Münster J. of Math. **8** (2015), 181–210
DOI 10.17879/65219674985
urn:nbn:de:hbz:6-65219675309

Münster Journal of Mathematics

Generalized spin representations

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(Communicated by Linus Kramer)

Abstract. We introduce the notion of a generalized spin representation of the maximal compact subalgebra $\mathfrak{k}$ of a symmetrizable Kac–Moody algebra $\mathfrak{g}$ in order to show that, if defined over a formally real field, every such $\mathfrak{k}$ has a nontrivial reductive finite-dimensional quotient. The appendix illustrates how to compute the isomorphism types of these quotients for the real E_n series. In passing this provides an elementary way of determining the isomorphism types of the maximal compact subalgebras of the semisimple split real Lie algebras of types E_6, E_7, E_8.

INTRODUCTION

During the last decade the family of Kac–Moody algebras of type $E_n(\mathbb{R})$ has received considerable attention because of its importance in M-theory; see [6, 11, 19, 26, 28]. By [4, 7] the (so-called) maximal compact subalgebra $\mathfrak{k} = \operatorname{Fix}\omega$ of the real split Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(E_{10})(\mathbb{R})$ with respect to the Cartan–Chevalley involution ω admits a 32-dimensional complex representation which extends the spin representation of its regular subalgebra $\mathfrak{so}_{10}(\mathbb{R})$. This implies that the (infinite-dimensional) Lie algebra $\mathfrak{k}$ has a nontrivial finite-dimensional quotient, in fact a semisimple finite-dimensional quotient (see Theorem 3.14). Since $\mathfrak{k}$ is anisotropic with respect to the invariant bilinear form of the Kac–Moody algebra $\mathfrak{g}$, it actually contains an ideal isomorphic to this finite-dimensional quotient.

In this article we show that the existence of nontrivial finite-dimensional representations is not peculiar to the maximal compact subalgebra of $\mathfrak{g}(E_{10})(\mathbb{R})$ but is shared by all maximal compact subalgebras of symmetrizable Kac–Moody algebras over arbitrary fields of characteristic 0. To this end we introduce the notion of a generalized spin representation (Definitions 3.6 and 3.13), which we inductively show to exist for arbitrary symmetrizable Kac–Moody algebras and which, in the case of formally real fields, affords a compact, whence reductive, and often even a semisimple image (Theorem 3.14).

Our results presented in this article are generalizations of the results concerning the $\frac{1}{2}$-spin representations described in [4, 7]. The key observation is Remark 3.7 that in the simply-laced case a $\frac{1}{2}$-spin representation can be described by linear operators A_i for each vertex i of the diagram that satisfy

(i) $A_i^2 = -\frac{1}{4} \cdot \mathrm{id}$,
(ii) $A_iA_j = A_jA_i$ if the vertices i, j do not form an edge of the diagram,
(iii) $A_iA_j = -A_jA_i$ if the vertices i, j form an edge of the diagram.

On the other hand, the $\frac{3}{2}$-spin representations of [4, 7] and the $\frac{5}{2}$- and $\frac{7}{2}$-spin representations of [18] are still elusive, as the algebraic identities that need to be satisfied by the corresponding linear operators are more involved.

Note that our terminology of *maximal compact subalgebra* is misleading. For one, in the infinite-dimensional situation there is no compact group associated to a maximal compact subalgebra. Rather, over the real numbers, the maximal compact subalgebra is related to the group K studied in [8, 15]. This group naturally carries a non-locally compact non-metrizable k_ω-topology (cp. [12]). Moreover, our construction only involves the Cartan–Chevalley involution and no field involution. Therefore, over the complex numbers, what we call a maximal compact subalgebra is not even anisotropic.

However, this terminology does not lead to serious ambiguities as our main focus lies on split Lie algebras over formally real fields. Our main structure-theoretic results in Section 3 below will consequently be obtained over formally real fields; the main future application of our result is over the real numbers.

1. PRELIMINARIES

In this section we collect several basic facts about Kac–Moody algebras. We refer the reader to [14, Chap. 1] and [22, Chap. 1] for proofs and further details.

1.1. **Kac–Moody algebras.** Let k be a field of characteristic 0, let $A = (a_{ij}) \in \mathbb{Z}^{n\times n}$ be a *generalized Cartan matrix* and let $\mathfrak{g} = \mathfrak{g}_A$ denote the corresponding *Kac–Moody algebra* over k. This means that

$$a_{ii} = 2, \quad a_{ij} \leq 0 \quad \text{and} \quad a_{ij} = 0 \Leftrightarrow a_{ji} = 0,$$

while $\mathfrak{g}$ is the quotient of the free Lie algebra over k generated by e_i, f_i, h_i, $i = 1, \dots, n$, subject to the relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j$$

for all $1 \leq i, j \leq n$, and

$$[e_i, f_j] = 0, \quad [e_i, f_i] = h_i, \quad (\operatorname{ad} e_i)^{-a_{ij}+1}(e_j) = 0, \quad (\operatorname{ad} f_i)^{-a_{ij}+1}(f_j) = 0$$

for all $i \neq j$.

A generalized Cartan matrix is called *simply laced* if the off-diagonal entries of A are either 0 or -1; it is called *symmetrizable* if there exists a diagonal matrix Λ such that ΛA is symmetric. By abuse of terminology, we will say that $\mathfrak{g}$ is simply laced, resp. symmetrizable, if its generalized Cartan matrix is simply laced, resp. symmetrizable.

Let $\mathfrak{h} := \langle h_1, \ldots, h_n \rangle$, $\mathfrak{n}_+ := \langle e_1, \ldots, e_n \rangle$ and $\mathfrak{n}_- := \langle f_1, \ldots, f_n \rangle$ denote the standard subalgebras of $\mathfrak{g}$. Then there is a decomposition as vector spaces

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

see [14, §1.3, p. 7]. The defining relations of $\mathfrak{g}$ imply that $\mathfrak{h}$ is n-dimensional abelian and normalizes $\mathfrak{n}_+$ and $\mathfrak{n}_-$. In fact, it acts by linear transformations on these vector spaces. Therefore, for each element $\alpha \in \mathfrak{h}^*$ of the dual space it is meaningful to define the eigenspaces

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

The relations $[h_i, e_j] = a_{ij} e_j$, $1 \leq i, j \leq n$, imply that each e_j is contained in such an eigenspace, which we denote by $\mathfrak{g}_{\alpha_j}$; the corresponding element of $\mathfrak{h}^*$ is denoted by α_j (cp. [14, §1.1]). Note that $\mathfrak{g}_{-\alpha_j}$ contains f_j.

The *diagram* of a simply-laced Kac–Moody algebra $\mathfrak{g}_A$ is the graph $D = (V, E)$ on vertices $\alpha_1, \ldots, \alpha_n$ with α_i and α_j connected by an edge if and only if $a_{ij} = -1$.

Let $Q := \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ denote a free $\mathbb{Z}$-module of rank n and $Q_+ := \bigoplus_{i=1}^n \mathbb{Z}_+\alpha_i$ the set of nonnegative integral linear combinations. By [14, Thm. 1.2 (d), Ex. 1.2] we have

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} g_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in Q \setminus \{0\}} g_\alpha = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} g_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in Q_+ \setminus \{0\}} g_\alpha.$$

Therefore, $\mathfrak{g}$ has a Q-grading by declaring

$$\deg h_i := 0, \quad \deg e_i := \alpha_i, \quad \deg f_i := -\alpha_i$$

for $i = 1, \ldots, n$, i.e.,

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \quad \text{and} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

Let $\Delta := \{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. Then $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ := \Delta \cap (Q_+ \setminus \{0\})$ and $\Delta_- := -\Delta_+$. An element $\alpha \in \Delta$ is called a *root* and $\mathfrak{g}_\alpha$ a *root space.* A root $\alpha \in \Delta$ is called *positive* if it belongs to Δ_+, otherwise *negative.* A root of the form $\alpha = \pm\alpha_i$ is called *simple.*

Since the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is integrable (see [14, §3.5]), the *extended Weyl group* $W^* \leq \mathrm{Aut}\,\mathfrak{g}$ can be defined as

$$W^* := \langle s_i^* \mid i = 1, \ldots, n \rangle,$$

where

$$s_i^* := s_i^{\mathrm{ad}} := \exp \mathrm{ad}\, f_i \cdot \exp \mathrm{ad}(-e_i) \cdot \exp \mathrm{ad}\, f_i,$$

cp. [14, §3.8]; note that $W^* \leq \mathrm{Aut}\,\mathfrak{g}$ by [14, Lem. 3.8 (b)]. For $\alpha \in \Delta$ and $w \in W^*$ there exists a unique $w \cdot \alpha \in \Delta$ such that $w(\mathfrak{g}_\alpha) = \mathfrak{g}_{w \cdot \alpha}$, by [14, Lem. 3.8 (a)]. A root α is called *real* if there is a $w \in W$ such that $w \cdot \alpha$ is simple, otherwise it is called *imaginary.* Let Δ^{re} denote the set of real roots.

For $\alpha = \sum_{i=1}^{n} a_i\alpha_i \in \Delta$, the *height* of α is defined as $\operatorname{ht}\alpha := \sum_{i=1}^{n} a_i$. For $n \in \mathbb{N}$ let

$$(\mathfrak{n}_+)_n := \bigoplus_{\alpha\in\Delta^+,\, \operatorname{ht}\alpha=n} \mathfrak{g}_\alpha \,.$$

This is a $\mathbb{Z}$-grading of $\mathfrak{n}_+$ and extends to a $\mathbb{Z}$-grading of $\mathfrak{g}$, the *principal grading* (cp. [14, §1.5]).

1.2. **The maximal compact subalgebra.** Let $\mathfrak{g}$ be a Kac–Moody algebra over a field k of characteristic 0. Let $\omega \in \operatorname{Aut}(\mathfrak{g})$ denote the *Cartan–Chevalley involution* characterized by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h_i) = -h_i$ (cp. [14, (1.3.4)]). Observe that $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.

Let $\mathfrak{k} := \mathfrak{k}(\mathfrak{g}) := \{X \in \mathfrak{g} \mid \omega(X) = X\}$ denote the fixed point subalgebra, which—in analogy to the situation of finite-dimensional semisimple split real Lie algebras—is called the *maximal compact subalgebra* of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, then $\omega(A) = -A^T$ and $\mathfrak{k} = \mathfrak{so}_n(\mathbb{R})$. In this case, $\mathfrak{so}_n(\mathbb{R})$ is the Lie algebra of the maximal compact subgroup $\mathrm{SO}_n(\mathbb{R})$ of $\mathrm{SL}_n(\mathbb{R})$. See also [20, §IV.4].

Over non-real closed fields, especially over the complex numbers, our terminology is a bit unfortunate and misleading. However, our main results in Section 3 below and future applications are over real closed fields.

A theorem of Berman [2] allows one to give a presentation of these. We point out that Berman's result in fact deals with a much more general class of so-called involutory algebras by also allowing other involutions of $\mathfrak{g}$ of the second kind (in the sense of [16, §4.6]). Note that Berman instead of our involution ω uses the involution η given by $\eta(e_i) = f_i$, $\eta(f_i) = e_i$, $\eta(h_i) = -h_i$ as the foundation of his investigations so that in order to apply his result one still has to relate the two involutions to one another.

Theorem 1.3 (cp. [2, Thm. 1.31]). *Let k be a field of characteristic 0. Let $A \in \mathbb{Z}^{n\times n}$ be a simply-laced generalized Cartan matrix, let $\mathfrak{g}_A$ denote the corresponding Kac–Moody algebra, and let $\mathfrak{k}$ denote the maximal compact subalgebra of $\mathfrak{g}$. Then $\mathfrak{k}$ is isomorphic to the quotient of the free Lie algebra over k generated by $X_1, \dots, X_n$ subject to the relations*

$$\begin{aligned} [X_i,[X_i,X_j]] &= -X_j && \textit{if the vertices } v_i, v_j \textit{ are connected by an edge,}\\ [X_i,X_j] &= 0 && \textit{otherwise} \end{aligned}$$

via the map $X_i \mapsto e_i - f_i$.

In Theorem 1.8 below we state and prove a general version of this result that applies to the maximal compact subalgebra of an arbitrary symmetrizable Kac–Moody algebra over a field of characteristic 0. Our motivation for splitting off the simply-laced case is that it is considerably easier to understand than the general case. Furthermore, the study of generalized spin representations in the simply-laced case is key to these representations in general.

Proof of Theorem 1.3. Let $\eta \in \operatorname{Aut}\mathfrak{g}$ denote the involution characterized by

$$\eta(e_i) = f_i, \quad \eta(f_i) = e_i \quad \text{and} \quad \eta(h_i) = -h_i$$

and let $\mathfrak{l} := \operatorname{Fix}\eta$ denote the subalgebra of fixed points of η. By [2, Thm. 1.31], the Lie algebra $\mathfrak{l}$ is isomorphic to the quotient of the free Lie algebra over k generated by $Y_1, \ldots, Y_n$ subject to the relations

$$\begin{aligned} [Y_i,[Y_i,Y_j]] &= Y_j && \text{if the vertices } v_i, v_j \text{ are connected by an edge,} \\ [Y_i,Y_j] &= 0 && \text{otherwise} \end{aligned}$$

via the map $Y_i \mapsto e_i + f_i$.

