SHORT COURSE ON HARMONIC SPACES

BASED ON W. BALLMANN'S LECTURE NOTES ON THE BLASCHKE CONJECTURE

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Description. For $m \geq 2$, Laplace's equation $\Delta f = 0$ has a global solution in $\mathbb{R}^m \setminus \{0\}$, given by:

$$f(x) = \|x\|^{2-m}, \text{ if } m > 2, \quad \text{and} \quad f(x) = \log\big(\|x\|\big), \text{ if } m = 2.$$

These functions are *radial* around 0, i.e., $f(x)$ depends only on the distance of x from 0.

Given a Riemannian manifold (M, g), let Δ_g denote its Laplacian operator, and let $p \in M$ be fixed. A function $f \colon M \to \mathbb{R}$ is *radial about* p if the value $f(q)$ depends only on the distance between p and q, i.e., if $f(q_1) = f(q_2)$ when $\mathrm{dist}(p, q_1) = \mathrm{dist}(p, q_2)$. It is then a natural question to ask whether an arbitrary Riemannian manifold admits radial solutions of the Laplace, equation, or if it admits radial eigenfunctions of Δ_g, etc.

Evidently, in order to answer these questions one of the first things to establish is whether the Laplacian $\Delta_g f$ of some radial function f is again a radial function. Elementary examples show that this is not always the case. Using a local expression of the Laplacian in polar coordinates, the reader will easily convince himself that a necessary and sufficient condition for this is that the Laplacian of the distance function itself must be radial.

This motivates the following definition. A Riemannian manifold (M, g) is harmonic at $p \in M$ if in some punctured geodesic ball $B_p(\delta) \setminus \{p\}$ around p one has $\Delta_g r = \phi(r)$, where $r(q) = \mathrm{dist}(p, q)$, and ϕ is a smooth function on $]0, \delta[$. The manifold (M, g) will be called *harmonic* if it is harmonic about every $p \in M$. This notion is due to Copson and Ruse, 1940. As expected, given any $\lambda > 0$ is the spectrum of the Laplace operator Δ_g of a harmonic space (M, g), and given any $p \in M$, there exists a nontrivial λ-eigenfunction of Δ_g which is radial about p.

Contents. Harmonic manifolds have an extremely rich geometry. During the course, we will discuss the proof of the following results.

Theorem (Copson-Ruse, 1940)**.** *Harmonic spaces are Einstein. In particular, a harmonic space of dimension* 2 *or* 3 *has constant sectional curvature.*

Recall that, given $k \in \{1, \dots, m-1\}$ and $L > 0$, a Riemannian manifold (M^m, g) is said to be *Allamigeon–Warner of type* (k, L) if for every point $p \in M$ and any (maximal) unit speed geodesic $\gamma : [0, b[\to M$ issuing from p, $\gamma(L)$ is the first point conjugate to p along γ, and its multiplicity is equal to k.

Theorem. *Compact harmonic spaces either are without conjugate points, or are Allamigeon–Warner.*

An old conjecture that harmonic spaces of any dimension should have constant sectional curvature turned out to be false.

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Theorem (Lichnerowicz, 1944)**.** *Non-flat harmonic spaces are irreducible.*[1]

This lead to the following:

Lichnerowicz conjecture. *Complete simply-connected harmonic spaces are either flat or symmetric spaces of rank* 1.

Lichnerowicz conjecture is true in dimension 4 (Lichnerowicz and Walker, 1945), but false in general. It is also true in the compact case.

Theorem (Szabó, 1990)**.** *If* (M, g) *is a compact simply-connected harmonic space, then* (M, g) *is a compact rank one symmetric space.*

We will not present a proof this theorem, but we will discuss the proof of Besse's immersion theorem, which is one of the main tools of of Szabó's proof.

References In this course I will present the material in Section 6 of W. Ballmann's note *Lectures on the Blaschke conjecture*, 2014. Lectures note of a course on differential geometry, taught jointly with K. Grove. This reference contains some results on the geometry of harmonic spaces, as well as the discussion of a proof of Besse's immersion theorem, which is one of the main tools in Szabó's proof of the Lichnerowicz conjecture in the compact case. For the proof presented during the lectures, I will employ results from the classical book of A. BESSE, *Einstein manifolds*, and from several papers, including D. M. DETURCK, J. KAZDAN, *Some regularity theorems in Riemannian geometry*, Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 3, 249–260, A. LICHNEROWICZ, *Sur les espaces riemanniens complètement harmoniques*, Bull. Soc. Math. France **72** (1944). 146–168, A. LICHNEROWICZ, A. G. WALKER, *Sur les espaces riemanniens harmoniques de type hyperbolique normal*, C.. R. Acad. Sci. Paris **221** (1945), 394–396, T. TAKAHASHI, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385, and Z. SZABÓ, *The Lichnerowicz conjecture on harmonic manifolds*, J. Differential Geom. **31** (1990), no. 1, 1–28.

[1] A Riemannian manifold is *irreducible* if admits no finite cover which is the Riemannian product of two Riemannian manifolds of lower dimension.