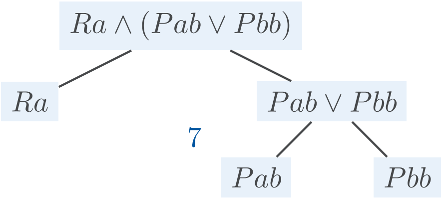

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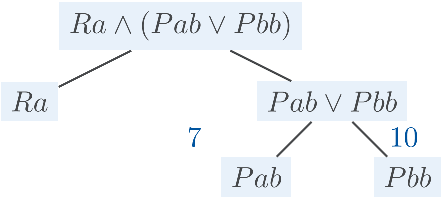

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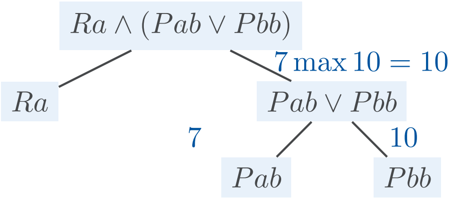

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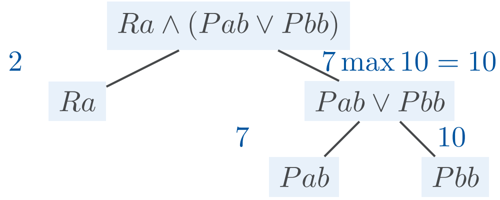

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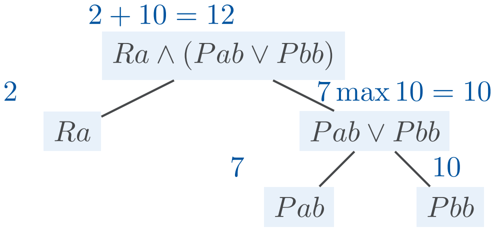

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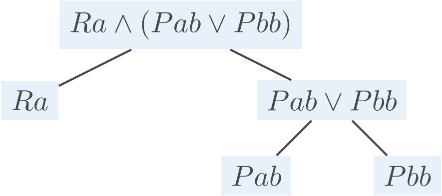

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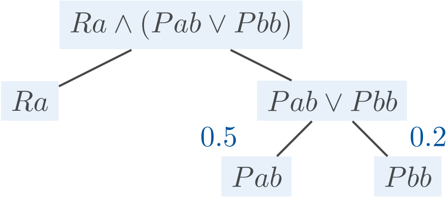

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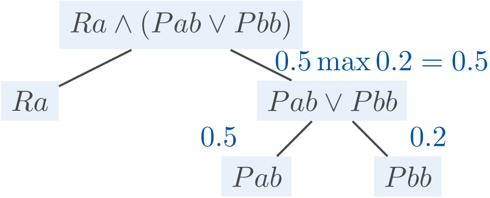

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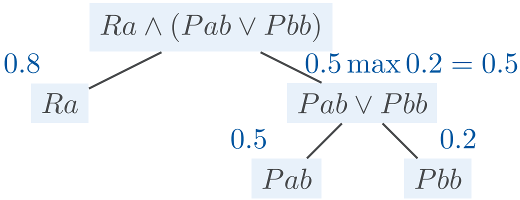

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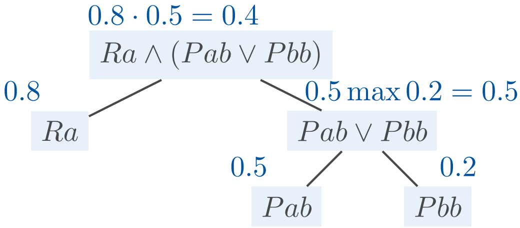

Goal

Abstract Reduction System

$$(A, \to)$$

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Provenance Analysis via Semirings

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Weighted Rewriting

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Weighted Rewriting

Uniform Framework for Analyzing Abstract Reduction Systems

- Termination Analysis
- Safety Analysis
- Cost Analysis
- Complexity Analysis
- Arbitrary Combinations
- Many more!

Semiring

Semiring $\mathbb{S} = (S, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with

- S the universe,

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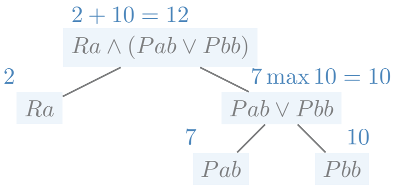
For example: $\mathbb{N} \subseteq \mathbb{N}^{\pm\infty}$ and $\sup \mathbb{N} = \top = \infty \in \mathbb{N}^{\pm\infty}$

One Framework Fits All

Weighted Abstract Reduction System $(A, \to, \mathbb{S}, \mathsf{f}_{\mathtt{NF}}, \mathsf{Aggr})$ with



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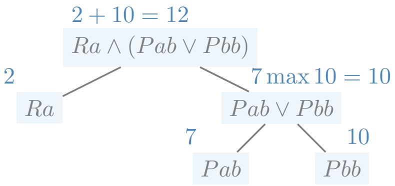
Costs in database:

Ra

$\psi \wedge \varphi \longrightarrow [\psi, \varphi]$

$\psi \vee \varphi \longrightarrow [\psi, \varphi]$



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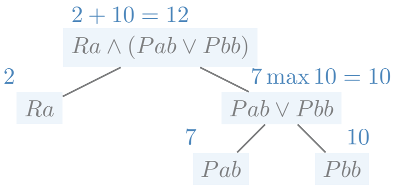

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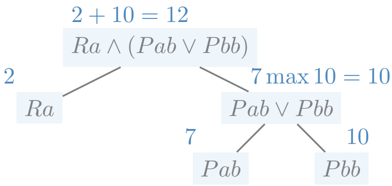

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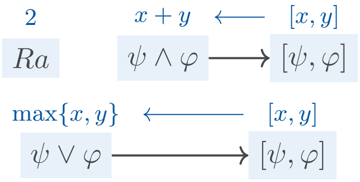

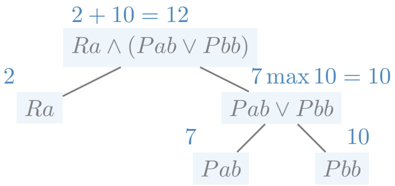

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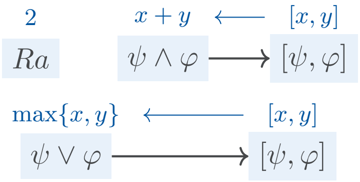

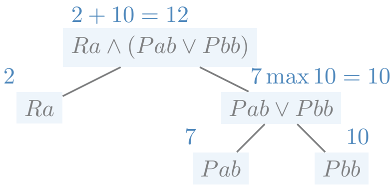

The *weight* of $a \in A$ is $[\![a]\!]$ with e.g. $[\![Ra \wedge (Pab \vee Pbb)]\!] = 12$.

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\text{false}, \text{true}\}, \vee, \wedge, \text{false}, \text{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more. . .