Let $I := \sqrt{-1}$ denote a square root of -1 and let $L := k(I)$, $\mathfrak{g}_L := \mathfrak{g} \otimes_k L$. There is a Lie algebra automorphism $\varphi \in \operatorname{Aut}(\mathfrak{g}_L)$ determined by

$$e_i \mapsto I \cdot e_i, \quad f_i \mapsto -I \cdot f_i \quad \text{and} \quad h_i \mapsto h_i.$$

This automorphism φ conjugates η to ω, i.e. $\omega = \varphi^{-1} \circ \eta \circ \varphi$, and hence the subalgebras $\operatorname{Fix}\eta$ and $\operatorname{Fix}\omega$ are isomorphic over L. As X_i is mapped to $I \cdot Y_i$ under this isomorphism, the claim follows. □

Remark 1.4. Suppose $k = \mathbb{C}$. We can exponentiate the subalgebra of $\mathfrak{g}$ spanned by e_i, f_i, h_i to a subgroup G_i of $\operatorname{Aut}\mathfrak{g}$ which is isomorphic to $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{PSL}_2(\mathbb{C})$. Then X_i identifies with $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in $\mathfrak{sl}_2$ and therefore $\exp(\xi X_i)$ is equal to the image of $\left(\begin{smallmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{smallmatrix}\right)$ in G_i. In particular, $\exp(-\frac{\pi}{2}X_i)$ is sent to s_i^*. It follows that s_i^* and ω are commuting automorphisms of $\mathfrak{g}$.

For the case of an arbitrary ground field, ω induces a Cartan–Chevalley involution on the standard type A_1 subgroup G_i of $\operatorname{Aut}\mathfrak{g}$ whose Lie algebra is spanned by e_i, f_i, h_i. The fixed point subgroup of G_i for the Cartan–Chevalley involution is either $\mathrm{SO}_2(k)$ or $\mathrm{SO}_2(k)/\{\pm I_2\}$, depending on whether G_i is isomorphic to SL_2 or PSL_2. Since this subgroup clearly contains s_i^*, it follows that s_i^* commutes with ω.

1.5. **Rank-two Kac–Moody algebras.** Let $\mathfrak{g}$ be the Kac–Moody algebra with Cartan matrix $\left(\begin{smallmatrix} 2 & -r \\ -s & 2 \end{smallmatrix}\right)$, where $r, s \in \mathbb{N}$. We map $\mathfrak{g}$ into a simply-laced Kac–Moody algebra as follows: Let D be a complete bipartite graph on r and s vertices, labelled $\alpha_1^{(i)}$ and $\alpha_2^{(j)}$ with $1 \leq i \leq r$, $1 \leq j \leq s$. Let $\tilde{\mathfrak{g}}$ be a Kac–Moody Lie algebra with simply-laced diagram D and label the generators correspondingly: $e_1^{(i)}$, $f_1^{(i)}$, $h_1^{(i)}$ and $e_2^{(j)}$, $f_2^{(j)}$, $h_2^{(j)}$. We remark that there is an action of $\mathrm{Sym}(r)$ (resp. $\mathrm{Sym}(s)$) on $\tilde{\mathfrak{g}}$ by permuting the roots $\alpha_1^{(i)}$ (resp. $\alpha_2^{(j)}$). Let

$$\begin{aligned} E_1 &= \sum_{i=1}^{r} e_1^{(i)}, \quad F_1 = \sum_{i=1}^{r} f_1^{(i)}, \quad H_1 = [E_1, F_1], \\ E_2 &= \sum_{j=1}^{s} e_2^{(j)}, \quad F_2 = \sum_{j=1}^{s} f_2^{(j)}, \quad H_2 = [E_2, F_2]. \end{aligned}$$

Then it is straight-forward to check that

$$\begin{aligned}
&[E_1, F_2] = 0 = [E_2, F_1] = [H_1, H_2],\\
&(\operatorname{ad} E_1)^{r+1}(E_2) = 0 = (\operatorname{ad} E_2)^{s+1}(E_1),\\
&(\operatorname{ad} F_1)^{r+1}(F_2) = (\operatorname{ad} F_2)^{s+1}(F_1) = 0.
\end{aligned}$$

Thus there is a well-defined Lie algebra homomorphism $\tilde{\varphi}$ from $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$, sending each of e_1, e_2, f_1, f_2, h_1, h_2 to its corresponding upper-case letter. Since $\mathfrak{g}$ has no nonzero ideals intersecting trivially with $\mathfrak{h}$, it follows that $\tilde{\varphi}$ is injective. It is clear from the definitions that $\tilde{\varphi}$ induces an injective homomorphism from the extended Weyl group of $\mathfrak{g}$ to that of $\tilde{\mathfrak{g}}$ by sending s_1^* to $(s_1^{(1)})^* \dots (s_1^{(r)})^*$, and similarly for s_2^*.

Remark 1.6. This construction is related to the notion of *pinning*[1] for split semisimple Lie algebras. Given a split semisimple Lie algebra $\tilde{\mathfrak{g}}$ over a field k of characteristic zero, let $\tilde{\mathfrak{h}}$ be a splitting Cartan subalgebra. A pinning of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ consists of a basis Π of the roots of $\tilde{\mathfrak{g}}$ relative to $\tilde{\mathfrak{h}}$, together with a choice $\{x_\alpha : \alpha \in \Pi\}$ of nonzero elements in each simple positive root space. If $\tilde{\mathfrak{g}}$ has a presentation as in Section 1.1 then we can take $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and $x_{\alpha_i} = e_i$ for $1 \leq i \leq n$. If a pinning of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is fixed, then a *pinned automorphism* is an automorphism which stabilizes $\tilde{\mathfrak{h}}$ and the Borel subalgebra of $\tilde{\mathfrak{g}}$ corresponding to Π, and which permutes the elements x_α, $\alpha \in \Pi$. Clearly, the group of pinned automorphisms is isomorphic to the group $\operatorname{Aut}(\Pi)$ of automorphisms of the Dynkin diagram of $\tilde{\mathfrak{g}}$. As follows from [3, VIII.3 Cor. 1 and VIII.4], the group $\operatorname{Aut}(\tilde{\mathfrak{g}})$ is the semi-direct product of $\operatorname{Aut}(\Pi)$ and $\tilde{G}(k)$, where $\tilde{G}$ is the adjoint type semisimple group with Lie algebra $\tilde{\mathfrak{g}}$. The corresponding result is also true in the Kac–Moody case [27, §6, Thm. 2 (c)]. When $\tilde{\mathfrak{g}}$ has generalized Cartan matrix $\left(\begin{smallmatrix} 2 & -r \\ -s & 2 \end{smallmatrix}\right)$, one obtains that the automorphism group is $(\operatorname{Sym}(r) \times \operatorname{Sym}(s)) \ltimes \tilde{G}$ if $r \neq s$ and is $(\operatorname{Sym}(r) \wr \operatorname{Sym}(2)) \ltimes \tilde{G}$ if $r = s$, where $\tilde{G}$ is an adjoint Kac–Moody group corresponding to $\tilde{\mathfrak{g}}$. (We exclude here the affine cases $r = s = 2$ and $\{r, s\} = \{1, 4\}$, where the picture is slightly more complicated.)

If $\tilde{\mathfrak{g}}$ has finite type, then there are no nontrivial pinned automorphisms unless $\tilde{\mathfrak{g}}$ is simply laced. Furthermore, a simple Lie algebra of type B_n (resp. C_n, F_4, G_2) can be realized as the fixed point subalgebra for a pinned automorphism of a Lie algebra of type D_{n+1} (resp. A_{2n-1}, E_6, D_4). In our case we can only say that $\mathfrak{g}$ is a *subalgebra* of the fixed-point subalgebra of $\tilde{\mathfrak{g}}$.

Let $\tilde{\omega}$ (resp. ω) denote the Cartan–Chevalley involution on $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$). Clearly $\tilde{\varphi} \circ \omega = \tilde{\omega} \circ \tilde{\varphi}$, so $\tilde{\varphi}$ induces a homomorphism from $\mathfrak{k} = \mathfrak{k}(\mathfrak{g})$ to $\tilde{\mathfrak{k}} = \mathfrak{k}(\tilde{\mathfrak{g}})$.

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[1] French “épinglage”, see [1, Exposé XXIII]. Although this is translated as “framing” in [3], it is clear from the footnote to [1, Exposé XXIII, Def. 1.1] (where a maximal torus is the body, and opposite Borel subgroups are the wings, of a butterfly) that “pinning” is more appropriate. It seems to have become the standard terminology in English.

Following the proof of Theorem 1.3, let

$$\begin{aligned} Y_1 &= e_1 + f_1, \quad Y_1^{(i)} = e_1^{(i)} + f_1^{(i)} \quad \text{for } 1 \le i \le r, \\ Y_2 &= e_2 + f_2, \quad Y_2^{(j)} = e_2^{(j)} + f_2^{(j)} \quad \text{for } 1 \le j \le s. \end{aligned}$$

Then

$$\tilde{\varphi}(Y_1) = \sum_{i=1}^{r} \tilde{Y}_1^{(i)}$$

and similarly for Y_2.

Since $\alpha_1^{(i)}$ and $\alpha_2^{(j)}$ are connected by a simple edge, we have

$$\big((\operatorname{ad} Y_1^{(i)})^2 - 1\big)(Y_2^{(j)}) = 0. \tag{1}$$

Now the space spanned by $Y_1^{(i)}$ for $1 \le i \le r$ is conjugate to the subspace of $\tilde{\mathfrak{h}}$ spanned by $h_1^{(i)}$ for $1 \le i \le r$. Thus equation (1) can be restated by saying that $Y_2^{(j)}$ is a sum of simultaneous eigenvectors for $\operatorname{ad} Y_1^{(i)}$, with each such eigenvalue being ±1. It follows that $Y_2^{(j)}$ is contained in the sum of eigenspaces for $\operatorname{ad} \tilde{\varphi}(Y_1)$ in $\tilde{\mathfrak{g}}$ with eigenvalues $r, r-2, \dots, -r$. Hence

$$\left(\prod_{i=0}^{r}\big(\operatorname{ad} \tilde{\varphi}(Y_1) - (r-2i)\big)\right)(\tilde{\varphi}(Y_2)) = 0.$$

Setting $X_i = e_i - f_i$ for $i = 1, 2$ and conjugating Y_i to X_i as in the proof of Theorem 1.3, we deduce that $P_r(\operatorname{ad} X_1)(X_2) = 0$ and $P_s(\operatorname{ad} X_2)(X_1) = 0$, where

$$P_m(t) = \begin{cases} (t^2 + m^2)(t^2 + (m-2)^2) \cdots (t^2 + 1) & \text{if } m \text{ is odd,} \\ (t^2 + m^2)(t^2 + (m-2)^2) \cdots (t^2 + 4)t & \text{if } m \text{ is even.} \end{cases}$$

1.7. **The general symmetrizable case.** Now suppose $\mathfrak{g}$ is an arbitrary symmetrizable Kac–Moody algebra with $n \times n$ generalized Cartan matrix $A = (a_{ij})_{1 \le i,j \le n}$. For $1 \le i \le n$ let $X_i = e_i - f_i \in \mathfrak{k}$. On restricting to the rank-two subalgebra of $\mathfrak{g}$ generated by e_i, e_j, f_i, f_j, where $1 \le i \ne j \le n$, we obtain the relation

$$P_{-a_{ij}}(\operatorname{ad} X_i)(X_j) = 0.$$

As in the simply-laced case, we can use Berman's theorem [2, Thm. 1.31] to prove that these relations generate all of the relations in $\mathfrak{k}$. For the sake of completeness, we reproduce a proof (which also applies in the simply-laced case).

Theorem 1.8. *The maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ has generators $X_1, \dots,$ X_n and, for any $1 \le i \ne j \le n$, the following relations:*

$$\big(P_{-a_{ij}}(\operatorname{ad} X_i)\big)(X_j) = 0.$$

Proof. By the Gabber–Kac theorem [14, Thm. 9.11] the ideal of relations satisfied by $e_1, \dots, e_n$ is generated by the terms $(\operatorname{ad} e_i)^{-a_{ij}+1}(e_j) = 0$. Let $\mathcal{L}$ be the Lie algebra on generators $x_1, \dots, x_n$ with relations $P_{-a_{ij}}(\operatorname{ad} x_i)(x_j) = 0$

for $1 \leq i \neq j \leq n$. Then there is a Lie algebra homomorphism $\pi : \mathcal{L} \to \mathfrak{k}$, sending x_i to $X_i = e_i - f_i$.

