Fixed points for actions of Aut($F_n$) on CAT(0) spaces

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Abstract. For $n \geq 4$ we discuss questions concerning global fixed points for isometric actions of Aut($F_n$), the automorphism group of a free group of rank $n$, on complete CAT(0) spaces. We prove that whenever Aut($F_n$) acts by isometries on complete $d$-dimensional CAT(0) space with $d < 2\left\lfloor \frac{n}{4} \right\rfloor - 1$, then it must fix a point. This property has implications for irreducible representations of Aut($F_n$), which are also presented here. For SAut($F_n$), the unique subgroup of index two in Aut($F_n$), we obtain similar results.

1. Introduction

In the mathematical world, this article is located in the area of geometric group theory, a field at the intersection of algebra, geometry and topology. Geometric group theory studies the interaction between algebraic and geometric properties of groups. One is interested in understanding on which 'nice' geometric spaces a given group can act in a reasonable way and how geometric properties of these spaces are reflected in the algebraic structure of the group. Here, the spaces will be CAT(0) metric spaces, while the groups will be Aut($F_n$) and SAut($F_n$). The questions we shall investigate are concerned with fixed point properties and the representation theory of these groups.

More precisely, let $\mathbb{Z}^n$ be the free abelian group and $F_n$ the free group of rank $n$. One goal for a group theorist is to understand the structure of their automorphism groups, GL$_n$(Z) resp. Aut($F_n$). The abelianization map $F_n \rightarrow \mathbb{Z}^n$ gives a natural epimorphism Aut($F_n$) $\rightarrow$ GL$_n$(Z). The special automorphism group of $F_n$, which we will denote by SAut($F_n$), is defined as the preimage of SL$_n$(Z) under this map. Much of the work on Aut($F_n$) and SAut($F_n$) is motivated by the idea that GL$_n$(Z) and Aut($F_n$) resp. SL$_n$(Z) and SAut($F_n$) should have many properties in common. Here we follow this idea and present analogies between these groups with respect to fixed point properties.

Let $\mathcal{X}$ be a class of metric spaces. A group $G$ is said to have property $F\mathcal{X}$ if any action of $G$ by isometries on any member of $\mathcal{X}$ has a fixed point. Let
\( \mathcal{A} \) be the class of simplicial trees, \( \mathcal{A}_d \) the class of complete CAT(0) spaces of covering dimension \( d \) and \( \mathcal{A}_s \) the class of finite dimensional complete CAT(0) spaces.

The starting point for our investigation is the study of group actions on simplicial trees which was initiated by Serre, see [18], [19]. He proved that \( \text{GL}_n(\mathbb{Z}) \) and \( \text{SL}_n(\mathbb{Z}) \) have property \( \text{FA} \) for \( n \geq 3 \). Regarding \( \text{Aut}(F_n) \) and \( \text{SAut}(F_n) \), Bogopolski was the first to prove that these groups also have property \( \text{FA} \), see [2].

A slight generalization of the class of simplicial trees is given by the class of metric trees, which we will denote by \( \mathcal{R} \). Different methods were developed by Culler and Vogtmann and later by Bridson to prove that \( \text{Aut}(F_n) \) and \( \text{SAut}(F_n) \) have property \( \text{FR} \), see [3], [8]. We obtain the fixed point property of \( \text{Aut}(F_n) \) and \( \text{SAut}(F_n) \) for a much larger class of higher dimensional complete CAT(0) spaces.

We present two results, Theorems A and B, regarding property \( \text{FA}_d \) for the groups \( \text{Aut}(F_n) \) and \( \text{SAut}(F_n) \). Using Bridson’s and Farb’s techniques from [3] and [10], we prove:

**Theorem A.** If \( n \geq 4 \) and \( d < \min \left\{ k \left\lfloor \frac{n}{k+2} \right\rfloor \mid k = 2, \ldots, d+1 \right\} \), then \( \text{Aut}(F_n) \) has property \( \text{FA}_d \). In particular, if \( n \geq 4 \) and \( d < 2 \left\lfloor \frac{n}{4} \right\rfloor - 1 \), then \( \text{Aut}(F_n) \) has property \( \text{FA}_d \).