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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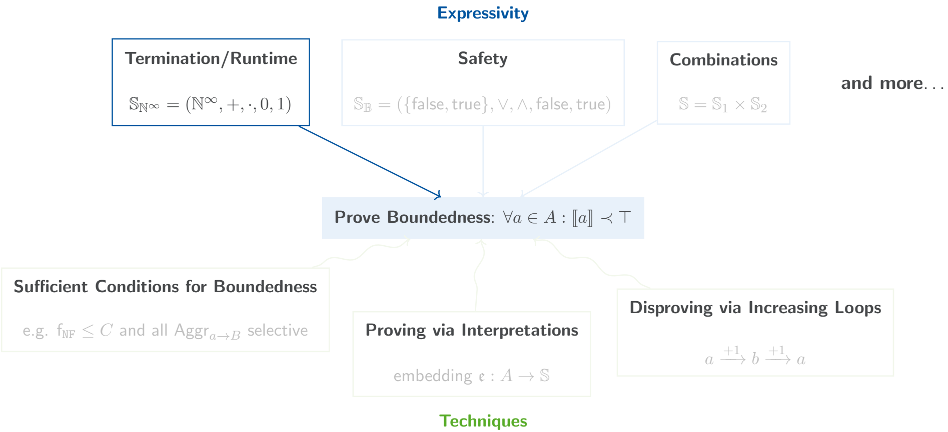
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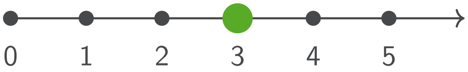

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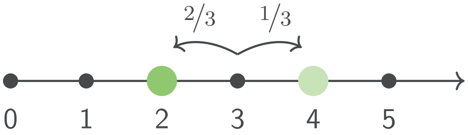

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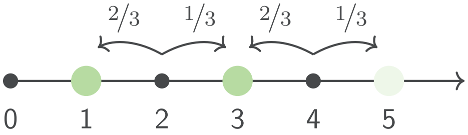

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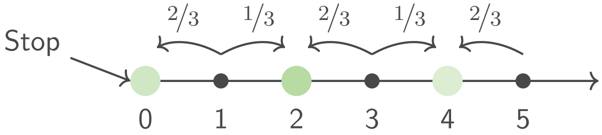

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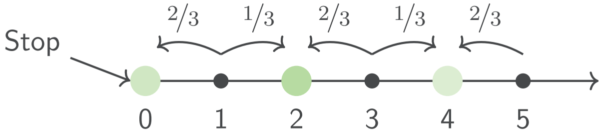
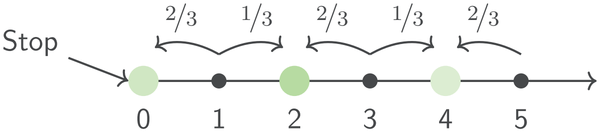


Sequence abstract reduction system $(\mathbb{N}, \to)$ with grammar

$$n + 1 \to [n, n + 2]$$

Complexity of the Random Walk



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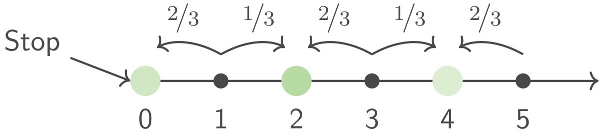
and normal form $\mathtt{NF}_{\to} = \{0\}$.

Complexity of the Random Walk



Sequence abstract reduction system $(\mathbb{N}, \to)$ with grammar

$$n + 1 \to [n, n + 2]$$

and normal form $\mathtt{NF}_{\to} = \{0\}$.

What is the probability to reach 0 in 3 steps? What about finitely many steps?
What is the expected runtime?

Complexity of the Random Walk

Probability to reach 0 in 3 steps?

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

0

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$1$$

$$0$$

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\overset{1}{0} \qquad n+1$$

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\overset{1}{0} \qquad n+1 \longrightarrow [n, n+2]$$

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

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Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

$$1 \qquad {}^2\!/\!_3 \cdot x + {}^1\!/\!_3 \cdot y \longleftarrow [x, y]$$

$$0 \qquad n + 1 \longrightarrow [n, n+2]$$

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

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$$3$$

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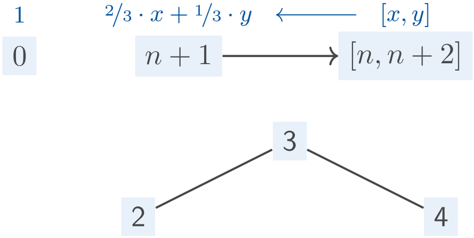

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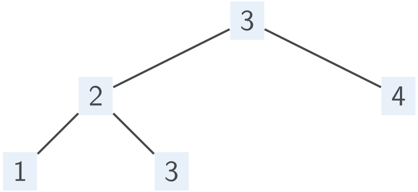

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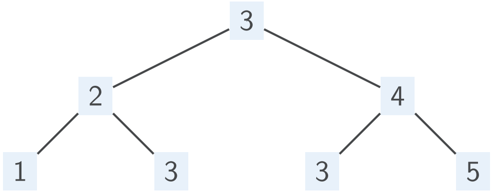

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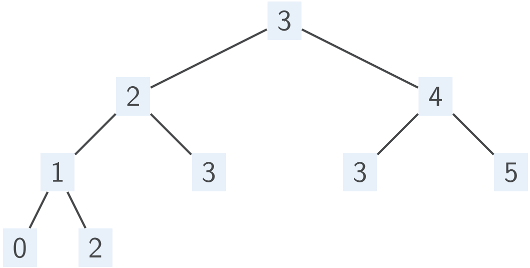

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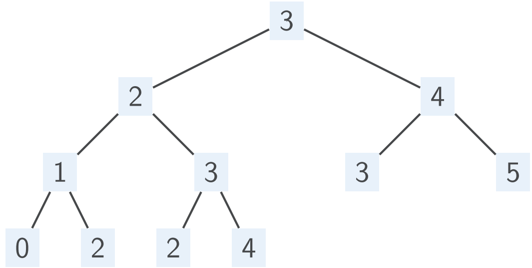

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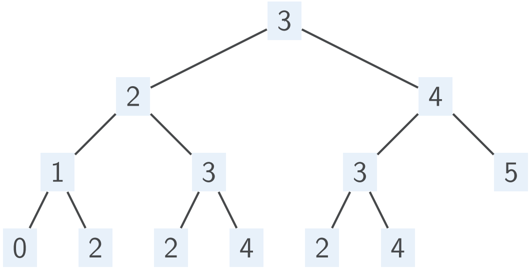

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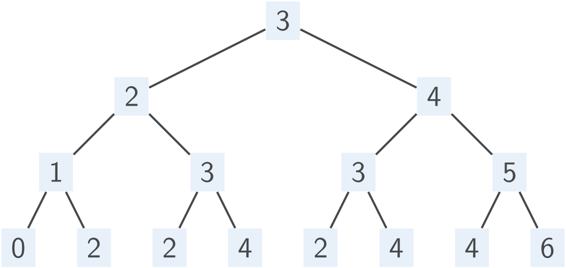

