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Equivariant K-homology of Bianchi groups in the case of nontrivial class group

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Abstract. We compute the equivariant K-homology of the groups PSL_2 of imaginary quadratic integers with trivial and nontrivial class group. This was done before only for cases of class number one.

We rely on reduction theory in the form of the Γ-CW-complex defined by Flöge. We show that the difficulty arising from the nonproper action of Γ on this complex can be overcome by considering a natural short exact sequence of C^*-algebras associated to the universal covering of the Borel–Serre compactification of the locally symmetric space associated to Γ. We use rather elementary C^*-algebraic techniques including a slightly modified Atiyah-Hirzebruch spectral sequence as well as several six-term sequences.

This computes the K-theory of the reduced and full group C^*-algebras of the Bianchi groups.

1. Introduction

This paper is concerned with the Bianchi groups, i.e. the class of groups defined by PSL_2 of the ring of integers of an imaginary quadratic number field $\mathbb{Q}[\sqrt{-m}]$ where m is a product of different primes. Any such group acts as a group of orientation-preserving isometries of hyperbolic three-space $\mathscr{H}$ by means of the right-coset identification $\mathscr{H} \cong \mathrm{PSL}(2,\mathbb{C})/\,\mathrm{PSU}(2)$. We will compute the left-hand side of the Baum–Connes-conjecture for these groups. Our main result is Theorem 4.1 which accomplishes the calculation in the case $m = 5$.

There is vast literature on the Bianchi groups [8, 6, 14] and the Baum–Connes conjecture [16]. The latter is concerned with the difficult problem of classifying stable isomorphism classes of finitely generated projectives modules (K-theory) over the reduced C^*-algebra of Γ. This C^*-algebra is defined as the closure in the operator norm of the left regular representation of Γ on $\ell^2\Gamma$. The conjecture asserts that the K-theory of $C^*_{red}\Gamma$ is isomorphic to equivariant K-homology with Γ-compact supports of a universal proper Γ-space $\underline{E}\Gamma$. We shall use the "official" definition

$$R\,\mathrm{KK}_*(\mathscr{C}_0(\underline{E}\Gamma),\mathbb{C}) := \lim_{\substack{X\subset \underline{E}\Gamma:\\ \Gamma\backslash X \text{compact}}} \mathrm{KK}^{\Gamma}_*(\mathscr{C}_0(X),\mathbb{C})$$

of equivariant K-theory with Γ-compact supports as in [2], where $\underline{E}\Gamma$ is a classifying space for proper actions of Γ, and KK^{Γ} is equivariant bivariant K-theory as defined in [13]. In the present context of a closed subgroup Γ of a semisimple Lie group G, the symmetric space $G/K = \mathscr{H}$ is a canonical model of a universal space $\underline{E}\Gamma$ [2], where K is a maximal compact subgroup of G. The conjecture states that the assembly map

$$R\,\mathrm{KK}_*(\mathscr{C}_0(\underline{E}H),\mathbb{C}) \to \mathrm{K}_*(C^*_{red}H),$$

defined in [2], is an isomorphism for all locally compact groups H. If H is an arbitrary connected Lie group, the conjecture has been proved [4]. Most often the Plancherel formula allows to understand the structure of the left regular representation of Lie groups, whereas for arithmetic groups H the group C^*-algebra $C^*_{red}H$ has turned out to be very hard to understand. One of the reasons is that for many arithmetic groups, the C^*-algebra is not of type I, i.e. there are irreducible representations not contained in the operator ideal of compact operators.

Hence, the knowledge of the K-theory of the C^*-algebra yields valuable information which seems not to be accessible by more elementary or more explicit approaches. Furthermore, the Baum–Connes conjecture is, so far, the method of choice for computing the K-theory whenever the structure of the C^*-algebra is unknown.

It often turns out, however, that the left-hand side is not immediately calculable either. One of the few previous explicit computations of equivariant K-homology of an arithmetic group is [23] which computes the left-hand-side for the group $\Gamma = \mathrm{SL}(3,\mathbb{Z})$; however, it is unknown if the assembly map is surjective in this case. In contrast, Julg and Kasparov verified the Baum–Connes conjecture for all discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ [11]. Since $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{PSO}(3,1)$, this readily implies that the assembly map is an isomorphism for all Bianchi groups. Therefore, the Bianchi groups are arguably the most interesting arithmetic groups for which complete knowledge of $\mathrm{K}_*(C^*_{red}\Gamma)$ is available.

The canonical way to understand arithmetic groups is via actions on retracts of the symmetric space, as provided by reduction theory. Here, we shall use (a suitable subset of) Flöge's CW-complex [9], a union of a two-dimensional retract of hyperbolic three-space and a countable subset of the Satake (spherical) boundary of $\mathscr{H}$. These boundary points of the complex are the Γ-orbits of so-called singular vertices of the Flöge complex, and such orbits are in bijection with the nontrivial elements of the class group [25]. The singular points are clearly visible in various realizations of fundamental domains in [9] and [20]. The contractibility of the Flöge complex has been shown by Rahm and the author in [20].

The equivariant K-homology of Bianchi groups has been calculated by Rahm in [19]. However, he considers only the cases of class number one, i.e. $m = 1, 2, 3, 7, 11, 19, 43, 67, 163$. The reason for this restriction is that in these cases there are no singular points. However, for higher class numbers the Flöge complex is not proper anymore, as the stabilizers of the singular points are parabolic. An algorithmic approach for computation of the complex has been introduced, described and exploited for the computation of the integral homology by A. Rahm and the author in [20]. Furthermore, a program for the calculation of fundamental domains in the computer algebra system Pari/GP [26] has been published as part of Rahm's PhD thesis [18].

As the presence of singular points in the Flöge complex leads to nonproper actions, the Flöge complex is not a model for the universal classifying space $\underline{E}\Gamma$ of proper actions. It is therefore not immediately suitable for calculation of equivariant K-homology for higher class numbers. Moreover, the Flöge complex is not locally finite since there are infinitely many edges emanating from the singular points. So, there is no obvious locally compact topology on it, which leads to severe difficulties in defining an associated commutative C^*-algebra.

There is a general construction for turning arbitrary, possibly nonproper, Γ-CW-complexes into proper ones [15] which could in principle be applied to the Flöge complex; however, the construction is not cofinite in general and therefore difficult to use for computational purposes.

In the present paper, we show how to overcome these difficulties. The approach pursued in this paper is more akin of classical C^*-algebraic techniques. The four main ingredients are: reduction theory in the form of the Flöge complex, the existence of the Borel–Serre compactification, a certain amount of Kasparov theory, and the Atiyah–Hirzebruch spectral sequence.