**Theorem B.** If \( n \geq 5 \) and \( d < \min \left\{ k \left\lfloor \frac{n-1}{k+2} \right\rfloor \mid k = 2, \ldots, d+1 \right\} \), then \( \text{SAut}(F_n) \) has property \( \text{FA}_d \). In particular, if \( n \geq 5 \) and \( d < 2 \left\lfloor \frac{n-1}{4} \right\rfloor - 1 \), then \( \text{SAut}(F_n) \) has property \( \text{FA}_d \).

Our proofs of Theorems A and B involve three ingredients. First we construct a generating set of \( \text{Aut}(F_n) \) such that each pair of elements generates a finite subgroup. Next, we need an extended version of Helly’s Theorem for higher dimensional CAT(0) spaces.

**Theorem.** Let \( X \) be a \( d \)-dimensional complete CAT(0) space and \( S \) a finite family of nonempty closed convex subspaces. If the intersection of \( (d+1) \)-elements of \( S \) is always nonempty, then \( \bigcap S \) is nonempty.

There exist several variations of this theorem in the literature, e.g. for finite families of convex open resp. closed subsets of a CAT(0) space, see [4, 3.2], [9, 2], [10, 3.2] and [13, 5.3]. Here we include a complete proof for the case of a finite family of closed convex subspaces.

Our main technique in the proofs of Theorems A and B is based on the following corollary. Indeed, it was Farb who discovered the connection between Helly’s Theorem and the combinatorics of generating sets for a large class of groups.

**Farb’s Fixed Point Criterion.** Let \( G \) be a group, \( Y \) a finite generating set of \( G \) and \( X \) a complete \( d \)-dimensional CAT(0) space. If \( \Phi : G \to \text{Isom}(X) \) is a homomorphism such that each \( (d+1) \)-element subset of \( Y \) has a fixed point in \( X \), then \( G \) has a fixed point in \( X \).
Farb used this criterion in [10] to obtain sharp results on property $F_{\mathcal{A}_d}$ for various groups. For example, he proved that $\text{SL}_n(\mathbb{Z}[1/p])$ has property $F_{\mathcal{A}_{n-2}}$ for semisimple actions, but not property $F_{\mathcal{A}_{n-1}}$, since it acts without a global fixed point on the affine building for $\text{SL}_n(\mathbb{Q}_p)$.

In a third step, we combine the extended version of Helly’s Theorem with the following theorem by Bridson to prove our results.

**Theorem.** [4, 3.6] Let $k$ and $l$ be in $\mathbb{N}_{>0}$ and let $X$ be a complete $d$-dimensional CAT(0) space with $d < k \cdot l$. Let $S$ be a subset of $\text{Isom}(X)$ and let $S_1, \ldots, S_l$ be conjugates of $S$ such that $[S_i, S_j] = 1$ holds for all $i, j = 1, \ldots, l$, $i \neq j$. If each $k$-element subset of $S$ has a fixed point in $X$, then each finite subset of $S$ has a fixed point in $X$.

Property $F_{\mathcal{A}_d}$ strongly affects the representation theory of groups. The following result by Farb, partially based on work by Bass, illustrates this fact.

**Theorem.** [10, 1.8] Let $K$ be an algebraically closed field and let $G$ be a group. If $G$ has property $F_{\mathcal{A}_d}$, then there are only finitely many conjugacy classes of irreducible representations

$$\rho : G \to \text{GL}_{d+1}(K).$$

As an application of our Theorems A and B, we obtain the following similar results for the representation theory of $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$.

**Corollary C.** Let $K$ be an algebraically closed field. If $n \geq 4$ and $d \leq 2\left\lfloor \frac{n}{4} \right\rfloor - 1$, then there are only finitely many conjugacy classes of irreducible representations

$$\rho : \text{Aut}(F_n) \to \text{GL}_d(K).$$

**Corollary D.** Let $K$ be an algebraically closed field. If $n \geq 5$ and $d \leq 2\left\lfloor \frac{n-1}{4} \right\rfloor - 1$, then there are only finitely many conjugacy classes of irreducible representations

$$\rho : \text{SAut}(F_n) \to \text{SL}_d(K).$$

**Remark.** A better bound for the complex representations of $\text{Aut}(F_n)$ is proved in [17, 3.1, 3.2]. If $n \geq 3$ and $d \leq 2 \cdot n - 2$, then there are only finitely many conjugacy classes of irreducible representations

$$\rho : \text{Aut}(F_n) \to \text{GL}_d(\mathbb{C}).$$

With Bridson’s and Vogtmann’s techniques from [6, 1.1] one can prove that the linear representations of $\text{SAut}(F_n)$ are very rigid. Let $K$ be a field of characteristic different from 2 and let

$$\rho : \text{SAut}(F_n) \to \text{SL}_d(K)$$

be a homomorphism. If $n \geq 3$ and $d < n$, then $\rho$ is trivial. In particular, if $n \geq 3$, then $\text{Aut}(F_n)$ has only finitely many conjugacy classes of irreducible representations in any dimension $\leq n - 1$. 