For $\alpha, \beta \in Q_+$ we write $\alpha \leq \beta$ when $\beta - \alpha \in Q_+$. We note that both $\mathcal{L}$ and $\mathfrak{k}$ are *filtered* by Q_+, that is, there exist subspaces $\mathcal{L}_{(\alpha)}$ of $\mathcal{L}$ such that

- $\mathcal{L} = \bigcup_{\alpha \in Q_+} \mathcal{L}_{(\alpha)}$,
- $\mathcal{L}_{(\alpha)} \subset \mathcal{L}_{(\beta)}$ whenever $\alpha \leq \beta$, and
- $[\mathcal{L}_{(\alpha)}, \mathcal{L}_{(\beta)}] \subseteq \mathcal{L}_{(\alpha+\beta)}$,

and similarly for $\mathfrak{k}$. Specifically, $\mathfrak{k}_{(\alpha)} = (\sum_{-\alpha \leq \beta \leq \alpha} \mathfrak{g}_\beta) \cap \mathfrak{k}$ and $\mathcal{L}_{(\alpha)}$ is the span of all commutators

$$[x_{i_1}, [x_{i_2}, [\ldots [x_{i_{r-1}}, x_{i_r}] \ldots]]],$$

where $\alpha_{i_1} + \cdots + \alpha_{i_r} \leq \alpha$. These filtrations are compatible, i.e. $\pi(\mathcal{L}_{(\alpha)}) \subset \mathfrak{k}_{(\alpha)}$. For $\alpha \in Q_+$, let $\mathcal{L}_{<\alpha} := \sum_{\beta < \alpha} \mathcal{L}_{(\beta)}$ and similarly for $\mathfrak{k}$. The *corresponding graded Lie algebra* of $\mathcal{L}$ is the vector space

$$\operatorname{gr} \mathcal{L} := \sum_{\alpha \in Q_+} \mathcal{L}_{(\alpha)} / \mathcal{L}_{<\alpha}$$

with the Lie bracket induced by that on $\mathcal{L}$. For $1 \leq i \leq n$ let $\overline{x}_i$ denote the image of x_i in $\mathcal{L}_{(\alpha_i)} / \mathcal{L}_{<\alpha_i} \subset \operatorname{gr} \mathcal{L}$. By the definition of the polynomials P_m, we have $(\operatorname{ad} \overline{x}_i)^{-a_{ij}+1}(\overline{x}_j) = 0$ for $1 \leq i \neq j \leq n$. It follows that there is a surjective homomorphism $\mathfrak{n}_+ \to \operatorname{gr} \mathcal{L}$ sending e_i to $\overline{x}_i$. On the other hand, $\mathfrak{k}_{(\alpha)} / \mathfrak{k}_{<\alpha}$ is spanned by $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$ so is of dimension $\dim \mathfrak{g}_\alpha$. (In fact, $\operatorname{gr} \mathfrak{k} \cong \mathfrak{n}_+$, see the remarks after Proposition 2.6 below.)

Now we can prove the theorem as follows. First of all, we claim that the homomorphism $\pi : \mathcal{L} \to \mathfrak{k}$ is surjective. To prove our claim it will suffice to show that $\pi(\mathcal{L}_{(\alpha)}) = \mathfrak{k}_{(\alpha)}$ for all $\alpha \in \Delta_+$. We note that $\mathfrak{g}_\alpha$ is spanned by elements of the form $y_\alpha = [e_i, y_{\alpha - \alpha_i}]$ where $y_{\alpha - \alpha_i} \in \mathfrak{g}_{\alpha - \alpha_i}$ and α_i can be any simple root. By an obvious induction hypothesis, we may assume that $\mathfrak{k}_{(\alpha - \alpha_i)} \subset \pi(\mathcal{L}_{(\alpha - \alpha_i)})$ and $\mathfrak{k}_{(\alpha - 2\alpha_i)} \subset \pi(\mathcal{L}_{(\alpha - 2\alpha_i)})$. Then

$$y_\alpha + \omega(y_\alpha) = [e_i - f_i, y_{\alpha - \alpha_i} + \omega(y_{\alpha - \alpha_i})] + [f_i, y_{\alpha - \alpha_i}] + \omega([f_i, y_{\alpha - \alpha_i}]).$$

Since

$$\begin{aligned} &[e_i - f_i, y_{\alpha - \alpha_i} + \omega(y_{\alpha - \alpha_i})] \in \pi([x_i, \mathcal{L}_{(\alpha - \alpha_i)}]), \\ &[f_i, y_{\alpha - \alpha_i}] + \omega([f_i, y_{\alpha - \alpha_i}]) \in \pi(\mathcal{L}_{(\alpha - 2\alpha_i)}), \end{aligned}$$

it follows that $y_\alpha + \omega(y_\alpha) \in \pi(\mathcal{L}_{(\alpha)})$. For injectivity, we remark that the inequalities

$$\dim \mathfrak{g}_\alpha \geq \dim \mathcal{L}_{(\alpha)} / \mathcal{L}_{<\alpha} \geq \dim \mathfrak{k}_{(\alpha)} / \mathfrak{k}_{<\alpha} = \dim \mathfrak{g}_\alpha$$

establish that $\ker \pi \cap \mathcal{L}_{(\alpha)} = \{0\}$. □

Remark 1.9. Suppose $A = \left(\begin{smallmatrix} 2 & -r \\ -s & 2 \end{smallmatrix}\right)$ where $r, s \neq 0$. It is easy to see that if we quotient $\mathfrak{k}$ by the ideal generated by $[X_1, [X_1, X_2]] + r^2 X_2$ and $[X_2, [X_2, X_1]] + s^2 X_1$ then we obtain an epimorphism $\mathfrak{k} \to \mathfrak{so}_3$. This corresponds to repeatedly applying Construction 2.8 (a) below to the complete bipartite graph to obtain a diagram of type A_2.

In what follows, we suppose that the generalized Cartan matrix A is indecomposable. Then there is a well-defined, unique up to scalar multiplication *length function* $|\cdot|$ on the simple roots such that $a_{ij}/a_{ji} = |\alpha_j|^2/|\alpha_i|^2$ whenever $a_{ij} \neq 0$. After scaling we may assume that $|\alpha_i|^2 \in \mathbb{N}$ for any i, and that the square lengths $|\alpha_i|^2$ have no common factor.

Definition 1.10. A *simply-laced cover diagram of* $\mathfrak{g}$ (or just a *cover diagram* for short) is a simply-laced diagram D with n_i vertices $\alpha_i^{(1)}, \dots, \alpha_i^{(n_i)}$ for each simple root α_i of $\mathfrak{g}$ (where n_i are some positive integers), and such that each $\alpha_i^{(k)}$ is connected to exactly $|a_{ij}|$ of the vertices $\alpha_j^{(l)}$ for $j \neq i$ and to none of the other vertices $\alpha_i^{(l)}$.

We remark that the n_i are related by the formula $n_i/n_j = a_{ij}/a_{ji}$ whenever $a_{ij} \neq 0$, hence $n_i = M/|\alpha_i|^2$ for some constant M. It follows that M is divisible by all $|\alpha_i|^2$. Moreover, each n_i must be divisible by any nonzero value $|a_{ij}|$, so that M is divisible by $\operatorname{lcm}_{j\neq k: a_{jk}\neq 0}(|\alpha_j|^2 \cdot |a_{jk}|)$. In the special case that $M = \operatorname{lcm}_{j\neq k: a_{jk}\neq 0}(|\alpha_j|^2 \cdot |a_{jk}|)$ we call the diagram to be of *minimal rank*.

Clearly, one can construct a minimal-rank simply-laced cover diagram for $\mathfrak{g}$ by setting

$$n_i = \frac{\operatorname{lcm}_{j\neq k: a_{jk}\neq 0}(|\alpha_j|^2 \cdot |a_{jk}|)}{|\alpha_i^2|}$$

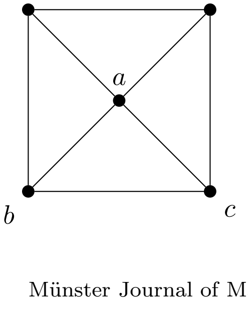
for all i and for each pair (i,j) with $a_{ij} < 0$, arbitrarily dividing the vertices $\alpha_i^{(1)}, \dots, \alpha_i^{(n_i)}$ (resp. $\alpha_j^{(1)}, \dots, \alpha_j^{(n_j)}$) into $m = n_i/|a_{ij}| = n_j/|a_{ji}|$ subsets $S_1, \dots, S_m$ (resp. $S'_1, \dots, S'_m$) of $|a_{ij}|$ (resp. $|a_{ji}|$) vertices with every vertex in S_k joined to every vertex in S'_k.

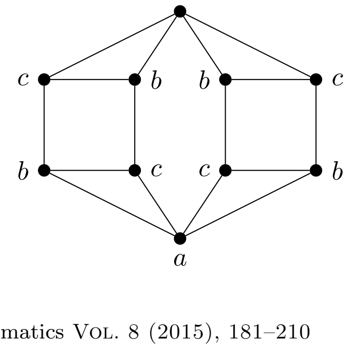
As the following examples show, not every connected cover diagram is of minimal rank, and two minimal-rank cover diagrams need not be isomorphic.

Example 1.11. (a) The Kac–Moody algebra which has generalized Cartan matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

has (at least) the following two simply-laced cover diagrams:





(b) If $\mathfrak{g}$ has symmetrizable Cartan matrix

$$\begin{pmatrix} 2 & -3 & -6 \\ -5 & 2 & -5 \\ -2 & -1 & 2 \end{pmatrix},$$

then under the assumptions above we have $|\alpha_1|^2 = 5$, $|\alpha_2|^2 = 3$ and $|\alpha_3|^2 = 15$. Thus $\mathrm{lcm}_{j \neq k : a_{jk} \neq 0}(|\alpha_j|^2 \cdot |a_{jk}|) = 30$ and therefore $n_1 = 6$, $n_2 = 10$, $n_3 = 2$. Note that $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$ are connected to all of the vertices $\alpha_1^{(1)}, \dots, \alpha_1^{(6)}$, but each to only half of $\alpha_2^{(1)}, \dots, \alpha_2^{(10)}$. Similarly, the vertices $\alpha_2^{(i)}$ also divide into two groups of five, each connecting to three of the vertices $\alpha_1^{(1)}, \dots, \alpha_1^{(6)}$. After renumbering we may assume that $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, $\alpha_1^{(3)}$ are connected to all of $\alpha_2^{(1)}, \dots, \alpha_2^{(5)}$. It is not hard to see that there are three isomorphism classes of minimal-rank cover diagrams for $\mathfrak{g}$, given by diagrams in which $\alpha_3^{(1)}$ connects to none, one or two of the vertices $\alpha_2^{(1)}, \dots, \alpha_2^{(5)}$.

Remark 1.12. If $\mathfrak{g}$ is of finite (resp. affine) type then there is a unique choice of connected simply-laced cover diagram for $\mathfrak{g}$, which is also finite (resp. affine). Specifically, for the finite type Lie algebras of type B_n, C_n, F_4 and G_2 one obtains simply-laced cover diagrams of type D_{n+1}, A_{2n-1}, E_6 and D_4, and similarly for the corresponding (untwisted) affine types. The twisted affine types all have simply-laced cover diagrams which are of affine type D except for the dual of affine F_4, which has simply-laced cover E_7^+. If $\mathfrak{g}$ is an arbitrary Kac–Moody Lie algebra of rank two then there exists a unique choice of simply-laced cover diagram, constructed in Section 1.5.

If the generalized Cartan matrix of $\mathfrak{g}$ is not indecomposable then a minimal-rank simply-laced cover diagram for $\mathfrak{g}$ is one which has the smallest possible number of vertices. Such a diagram can be constructed as the union of the (minimal-rank) simply-laced cover diagrams for the simple summands of $\mathfrak{g}$.

Let $\mathfrak{g}$ be an arbitrary symmetrizable Kac–Moody algebra and let $\tilde{\mathfrak{g}}$ be the Kac–Moody algebra associated to some simply-laced cover diagram for $\mathfrak{g}$. Let $e_i^{(k)}$, $f_i^{(k)}$, $h_i^{(k)}$ be the simple root elements corresponding to the vertex $\alpha_i^{(k)}$, for $1 \leq k \leq n_i$. As in the rank-two case there is a natural embedding $\tilde{\varphi} : \mathfrak{g} \to \tilde{\mathfrak{g}}$ which sends e_i (resp. f_i) to $\sum_{k=1}^{n_i} e_i^{(k)}$ (resp. $\sum_{k=1}^{n_i} f_i^{(k)}$) and which induces a map from the extended Weyl group of $\mathfrak{g}$ to that of $\tilde{\mathfrak{g}}$. Clearly, there is also a corresponding embedding $\mathfrak{k} \hookrightarrow \tilde{\mathfrak{k}}$.

2. SOME ALGEBRAIC PROPERTIES OF $\mathfrak{k}$

In this section we collect some consequences of Berman's presentation of the maximal compact subalgebra of a Kac–Moody algebra.