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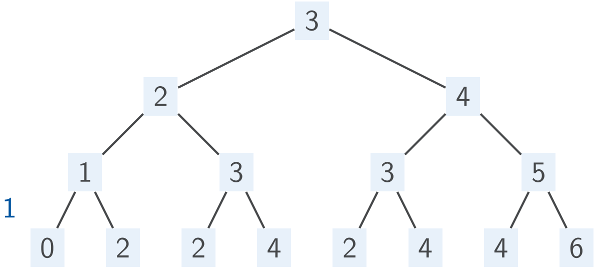

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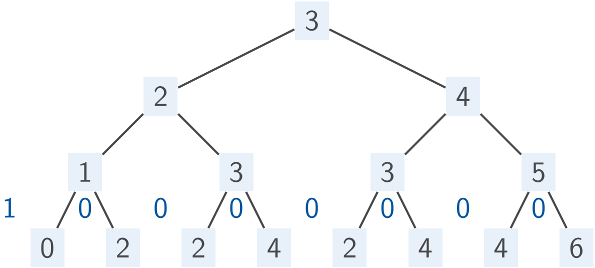

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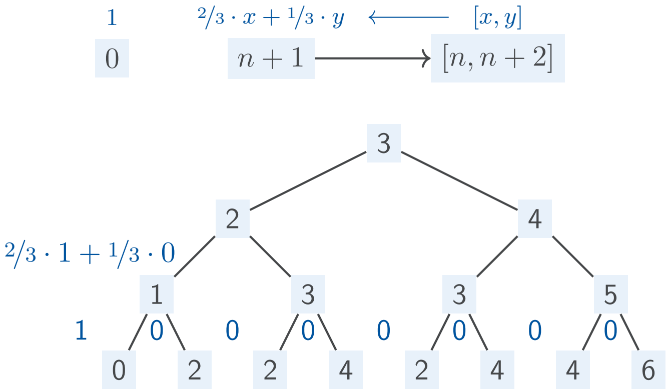

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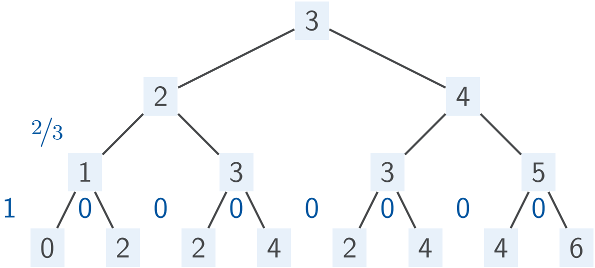

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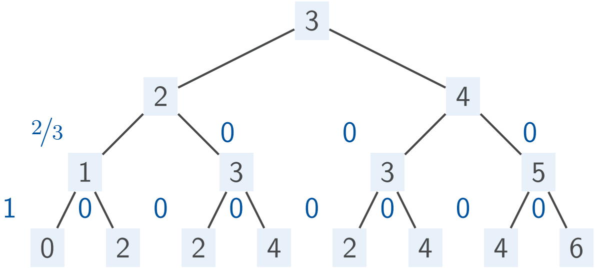

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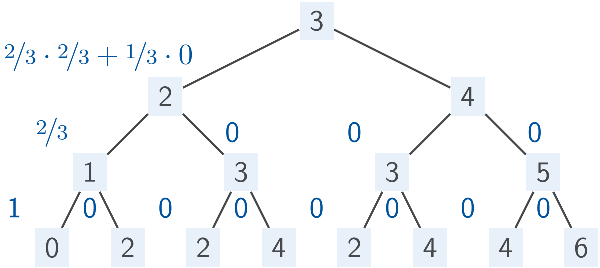

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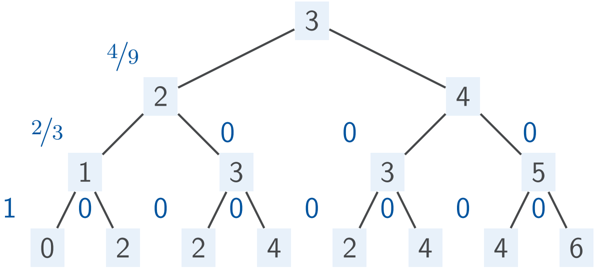

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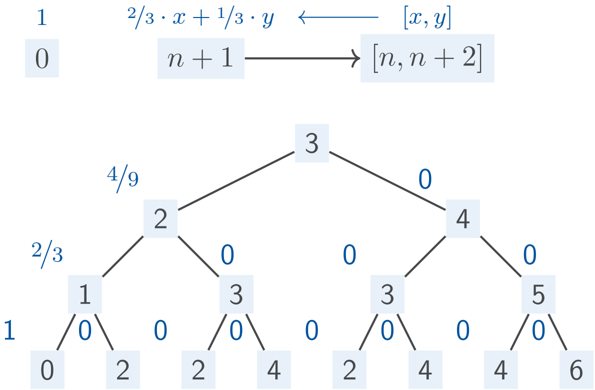

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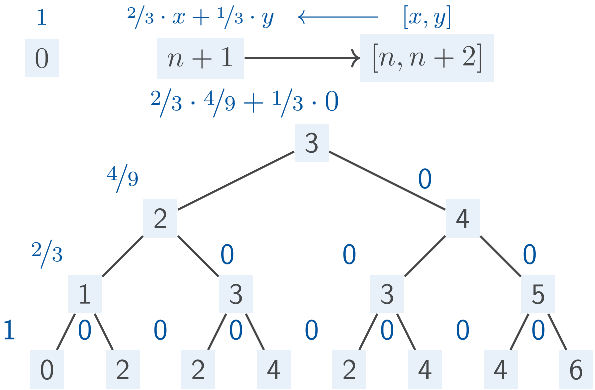

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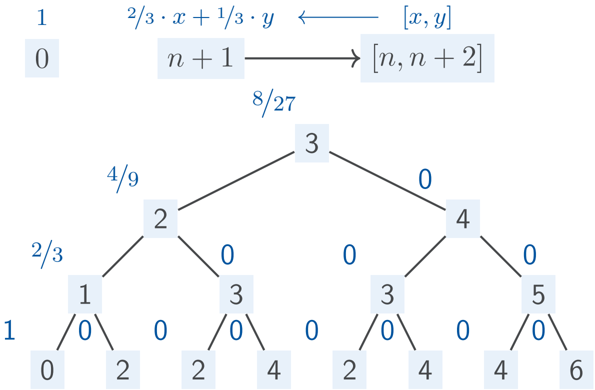

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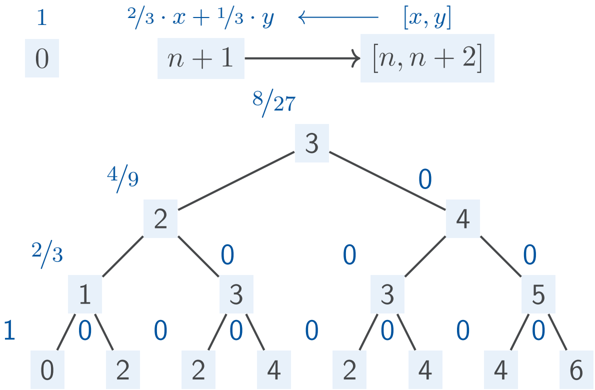

Probability to reach 0 ?