2. A generating set of $\text{Aut}(F_n)$

The purpose of this section is to construct a generating set of the group $\text{Aut}(F_n)$ such that each pair of its elements generates a finite subgroup. Although it may seem awkward at first glance, it is convenient and standard to work with the right action of $\text{Aut}(F_n)$ on $F_n$.

**Convention 2.1.** For $\alpha, \beta$ in $\text{Aut}(F_n)$ the automorphism $\alpha\beta$ is the composite where $\alpha$ acts before $\beta$.

Let us first introduce a notations for some elements of $\text{Aut}(F_n)$. We define the *right Nielsen automorphism* $\rho_{ij}$, involutions $(x_i, x_j)$ and $e_i$ for $i, j = 1, \ldots, n, i \neq j$ as follows:

$$\rho_{ij}(x_k) := \begin{cases} x_i x_j & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}$$

$$(x_i, x_j)(x_k) := \begin{cases} x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \\ x_k & \text{if } k \neq i, j. \end{cases}$$

$$e_i(x_k) := \begin{cases} x_i^{-1} & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}$$

It is easy to see that the image of $X = \{x_1, \ldots, x_n\}$ under any of these maps is another basis of $F_n$, therefore these elements are automorphisms. It was proven by Nielsen in [16, p. 173]) that for $n \geq 3$ the group $\text{Aut}(F_n)$ is generated by the set

$$Y_1 := \{\rho_{12}, e_1, (x_1, x_2), (x_1, x_2, \ldots, x_n)\},$$

where $(x_1, x_2, \ldots, x_n)$ denotes the composite $(x_{n-1}, x_n)(x_{n-2}, x_{n-1}) \cdots (x_1, x_2)$.

Our strategy in this section is to modify the set $Y_1$ such that each pair of elements in the new generating set generates a finite group, compare [3, 1.1, 1.2].

**Proposition 2.2.** Let $n \geq 3$.

(i) The group $\text{Aut}(F_n)$ is generated by

$$Y_2 := \{(x_1, x_2)e_1e_2, (x_2, x_3)e_1, (x_i, x_{i+1}), e_2\rho_{12}, e_n \mid i = 3, \ldots, n-1\}.$$

(ii) The subgroup generated by $Y_2 - \{e_2\rho_{12}\}$ is finite.

(iii) For $\alpha, \beta$ in $Y_2$ the subgroup generated by $\{\alpha, \beta\}$ is finite.

**Proof.** Let us denote by $\Sigma_n \subseteq \text{Aut}(F_n)$ the group of automorphisms which permute the basis $X$. The conjugation by $\sigma \in \Sigma_n$ sends $e_i$ to $e_{\sigma(i)}$: $\sigma^{-1} e_i \sigma = e_{\sigma(i)}$, therefore $\text{Aut}(F_n)$ is generated by the set $\{\rho_{12}, e_n, \Sigma_n\}$. It is a well-known result that the group $\Sigma_n$ is generated by the involutions $(x_i, x_{i+1})$ with $i = 1, \ldots, n-1$. We can further replace $\rho_{12}$ by the involution $e_2\rho_{12}$ and we obtain the following generating set of $\text{Aut}(F_n)$: $\{e_2\rho_{12}, e_n, (x_i, x_{i+1}) \mid i = 1, \ldots, n-1\}$. To see that $Y_2$ is a generating set of $\text{Aut}(F_n)$, we must show that the involutions $(x_1, x_2)$ and $(x_2, x_3)$ are in $\langle Y_2 \rangle$. First we show this results for $n = 3$. We have

$$e_2 = (x_2, x_3)e_1e_3 \in \langle Y_2 \rangle$$

and therefore \((x_1, x_3) = (x_2, x_3)e_1(x_1, x_2)e_1e_2e_2(x_2, x_3)e_1e_3\) is contained in \((Y_2)\). Using \((x_1, x_3, x_2) = (x_1, x_2)e_1e_2e_2(x_2, x_3)e_1 \in (Y_2)\) we obtain
\[
(x_1, x_2) = (x_1, x_3)\left(x_1, x_3, x_2\right) \in (Y_2),
\]
\[
(x_2, x_3) = (x_1, x_2)\left(x_1, x_3, x_2\right) \in (Y_2).
\]