2.1. **Automorphisms.** For $i = 1, \dots, n$ let $\varepsilon_i \in \{\pm 1\}$. Then there is an automorphism φ_ε of $\mathfrak{k}$ characterized by $\varphi(X_i) = \varepsilon_i X_i$, called a *sign automorphism.*

If $\pi \in \mathrm{Sym}(n)$ is a permutation which preserves the generalized Cartan matrix of $\mathfrak{g}$ (i.e., $a_{\pi(i)\pi(j)} = a_{ij}$ for all i, j) then there is an induced automorphism φ_π of $\mathfrak{k}$ satisfying $\varphi_\pi(X_i) = X_{\pi(i)}$. Such an automorphism is called a *graph automorphism.* (In the simply-laced case π corresponds exactly to an automorphism of the diagram of $\mathfrak{g}$, i.e., a permutation of the vertices which preserves adjacency.)

Lemma 2.2. *Let $\mathfrak{g}$ be a Kac–Moody algebra over a field k of characteristic* 0.

(a) *For $i = 1, \dots, n$, the element $s_i^* \in W^*$ commutes with ω.*
(b) *Every $w \in W^*$ induces an automorphism $\pi(w)$ of $\mathfrak{k}$.*
(c) *If the Kac–Moody algebra $\mathfrak{g}$ is simply laced, the automorphism $\pi(s_i^*)$ induced by s_i^* via the isomorphism given in Theorem 1.3 satisfies*

$$X_i \mapsto X_i, \quad X_j \mapsto \begin{cases} X_j & \text{if } (i,j) \notin E, \\ [X_i, X_j] & \text{if } (i,j) \in E. \end{cases}$$

Proof. Statement (a) has been proved in Remark 1.4. By (a), each s_i^* stabilizes $\mathfrak{k}$. Statement (b) therefore follows immediately from [14, Lem. 3.8 (b)].

Concerning (c), a calculation in $\mathfrak{sl}_2(k)$ shows that $s_i^*(e_i) = -f_i$. A calculation in $\mathfrak{sl}_3(k)$ shows $s_i^*(e_j) = [e_i, e_j]$ if $(i,j) \in E$, and a calculation in $\mathfrak{sl}_2(k) \oplus \mathfrak{sl}_2(k)$ shows $s_i^*(e_j) = e_j$ if $(i,j) \notin E$. More calculations (or the use of assertion (a)) show, furthermore, $s_i^*(f_i) = -e_i$ and $s_i^*(f_j) = -[f_i, f_j]$ if $(i,j) \in E$, and $s_i^*(f_j) = f_j$ if $(i,j) \notin E$. In particular,

$$s_i^*(e_j - f_j) = s_i^*(e_j) - s_i^*(f_j) = [e_i, e_j] + [f_i, f_j] = [e_i - f_i, e_j - f_j].$$

Statement (c) follows. ☐

For $w \in W^*$, the induced automorphism $\pi(w) \in \operatorname{Aut} \mathfrak{k}$ is called a *Weyl group automorphism.*

Remark 2.3. (a) Let

$$\varphi_+ : \mathfrak{n}_+ \to \mathfrak{k}, \quad x \mapsto x + \omega(x)$$

be the canonical k-linear bijection (cp. [2, p. 3169]), and write $\mathfrak{k}_\alpha := \varphi_+(\mathfrak{g}_\alpha)$. Observe that for the analogous k-linear bijection

$$\varphi_- : \mathfrak{n}_- \to \mathfrak{k}, \quad x \mapsto x + \omega(x)$$

one has $\mathfrak{k}_\alpha = \varphi_+(\mathfrak{g}_\alpha) = \varphi_-(\mathfrak{g}_{-\alpha}) = \mathfrak{k}_{-\alpha}$.

It follows from Lemma 2.2 (a) that $\pi(s)(\mathfrak{k}_\alpha) = \mathfrak{k}_{s \cdot \alpha}$. Hence, by induction and by the definition of the set of real roots, for any positive real root $\alpha \in \Delta_+$ there is a Weyl group automorphism $\pi(w)$ and a positive simple root α_i such that $\pi(w)(\mathfrak{k}_\alpha) = \mathfrak{k}_{\alpha_i} = kX_i$.

(b) The set of subspaces $\{\mathfrak{k}_\gamma \mid \gamma \in \Delta^{\mathrm{re}} \cap \Delta_+\}$ is invariant under the action of the group of Weyl group automorphisms. It can be identified with the walls of the Coxeter complex of the Weyl group W (cp. [14, Rem. 3.8]).

Remark 2.4. If $\mathfrak{g}$ is simply laced then for i, j in the same connected component of the diagram of $\mathfrak{k}$ there is an automorphism such that $\varphi(X_i) = X_j$.

This is because, if (i,j) is an edge, then

$$\begin{aligned}\pi(s_i^* s_j^*)(X_i) &= \pi(s_i^*)([X_j, X_i]) \\ &= [\pi(s_i^*)(X_j), \pi(s_i^*)(X_i)] \\ &= [[X_i, X_j], X_i] \\ &= X_j,\end{aligned}$$

the first and third equation being consequences of Lemma 2.2, and the last one of Theorem 1.3. Thus, the claim follows by induction.

This can be used as follows: Let $\mathfrak{k}$ be the maximal compact subalgebra of a Kac–Moody algebra of type AE_4 (see Section 4). Then the generator X_4 is contained in a subalgebra isomorphic to the maximal compact subalgebra of a Kac–Moody algebra of type A_2^+. Indeed, let φ be a Weyl group automorphism such that $\varphi(X_3) = X_4$. Then $\varphi(\langle X_1, X_2, X_3\rangle)$ is as required, as by Theorem 1.3 the Lie algebra $\langle X_1, X_2, X_3\rangle$ equals the maximal compact subalgebra of the Kac–Moody algebra with positive simple roots α_1, α_2, α_3.

2.5. **A contraction of $\mathfrak{k}$.** Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over $\mathbb{R}$ with Chevalley generators e_i, f_i, h_i, $i = 1, \dots, n$. For $\varepsilon > 0$ define ω_ε to be the Lie algebra automorphism satisfying

$$\omega_\varepsilon(e_i) = -\varepsilon f_i, \quad \omega_\varepsilon(f_i) = -\frac{1}{\varepsilon} e_i, \quad \omega_\varepsilon(h_i) = -h_i;$$

moreover, set $\mathfrak{k}_\varepsilon := \operatorname{Fix}\omega_\varepsilon$. Observe that $\mathfrak{k} = \mathfrak{k}_1$ and that $X_i^\varepsilon := e_i - \varepsilon f_i \in \mathfrak{k}_\varepsilon$ for $i = 1, \dots, n$. Moreover, the automorphism θ_ε of $\mathfrak{g}$ given by $e_i \mapsto \frac{1}{\sqrt{\varepsilon}} e_i$ and $f_i \mapsto \sqrt{\varepsilon} f_i$ for all i satisfies

$$\theta_\varepsilon(X_i) = \frac{1}{\sqrt{\varepsilon}} X_i^\varepsilon, \quad \omega_\varepsilon = \theta_\varepsilon^2 \circ \omega = \theta_\varepsilon \circ \omega \circ \theta_\varepsilon^{-1}.$$

Thus θ_ε maps $\mathfrak{k}$ isomorphically onto $\mathfrak{k}_\varepsilon$. By applying θ_ε to $P_{-a_{ij}}(\operatorname{ad} X_i)(X_j)$ (using the notation of Theorem 1.8), we obtain the relations

$$P^\varepsilon_{-a_{ij}}(\operatorname{ad} X_i^\varepsilon)(X_j^\varepsilon) = 0 \quad \text{where} \quad P^\varepsilon_m(t) = \varepsilon^{\frac{m+1}{2}} P_m\Big(\frac{t}{\sqrt{\varepsilon}}\Big)$$

that is,

$$P^\varepsilon_m(t) = \begin{cases} (t^2 + m^2\varepsilon)\cdots(t^2+\varepsilon) & \text{for } m \text{ odd,} \\ (t^2 + m^2\varepsilon)\cdots(t^2+4\varepsilon)t & \text{for } m \text{ even.} \end{cases}$$

In particular, $[X_i^\varepsilon, [X_i^\varepsilon, X_j^\varepsilon]] = -\varepsilon X_j^\varepsilon$ if $a_{ij} = -1$.

Since θ_ε maps $\mathfrak{k}$ isomorphically onto $\mathfrak{k}_\varepsilon$, we have:

Proposition 2.6. *The subalgebra $\mathfrak{k}_\varepsilon$ is isomorphic to the quotient of the free Lie algebra over k generated by $X_1, \dots, X_n$ subject to the relations*

$$P^\varepsilon_{-a_{ij}}(\operatorname{ad} X_i)(X_j) = 0$$

via the map $X_i \mapsto e_i - \varepsilon f_i$.

Note that, if we set $\varepsilon = 0$ in the above presentation, the resulting algebra is isomorphic to $\mathfrak{n}_+$ by the Gabber–Kac theorem [14, Thm. 9.11]. This means that $\mathfrak{n}_+$ is a *contraction* of the maximal compact subalgebra $\mathfrak{k} = \mathfrak{k}_1$ in the sense of [9].

2.7. **Quotients.** Let k be a field of characteristic 0 and $\mathfrak{g}$ a Kac–Moody algebra over k with simply-laced diagram D. Due to the Coxeter-like presentation of the maximal compact subalgebra $\mathfrak{k}$ it is possible to exhibit quotients of $\mathfrak{k}$ if D has a certain shape.

For a graph D, let $\mathfrak{k}(D)$ denote the maximal compact subalgebra of the Kac–Moody algebra $\mathfrak{g}$ over k with diagram D.

Construction 2.8. Suppose that there are distinct vertices v_i, v_j of the diagram D such that any vertex v_r distinct from v_i, v_j is connected to v_i if and only if v_r is connected to v_j.

(a) If v_i and v_j are not connected by an edge, let D' be the diagram obtained from D by deleting the vertex v_j. Let $\mathfrak{k}' := \mathfrak{k}(D')$ and $X'_1, \dots, X'_n$ its Berman generators. Then there is a well-defined epimorphism of Lie algebras $\varphi\colon \mathfrak{k} \to \mathfrak{k}'$ determined by $\varphi(X_r) := X'_r$ for $r \neq j$ and $\varphi(X_j) := X'_i$.

(b) If v_i and v_j are connected by an edge, let D' be the diagram obtained from D by deleting all edges emanating from v_j except for the edge (v_i, v_j). As above, let $\mathfrak{k}' := \mathfrak{k}(D')$ and $X'_1, \dots, X'_n$ its Berman generators. Then there is a well-defined epimorphism of Lie algebras $\varphi\colon \mathfrak{k} \to \mathfrak{k}'$ determined by $\varphi(X_r) := X'_r$ for $r \neq j$ and $\varphi(X_j) := [X'_i, X'_j]$.

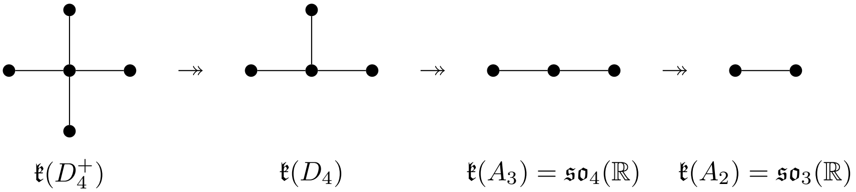
This can be checked by using the Weyl automorphisms introduced in Lemma 2.2. For instance, for all $r \neq i, j$ with $(v_r, v_j) \in E_D$ (which is equivalent to $(v_r, v_i) \in E_D$), one has

$$\begin{aligned}
\big[\varphi(X_j), [\varphi(X_j), \varphi(X_r)]\big] &= \big[[X'_i, X'_j], [[X'_i, X'_j], X'_r]\big] \\
&= \big[-\pi(s_j^*)(X'_i), [-\pi(s_j^*)(X'_i), \pi(s_j^*)(X'_r)]\big] \\
&= \pi(s_j^*)[X'_i, [X'_i, X'_r]] \\
&= \pi(s_j^*)(-X'_r) \\
&= -X'_r \\
&= \varphi(-X_r) \\
&= \varphi[X_j, [X_j, X_r]],
\end{aligned}$$

the second and fifth equation being consequences of Lemma 2.2, and the fourth and seventh of Theorem 1.3.

Case (a) (resp. (b)) of Construction 2.8 corresponds to factoring $\mathfrak{k}$ modulo the ideal generated by $(X_i - X_j)$ (resp. by all terms of the form $[X_r, [X_i, X_j]]$ where $r \neq i, j$).

Example 2.9. (a) The preceding discussion gives a sequence of epimorphisms of real Lie algebras $\mathfrak{k}(D_4^+) \twoheadrightarrow \mathfrak{k}(D_4) \twoheadrightarrow \mathfrak{k}(A_3) = \mathfrak{so}_4(\mathbb{R}) \twoheadrightarrow \mathfrak{k}(A_2) = \mathfrak{so}_3(\mathbb{R})$.