Let us give a short summary of the paper. Section 2 introduces the topological objects under study. We identify the universal covering of the Borel–Serre compactification of the locally symmetric space associated to Γ as a universal Γ-space (Lemma 2.8). Note that for torsion-free subgroups of Bianchi groups, the Borel–Serre compactification of the quotient $\Gamma\backslash\mathscr{H}$ (which is a manifold) has already been mentioned in [10]. There is an extension (2) of C^*-algebras naturally associated to the inclusion of the boundary component into this universal space. We use this to show that the equivariant K-homology of Γ fits into a six-term exact sequence (5) relating it to topological K-homology of a disjoint sum of tori and the Kasparov K-homology of the crossed product $\Gamma \ltimes \mathscr{C}_0(\mathscr{H})$. The latter is determined (Lemma 2.11) by the K-homology of the subset $X_\circ := X \cap \mathscr{H}$ of the Flöge complex X. It is necessary to work with $X_\circ$ instead of X in order to overcome the problem that X is not locally finite. The space $X_\circ$ is naturally endowed with a locally compact topology, so it is possible to associate a well-behaved commutative C^*-algebra to it. The goal of Section 3 is to introduce the spectral sequence used to compute the K-homology of $X_\circ$. In principle, the Atiyah–Hirzebruch spectral sequence starting from Bredon homology associated to a group action on a complex is the classical tool to compute its equivariant K-homology. That spectral sequence is explained in [16] and was used in [23, 18, 19]. Here, we have to consider a slightly more general form of the spectral sequence (Lemma 3.5) in order to be able to treat the space $X_\circ$ instead of X. This generalization is achieved by identifying the Atiyah–Hirzebruch spectral sequence for the equivariant K-homology of a complex as the Schochet spectral sequence associated to a filtration by closed ideals of a C^*-algebra. In our case, these ideals are obtained from the intersections of the skeleta of X with $\mathscr{H}$. At this point, all technical difficulties are resolved, and we can illustrate the computation by means of the concrete example $m = 5$ in Section 4. This leads to the main Theorem 4.1.

The Bianchi groups, although far from being amenable, are K-amenable [5], as they are closed subgroups of the K-amenable group $\mathrm{SO}(3,1)$ [12]. As a consequence, there is a KK-equivalence between the reduced and full group C^*-algebra of Γ. Therefore, by calculating the K-theory of the former we simultaneously calculate that of the latter. Hence, the Bianchi groups may also be the most interesting lattices for which complete knowledge of the K-theory of the full algebra is now available.

2. Flöge's complex, equivariant K-homology and the Borel–Serre boundary

2.1. **Flöge's complex.** Briefly, the Flöge complex $X^\bullet$ is defined as follows, see [9, 20] for details and references. Let $m \in \mathbb{N}$ be square-free, let $R = \mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}$ and let $\Gamma = \mathrm{PSL}(2, R)$. Denote the hyperbolic three-space by $\mathscr{H} = \mathbb{C} \times \mathbb{R}_+^*$. The space $\mathscr{H}$ is a homogeneous space under the Lie group $G = \mathrm{PSL}(2, \mathbb{C})$ acting by Möbius transformations (the formula is given in [20], for instance). The Satake (or spherical) boundary $b\mathscr{H}$ of $\mathscr{H}$ is $\mathbb{C}P^1$, and the action of G, and hence that of Γ, extend continuously to actions on the boundary by Möbius transformations [22, Sec. 12]. A point $D \in \mathbb{C}P^1 - \{\infty\}$ is called a *singular point* if for all $c, d \in R, c \neq 0, Rc + Rd = R$, we have $|cD - d| \geqslant 1$.

We shall denote the union of all orbits of singular points by S. Note that the point $\infty \in \mathbb{C}P^1$ is never in the orbit of a singular point. The set S can either be viewed as a subset of $\mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{R}_{\geqslant 0}$, or of $\mathbb{C}P^1 - \infty$, or of $b\mathscr{H} - \infty$.

The set S is a subset of the set of cusps of Γ as defined in [25]. Serre considers the set P of rational boundary points defined as $\mathbb{Q}[\sqrt{-m}] \cup \{\infty\}$ as a subset of the boundary viewed as $\mathbb{C}P^1$. The action of Γ on P falls into orbits which are in natural bijection with the class group, and the orbit corresponding to the trivial element is the orbit of ∞, whereas the set S is the union of all orbits corresponding to nontrivial elements of the class group.

Furthermore, one considers the union of all hemispheres

$$\left\{(z, r) \;\middle|\; \left|z - \frac{\lambda}{\mu}\right|^2 + r^2 = \frac{1}{|\mu|^2}\right\} \subset \mathscr{H},$$

for any two μ, λ with $R\mu + R\lambda = R$, as well as the "space above the hemispheres"

$$B := \{(z, r) \mid |cz - d|^2 + r^2 |c|^2 \geqslant 1 \text{ for all } c, d \in R, c \neq 0 \\ \text{such that } Rc + Rd = R\}.$$

Let $|\cdot|$ denote the modulus in $\mathbb{C}$.

Proposition and Definition 2.2 ([9]). *As a set, $\widehat{\mathscr{H}} \subset \mathbb{C} \times \mathbb{R}^{\geqslant 0}$ is the union $\mathscr{H} \cup S$ (where we omit, for simplicity, the standard identification $\mathbb{C} \times \{0\} \cong \mathbb{C}P^1 - \infty$). The topology of $\widehat{\mathscr{H}}$ is generated by the topology of $\mathscr{H}$ together with the following neighborhoods of the translates $D \in S$ of singular points:*

$$\widehat{U}_\epsilon(D) := \{D\} \cup \left\{(z, r) \in \mathscr{H} \;\middle|\; |z - D|^2 + \left(r - \frac{\epsilon}{2}\right)^2 < \frac{\epsilon^2}{4}\right\}.$$

This definition makes S a closed subset of $\widehat{\mathscr{H}}$.

There is a retraction ρ from $\widehat{\mathscr{H}}$ onto the set $X := S \cup \Gamma \cdot \partial B \subset \widehat{\mathscr{H}}$ of the union S with all Γ-translates of ∂B, i.e. there is a continuous map $\rho : \widehat{\mathscr{H}} \to X$ such that $\rho(p) = p$ for all $p \in X$. The set X admits a natural structure as a cell complex $X^\bullet$, such that Γ acts cellularly on $X^\bullet$. We shall refer to the complex thus defined, as well as to its structure as a Γ-subset of $\widehat{\mathscr{H}}$, as to the "Flöge complex".

We shall not make use of any topology on X, only of the structure as a subset of $\widehat{\mathscr{H}}$ and of the combinatorial structure. Since B comprises Bianchi's fundamental polyhedron (which is called D in [9]), we have $\Gamma \cdot B = \mathscr{H}$ and hence this definition of X coincides with Flöge's original definition as the Γ-closure of $B \cup \{\text{singular points}\}$. Note that the topology of $\widehat{\mathscr{H}}$ is not the one inherited from the Satake (spherical) compactification of $\mathscr{H}$. However, $\widehat{\mathscr{H}}$ coincides on $\mathscr{H}$ with the usual topology; furthermore, it is path-connected, locally path-connected and simply connected [9, Satz 1], and contractible [20, Lem. 8], as is the cell complex $X^\bullet$ [20, Cor. 7].