Complexity of the Random Walk

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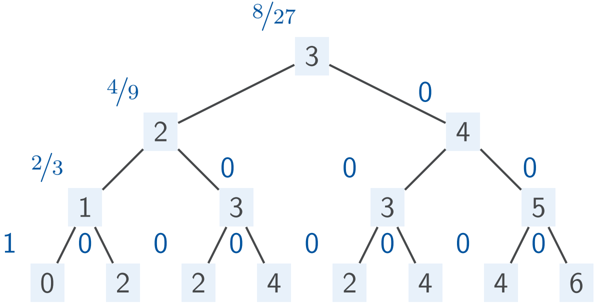
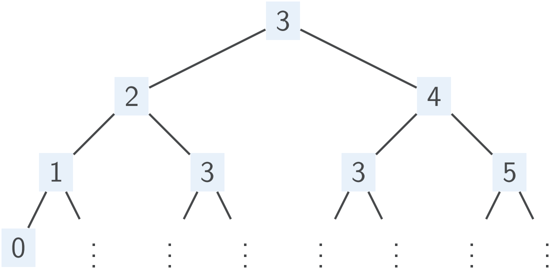


Probability to reach 0 $= \sup_{n \in \mathbb{N}}\{$ Probability to reach 0 in (at most) n steps $\}$ (Forward Unrolling)

Complexity of the Random Walk

What is the expected runtime?



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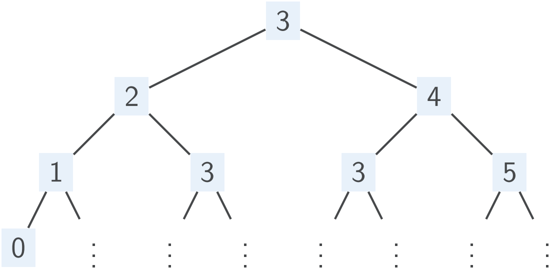

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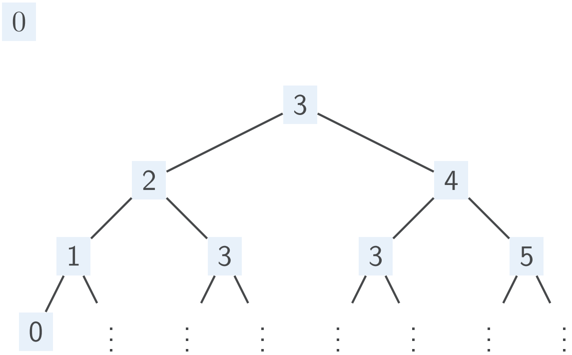

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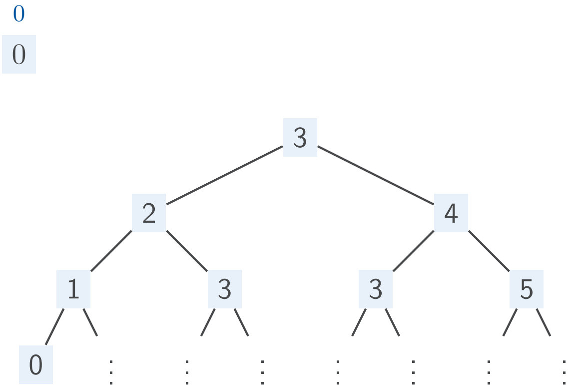
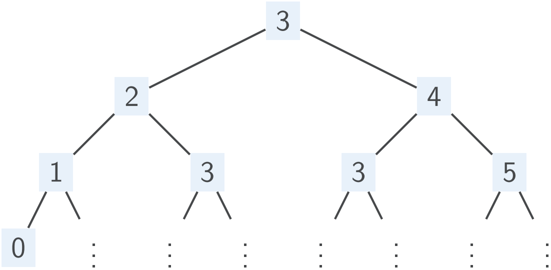


Complexity of the Random Walk

What is the expected runtime? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

0

0 $n+1$



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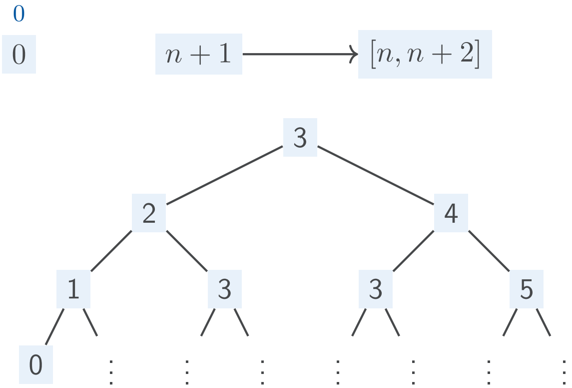
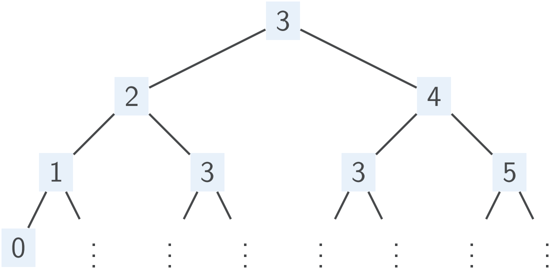


Complexity of the Random Walk

What is the expected runtime? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

$$\overset{0}{0} \qquad\qquad n+1 \xrightarrow{[x,y]} [n, n+2]$$



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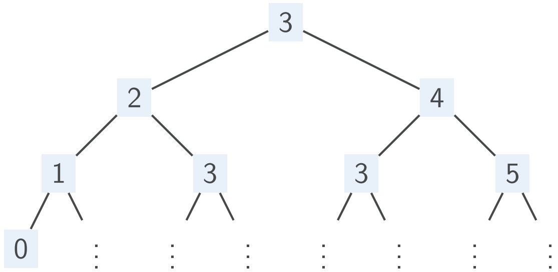

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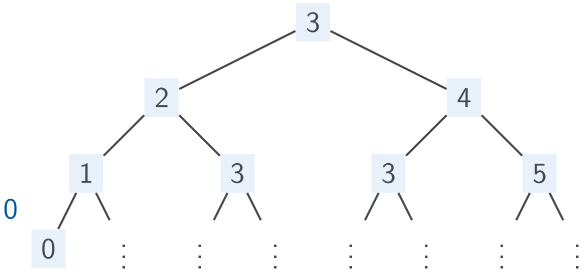

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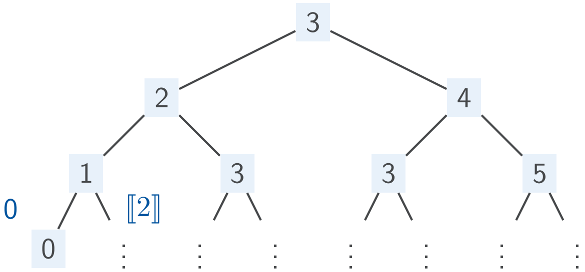