This sequence can be extended further: Let

$$\Gamma_n = \big(\{1, \ldots, n\}, \{(1, k) \mid 2 \leq k \leq n\}\big)$$

denote the star diagram on n vertices and let $\mathfrak{k}_n$ denote the maximal compact subalgebra of the Kac–Moody algebra $\mathfrak{g}_n$ with Dynkin diagram Γ_n. Then there are epimorphisms $\mathfrak{k}_n \to \mathfrak{k}_{n-1}$.

(b) Denoting by K_4 the complete graph on four vertices, there similarly is a sequence of epimorphisms $\mathfrak{k}(K_4) \twoheadrightarrow \mathfrak{k}(AE_4) \twoheadrightarrow \mathfrak{k}(A_4)$.

3. GENERALIZED SPIN REPRESENTATIONS

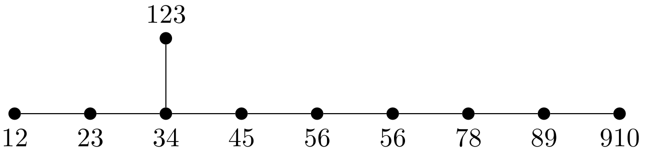
3.1. **Generalized spin representations of $\mathfrak{k}(E_{10}(\mathbb{R}))$.** Let us recall the extension of the spin representation of $\mathfrak{k}(\mathfrak{sl}_{10}(\mathbb{R}))$ to $\mathfrak{k}(E_{10})(\mathbb{R})$ as described in [4, 7] (also [17]).

Example 3.2. Let V be a k-vector space and $q\colon V \to k$ a quadratic form with associated bilinear form b. Then the *Clifford algebra* $C := C(V, q)$ is defined as $C := T(V)/\langle vw + wv - 2b(v, w)\rangle$ where $T(V)$ is the tensor algebra of V.

Now let $V = \mathbb{R}^{10}$ with standard basis vectors v_i, let $q = x_1^2 + \cdots + x_{10}^2$ and let $C = C(V, q)$. Then in C we have

$$v_i^2 = 1 \quad \text{and} \quad v_i v_j = -v_j v_i.$$

Since C is an associative algebra, it becomes a Lie algebra by setting $[A, B] := AB - BA$. Let the diagram of $\mathfrak{g}(E_{10})(\mathbb{R})$ be labelled as



and define a Lie algebra homomorphism $\rho : \mathfrak{k} \to C$ using these labels, i.e., via

$$\begin{aligned}
&X_1 \mapsto \frac{1}{2}v_1v_2, \quad X_2 \mapsto \frac{1}{2}v_1v_2v_3, \quad X_3 \mapsto \frac{1}{2}v_2v_3,\\
&X_4 \mapsto \frac{1}{2}v_3v_4, \quad X_5 \mapsto \frac{1}{2}v_4v_5, \quad X_6 \mapsto \frac{1}{2}v_5v_6,\\
&X_7 \mapsto \frac{1}{2}v_6v_7, \quad X_8 \mapsto \frac{1}{2}v_7v_8, \quad X_9 \mapsto \frac{1}{2}v_8v_9, \quad X_{10} \mapsto \frac{1}{2}v_9v_{10},
\end{aligned}$$

where X_i denotes the Berman generator corresponding to the root α_i, enumerated in Bourbaki style as in Section 4. Observe that each $A_i := \rho(X_i)$ satisfies $A_i^2 = -\frac{1}{4}\,\mathrm{id}$. Here we would like to remark that $(v_1v_2v_3)^2 = (v_2v_3)^2 = -1$

depends on $v_i^2 = 1$; for parity reasons, this would not be true in the Clifford algebra $C(V, -q)$, as then $(v_1v_2v_3)^2 = -(v_2v_3)^2 = 1$.

Using the criterion established in Remark 3.7 below, one checks easily that ρ indeed is a Lie algebra homomorphism, i.e., that the defining relations of $\mathfrak{k}$ from Theorem 1.3 are respected. Indeed, one just needs to establish

(i) $A_i^2 = -\frac{1}{4} \cdot \mathrm{id}_s$,
(ii) $A_iA_j = A_jA_i$ if $(i,j) \notin E$,
(iii) $A_iA_j = -A_jA_i$ if $(i,j) \in E$.

We have already observed (i). Assertions (ii) and (iii) are obvious for $i, j \neq 2$. Moreover, one quickly computes

$$(v_1v_2v_3)(v_3v_4) = -(v_3v_4)(v_1v_2v_3)$$

and

$$(v_1v_2v_3)(v_{k_1}v_{k_2}) = (v_{k_1}v_{k_2})(v_1v_2v_3)$$

if $\{k_1, k_2\}$ is a set of two elements that is either a subset of $\{1,2,3\}$ or disjoint from $\{1,2,3\}$. Assertions (ii) and (iii) follow.

By [10, Lem. 20.9] and [25, Prop. 2.4] the Clifford algebra C splits over $\mathbb{C}$ as $C \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{32\times 32}$. Hence ρ affords a 32-dimensional complex representation of $\mathfrak{k}(E_{10})(\mathbb{R})$. The restriction of this representation to the maximal compact subalgebra of the A_9-subdiagram, $\mathfrak{k}(A_9)(\mathbb{R}) = \mathfrak{so}_{10}(\mathbb{R})$, coincides with the spin representation of $\mathfrak{so}_{10}$ (see, e.g., [10, Chap. 20]), i.e., ρ extends the classical spin representation.

Let $\iota \in \operatorname{Aut} C$ denote the involution (known as parity automorphism) induced by $V \to V : v \mapsto -v$. Let $C_0 := \operatorname{Fix} \iota$ and $C_1 := \{w \in C \mid \iota(w) = -w\}$ denote the even and the odd part of C. Then C_0 and C_1 are invariant subspaces under the spin representation of $\mathfrak{so}_{10}$ since $\operatorname{im} \rho \subseteq C_0$ (multiplication with a product of the v_i of even length does not change the parity) and these subspaces are irreducible nonisomorphic representations of $\mathfrak{so}_{10}$ (see [10, Chap. 20]).

The remaining Berman generator X_2 of $\mathfrak{k}(E_{10})$ is sent to an element which interchanges C_0 and C_1.

Remark 3.3. A calculation shows that $\operatorname{im} \rho$ is the linear span of all elements of the form $v_{i_1} \cdots v_{i_k}$, where $\{i_1, \dots, i_k\} = I \subseteq \{1, \dots, 10\}$ with $|I| \in \{2,3,6,7,10\}$. Therefore, $\dim \operatorname{im}(\rho) = 45 + 120 + 210 + 120 + 1 = 496$. Since $\operatorname{im}(\rho) \leq C \cong \mathbb{R}^{32\times 32}$ by [25, §2.2.3] and since $\operatorname{im}(\rho)$ is compact and semisimple by Theorem 3.14, this dimension $\dim \operatorname{im}(\rho) = 496$ implies $\operatorname{im}(\rho) \cong \mathfrak{so}_{32}(\mathbb{R})$ (see also [5]).

The existence of Example 3.2 is not peculiar to the diagram E_{10}, it can be generalized to arbitrary diagrams E_n in the obvious way. A careful analysis of dimensions combined with the Cartan–Bott periodicity of Clifford algebras allows one to determine the isomorphism types of the quotients for the whole E_n series. This is carried out in the appendix. A key observation is that the cardinality $|I|$ from above in general has to be equal to 2 or 3 modulo 4 (see Lemma A.6).

Remark 3.4. Let $\rho\colon \mathfrak{so}_{10}(\mathbb{R}) \to \mathbb{C}^{n\times n}$ be a representation. To extend ρ to a representation of $\mathfrak{k}(E_{10})$, it suffices to find a matrix $X \in \mathbb{C}^{n\times n}$ such that for $A_i := \rho(X_i)$, $1 \leq i \leq 10$, $i \neq 2$, the following equations are satisfied (where we again use the labelling of the diagram E_{10} as given in Section 4):

$$\begin{aligned} [A_i, X] &= 0 \quad \text{for } 1 \leq i \leq 10,\ i \neq 2, 4,\\ [A_4, [A_4, X]] &= -X,\\ [X, [X, A_4]] &= -A_4. \end{aligned}$$

Theorem 1.3 implies that ρ can be extended to $\mathfrak{k}(E_{10})$ by setting $\rho(X_2) := X$.

The first two sets of equations define a linear subspace, the third set of equations yields a family of quadratic equations. With the help of a Gröbner basis one can compute that in case of the spin representation, this variety is isomorphic to $\mathbb{C}^{\times}$, i.e., the extension is unique up to a scalar.

3.5. **Generalized spin representations for the simply-laced case.** Throughout this section, let k be a field of characteristic 0, let $\mathfrak{g}$ be a Kac–Moody algebra over k with simply-laced diagram and let $\mathfrak{k}$ be its maximal compact subalgebra.

Let $L := k(I)$, where I is a square root of -1. Denote by $\mathrm{id}_s \in L^{s\times s}$ the identity matrix.

Definition 3.6. A representation $\rho\colon \mathfrak{k} \to \mathrm{End}(L^s)$ is called a *generalized spin representation* if the images of the Berman generators from Theorem 1.3 satisfy

$$\rho(X_i)^2 = -\frac{1}{4}\,\mathrm{id}_s \quad \text{for } i = 1, \ldots, n.$$

Remark 3.7. (a) Since ρ is assumed to be a representation, it follows from the defining relations that $\rho(X_i)$ and $\rho(X_j)$ commute if $(i,j) \notin E$. On the other hand, if $(i,j) \in E$, then $A := \rho(X_i)$ and $B := \rho(X_j)$ anticommute. Indeed, we have

$$-B = [A, [A, B]] = A^2B - 2ABA + BA^2 = -\frac{1}{2}B - 2ABA,$$

where the first equation is due to Theorem 1.3. The claim now follows after multiplying with $A^{-1} = -4A \iff A^2 = -\frac{1}{4}\,\mathrm{id}_s$.

(b) Conversely, suppose that there are matrices $A_i \in L^{s\times s}$ satisfying

(i) $A_i^2 = -\frac{1}{4}\cdot \mathrm{id}_s$,
(ii) $A_iA_j = A_jA_i$ if $(i,j) \notin E$,
(iii) $A_iA_j = -A_jA_i$ if $(i,j) \in E$.

Then, by reversing the argument in the above computation, the assignment $X_i \mapsto A_i$ gives rise to a representation of $\mathfrak{k}$.

Remark 3.8. Let ρ be a generalized spin representation of $\mathfrak{k}$ and set $S_i := 2I \cdot \rho(X_i)$. Let W be a Coxeter group defined by the presentation

$$W = \langle s_1, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1\rangle,$$

where $m_{ii} = 1$ and $m_{ij} = 2$ if $(i,j) \notin E$, while $m_{ij} \in \{3,4\}$ if $(i,j) \in E$. Then the assignment $s_i \mapsto S_i$ gives a representation of W.

Write $\mathfrak{k}_{\leq r} := \langle X_1, \ldots, X_r \rangle$.

Theorem 3.9. *Let* $1 \leq r < n$. *Let* $\rho : \mathfrak{k}_{\leq r} \to \operatorname{End}(L^s)$ *be a generalized spin representation.*

(a) *If* X_{r+1} *centralizes* $\mathfrak{k}_{\leq r}$, *then* ρ *can be extended to a generalized spin representation* $\rho' : \mathfrak{k}_{\leq r+1} \to \operatorname{End}(L^s)$ *by setting* $\rho'(X_{r+1}) := \frac{1}{2} I \cdot \mathrm{id}_s$.

(b) *If* X_{r+1} *does not centralize* $\mathfrak{k}_{\leq r}$, *then* ρ *can be extended to a generalized spin representation* $\rho' \colon \mathfrak{k}_{\leq r+1} \to \operatorname{End}(L^s \oplus L^s)$ *as follows. Define the sign automorphism* $s_0 : \mathfrak{k}_{\leq r} \to L^s$ *via*

$$s_0(X_i) := \begin{cases} X_i & \text{if } (i, r+1) \notin E, \\ -X_i & \text{if } (i, r+1) \in E, \end{cases}$$

let

$$\rho'|_{\mathfrak{k}_{\leq r}} := \rho \oplus \rho \circ s_0$$

and

$$\rho'(X_{r+1}) := \frac{1}{2} I \cdot \mathrm{id}_s \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. If X_{r+1} centralizes $\mathfrak{k}_{\leq r}$, it is clear that ρ' is well-defined and that $\rho'(X_{r+1})^2 = -\frac{1}{4} \mathrm{id}_s$.

In the second case it is clear that $\rho'|_{\mathfrak{k}_{\leq r}}$ is a generalized spin representation of $\mathfrak{k}_{\leq r}$ which extends ρ. It is easy to check that $\rho'(X_i)$ commutes with $\rho'(X_{r+1})$ if $(i, r+1) \notin E$, and that $\rho'(X_i)$ anticommutes with $\rho'(X_{r+1})$ if $(i, r+1) \in E$. Remark 3.7 therefore implies that ρ' is a generalized spin representation. ☐

For a graph $G = (V, E)$, a subset $M \subseteq V$ is called a *coclique* if the subgraph of G induced on M does not contain any edges, i.e., if no two elements m_1, m_2 in M are connected by an edge.