Since X is not locally finite at the points of S, we shall not work with X directly, but only with the intersection $X \cap \mathscr{H} = X - S$ of X with $\mathscr{H}$. On $X - S$, the cellular topology coincides with the topology inherited from $\mathscr{H}$, and is locally compact (unlike that of $\widehat{\mathscr{H}}$), whence there are no difficulties in associating a C^*-algebra $\mathscr{C}_0(X - S)$. Although $X - S$ is not a complex anymore, the C^*-algebra still possesses a filtration by closed ideals, defined by functions that vanish on the intersections of the skeleta of X with $\mathscr{H}$. The associated Schochet-spectral sequence will be one of the key tools in the calculation of the equivariant K-homology of Γ. In the following, the punctured cell complex $X - S = X \cap \mathscr{H}$ endowed with the subset topology of $\mathscr{H}$ shall be denoted by $X_\circ$, and its punctured skeleta by $X_\circ^q := X^q \cap X_\circ$.

It is important to note that any two edges adjacent to a common singular point in X define half-open intervals that are *disjoint* from each other as subsets of $X_\circ$. Furthermore, the space $X_\circ$ is, as subset of $\mathscr{H}$, locally compact. This contrasts the failure of local finiteness of X. As a warning, the space $X_\circ$ does not necessarily have the same $\mathrm{KK}_*^\Gamma(\mathscr{C}_0(-),\mathbb{C})$-theory as the subcomplex of X that consists of all cells that do not touch a singular point although these spaces are equivariantly homotopy equivalent. The reason is as follows. These homotopies are not proper since the singular points are at infinity, for the geometry of $X_\circ$, but $\mathrm{KK}_*^\Gamma(\mathscr{C}_0(-),\mathbb{C})$ is only invariant under proper equivariant homotopies (see end of Section 2.6).

2.3. **The Borel–Serre boundary.** Apart from the Flöge construction $\widehat{\mathscr{H}}$, we shall need another enlargement of $\mathscr{H}$, the Borel–Serre construction $\mathscr{H}(P)$ for the set $P = S \cup \Gamma \cdot \{\infty\}$ of rational boundary points. Very roughly speaking, the space $\mathscr{H}(P)$ is obtained from $\mathscr{H}$ by gluing copies of $\mathbb{R}^2$ onto each $D \in P$ (whereas $\widehat{\mathscr{H}}$ was obtained from $\mathscr{H}$ by merely gluing copies of a point onto each $D \in S$). Let us recall Serre's setup and notation of the Borel–Serre boundary for linear algebraic groups of $\mathbb{R}$-rank one [25, Appendix 1].

Serre's notation translates as follows. The space $\mathscr{H}$ is hyperbolic space, $b\mathscr{H}$ is the ordinary spherical (Satake) boundary of $\mathscr{H}$, D is a translate of a singular point or of the point ∞, the subgroup Q_D is its stabilizer inside G. We are going to view $\mathscr{H}$ as a space acted upon by G from the right; thus, there is the natural identification $\mathscr{H} \cong K\backslash G$. The group Q_D is a minimal parabolic subgroup, and the singular points defined by Flöge naturally embed into the Satake boundary. For instance, consider the simplest choice where D is the origin of the Poincaré plane. Then $Q_D \cap \Gamma$ is the set of upper triangular matrices in Γ, hence isomorphic to $R = \mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}$. Even though there are several orbits of singular points, all stabilizer groups $Q_D \cap \Gamma$ are free abelian of rank two [20].

The group N_D is conjugated inside $G = \mathrm{PSL}(2,\mathbb{C})$ to the group $\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$ whence Q_D/N_D is isomorphic to $\mathbb{C}^*$, the multiplicative group of nonzero complex numbers. The space $\mathscr{H}_D$ is defined as the union of hyperbolic space with a boundary component Y_D, defined as the space of rank one-tori of Q_D. In our case, all Y_D are diffeomorphic to $\mathbb{R}^2$. More specifically, we obtain for $D = \infty$, the regular cusp at infinity of the Riemann sphere:

$$\begin{aligned} Q_\infty &= \left\{ \left(\begin{smallmatrix} z & * \\ 0 & z \end{smallmatrix}\right) \mid z \in \mathbb{C} \right\} \\ T_{\mathrm{PSU}(2),\infty} &:= \left\{ \left(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right) \mid t \in \mathbb{R}_+^* \right\} \\ Y_\infty &= \left\{ \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \mid z \in \mathbb{C} \right\}. \end{aligned}$$

Furthermore, there is a unique fixed point S_K of the Cartan involution on Y_D defined by K. This defines an Iwasawa decomposition $G = K \cdot A_K \cdot N_D$ associated to the subgroup K, where A_K is the neutral connected component of S_K, and $\cdot\cdot$ denotes the ordinary multiplication in the group G. The group A_K identifies with $\mathbb{R}_+^*$ in a canonical way (induced by the positive root of S_K with respect to the order relation coming from N_D), and for $t \in \mathbb{R}_+^*$, the associated element in A_K is denoted by t_K. These observations permit to topologize the space $\mathscr{H}_D$ by means of the family of bijective maps $f_K : \mathbb{R}_+ \times N_D \to \mathscr{H}_D$, defined by

$$f_K(t,n) = \begin{cases} (K) \cdot t_K \cdot n \in K\backslash G \cong \mathscr{H}, & t > 0,\, n \in N_D, \\ n^{-1} \cdot S_K \cdot n \in Y_D, & t = 0,\, n \in N_D. \end{cases}$$

where K is a maximal compact subgroup of G. These maps are compatible for different choices of K and therefore define a unique structure of manifold with boundary on $\mathscr{H}_D$, independent of K.

Let now P be the set of cusps of Γ, i.e. the set of orbits of singular points together with the orbits of the trivial cusp at ∞. In Serre's terminology, the space $\mathscr{H}(P)$ is the union

$$\mathscr{H}(P) = \mathscr{H} \cup \bigcup_{D \in P} Y_D.$$

defined by gluing the spaces $\mathscr{H}_D$ together along their common intersection $\mathscr{H}$. Hence, as a set, $\mathscr{H}(P)$ is the disjoint union $\mathscr{H} \cup (P \times \mathbb{R}^2)$. Serre shows that $\mathscr{H}(P)/\Gamma$ is compact, and called the Borel–Serre compactification of $\mathscr{H}/\Gamma$. The space $\mathscr{H}(P)$ is its universal covering.

Of course, it is also possible to consider a slightly smaller Γ-space, defined by considering only the subset S of P,

$$\mathscr{H}(S) = \mathscr{H} \cup \bigcup_{D \in S} Y_D \subset \mathscr{H}(P).$$

in order to obtain a space that surjects onto $\widehat{\mathscr{H}}$. In fact, one has

Observation 2.4. There is a natural map $\mathscr{H}(S) \to \widehat{\mathscr{H}}$ defined by collapsing each boundary component to the corresponding translate of a singular point. This map is continuous and surjective.