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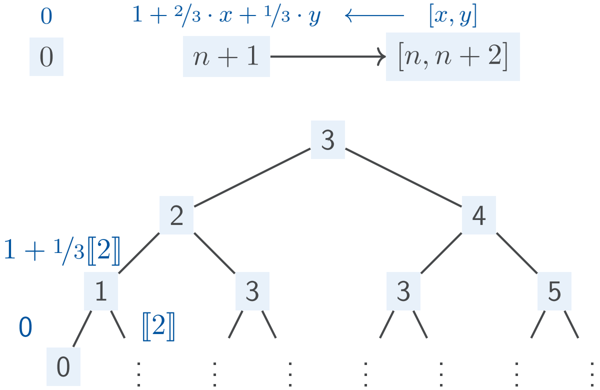

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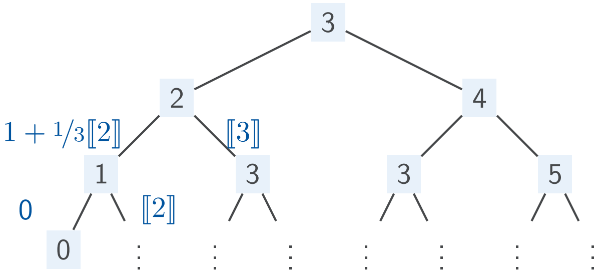

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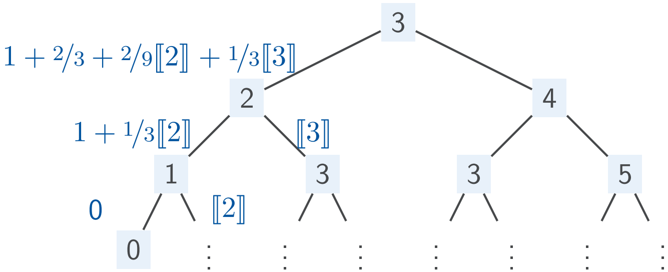

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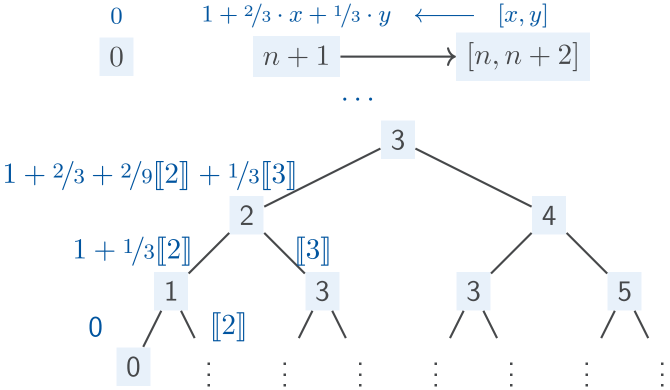

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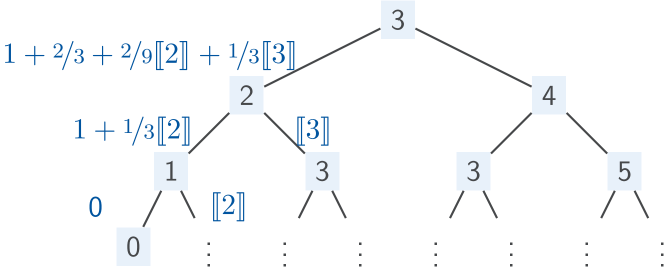
$$\dots$$

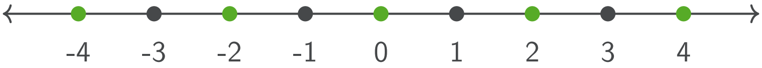


Is $[\![n]\!] \prec \infty$ for all $n \in \mathbb{N}$?

Termination and Runtime Complexity

Abstract reduction system $(\mathbb{Z}, \to)$ with

$$
\begin{aligned}
z \text{ is even and } z \leq -2: &\quad z \to z + 2 \\
z \text{ is even and } z \geq 2: &\quad z \to z - 2 \\
z \text{ is odd}: &\quad z \to -z
\end{aligned}
$$



Abstract reduction system $(\mathbb{Z}, \to)$ with

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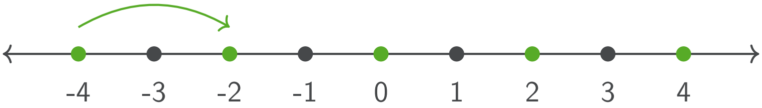

Termination and Runtime Complexity

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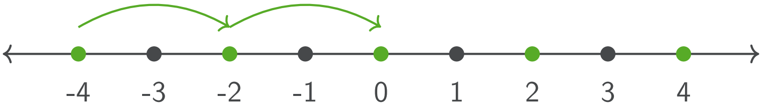

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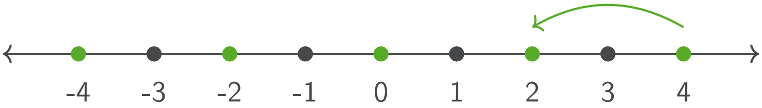

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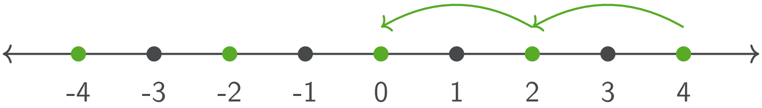

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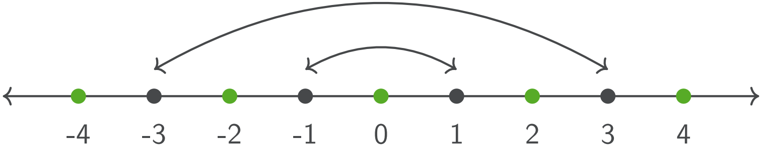

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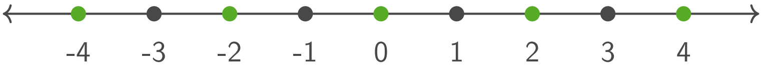

Do all reductions terminate?

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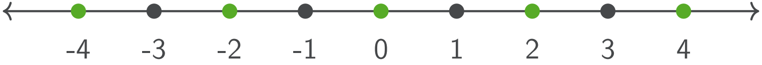

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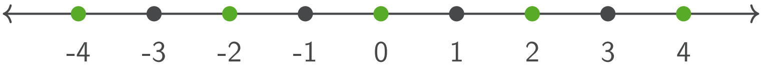

Do all reductions terminate? Choose $\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$ and count steps. No!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\text{false}, \text{true}\}, \vee, \wedge, \text{false}, \text{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more...

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

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embedding $\mathfrak{e} : A \to \mathbb{S}$

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$a \xrightarrow{+1} b \xrightarrow{+1} a$

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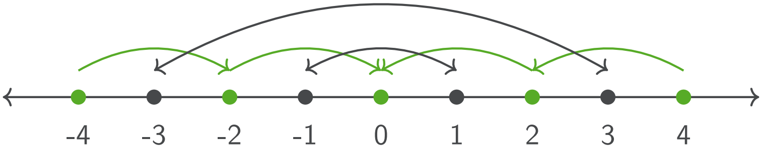

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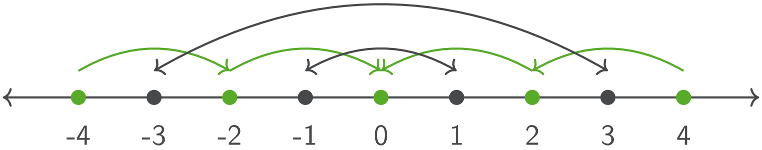

Hitting an even number is "unsafe". Are all reductions safe?