Corollary 3.10. *Let* n *be the cardinality of the diagram of* $\mathfrak{g}$ *and let* r *be the size of a maximal coclique of that diagram. Then there exists a* 2^{n-r}-*dimensional generalized spin representation of* $\mathfrak{k}$. *Furthermore, if the diagram is irreducible, then there exists a* 2^{n-1}-*dimensional* maximal *generalized spin representation of* $\mathfrak{k}$.

Proof. Up to a change of labelling the set $M := \{\alpha_1, \ldots, \alpha_r\}$ forms a maximal coclique. The map

$$\rho : \mathfrak{k}_{\leq r} \to \operatorname{End}(L^1), \quad X_i \mapsto \frac{1}{2} I \cdot \mathrm{id}_1$$

is a generalized spin representation. By Theorem 3.9, the representation ρ can be extended inductively to a generalized spin representation of $\mathfrak{k}$; the dimension doubles at each step because M was assumed to be a maximal coclique.

For the second claim it suffices to order the vertices of the diagram in such a way that two consecutive vertices are adjacent. ☐

Remark 3.11. An inductive construction of the basic spin representations of the symmetric group similar to the one in Theorem 3.9 has independently been obtained by Maas [24]. It is likely that by a combination of the methods of [24] and of the present article, a similar construction of generalized (basic) spin representations is possible for any (simply-laced) Coxeter group.

3.12. Generalized spin representations for symmetrizable Kac–Moody algebras.

In this section let $\mathfrak{g}$ be an arbitrary symmetrizable Kac–Moody Lie algebra with maximal compact subalgebra $\mathfrak{k}$, and let n_i be the number of vertices associated to the root α_i in a minimal-rank simply-laced cover diagram for $\mathfrak{g}$. As above, we assume the ground field k has characteristic zero.

Definition 3.13. A generalized spin representation for $\mathfrak{k}$ is a Lie algebra homomorphism $\rho : \mathfrak{k} \to \mathrm{End}(L^s)$ such that each of the Berman generators X_i (see Theorem 1.8) satisfies

$$\Big(\rho(X_i)^2 + \frac{n_i^2}{4}\,\mathrm{id}_s\Big)\Big(\rho(X_i)^2 + \frac{(n_i-2)^2}{4}\,\mathrm{id}_s\Big)\dots\big(\rho(X_i)^2 + \mathrm{id}_s\big)\rho(X_i) = 0$$

if n_i is even, and

$$\Big(\rho(X_i)^2 + \frac{n_i^2}{4}\,\mathrm{id}_s\Big)\Big(\rho(X_i)^2 + \frac{(n_i-2)^2}{4}\,\mathrm{id}_s\Big)\dots\Big(\rho(X_i)^2 + \frac{1}{4}\,\mathrm{id}_s\Big) = 0$$

if n_i is odd; i.e., $P_{n_i}^{1/4}(\rho(X_i)) = 0$ (in the notation of Proposition 2.6).

Another way of saying this is that $\rho(X_i)$ is semisimple with eigenvalues belonging to the set $\{(n_i - 2j)/2\,I \mid 0 \le j \le n_i\}$. When the generalized Cartan matrix of $\mathfrak{g}$ is simply laced, this definition clearly coincides with Definition 3.6.

Theorem 3.14. *Let $L = k(I)$ where $I^2 = -1$. Let $\mathfrak{g}$ be an arbitrary symmetrizable Kac–Moody Lie algebra with maximal compact subalgebra $\mathfrak{k}$. Then there exists a generalized spin representation $\rho : \mathfrak{k} \to \mathrm{End}(L^s)$.*

Moreover, if k is formally real, then ρ can be considered as a representation $\mathfrak{k} \to \mathrm{End}(k^{2s})$ with $\mathrm{im}\,\rho$ compact and, therefore, reductive. Furthermore, in this case $\mathrm{im}\,\rho$ is semisimple, if for all i there exists $j \neq i$ such that a_{ji} is odd. Finally, in this case $\mathfrak{k} \cong \ker\rho \oplus \mathrm{im}\,\rho$.

Note that the condition in the next-to-final sentence of the theorem is satisfied if, for example, $\mathfrak{g}$ has a simply-laced diagram which has no isolated nodes. It will follow from the proof that the theorem is actually applicable to all generalized spin representations discussed in Theorem 3.9 and Corollary 3.10, in particular the standard generalized spin representation from Example 3.2.

Proof. To see that $\mathfrak{k}$ has a generalized spin representation, let $\tilde{\mathfrak{g}}$ be the Kac–Moody algebra associated to some minimal-rank simply-laced cover diagram for $\mathfrak{g}$ and let $\tilde{\varphi} : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be the Lie algebra embedding described in Section 1.7. Then it is clear from the earlier discussion that, if $\tilde{\rho} : \tilde{\mathfrak{k}} \to \mathrm{End}(L^s)$ is a generalized spin representation for $\tilde{\mathfrak{k}}$, then $\rho = \tilde{\rho} \circ \tilde{\varphi}|_{\mathfrak{k}}$ is a generalized spin representation for $\mathfrak{k}$. (It is, however, not clear that any generalized spin representation for $\mathfrak{k}$ arises in this way.) Thus the first statement follows immediately from Corollary 3.10.

For the second statement it will suffice to prove that there exists a generalized spin representation $\rho : \mathfrak{k} \to \mathrm{End}(L^s)$ such that, with respect to an appropriate choice of k-basis for L^s, each of the images $\rho(X_i)$ is a skew-symmetric $2s \times 2s$ matrix over k and, thus, ρ can be interpreted as a homomorphism $\mathfrak{k} \to \mathfrak{so}_{2s}(k)$. Since we can construct generalized spin representations for $\mathfrak{k}$ by restricting from those for the Lie algebra associated to a simply-laced cover diagram, it will clearly suffice to show that the representation constructed in Theorem 3.9 can be realized by using skew-symmetric matrices only. For the extension of the representation in part (a) of Theorem 3.9 this is obvious, as

$$L \cong \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in k \right\}$$

as k-algebras, whence I is represented by the skew-symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For the extension of the representation in part (b) of Theorem 3.9, observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that after a change of basis we have instead

$$\rho'(X_{r+1}) = \frac{1}{2} \,\mathrm{id}_s \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(while $\rho'|_{\mathfrak{k}_{\leq r}}$ remains unchanged). Therefore, if the representation of $\mathfrak{k}_{\leq r}$ consists of skew-symmetric matrices over k, one can ensure that the representation of $\mathfrak{k}_{\leq r+1}$ also consists of skew-symmetric matrices over k. Thus $\mathrm{im}(\rho)$ is compact, whence reductive.

For the statement concerning semisimplicity observe that $\mathfrak{k}$ is perfect. Indeed, by hypothesis, for each generator X_i of $\mathfrak{k}$, there is some j such that a_{ji} is odd, and therefore the constant term in the polynomial $P_{-a_{ji}}$ is nonzero. Since $P_{-a_{ji}}(\mathrm{ad}\, X_j)(X_i) = 0$ by Theorem 1.8, it follows that X_i is contained in the linear span of $(\mathrm{ad}\, X_j)^{2l}(X_i)$, $l \geq 1$. Thus, the image $\mathrm{im}(\rho)$ is perfect and, by the above, reductive. The claim is now obvious, as a perfect direct sum of a semisimple and an abelian Lie algebra necessarily is semisimple.

For the final statement observe that $\mathfrak{k}$ is anisotropic with respect to the invariant bilinear form of the Kac–Moody algebra $\mathfrak{g}$ and so $(\ker \rho)^{\perp} \cong \mathrm{im}\, \rho$ is an ideal of $\mathfrak{k}$, where $\perp$ denotes the orthogonality relation with respect to the invariant bilinear form. ☐

Let $\mathcal{C}$ denote the class of all generalized spin representations of $\mathfrak{k}$. We check some closure properties of $\mathcal{C}$.

Proposition 3.15. (a) *$\mathcal{C}$ is closed under direct sums, quotients, duals and taking subrepresentations.*

(b) *If the generalized Cartan matrix of $\mathfrak{g}$ is simply laced and $\rho_1, \rho_2, \rho_3 \in \mathcal{C}$, then so is $\rho\colon X_i \mapsto 4\rho_1(X_i) \otimes \rho_2(X_i) \otimes \rho_3(X_i)$.*

(c) *More generally, if the generalized Cartan matrix of $\mathfrak{g}$ is simply laced and $\rho_1, \rho_2 \in \mathcal{C}$, then so is $\rho := 2I\rho_1 \otimes \rho_2$, where I is a primitive fourth root of unity.*

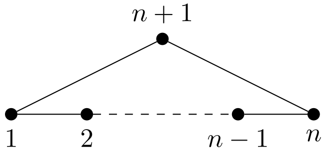
(d) *If $\rho \in \mathcal{C}$ and φ is either a sign, graph or Weyl group automorphism of $\mathfrak{k}$, then $\rho \circ \varphi \in \mathcal{C}$.*

Proof. The first three assertions can be easily verified. The fourth assertion is clear if φ is a graph or a sign automorphism. The remaining claim follows from Remark 1.4, since if $\rho(X_j)$ has eigenvalues $\frac{rI}{2}, \frac{(r-2)I}{2}, \ldots, -\frac{rI}{2}$ then so does $\rho(\exp(\xi \operatorname{ad} X_i)(X_j)) = \exp(\xi\rho(X_i))(\rho(X_j))$. $\square$

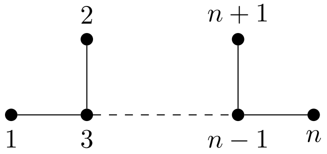
4. SOME DYNKIN DIAGRAMS

We give the list of relevant Dynkin diagrams we use in the main text.

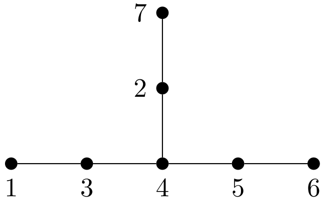
A_n^+



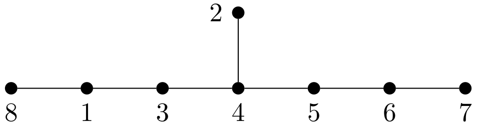
D_n^+

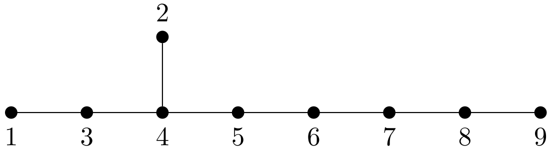


E_6^+

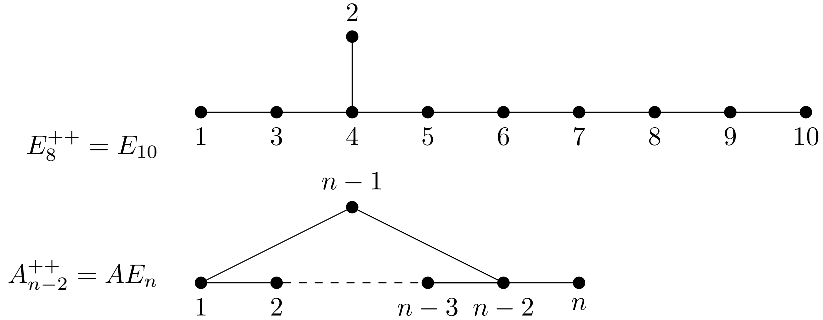


E_7^+





$E_8^+ = E_9$



Acknowledgements. We thank Pierre-Emmanuel Caprace for pointing out to us the 32-dimensional representation of the maximal compact subalgebra of $E_{10}(\mathbb{R})$, thus triggering our research. We also thank Kay Magaard for bringing our attention to [24] and Thibault Damour, David Ghatei, Axel Kleinschmidt, Karl-Hermann Neeb, Sebastian Weiß and especially Max Horn and two anonymous referees for valuable comments on preliminary versions of this work. This research has been partially funded by the EPRSC grants EP/H02283X and EP/K022997/1. The second author gratefully acknowledges the hospitality of the IHES at Bures-sur-Yvette and of the Albert Einstein Institute at Golm.

Appendix: Cartan–Bott periodicity for the real E_n series

(by Max Horn and Ralf Köhl)

In this appendix we continue the investigation of the generalized spin representations introduced in the main text. We focus on the E_n series and use the original description of the generalized spin representation from [4, 7] via Clifford algebras (see Example 3.2). The E_n series is traditionally only defined for $n \in \{6, 7, 8\}$. However, using the Bourbaki style labeling shown in Figure 1, it naturally extends to arbitrary $n \geq 3$. Using this description, one has $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, $E_5 = D_5$ (see Figure 2).

An elementary combinatorial counting argument using binomial coefficients allows us to determine lower bounds for the $\mathbb{R}$-dimension of the images of the generalized spin representation. These images have to be compact, whence reductive by Theorem 3.14 and even semisimple, if the diagram is irreducible. One therefore obtains an upper bound for their $\mathbb{R}$-dimension via the maximal compact Lie subalgebras of the Clifford algebras. As it turns out, the lower and the upper bounds coincide, providing the following Cartan–Bott periodicity.