This is proved by checking directly that preimages of the open sets that define the topology on $\widehat{\mathscr{H}}$ are open in $\mathscr{H}(S)$. The relation between the spaces $\mathscr{H}, \widehat{\mathscr{H}}, \mathscr{H}(S)$ and $\partial\mathscr{H}(S)$ can be summarized by the equivariant diagram

$$\begin{array}{ccc} \mathscr{H}(S) & \longleftarrow & \partial\mathscr{H}(S) \\ \uparrow & \searrow & \downarrow \\ \mathscr{H} & \longrightarrow & \widehat{\mathscr{H}}. \end{array}$$

The following lemma uses the notion of amenable transformation group, discussed in detail in [1].

Lemma 2.5. *Let $C = \mathscr{C}_0(Y)$ with $Y \in \{\mathscr{H}, X_\circ, \mathscr{H} - X_\circ, \partial\mathscr{H}(P), \mathscr{H}(P), \partial\mathscr{H}(S), P\}$. Then the transformation group (Y, Γ) is amenable. In particular, the C^*-algebraic crossed product $\Gamma \ltimes C$ is nuclear and unique, i.e. the quotient map from the maximal to the reduced crossed product is an isomorphism.*

Proof. By [1, Thm. 5.3], it suffices to show amenability of the transformation group.

For $C = \mathscr{C}_0(\mathscr{H})$, write the symmetric space as a coset space $\mathscr{H} = G/K$ where K is a maximal compact subgroup of G. For two closed subgroups Γ and K of a locally compact group G, the associated transformation group $(G/K, \Gamma)$ is an amenable transformation group [1, Ex. 2.7(5)].

For $C = \mathscr{C}_0(\partial\mathscr{H}(P))$, observe that the action of Γ on $\partial\mathscr{H}(P)$ is proper in the sense that the map

$$\Gamma \times \partial\mathscr{H}(P) \to \partial\mathscr{H}(P) \times \partial\mathscr{H}(P)$$

defined by $(\gamma, x) \mapsto (\gamma x, x)$ is topologically proper because the action is free and cocompact. Any proper action defines an amenable transformation group. In fact, the functions $g_i(x, \gamma) = h(\gamma^{-1}x)$ satisfy the conditions of [1, Prop. 2.2(2)] where h is a continuous nonnegative function on $\partial\mathscr{H}(P)$ such that $\sum_{\gamma\in\Gamma} h(\gamma^{-1}x) = 1$ for all $x \in \partial\mathscr{H}(P)$. This also shows the assertion for $Y \in \{X_\circ, \mathscr{H} - X_\circ\}$.

Form the associated extension of maximal crossed products associated to the invariant ideal $\mathscr{C}_0(\mathscr{H})$ in $\mathscr{C}_0(\mathscr{H}(P))$ (for maximal crossed products, exactness is automatic). We have shown that the ideal and the quotient are nuclear (and therefore coincide with minimal crossed product). Nuclearity for $Y = \mathscr{H}(P)$ then follows from the fact that nuclearity is stable under extensions [17]. Moreover, since Γ is discrete, it has property (W) [1, Ex. 4.4(3)], and therefore also the transformation groups $(\partial\mathscr{H}(P), \Gamma)$ and $(\mathscr{H}(P), \Gamma)$ are amenable [1, Thm. 5.8].

For $C = \mathscr{C}_0(P)$, write P as the disjoint union of orbits $P = \cup P_i$ (this is a decomposition in finitely, namely k components, where k is the class number). Any crossed product splits into a direct sum over these components, so it is enough to prove the statement for each of them separately. On such a component, we can apply [1, Ex. 2.7(5)] again, this time to the ambient group Γ (which is locally compact) and the two closed subgroups given by Γ itself and the stabilizer Γ_i, since the orbit S_i is then equal to the Γ-space Γ/Γ_i, and $\Gamma_i \cong \mathbb{Z}^2$ is abelian, hence amenable. □

2.6. **A classifying space.** Lemma 2.8 below uses the following simple criterion, well-known in the literature, for a space to be universal, i.e. to serve as a model for the classifying space for proper actions for Γ. Note that whereas $\mathscr{H}$ is both a universal Γ- and G-space, the space $\mathscr{H}(P)$ is not acted upon by G.

Fact 2.7 ([7, after Def. 3])**.** Any free and proper Γ-space is universal if and only if it is H-equivariantly contractible for any finite subgroup $H \subset \Gamma$.

Lemma 2.8. *For any Γ-closed subset $S \subset P$, the universal covering $\mathscr{H}(P)$, as constructed in* [25, Appendix 1]*, of the Borel–Serre compactification of $\Gamma\backslash\mathscr{H}$ is a universal proper Γ-space.*

Proof. By construction, the action of Γ on $\partial\mathscr{H}(P)$ is free. Moreover, for each boundary component we can choose an isomorphism of its (nonpointwise) stabilizer with $\mathbb{Z}^2$ such that its action on the boundary component is equivariantly homeomorphic to the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$, hence proper. Thus, the action on $\partial\mathscr{H}(P)$ is proper. It ensues that $\mathscr{H}(P)$ is proper. Let H be a finite subgroup of Γ. The quotient $H\backslash\mathscr{H}(P)$ is an orbifold with boundary which has the same homotopy type as its interior $H\backslash\mathscr{H}$. The latter is contractible because $\mathscr{H}$ is a universal proper Γ-space [2, Sec. 2]. So $H\backslash\mathscr{H}(P)$ is contractible. Moreover, since $\mathscr{H}(P)$ is contractible, the contraction of $H\backslash\mathscr{H}(P)$ thus obtained lifts to an H-equivariant contraction of $\mathscr{H}(P)$ whence the assertion. □

In fact, the space $\mathscr{H}$ itself is the typical example of a universal proper Γ-space, so $\mathscr{H}$ and $\mathscr{H}(P)$ can both be used as models for $\underline{E}\Gamma$ and for computation of equivariant K-homology. An isomorphism $R\,\mathrm{K}_*^\Gamma(\mathscr{H}) \to R\,\mathrm{K}_*^\Gamma(\mathscr{H}(P))$ is induced by the inclusion map $\mathscr{H} \to \mathscr{H}(P)$. However, $\mathscr{H}(P)$ has, by construction, the advantage that it is cocompact unlike $\mathscr{H}$. Therefore, we can pass from $R\,\mathrm{KK}$ to KK as follows.

$$
\begin{aligned}
R\,\mathrm{KK}^\Gamma(\mathscr{C}_0(\underline{E}\Gamma), \mathbb{C}) &\cong R\,\mathrm{KK}^\Gamma(\mathscr{C}_0(\mathscr{H}(P)), \mathbb{C}) \\
&\cong \mathrm{KK}^\Gamma(\mathscr{C}_0(\mathscr{H}(P)), \mathbb{C}) \\
&\cong \mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(\mathscr{H}(P))).
\end{aligned}
\tag{1}
$$

The last isomorphism is the dual Green–Julg theorem [3, Thm. 20.2.7(b)]. Since the crossed product is nuclear, every ideal is semisplit [3, Thm. 15.8.3].