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0 $\quad$ z is even: $\quad z \longrightarrow z \pm 2$ $\quad$ z is odd:

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4

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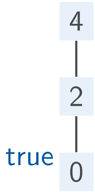

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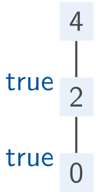

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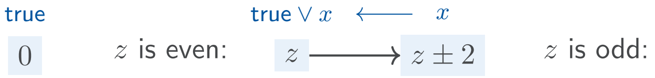

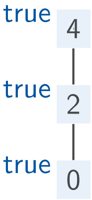

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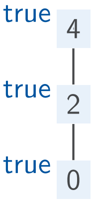
z is even: $z \xrightarrow{} z \pm 2$, with weight $\mathsf{true} \vee x \longleftarrow x$

z is odd: z



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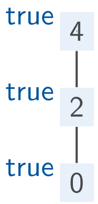

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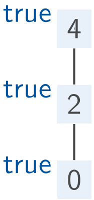

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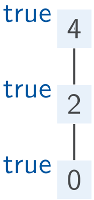

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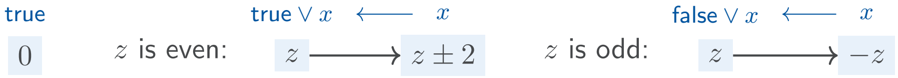

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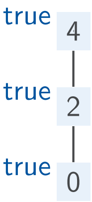

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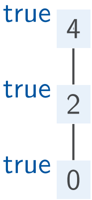

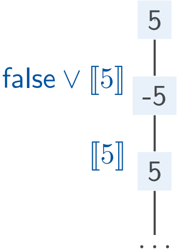

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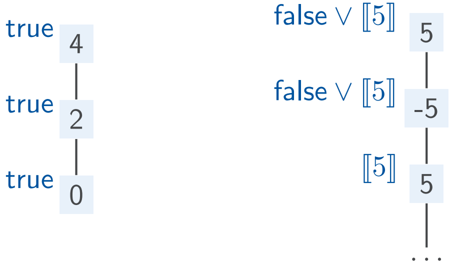

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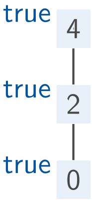

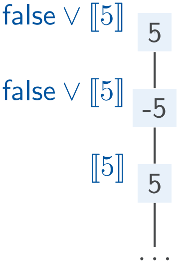

Is $[\![z]\!] < \mathsf{true}$ for all $z \in \mathbb{Z}$?

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true 4

true 2

true 0

false $\vee$ $[\![5]\!]$ 5

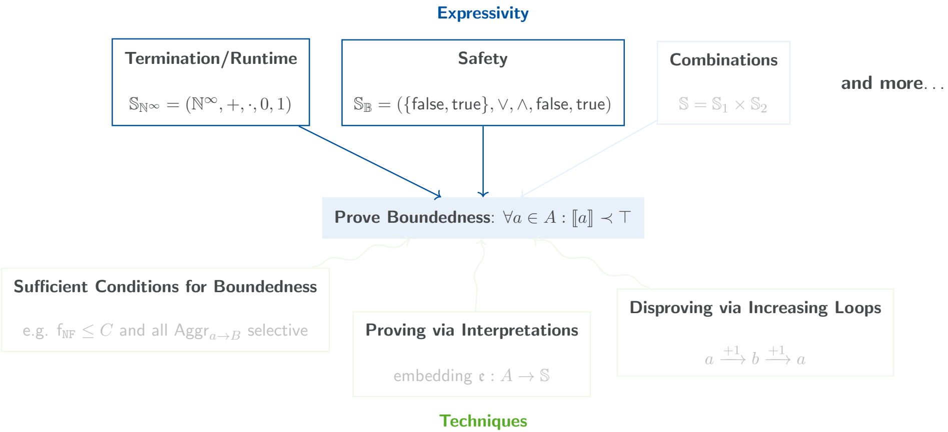
false $\vee$ $[\![5]\!]$ -5

$[\![5]\!]$ 5

...

Is $[\![z]\!] < \text{true}$ for all $z \in \mathbb{Z}$? No!

Overview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Techniques

Combining Complexity and Safety

Abstract reduction system $(\mathbb{Z}, \to)$ with

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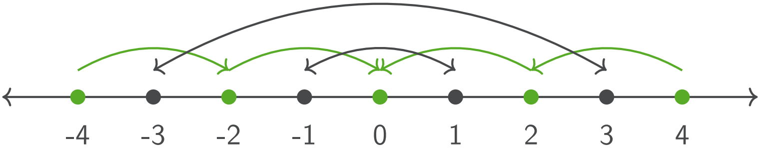

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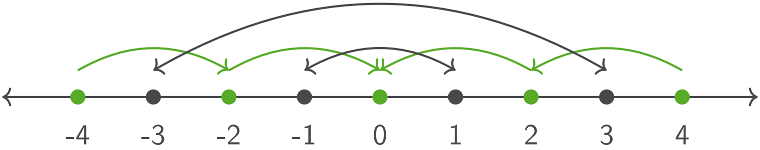

Are all runs terminating? Are all runs safe?

Combining Complexity and Safety

Abstract reduction system $(\mathbb{Z}, \to)$ with

$$\begin{aligned} z \text{ is even and } z \leq -2: &\quad z \to z+2 \\ z \text{ is even and } z \geq 2: &\quad z \to z-2 \\ z \text{ is odd}: &\quad z \to -z \end{aligned}$$

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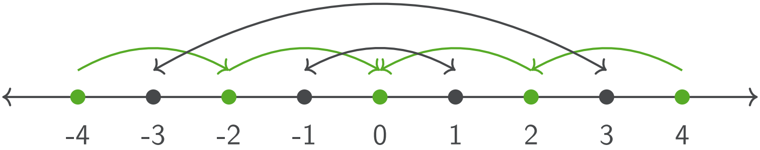


Are all runs terminating? Are all runs safe?

Now: Are all runs terminating or safe?

Combining Complexity and Safety

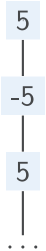
Are all runs terminating or safe?