Theorem A (Cartan–Bott periodicity of the E_n series)**.** *Let $n \in \mathbb{N}$ with $n \geq 4$, let $\mathfrak{k}$ be the maximal compact Lie subalgebra of the split real Kac–Moody Lie algebra of type E_n, let $C = C(\mathbb{R}^n, q)$ be the Clifford algebra with respect to the standard positive definite quadratic form q, and let $\rho : \mathfrak{k} \to C$ be the standard generalized spin representation. Then* $\mathrm{im}(\rho)$ *is isomorphic to*

(a) $\mathfrak{so}(2^{\frac{n}{2}}) \leq \mathbb{R} \otimes_{\mathbb{R}} M(2^{\frac{n}{2}}, \mathbb{R})$ *if* $n \equiv 0 \pmod 8$,
(b) $\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}}) \leq (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-1}{2}}, \mathbb{R})$ *if* $n \equiv 1 \pmod 8$,
(c) $\mathfrak{so}(2^{\frac{n}{2}}) \leq M(2, \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-2}{2}}, \mathbb{R})$ *if* $n \equiv 2 \pmod 8$,
(d) $\mathfrak{su}(2^{\frac{n-1}{2}}) \leq M(2, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-3}{2}}, \mathbb{R})$ *if* $n \equiv 3 \pmod 8$,
(e) $\mathfrak{sp}(2^{\frac{n-2}{2}}) \leq M(2, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-4}{2}}, \mathbb{R})$ *if* $n \equiv 4 \pmod 8$,
(f) $\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}}) \leq (M(2, \mathbb{H}) \oplus M(2, \mathbb{H})) \otimes_{\mathbb{R}} M(2^{\frac{n-5}{2}}, \mathbb{R})$ *if* $n \equiv 5 \pmod 8$,
(g) $\mathfrak{sp}(2^{\frac{n-2}{2}}) \leq M(4, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-6}{2}}, \mathbb{R})$ *if* $n \equiv 6 \pmod 8$,
(h) $\mathfrak{su}(2^{\frac{n-1}{2}}) \leq M(8, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-7}{2}}, \mathbb{R})$ *if* $n \equiv 7 \pmod 8$,

i.e., $\mathrm{im}(\rho)$ *is a semisimple maximal compact Lie subalgebra of* C.

Along the way we arrive at a structural explanation for the well-known isomorphism types of the maximal compact Lie subalgebras of the semisimple split real Lie algebras of types $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, $E_5 = D_5$, E_6, E_7, E_8 (cp., e.g., [13, p. 518, Tab. V]).

Theorem B. *The maximal compact Lie subalgebras of the semisimple split real Lie algebras of types* $A_2 \oplus A_1$, A_4, D_5, E_6, E_7, E_8 *are isomorphic to* $\mathfrak{u}(2)$, $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, $\mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \cong \mathfrak{so}(5) \oplus \mathfrak{so}(5)$, $\mathfrak{sp}(4)$, $\mathfrak{su}(8)$, $\mathfrak{so}(16)$, *respectively.*

A.1. **Cartan–Bott periodicity of Clifford algebras.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, and let $\mathbb{R}$, $\mathbb{C}$, resp. $\mathbb{H}$ denote the reals, complex numbers, resp. quaternions. For $n \in \mathbb{N}$ and a division ring $\mathbb{D}$, denote by $M(n, \mathbb{D})$ the $\mathbb{D}$-algebra of $n \times n$ matrices over $\mathbb{D}$.

Let V be an $\mathbb{R}$-vector space and $q \colon V \to \mathbb{R}$ a quadratic form with associated bilinear form b. Then the *Clifford algebra* $C(V, q)$ is defined as

$$C(V, q) := T(V)/\langle vw + wv - 2b(v, w)\rangle,$$

where $T(V)$ is the tensor algebra of V; cp. [21, §4.3], [23, Chap. 1, §1].

Let $V = \mathbb{R}^n$ with standard basis vectors v_i, let $q = x_1^2 + \cdots + x_n^2$. Then in $C(V, q)$ we have $v_i^2 = 1$ and $v_i v_j = -v_j v_i$.

Proposition A.2 (Cartan–Bott periodicity)**.** *For* $n \geq 2$, *the Clifford algebra* $C(\mathbb{R}^n, q)$ *is isomorphic to the following algebra:*

(a) $\mathbb{R} \otimes_{\mathbb{R}} M(2^{\frac{n}{2}}, \mathbb{R})$ *if* $n \equiv 0 \pmod 8$,
(b) $(\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-1}{2}}, \mathbb{R})$ *if* $n \equiv 1 \pmod 8$,
(c) $M(2, \mathbb{R}) \otimes_{\mathbb{R}} M(2^{\frac{n-2}{2}}, \mathbb{R})$ *if* $n \equiv 2 \pmod 8$,
(d) $M(2, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-3}{2}}, \mathbb{R})$ *if* $n \equiv 3 \pmod 8$,
(e) $M(2, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-4}{2}}, \mathbb{R})$ *if* $n \equiv 4 \pmod 8$,
(f) $(M(2, \mathbb{H}) \oplus M(2, \mathbb{H})) \otimes_{\mathbb{R}} M(2^{\frac{n-5}{2}}, \mathbb{R})$ *if* $n \equiv 5 \pmod 8$,
(g) $M(4, \mathbb{H}) \otimes_{\mathbb{R}} M(2^{\frac{n-6}{2}}, \mathbb{R})$ *if* $n \equiv 6 \pmod 8$,
(h) $M(8, \mathbb{C}) \otimes_{\mathbb{R}} M(2^{\frac{n-7}{2}}, \mathbb{R})$ *if* $n \equiv 7 \pmod 8$.

Proof. See, e.g., [21, Prop. 4.4.1, Tab. 4.4.1]. □

Since $C(V, q)$ is an associative algebra, it becomes a Lie algebra by setting $[A, B] := AB - BA$. With this in mind, Proposition A.2 implies the following.

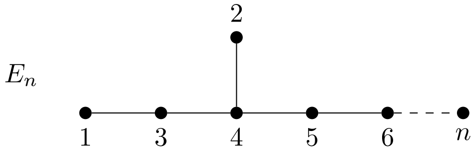


Figure 1. The Dynkin diagram of type E_n.

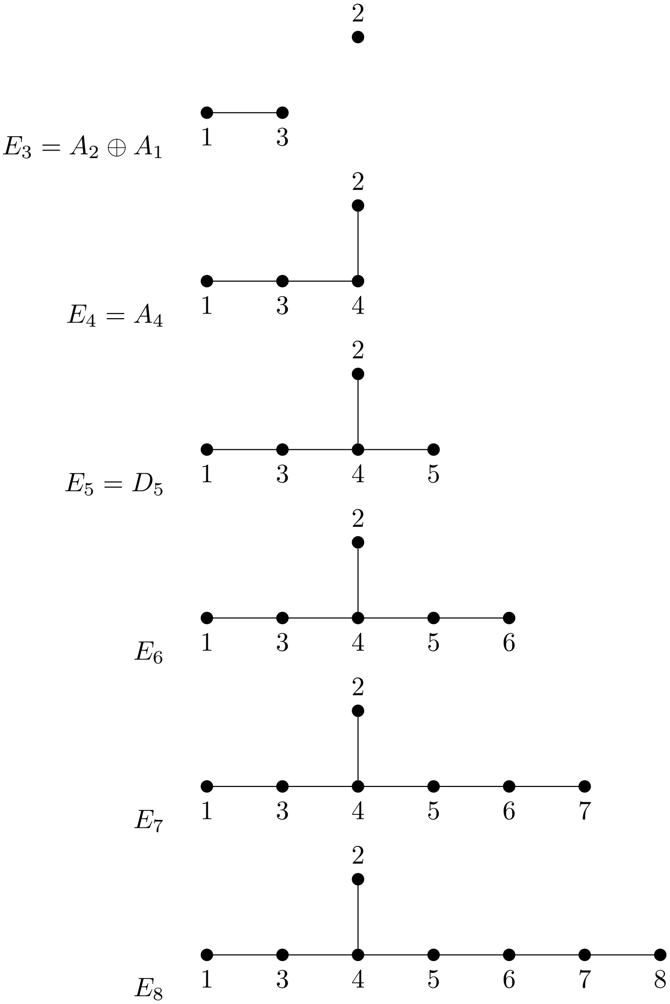


Figure 2. The Dynkin diagrams of types E_3 to E_8.

Corollary A.3. *For $n \geq 2$, the maximal semisimple compact Lie subalgebra of the Clifford algebra $C(\mathbb{R}^n, q)$ is isomorphic to the following Lie algebra:*
(a) $\mathfrak{so}(2^{\frac{n}{2}})$ *if* $n \equiv 0 \pmod 8$,
(b) $\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})$ *if* $n \equiv 1 \pmod 8$,
(c) $\mathfrak{so}(2^{\frac{n}{2}})$ *if* $n \equiv 2 \pmod 8$,
(d) $\mathfrak{su}(2^{\frac{n-1}{2}})$ *if* $n \equiv 3 \pmod 8$,
(e) $\mathfrak{sp}(2^{\frac{n-2}{2}})$ *if* $n \equiv 4 \pmod 8$,
(f) $\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}})$ *if* $n \equiv 5 \pmod 8$,
(g) $\mathfrak{sp}(2^{\frac{n-2}{2}})$ *if* $n \equiv 6 \pmod 8$,
(h) $\mathfrak{su}(2^{\frac{n-1}{2}})$ *if* $n \equiv 7 \pmod 8$.

A.4. **A lower bound on the dimension of a subalgebra.**

Definition A.5. For $n \geq 3$ let $\mathfrak{m}$ be the Lie subalgebra of $C(\mathbb{R}^n, q)$ generated by $v_1v_2v_3$ and by v_iv_{i+1}, $1 \leq i < n$.

Lemma A.6. *Let $n \geq 3$. Then $\mathfrak{m}$ contains all products of the form $v_{j_1}v_{j_2}\cdots v_{j_k}$ for $2 \leq k \leq n$ and $k \equiv 2, 3 \pmod 4$ with pairwise distinct $j_t \in \{1, \ldots, n\}$, with the possible exception of $v_1v_2\cdots v_n$ if $n \equiv 3 \pmod 4$.*

Proof. It is well known that all products $v_{j_1}v_{j_2}$, $j_1 \neq j_2$, are contained in $\mathfrak{m}$: Indeed, $\Lambda^2\mathbb{R}^n \cong \mathfrak{so}(n)$ (cp., e.g., [23, Prop. 6.1]) is generated as a Lie algebra by the v_iv_{i+1}, $1 \leq i < n$ (cp., e.g., [2, Thm. 1.31] and Theorem 1.3 of the main text).

Moreover, for pairwise distinct j_t, $1 \leq t \leq k+1$, one has

$$[v_{j_1}v_{j_2}, v_{j_2}v_{j_3}\cdots v_{j_{k+1}}] = 2v_{j_1}v_{j_3}\cdots v_{j_{k+1}}.$$

Since re-ordering of the factors simply yields scalar multiples, this shows inductively that, as long as $k+1 \leq n$, once an arbitrary factor of the form $v_{j_1}v_{j_2}\cdots v_{j_k}$ is contained in the Lie subalgebra, all factors of that form are contained in the Lie subalgebra. This statement is also true in the situation $k = n$, because in that case all factors of that form are scalar multiples of one another.

We prove the claim of the lemma by induction over k. For $k = 2$ and $k = 3$, this is obvious. Suppose the claim holds for $k \equiv 3 \pmod 4$, so that the next value for k to consider is $k + 3 \equiv 2 \pmod 4$. By induction hypothesis $v_4v_5\cdots v_{k+3} \in \mathfrak{m}$ and

$$0 \neq [v_1v_2v_3, v_4v_5\cdots v_{k+3}] = 2v_1v_2v_3v_4\cdots v_{k+3}.$$

If on the other hand the claim holds for $k \equiv 2 \pmod 4$, then the next value for k to consider is $k+1 \equiv 3 \pmod 4$. If $k+2 \leq n$, then by induction hypothesis $v_3v_4\cdots v_{k+2} \in \mathfrak{m}$ and

$$0 \neq [v_1v_2v_3, v_3v_4\cdots v_{k+2}] = 2v_1v_2v_4\cdots v_{k+2}.$$

That is, the presence of all elements of the form $v_{j_1}v_{j_2}v_{j_3}$ with pairwise distinct $j_t \in \{1, \ldots, n\}$ inductively allows us to construct all elements of the form $v_{j_1}v_{j_2}\cdots v_{j_k}$ for $k \equiv 2, 3 \pmod 4$ with pairwise distinct $j_t \in \{1, \ldots, n\}$ for all

$k \leq n$, with the possible exception of the situation $k = n \equiv 3 \pmod 4$, as the element v_{k+2} does not exist in that case. ☐

Remark A.7. It will turn out later, as a consequence of the proof of Theorem A based on dimension arguments, that the above elements in fact generate $\mathfrak{m}$ as an $\mathbb{R}$-vector space and that for $n \equiv 3 \pmod 4$ the element $v_1 v_2 \cdots v_n$ indeed is not contained in $\mathfrak{m}$, unless of course $n = 3$.