Throughout the paper, the reader should bear in mind that the notations $\mathrm{KK}^*(-,-)$ and $\mathrm{K}^*(-)$ refer to the "original" Kasparov KK-groups instead of the compact-support group $R\,\mathrm{KK}$. We shall prefer to work with the former. For a commutative C^*-algebra $\mathscr{C}_0(X)$, the group $\mathrm{K}^*(\mathscr{C}_0(X))$ is K-homology with locally finite support, rather than the usually considered group $R\,\mathrm{K}_*(X)$ with compact support as in [16], for instance. The analogous remark holds for the equivariant theory, with "Γ-compact" instead of "compact".

2.9. ***C*-extensions.*** We can now make use of six-term sequences which are available in KK, in contrast to $R\,\mathrm{KK}$. The space $X_\circ$ was defined in Section 2.1. The sequences

$$
\begin{gathered}
0 \to \mathscr{C}_0(\mathscr{H} - X_\circ) \to \mathscr{C}_0(\mathscr{H}) \to \mathscr{C}_0(X_\circ) \to 0 \\
0 \to \mathscr{C}_0(\mathscr{H}) \to \mathscr{C}_0(\mathscr{H}(P)) \to \mathscr{C}_0(\partial\mathscr{H}(P)) \to 0,
\end{gathered}
\tag{2}
$$

defined by the respective evaluation maps, are exact and of course equivariant. Note that in the latter sequence we choose to work with $\mathscr{H}(P)$ instead of $\mathscr{H}(S)$ because only $\mathscr{H}(P)$ is cocompact, whence leading to equivariant K-homology.

Note that none of these algebras have a unit. Of each short exact sequence, we can take maximal crossed products by Γ, which is an exact functor. As we have shown in Lemma 2.5 that they coincide with the reduced ones, we arrive at a short exact sequence of reduced crossed products with the sequences (2).

These induce exact six-term sequences in K-homology $\mathrm{KK}(-,\mathbb{C})$. For this, note that all algebras occurring in extensions throughout this paper are nuclear, and therefore all extensions are semisplit, so excision in KK-theory holds.

The K-homology of the quotient C^*-algebras is easy to compute. To start with the second extension, the action of Γ on $\partial\mathscr{H}(P)$ is free and proper. (This is not a contradiction to Γ having torsion because $\partial\mathscr{H}(P)$ is homeomorphic to a countable union of disjoint copies of $\mathbb{R}^2$.) There is a strong Morita equivalence $\Gamma \ltimes \mathscr{C}_0(\partial\mathscr{H}(P)) \sim \mathscr{C}(\Gamma\backslash\partial\mathscr{H}(P))$. Let k denote the class number of the underlying number field. Then the number of orbits of singular points is k. Hence, the quotient space is a disjoint union of k compact two-tori [25], so we have

$$\mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(\partial\mathscr{H}(P))) \cong \begin{cases} \mathbb{Z}^{2k}, & * = 0, \\ \mathbb{Z}^{2k}, & * = 1. \end{cases} \tag{3}$$

Hence, the second short exact sequence of (2) allows to compute the equivariant K-homology of to compute that of $\mathscr{H}(P)$ from $\mathscr{H}$.

The former short exact sequence of (2), in turn, allows to compute the K-homology of $\mathscr{H}$ from that of $X_\circ$. As we have shown in Lemma 2.8, the space $\mathscr{H}(P)$ is a classifying space, so this will achieve the computation.

The interior of Bianchi's fundamental polyhedron has trivial stabilizer [20]. Therefore, the action on $\mathscr{H} - X_\circ$ is free and proper so $\Gamma \ltimes \mathscr{C}_0(\mathscr{H} - X_\circ)$ is Morita equivalent to $\mathscr{C}_0(\Gamma\backslash(\mathscr{H} - X_\circ))$ which is isomorphic to $\mathscr{C}_0$ of an open three-cell, i.e. $\mathscr{C}_0(\mathbb{R}^3)$. Therefore,

$$\mathrm{KK}_*(\Gamma \ltimes \mathscr{C}_0(\mathscr{H} - X_\circ), \mathbb{C}) \cong \begin{cases} 0, & * = 0, \\ \mathbb{Z}, & * = 1. \end{cases}$$

Thus, we can compute the left-hand side $R\,\mathrm{K}_*^\Gamma(\underline{E}\Gamma) = \mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(\mathscr{H}(P)))$ via successive computation of the groups $\mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(X_\circ))$ and $\mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(\mathscr{H}))$ instead of the analogous homology groups with compact support, by the exact six-term sequences

$$\begin{array}{ccccc} \mathrm{K}^0(\Gamma \ltimes \mathscr{C}_0(X_\circ)) & \longrightarrow & \mathrm{K}^0(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) & \longleftarrow & \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(X_\circ)) \end{array} \tag{4}$$

and, using (3),

$$\begin{array}{ccccc} \mathbb{Z}^{2k} & \longrightarrow & R\,\mathrm{K}_0^\Gamma(\underline{E}\Gamma) & \longrightarrow & \mathrm{K}^0(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) \\ \uparrow & & & & \downarrow \\ \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) & \longleftarrow & R\,\mathrm{K}_1^\Gamma(\underline{E}\Gamma) & \longleftarrow & \mathbb{Z}^{2k}, \end{array} \tag{5}$$

which paves the way to reduce the computation of the equivariant K-homology of Γ to that of the K-homology of the reduced crossed product and that of the boundary tori.

Remark 2.10. Denote the Kasparov representation ring $\mathrm{KK}^{\Gamma}_{*}(\mathbb{C},\mathbb{C})$ by $R_*(\Gamma)$. There is an invertible in $\mathrm{KK}^{\Gamma}_{1}(\mathbb{C},\mathscr{C}_0(\mathscr{H}))$, namely Kasparov's dual-Dirac element [11], leading to an isomorphism $\mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) \cong \mathrm{K}^{*+1}(C^*_{max}\Gamma) = R_{*+1}(\Gamma)$. Therefore, rewriting the six-term sequence (5) and using the fact that the assembly map is an isomorphism, we arrive at the following exact six-term-sequence.

$$\begin{array}{ccccc} \mathbb{Z}^{2k} & \longrightarrow & \mathrm{K}_0(C^*_{red}\Gamma) & \longrightarrow & R_1(\Gamma) \\ \uparrow & & & & \downarrow \\ R_0(\Gamma) & \longleftarrow & \mathrm{K}_1(C^*_{red}\Gamma) & \longleftarrow & \mathbb{Z}^{2k}. \end{array} \tag{6}$$

This is remarkable since there are rarely exact sequences available that connect K-theory and K-homology of the same algebra, in view of

$$R(\Gamma) = \mathrm{KK}(C^*_{max}\Gamma, \mathbb{C}) = \mathrm{K}^*(C^*_{red}\Gamma)$$

by K-amenability.