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

z is even:

z is odd:



Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

0

z is even:

z is odd:

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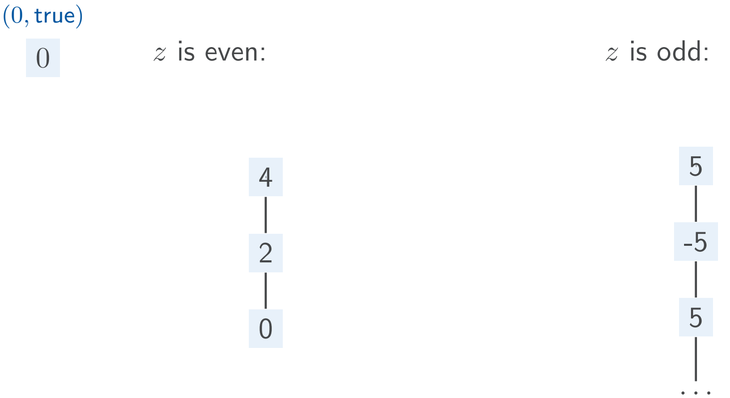

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even: z

z is odd:

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even: $z \longrightarrow z \pm 2$ z is odd:

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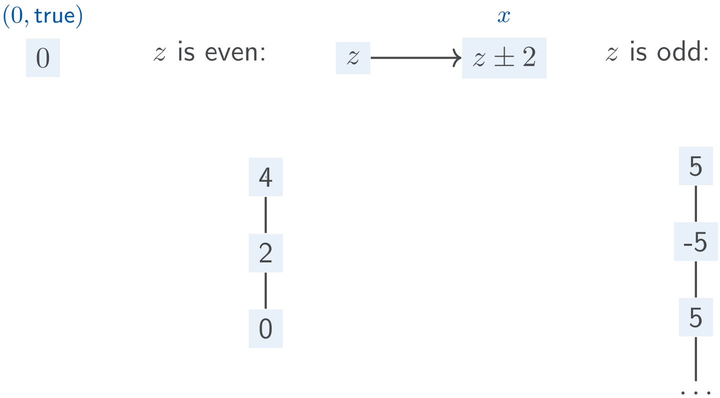

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$ for 0

$(1, \mathsf{true}) \oplus x \longleftarrow x$ for $z \longrightarrow z \pm 2$

z is even:

z is odd:

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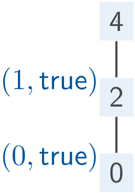

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0 $\quad$ z is even: $\quad z \longrightarrow z \pm 2$ $\quad$ z is odd:

$(2, \mathsf{true})$ 4 — $(1, \mathsf{true})$ 2 — $(0, \mathsf{true})$ 0

5 — -5 — 5 — . . .

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$$\overset{(0,\mathsf{true})}{0} \qquad z \text{ is even: } z \xrightarrow{(1,\mathsf{true}) \oplus x \;\longleftarrow\; x} z \pm 2 \qquad z \text{ is odd: } z \xrightarrow{x} -z$$

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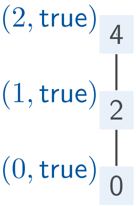

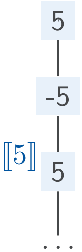

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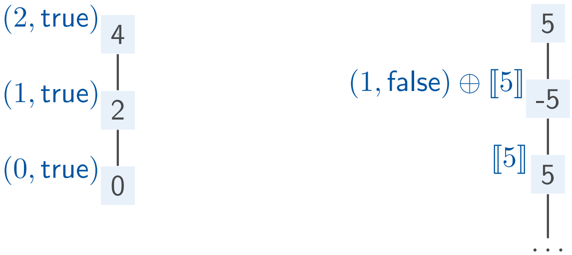

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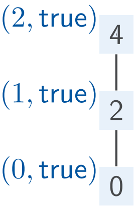

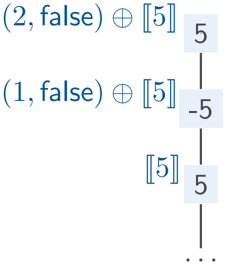

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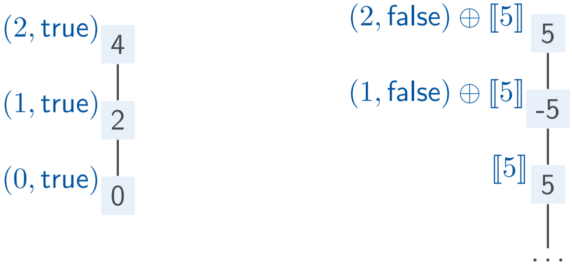

Is $[\![z]\!] < (\infty, \mathsf{true})$ for all $z \in \mathbb{Z}$?

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

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z is even: $z \to z \pm 2$ with $(1, \text{true}) \oplus x \longleftarrow x$

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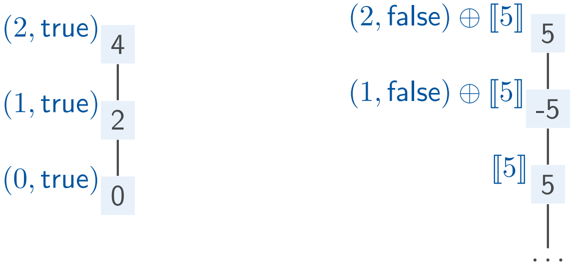

Is $[\![z]\!] < (\infty, \text{true})$ for all $z \in \mathbb{Z}$? Yes!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\mathsf{false}, \mathsf{true}\}, \vee, \wedge, \mathsf{false}, \mathsf{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more. . .

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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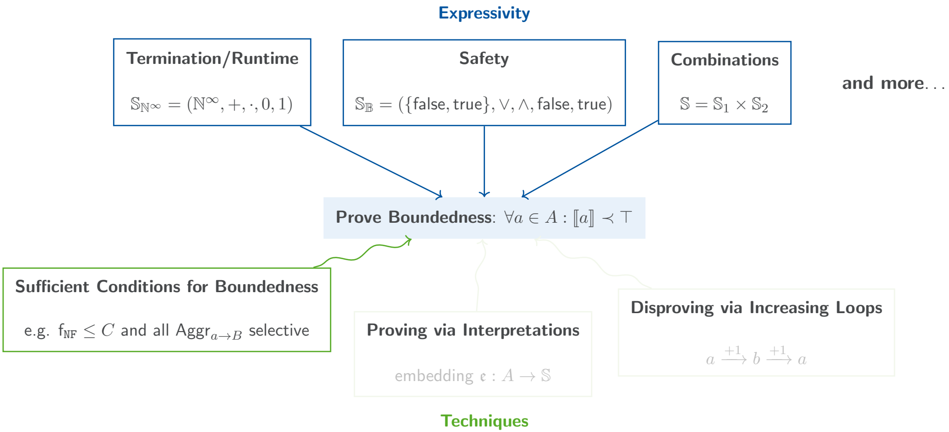
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$$0 \qquad\qquad 1 + {}^{2}/_{3} \cdot x + {}^{1}/_{3} \cdot y \longleftarrow [x, y]$$

$$0 \qquad\qquad n + 1 \longrightarrow [n, n+2]$$

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Choose $\mathfrak{e} : n \mapsto 3 \cdot n$ such that

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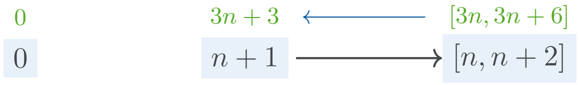

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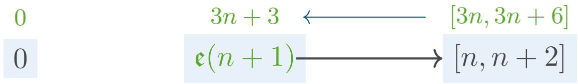

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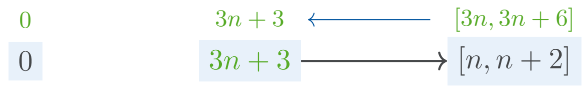
- $\mathfrak{e}(a) \succeq [\![a]\!] = \mathsf{f}_{\mathtt{NF}}(a)$ for all normal forms $a \in A$.
- $\mathfrak{e}(a) \succeq \mathsf{Aggr}_{a \to [b_1, \ldots, b_n]}(\mathfrak{e}(b_1), \ldots, \mathfrak{e}(b_n))$ for all $a \to [b_1, \ldots, b_n]$.