Definition A.8. For $k \in \{0,1,2,3\}$, let

$$\delta_k : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \sum_{\substack{i=0, \\ i \equiv k \,(\mathrm{mod}\, 4)}}^{n} \binom{n}{i}.$$

Consequence A.9. *Let $n \geq 3$. Then*

$$\dim \mathfrak{m} \geq \begin{cases} \delta_2(n) + \delta_3(n) & \text{if } n \not\equiv 3 \pmod 4, \\ \delta_2(n) + \delta_3(n) - 1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

A.10. **Combinatorics of binomial coefficients.** We now turn the lower bound from Consequence A.9 into a numerically explicit bound by deriving a closed formula in n for the functions δ_k.

Proposition A.11. *Let $n \in \mathbb{N}$ and $k \in \{0,1,2,3\}$.*

(a) *If $n \equiv 0 \pmod 4$, then*

$$\delta_k(n) = \begin{cases} 2^{n-2} & \text{for } k \in \{1,3\}, \\ 2^{n-2} + (-1)^{\frac{n}{4}+\frac{k}{2}} 2^{\frac{n}{2}-1} & \text{for } k \in \{0,2\}. \end{cases}$$

(b) *If $n \equiv 1 \pmod 4$, then*

$$\delta_k(n) = \begin{cases} 2^{n-2} + (-1)^{\frac{n-1}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{0,1\}, \\ 2^{n-2} - (-1)^{\frac{n-1}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{2,3\}. \end{cases}$$

(c) *If $n \equiv 2 \pmod 4$, then*

$$\delta_k(n) = \begin{cases} 2^{n-2} & \text{for } k \in \{0,2\}, \\ 2^{n-2} + (-1)^{\frac{n-2}{4}+\frac{k-1}{2}} 2^{\frac{n}{2}-1} & \text{for } k \in \{1,3\}. \end{cases}$$

(d) *If $n \equiv 3 \pmod 4$, then*

$$\delta_k(n) = \begin{cases} 2^{n-2} - (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{0,3\}, \\ 2^{n-2} + (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}} & \text{for } k \in \{1,2\}. \end{cases}$$

Proof. For $a, n \in \mathbb{N}$ the binomial theorem implies

$$(1 + i^a)^n = \sum_{k=0}^{3} i^{ak} \delta_k(n),$$

where $i \in \mathbb{C}$ denotes the imaginary unit. Evaluation of this formula for $a \in \{0,1,2,3\}$ yields the following system of four identities:

$$\delta_0(n) + \delta_1(n) + \delta_2(n) + \delta_3(n) = 2^n, \tag{2}$$

$$\delta_0(n) + i\delta_1(n) - \delta_2(n) - i\delta_3(n) = (1+i)^n = 2^{\frac{n}{2}} \cdot e^{\frac{n2\pi i}{8}}, \tag{3}$$

$$\delta_0(n) - \delta_1(n) + \delta_2(n) - \delta_3(n) = 0, \tag{4}$$

$$\delta_0(n) - i\delta_1(n) - \delta_2(n) + i\delta_3(n) = (1-i)^n = 2^{\frac{n}{2}} \cdot e^{-\frac{n2\pi i}{8}}. \tag{5}$$

These four identities imply

$$\delta_0(n) + \delta_2(n) = 2^{n-1} \qquad \text{by } ((2)+(4))/2, \tag{6}$$

$$\delta_0(n) - \delta_2(n) = 2^{\frac{n-2}{2}}(e^{\frac{n2\pi i}{8}} + e^{-\frac{n2\pi i}{8}}) \qquad \text{by } ((3)+(5))/2, \tag{7}$$

$$\delta_1(n) + \delta_3(n) = 2^{n-1} \qquad \text{by } ((2)-(4))/2, \tag{8}$$

$$\delta_1(n) - \delta_3(n) = -2^{\frac{n-2}{2}} i(e^{\frac{n2\pi i}{8}} - e^{-\frac{n2\pi i}{8}}) \qquad \text{by } ((3)-(5))/(2i). \tag{9}$$

One readily computes $\delta_0(n)$, $\delta_2(n)$ from (6), (7) and $\delta_1(n)$, $\delta_3(n)$ from (8), (9). □

Combining this with Consequence A.9 yields the following result.

Consequence A.12. *Let $n \in \mathbb{N}$ and $n \geq 2$.*

(a) *If $n \equiv 0 \pmod 8$, then*

$$\begin{aligned}\dim \mathfrak{m} \geq \delta_2(n) + \delta_3(n) &= 2^{n-2} - 2^{\frac{n}{2}-1} + 2^{n-2} = 2^{\frac{n-2}{2}}(2^{\frac{n}{2}} - 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{so}(2^{\frac{n}{2}})).\end{aligned}$$

(b) *If $n \equiv 1 \pmod 8$, then*

$$\begin{aligned}\dim \mathfrak{m} \geq \delta_2(n) + \delta_3(n) &= 2(2^{n-2} - 2^{\frac{n-3}{2}}) = 2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}} - 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})).\end{aligned}$$

(c) *If $n \equiv 2 \pmod 8$, then*

$$\begin{aligned}\dim \mathfrak{m} \geq \delta_2(n) + \delta_3(n) &= 2^{n-2} + 2^{n-2} - 2^{\frac{n}{2}-1} = 2^{\frac{n-2}{2}}(2^{\frac{n}{2}} - 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{so}(2^{\frac{n}{2}})).\end{aligned}$$

(d) *If $n \equiv 3 \pmod 8$, then*

$$\begin{aligned}\dim \mathfrak{m} + 1 \geq \delta_2(n) + \delta_3(n) &= 2^{n-2} + 2^{\frac{n-3}{2}} + 2^{n-2} - 2^{\frac{n-3}{2}} = 2^{n-1} \\ &= \dim_{\mathbb{R}}(\mathfrak{su}(2^{\frac{n-1}{2}})) + 1.\end{aligned}$$

(e) *If $n \equiv 4 \pmod 8$, then*

$$\begin{aligned}\dim \mathfrak{m} \geq \delta_2(n) + \delta_3(n) &= 2^{n-2} + 2^{\frac{n}{2}-1} + 2^{n-2} = 2^{\frac{n-2}{2}}(2^{\frac{n}{2}} + 1) \\ &= \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-2}{2}})).\end{aligned}$$

(f) *If* $n \equiv 5 \pmod 8$, *then*

$$\begin{aligned}\dim \mathfrak{m} \geq \delta_2(n) + \delta_3(n) &= 2(2^{n-2} + 2^{\frac{n-3}{2}}) = 2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}} + 1)\\ &= \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}})).\end{aligned}$$

(g) *If* $n \equiv 6 \pmod 8$, *then*

$$\begin{aligned}\dim \mathfrak{m} \geq \delta_2(n) + \delta_3(n) &= 2^{n-2} + 2^{n-2} + 2^{\frac{n}{2}-1} = 2^{\frac{n-2}{2}}(2^{\frac{n}{2}} + 1)\\ &= \dim_{\mathbb{R}}(\mathfrak{sp}(2^{\frac{n-2}{2}})).\end{aligned}$$

(h) *If* $n \equiv 7 \pmod 8$, *then*

$$\begin{aligned}\dim \mathfrak{m} + 1 \geq \delta_2(n) + \delta_3(n) &= 2^{n-2} - 2^{\frac{n-3}{2}} + 2^{n-2} + 2^{\frac{n-3}{2}} = 2^{n-1}\\ &= \dim_{\mathbb{R}}(\mathfrak{su}(2^{\frac{n-1}{2}})) + 1.\end{aligned}$$

A.13. **Generalized spin representations of the split real E_n series and the resulting quotients.** The example of a generalized spin representation of the maximal compact subalgebra of the split real Kac–Moody Lie algebra of type E_{10} described in [4] and [7] (see Example 3.2 in the main text) generalizes directly to the whole E_n series as follows.

Let $n \in \mathbb{N}$, let $\mathfrak{g}$ be the split real Kac–Moody Lie algebra of type E_n, let $\mathfrak{k}$ be its maximal compact subalgebra, and let X_i, $1 \leq i \leq n$, be the Berman generators of $\mathfrak{k}$ (cp. [2, Thm. 1.31] and Theorem 1.3 in the main text) enumerated in Bourbaki style as shown in Figure 1, i.e., X_1, X_3, $X_4, \ldots, X_n$ belong to the A_{n-1} subdiagram, generating $\mathfrak{so}(n)$, and X_2 to the additional node. As in Section A.1 let q be the standard positive definite quadratic form on $\mathbb{R}^n$ and let $C = C(\mathbb{R}^n, q)$ be the corresponding Clifford algebra, considered as a Lie algebra.

Proposition A.14. *Let* $n \geq 3$. *The assignment*

$$X_j \mapsto \begin{cases} \frac{1}{2}v_1v_2 & \text{for } j = 1,\\ \frac{1}{2}v_1v_2v_3 & \text{for } j = 2,\\ \frac{1}{2}v_{j-1}v_j & \text{for } 3 \leq j \leq n \end{cases}$$

defines a Lie algebra homomorphism ρ *from* $\mathfrak{k}$ *to the Lie subalgebra* $\mathfrak{m}$ *of* C *generated by* $v_1v_2v_3$ *and by* v_iv_{i+1}, $1 \leq i < n$, *called the* standard generalized spin representation of $\mathfrak{k}$.

Proof. The proof is based on the criterion established in Remark 3.7 and is exactly the same as in the E_{10} case discussed in Example 3.2. ☐

Proof of Theorem A. By Theorem 3.14 and since E_n is simply laced and connected for $n \geq 4$, the image $\mathfrak{m}$ of ρ is semisimple and compact. By Lemma A.6 and Consequence A.12, the dimension $\dim_{\mathbb{R}}(\mathfrak{m})$ is at least as large as the dimension of the maximal semisimple compact Lie subalgebra of C as given in Corollary A.3. The claim follows. ☐

Proof of Theorem B. Let $\mathfrak{g}$ be a semisimple split real Lie algebra of type $E_4 = A_4$, $E_5 = D_5$, E_6, E_7 or E_8, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be its Iwasawa decomposition. Since $\dim_{\mathbb{R}}(\mathfrak{k}) = \dim_{\mathbb{R}}(\mathfrak{n})$, from the combinatorics of the respective root system we conclude that the maximal compact Lie subalgebra $\mathfrak{k}$ has dimension

$$\begin{aligned}
10 &= \frac{4 \cdot 5}{2} = \frac{2^{\frac{4}{2}} \cdot (2^{\frac{4}{2}} + 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{sp}(2)) = \dim_{\mathbb{R}}(\mathfrak{so}(5)) && \text{if } n = 4,\\
20 &= 2 \cdot 10 = \dim_{\mathbb{R}}(\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)) = \dim_{\mathbb{R}}(\mathfrak{so}(5) \oplus \mathfrak{so}(5)) && \text{if } n = 5,\\
36 &= 4 \cdot 9 = 2^{\frac{6-2}{2}} \cdot (2^{\frac{6}{2}} + 1) = \dim_{\mathbb{R}}(\mathfrak{sp}(4)) && \text{if } n = 6,\\
63 &= 2^6 - 1 = \dim_{\mathbb{R}}(\mathfrak{su}(8)) && \text{if } n = 7,\\
120 &= \frac{16 \cdot 15}{2} = \frac{2^{\frac{8}{2}} \cdot (2^{\frac{8}{2}} - 1)}{2} = \dim_{\mathbb{R}}(\mathfrak{so}(16)) && \text{if } n = 8.
\end{aligned}$$

For $n \geq 4$ we may now apply Theorem A and deduce that the standard generalized spin representation ρ has to be injective in these cases.

This leaves the case $E_3 = A_2 \oplus A_1$. Since this diagram is not irreducible, Theorem 3.14 only implies that $\operatorname{im}(\rho) = \mathfrak{m}$ is compact but not that it is semisimple (and, indeed, it is not). However, $n = 3$ is also an exceptional case for Lemma A.6: In this case $\dim_{\mathbb{R}}(\mathfrak{m}) = 4$, as v_1v_2, v_1v_3, v_2v_3, $v_1v_2v_3$ form an $\mathbb{R}$-basis of $\mathfrak{m}$. On the other hand, the Clifford algebra C is isomorphic to $M(2, \mathbb{C})$, hence $\mathfrak{k} \cong \mathfrak{u}(2)$, and this has dimension 4. Thus ρ is also injective when $n = 3$. The claim follows. ☐

Acknowledgements for the appendix. We thank Klaus Metsch for pointing out to us the identity of sums of binomial coefficients in Proposition A.11 and one of the referees for relaying the elegant version of its proof given in this appendix. This research has been partially funded by the EPRSC grant EP/H02283X. The second author gratefully acknowledges the hospitality of the IHES at Bures-sur-Yvette and of the Albert Einstein Institute at Golm.

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Received March 20, 2014; accepted March 23, 2015

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