Lemma 2.11. *The connecting homomorphism of the six-term-sequence* (4) *is zero, so there is an isomorphism* $\mathrm{K}^0(\Gamma \ltimes \mathscr{C}_0(X_\circ)) \cong \mathrm{K}^0(\Gamma \ltimes \mathscr{C}_0(\mathscr{H}))$ *and a short exact sequence*

$$0 \to \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(X_\circ)) \to \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) \to \mathbb{Z} \to 0.$$

Proof. Let $\mathcal{D} \subset \mathscr{H}$ denote the interior of the Bianchi fundamental polyhedron which is called D in [9] (in the present manuscript, the letter D is already occupied); hence, there are homeomorphisms $\mathcal{D} = \mathbb{R}^3$ and $\mathscr{H} - X_\circ = \Gamma \times \mathcal{D}$ because the interior of $\mathcal{D}$ is trivially stabilized. It is well known that the KK-equivalence coming from the strong Morita–Rieffel equivalence

$$\Gamma \ltimes \mathscr{C}_0(\mathscr{H} - X_\circ) \sim \mathscr{C}_0(\mathcal{D})$$

is induced by the inclusion of C^*-algebras $\mathscr{C}_0(\mathcal{D}) \to \Gamma \ltimes \mathscr{C}_0(\mathscr{H} - X_\circ)$ where the first algebra is viewed as a crossed product with the trivial group, and a function on $\mathcal{D}$ is viewed as a function on $\mathscr{H} - X_\circ$ by setting it to zero outside $\mathcal{D}$.

Consider the map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C}_0(\mathcal{D}) & \xrightarrow{=} & \mathscr{C}_0(\mathcal{D}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma \ltimes \mathscr{C}_0(\mathscr{H} - X_\circ) & \longrightarrow & \Gamma \ltimes \mathscr{C}_0(\mathscr{H}) & \longrightarrow & \Gamma \ltimes \mathscr{C}_0(X_\circ) & \longrightarrow & 0. \end{array} \tag{7}$$

The vertical arrow in the middle induces a map

$$K^1(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) \to \mathrm{K}^1(\mathscr{C}_0(\mathbb{R}^3)) = \mathrm{K}_1(\mathbb{R}^3) \cong \mathbb{Z}$$

in K-homology. This map is part of the six-term-sequence (4) and the assertion follows as soon as we have shown that it is surjective.

Consider the following diagram of $\mathrm{KK}(-,\mathbb{C})$-theory groups

$$\begin{array}{ccc} & & \mathrm{K}^1(\mathscr{C}_0(\mathcal{D})) \\ & \swarrow & \uparrow \\ \mathrm{K}^0(\mathbb{C}) & \longleftarrow & \mathrm{K}^1(\mathscr{C}_0(\mathscr{H})) \\ \uparrow & & \uparrow \\ \mathrm{K}^0(C^*_{max}\Gamma) & \longleftarrow & \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})), \end{array}$$

where all vertical arrows are induced by C^*-inclusions. The diagonal arrow and the upper horizontal arrow are the K-homological Bott isomorphisms associated to the standard K-orientations of $\mathbb{R}^3$ and of $\mathscr{H}$, respectively. The lower horizontal arrow is left-multiplication with $j_{max}(\beta)$, where

$$j_{max} : \mathrm{KK}^\Gamma_*(\mathbb{C}, \mathscr{C}_0(\mathscr{H})) \to \mathrm{KK}_*(C^*_{max}\Gamma, \Gamma \ltimes \mathscr{C}_0(\mathscr{H}))$$

is the maximal version of the descent homomorphism, and $\beta \in \mathrm{KK}^\Gamma_1(\mathbb{C}, \mathscr{C}_0(\mathscr{H}))$ is Kasparov's dual-Dirac element. The commutativity of the triangle is inherent in the definition of the Bott element, and that of the square follows from the fact that the forgetful homomorphism $\mathrm{KK}^\Gamma_1(\mathbb{C}, \mathscr{C}_0(\mathscr{H})) \to \mathrm{KK}_1(\mathbb{C}, \mathscr{C}_0(\mathscr{H}))$ sends β to the Bott element.

All arrows except the two lower vertical arrows are isomorphisms because β and hence also $j_{max}(\beta)$ are invertible. Moreover, the composition

$$\mathbb{C} \to C^*_{max}\Gamma \to \mathbb{C}$$

of the inclusion of the unit with the trivial representation is, of course, the identity on $\mathbb{C}$, so the induced composition $\mathrm{K}^0(\mathbb{C}) \to \mathrm{K}^0(C^*_{max}\Gamma) \to \mathrm{K}^0(\mathbb{C})$ is the identity on $\mathrm{K}^0(\mathbb{C})$ and the left-hand vertical arrow is surjective. Summarizing, the diagram shows that the map $\mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) \to \mathrm{K}^1(\mathscr{C}_0(\mathcal{D}))$ is surjective. This completes the proof. □

3. The Atiyah–Hirzebruch spectral sequence

3.1. **Bredon homology.** The importance of Bredon homology for equivariant K-homology lies in the fact that the Atiyah–Hirzebruch's spectral sequence's E^2-term is Bredon homology [16]. Let us recall the definition of Bredon homology of a proper locally finite oriented $\Gamma - CW-$complex, as it was used in [23, 18, 19]. Out of each orbit of q-cells, choose a representative q-cell σ^q for all q. Denote a finite group's complex representation ring by $R-$, and the stabilizer of σ^q by Γ_{σ^q}. Then there is a chain complex

$$\ldots \longrightarrow \bigoplus_{\sigma^q} R\Gamma_{\sigma^q} \longrightarrow \bigoplus_{\sigma^{q-1}} R\Gamma_{\sigma^{q-1}} \longrightarrow \ldots$$

where the sum in degree q extends over all orbit representatives σ^q of dimension q. The differential is defined as follows.

Definition and Observation 3.2. Let $\partial\sigma^q = \cup\gamma_i\sigma_i^{q-1}$, and let $\delta_i \in \mathbb{Z}$ be the degree of the map $\mathbb{S}^{q-1} \cong \partial\sigma^q \to X^{q-1} \to X^{q-1}/(X^{q-1} - \sigma_i^{q-1}) \cong \mathbb{S}^{q-1}$ given by composition of the attaching map with the quotient map. The group $\Gamma\sigma^q$ is a subgroup of each of the groups $\Gamma_{\gamma_i\sigma_i^{q-1}}$, and δ_i times the composition of induction from that subgroup with the isomorphism $R\Gamma_{\gamma_i\sigma_i^{q-1}} \cong R\Gamma_{\sigma_i^{q-1}}$ induced by the conjugation isomorphism $\Gamma_{\gamma_i\sigma_i^{q-1}} \cong \Gamma_{\sigma_i^{q-1}}, \gamma \mapsto \gamma_i\gamma\gamma_i^{-1}$ defines a map $R\Gamma_{\sigma^q} \to R\Gamma_{\sigma^{q-1}}$ and thus a map $d : \bigoplus_{\sigma^q} R\Gamma_{\sigma^q} \to \bigoplus_{\sigma^{q-1}} R\Gamma_{\sigma^{q-1}}$. The map d thus defined satisfies $d^2 = 0$ and therefore defines a differential. Furthermore, the chain complex is well-defined, i.e. does not depend on the choice of representatives or that of intertwining group elements γ_i. The complex thus defined is called the Bredon chain complex, and its homology is called Bredon homology.