Proving via Interpretations

$(A, \to, \mathbb{S}, \mathsf{f}_{\mathtt{NF}}, \mathsf{Aggr})$ is bounded $:\Leftrightarrow$ for all $a \in A$: $[\![a]\!] \prec \top$

- Interpretation method via embedding $\mathfrak{e} : A \to \mathbb{S} \setminus \{\top\}$ (sound, for ω-continuous semirings complete)

Random Walk: Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and



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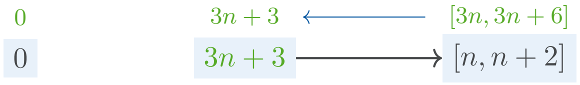

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Is the expected runtime finite? Yes, $[\![n]\!] \leq 3 \cdot n \prec \infty$!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\mathsf{false}, \mathsf{true}\}, \vee, \wedge, \mathsf{false}, \mathsf{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more. . .

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathtt{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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Techniques

Disproving via Increasing Loops

$(A, \to, \mathbb{S}, \mathrm{f}_{\mathtt{NF}}, \mathrm{Aggr})$ is unbounded $:\Leftrightarrow$ there exists an $a \in A$: $[\![a]\!] = \top$

- Show unboundedness via induced weight polynomial $\mathcal{P}_a(X)$ of loops $a \to^* a$

Disproving via Increasing Loops

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$$\begin{array}{ccc} 0 & \qquad 1 + x \longleftarrow x & \qquad 1 + x \longleftarrow x \\ 0 & \qquad z \longrightarrow -z & \qquad z \longrightarrow z \pm 2 \end{array}$$

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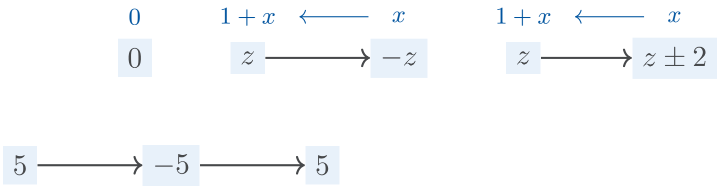

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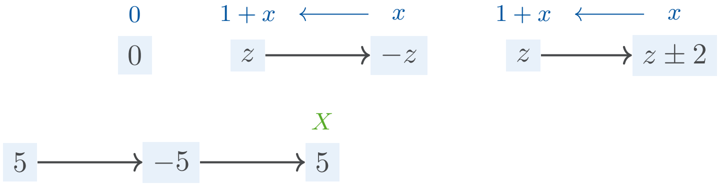

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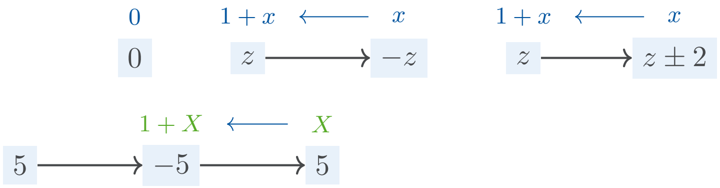

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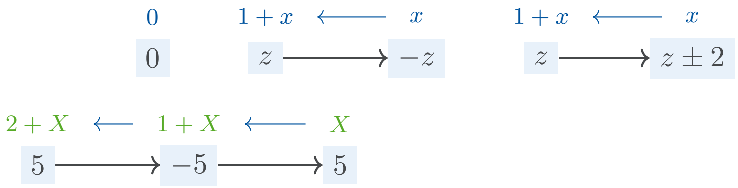

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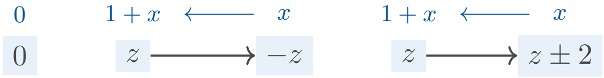

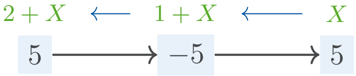

$$\mathcal{P}_5(X) = 2 + X,$$

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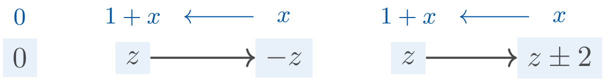

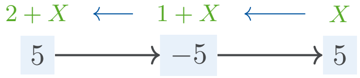

$\mathcal{P}_5(X) = 2 + X, \qquad \bigoplus_{i \in \mathbb{N}} 2 = \infty$ ✓

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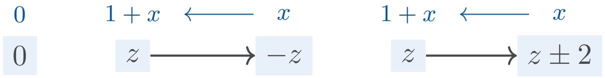

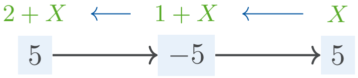

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Are all reductions terminating? No!

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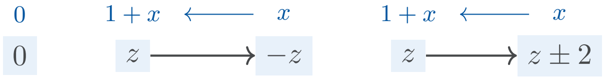

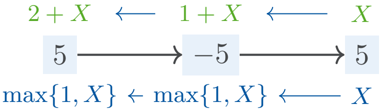

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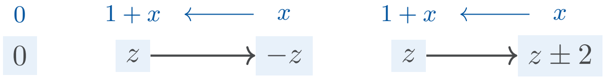

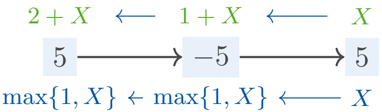

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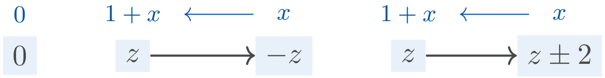

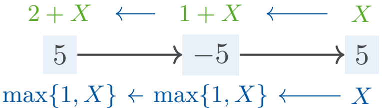

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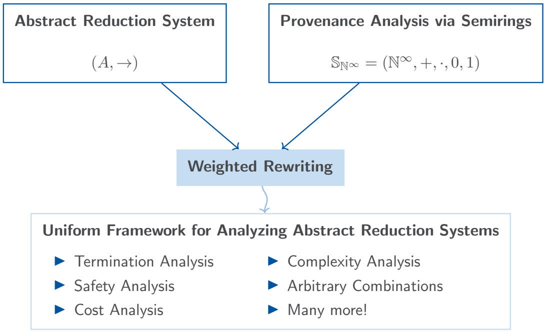
Techniques

One Framework Fits All

- Framework to unify research questions
- Lifts techniques from one area to another
- Future Work:
  - Lift more techniques to the general setting
  - Increase expressivity further
    - Starvation Freedom
  - Automate the techniques



Emma Ahrens, Jan-Christoph Kassing, Jürgen Giesl, Joost-Pieter Katoen:
Weighted Rewriting: Semiring Semantics for Abstract Reduction Systems.