Bredon homology reduces to cellular homology in the trivial case $\Gamma = \{1\}$. In [18], a program for automatic computation of Bredon homology of the Flöge complex of Bianchi groups in the case of trivial class group was described and exploited, based on the strategy of [23] for $\mathrm{SL}(3, \mathbb{Z})$: For each occurring isomorphism type of stabilizer group, the character table is considered. Furthermore, for each occurring boundary inclusion of cells in the fundamental domain, the conjugation map is determined, and the induction map of representation rings as well as the conjugation map is explicitly computed in terms of these character tables. The essential information for this is the information on stabilizer groups of cells in the fundamental domain and intertwining matrices between cells in the same orbit occurring more than once in the fundamental domain, as well as the Frobenius reciprocity theorem for determination of the induction map. Computation of the Flöge complex' Bredon homology is tedious but still possible because of the small number of occurring group inclusion isomorphism types.

3.3. **The Schochet spectral sequence for a punctured cell complex.** Lemma 2.11 indicates the significance of the K-homology of $\Gamma \ltimes \mathscr{C}_0(X_\circ)$. In the usual setting of a proper cell complex, there is the Atiyah–Hirzebruch spectral sequence computing equivariant K-homology. Its E^1-term is the Bredon complex, and its E^2-term is Bredon homology. For the present purpose, we modify this setting in the following way. Note that $X_\circ$ is not a cell complex. However, there is still a natural notion of p-cell, with the difference that there are one-cells that are adjacent to only one vertex. In the cell complex X, there are one- and two-cells σ adjacent to singular points. These one- and two-cells are compact subsets of X. However, their intersection $\sigma \cap \mathscr{H} \subset X_\circ$ is, of course, not compact in $X_\circ$. However, this makes no difference with regard to the combinatorial structure: They are still adjacent to the same edges. Since the definition of Bredon homology only takes into account the combinatorial structure, we may still write down a combinatorial Bredon complex for $X_\circ$.

This is encompassed by the following definition. Note that Bredon homology is only defined for proper complexes, so its construction is not applicable to the Flöge complex.

Definition and Observation 3.4. Choose a representative out of each orbit of q-simplices in the Flöge complex X. Denote the set of representatives by $\Gamma\backslash X^q$. Define $\Gamma\backslash X^q_\circ$ as the set representatives of nonsingular vertex orbits for $q = 0$, and as $\Gamma\backslash X^q$ for $q \neq 0$. Consider the chain complex

$$0 \longrightarrow \bigoplus_{\sigma\in\Gamma\backslash X^2} R\Gamma_\sigma \longrightarrow \bigoplus_{\sigma\in\Gamma\backslash X^1} R\Gamma_\sigma \longrightarrow \bigoplus_{\sigma\in\Gamma\backslash X^0:\sigma \text{ nonsingular}} R\Gamma_\sigma \longrightarrow 0,$$

where the differential is defined by conjugation and subgroup induction, analogously to the Bredon differential. The definition does, up to isomorphism, not depend on the choice of representatives or that of intertwining group elements, and the differential satisfies $d^2 = 0$ as well. Therefore, the complex is well-defined. It is called the *modified Bredon complex*, its homology is called the *modified Bredon homology* and denoted by $\mathrm{H}_*(X_\circ)$.

Therefore, in dimension $q = 0$ there are only summands associated to the nonsingular vertex orbits. On a summand $R\Gamma_e$ belonging to the representative of an edge e touching a singular point in X, there is only an induction map to $R\Gamma_v$, where v is the representative of the interior bounding vertex of e.

The following lemma says, roughly speaking, that this modification of Bredon homology is not only a modification of the E^1 term, but that in fact this modified E^1-term fits into an entire modified spectral sequence whose E^1-term is the modified E^1-term. Furthermore, this modified spectral sequence computes "the right thing", namely the equivariant K-homology of $X_\circ$.

For a finite group H, let us write

$$M_q H := \mathrm{KK}_q(\mathbb{C}H, \mathbb{C}) = \begin{cases} RH, & q \text{ even}, \\ 0, & q \text{ odd}. \end{cases}$$

$M_q H$ is an example of a Bredon module, but we do not need that fact.

Lemma 3.5. *There is a homological spectral sequence computing equivariant K-homology,*

$$E^1_{p,q} = \bigoplus_{\sigma\in\Gamma\backslash X^p_\circ} M_q\Gamma_\sigma \implies \mathrm{K}^{p+q}(\Gamma \ltimes \mathscr{C}_0(X_\circ)),$$

where the sum extends over representatives σ of orbits of p-cells in $X_\circ$, and Γ_σ is the respective stabilizer isomorphism type. It is concentrated in the first and fourth quadrants, and two-periodic in the index q. Its differential d^n has bidegree $(-n, n-1)$.

The summand of the E^1-term belonging to the cells of $X_\circ$ that do not touch a singular point is equal to the usual Bredon complex of these cells. The summand of the E^1-term belonging to the one-cells of $X_\circ$ that touch a singular point has only a differential to one vertex orbit. The summand of the E^1-term belonging to the two-cells of $X_\circ$ that touch a singular point is equal to the usual Bredon *complex associated to the combinatorial structure. Therefore, the E^2-term is* $\mathrm{H}_p(X_\circ; M_q)$.

Proof. This is a slight generalization of the Atiyah–Hirzebruch spectral sequence. It is constructed as the spectral sequence associated as in [24] to the three-step filtration by closed ideals

$$0 \subset \Gamma \ltimes \{f \mid f|X_\circ^1 = 0) \subset \Gamma \ltimes \{f \mid f|X_\circ^0 = 0) \subset \Gamma \ltimes \mathscr{C}_0(X_\circ)$$

of functions vanishing on the punctured skeleta.

The sub-quotient

$$(\Gamma \ltimes \{f \mid f|X_\circ^{q-1} = 0))/(\Gamma \ltimes \{f \mid f|X_\circ^q = 0)) = \Gamma \ltimes \mathscr{C}_0(X_\circ^q - X_\circ^{q-1})$$

is a direct sum over the orbits of q-cells, and a summand corresponding to the orbit of a cell c stabilized by the (necessarily finite) subgroup $\Gamma_c \subset \Gamma$ is isomorphic to $\mathscr{C}_0(\mathbb{R}^q) \otimes (\Gamma \ltimes \mathscr{C}_0(\Gamma c))$ which is strongly Morita equivalent to the q-fold suspension of the complex group ring $\mathbb{C}\Gamma_c$ whose K-homology is isomorphic to $R\Gamma_c$. This yields the Atiyah–Hirzebruch spectral sequence for equivariant K-homology, upon implementing the q-fold dimension shifts pertaining to the cells' dimensions. Its E^1 computes (and, therefore, its E^2-term is) Bredon homology with respect to finite subgroups [16], as it is considered in [23] for computation of equivariant K-homology. □

The fact that we deal with punctured skeleta introduces no peculiarities since on the level of sub-quotients it only means to pay attention to the fact that for each edge e touching a singular point, there is only one inclusion of vertex stabilizers instead of two (Recall that each edge touches at most one singular point).

Example 3.6. It is easy to visualize the modified Atiyah–Hirzebruch spectral sequence thus defined for the very simple case of a single edge acted upon by the trivial group, and to compare it with the Schochet spectral sequence associated to a half-open interval with the filtration $0 \subset \mathscr{C}_0(\mathbb{R}_+^*) \subset \mathscr{C}_0(\mathbb{R}_{\geqslant 0})$, where the endpoint is the single vertex. The latter's E^1-term takes the form $\mathbb{Z} \leftarrow \mathbb{Z}$ in even rows, and zero in odd rows, according to a single edge and a single point. The homology of this vanishes, in accordance with $\mathrm{K}^*(\mathscr{C}_0(\mathbb{R}_{\geqslant 0})) = 0$. Bearing this example in mind might help to understand the slight generalization of the Atiyah–Hirzebruch spectral sequence to punctured cell complexes, considered here. Roughly speaking, "it is enough to omit the missing points from the spectral sequence".

3.7. **Short exact sequences.** Let us write down how to pass, in the case we are interested in, from E^∞ to K-homology using the edge homomorphisms.

Corollary 3.8. *The odd rows of the E^2-term vanish since* K_1 *of a finite-dimensional C^*-algebra such as a finite group's ring necessarily vanishes. Since dim $X_\circ = 2$, the spectral sequence is concentrated in the columns $0 \leqslant p \leqslant 2$. Therefore, $E^2 = E^\infty$, and there is an isomorphism*

$$E^2_{1,0} \cong \mathrm{K}^1(\Gamma \ltimes \mathscr{C}_0(X_\circ)) \tag{8}$$

and a short exact sequence

$$0 \to E^2_{0,2} \to \mathrm{K}^0(\Gamma \ltimes \mathscr{C}_0(X_\circ)) \to E^2_{2,0} \to 0. \tag{9}$$

4. Explicit calculations

Let us go through all computations for $m = 5$. Details, including a picture, of a simple fundamental domain for the action of X are given in [9] and [20, Sec. 3.2].We shall use the notations therein.

Recall that we set out to calculate the spectral sequence, which is essentially associated to the skeletal filtration of a cell complex. However, there is "one vertex orbit missing" in $X_\circ$. So, there are only four orbits of vertices, namely those of a, b, u, v, but the same numbers of edges and faces as in [20, Sec. 3.2], namely seven orbits of edges, those of $ba, vv_1, a_3u, ub, u_1b, av, as$, and three orbits of faces. Stabilizers are given in explicit form in the same references. There is only one orbit of singular points, since the class number k of $\mathbb{Q}[\sqrt{-5}]$ is two.

All information on the representation rings $\mathrm{K}^0(C^*_{max}\Gamma_\sigma) = R\Gamma_\sigma$ of finite stabilizers of vertices σ is given in [18] and used as explained in Section 3.1.

The E^1-term is a complex

$$d^1 : \mathbb{Z}^{4+4+2+3} \xleftarrow{d^1_{1,0}} \mathbb{Z}^{2+3+2+2+2+1+1} \xleftarrow{d^1_{2,0}} \mathbb{Z}^{1+1+1}$$

in even rows, and zero in odd rows. The d^1-matrices are displayed in Tables 1 and 2. Their elementary divisors are easily determined (for instance by the Pari/GP software [26]) by calculating the Smith normal forms, yielding

$$d^1_{1,0} \sim \mathrm{diag}(0,0,0,0,0,2,1,1,1,1,1,1,1)$$

and

$$d^1_{2,0} \sim \mathrm{diag}(0,0,0,0,0,0,0,0,0,0,0,1,1).$$

Therefore, the modified Bredon homology is

$$\begin{aligned} E^2_{0,0} &\cong \mathbb{Z}^5 \oplus \mathbb{Z}/2 \\ E^2_{0,1} &\cong \mathbb{Z}^3 \\ E^2_{0,2} &\cong \mathbb{Z}, \end{aligned} \tag{10}$$

where we write $\mathbb{Z}/g$ for $\mathbb{Z}/g\mathbb{Z}$. (This leads to the correct rational Euler characteristic $13 - 13 + 3 = 5 - 3 + 1$, as it should.) Using (8) and (9), we obtain

$$\mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(X_\circ)) = \begin{cases} \mathbb{Z}^6 \oplus \mathbb{Z}/2, & * = 0, \\ \mathbb{Z}^3, & * = 1. \end{cases}$$

Using Lemma 2.11, we have

$$\mathrm{K}^*(\Gamma \ltimes \mathscr{C}_0(\mathscr{H})) = \begin{cases} \mathbb{Z}^6 \oplus \mathbb{Z}/2, & * = 0, \\ \mathbb{Z}^4, & * = 1. \end{cases}$$

We can now state the main theorem.

	(b,a)	(b,a)	(v,v_1)	(v,v_1)	(v,v_1)	(a_3,u)	(a_3,u)	(u,b)	(u,b)	(u_1,b)	(u_1,b)	(a,v)	(a,s)
large cell	-1	-1	-1	-1	-1	-1	-1			-1	-1		
mid-size cell	1	1				1	1	1	1				
small cell													

Table 1. The transpose of the differential $d^1_{2,0}$ for $m = 5$, with zeroes omitted. There are multiple column names because they were chosen to indicate only the originating cell instead of including the full character information, in order to save space. Rank is two, elementary divisors are zero and one. The small cell is in the kernel of $d^1_{2,0}$.

	(b,a)	(b,a)	(v,v_1)	(v,v_1)	(v,v_1)	(a_3,u)	(a_3,u)	(u,b)	(u,b)	(u_1,b)	(u_1,b)	(a,v)	(a,s)
b	-1							1		1			
		-1							1	1			
	-1								1		1		
b		-1						1			1		
u						1		-1		-1			
							1		-1	-1			
							1	-1			-1		
u						1			-1		-1		
a	1					-1						-1	-1
a		1					-1					-1	-1
v												1	
												1	
v												1	

Table 2. The differential $d^1_{1,0}$ for $m = 5$. Row and columns information is shortened as in Table 1, whence the occurrence of multiple row and column names. The elementary divisor one occurs with multiplicity 7, and elementary divisor two occurs with multiplicity one.

Theorem 4.1. *The equivariant K-homology of the Bianchi group* $\mathrm{PSL}_2(\mathcal{O}(\mathbb{Q}[\sqrt{-5}]))$ *is*

$$R\,\mathrm{K}_*^\Gamma(\underline{E}\Gamma) \cong \begin{cases} \mathbb{Z}^6 \oplus \mathbb{Z}/2, & * = 0, \\ \mathbb{Z}^4, & * = 1. \end{cases}$$

Proof. It only remains to solve the six-term extension problem (5). Recall that the class number k is two. The strategy for solving this six-term problem is similar to the one used in [21] where the analogous long term sequence in homology was computed. □

As stated in the introduction, this simultaneously computes $\mathrm{K}_*(C^*_{red}\Gamma)$ and $\mathrm{K}_*(C^*_{max}\Gamma)$.

Analogous computations for other m should present no fundamentally new difficulties. However, they are beyond the scope of the present paper.

Another interesting problem is to determine the minimal size of idempotent matrices over $C^*_{red}\Gamma$ realizing a given K-theory class on the list we computed.

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