For \(n \geq 4\) we have \(e_3 = (x_3, x_n)e_ne_3(x_3, x_n) \in (Y_2)\). The same arguments as above show that the involutions \((x_1, x_2)\) and \((x_2, x_3)\) are contained in \((Y_2)\). This finishes the proof of statement (i).

It easy to verify that the subgroup \(Y_2 - \{e_2\rho_{12}\}\) of \(\text{Aut}(F_n)\) is isomorphic to the semidirect product \(\text{Sym}(n) \ltimes \mathbb{Z}_2^n\), where we denote by \(\mathbb{Z}_2\) the cyclic group of order 2, and therefore finite.

Now we prove the last statement of the proposition. If \(\{\alpha, \beta\}\) is a subset of \(Y_2 - \{e_2\rho_{12}\}\) then the statement is obvious. Otherwise we compute the order of \(e_2\rho_{12}\alpha\) for \(\alpha \in Y_2\) in detail. The involution \(e_2\rho_{12}\) commutes with \((x_i, x_{i+1})\) for \(i = 3, \ldots, n\) and with \(e_n\). It follows that the order of \(e_2\rho_{12}(x_i, x_{i+1})\) and of \(e_2\rho_{12}e_n\) is equal to 2 and therefore the subgroups \(\{e_2\rho_{12}, (x_i, x_{i+1})\}\) for \(i = 3, \ldots, n\) and \(\{e_2\rho_{12}, e_n\}\) are isomorphic to \(\mathbb{Z}_2 \ltimes \mathbb{Z}_2\). The order of \(e_2\rho_{12}(x_1, x_2)e_1e_2\) is equal to 3. It follows that the dihedral group \(D_3\) of order 6 has an epimorphism onto \(\{e_2\rho_{12}, (x_1, x_2)e_1e_2\}\) and this group is therefore finite. The order of \(e_2\rho_{12}(x_2, x_3)e_1\) is equal to 4, and it follows that the dihedral group \(D_4\) of order 8 has an epimorphism onto \(\{e_2\rho_{12}, (x_2, x_3)e_1\}\) and this group is therefore finite. \(\square\)

3. A generating set of SAut\((F_n)\)

In this section we construct a generating set for the group SAut\((F_n)\) with the same finiteness property as the set \(Y_2\) in the previous section.

For \(i, j = 1, \ldots, n, \ i \neq j\) we define the left Nielsen automorphism \(\lambda_{ij}\) as follows:
\[
\lambda_{ij}(x_k) := \begin{cases} x_jx_i & \text{if } k = i, \\ x_k & \text{if } k \neq i. \end{cases}
\]
The group SAut\((F_n)\) is generated by the set \(\{\rho_{ij}, \lambda_{ij} | \ i, j = 1, \ldots, n, \ i \neq j\}\) for \(n \geq 3\), see [11, 2.8]. An easy calculation shows that the commutator of \(\rho_{ij}\) and \(\rho_{jk}\) is equal to \(\rho_{ik}\) and that the commutator of \(\lambda_{ij}\) and \(\lambda_{jk}\) is equal to \(\lambda_{ik}\) for \(i, j, k = 1, \ldots, n\) distinct, therefore SAut\((F_n)\) is generated by the set
\[
Y_3 = \{\rho_{i(i+1)}, \rho_{n1}, \lambda_{i(i+1)}, \lambda_{n1} | \ i = 1, \ldots, n - 1\}.
\]

Our strategy in this section is to modify the set $Y_3$ to obtain a new generating set of $\SAut(F_n)$ which will have the additional property that each group generated by any two of its elements is finite.

**Proposition 3.1.** Let $n \geq 4$.

(i) The group $\SAut(F_n)$ is generated by

$$Y_4 := \{(x_1, x_2)e_1e_2e_3, (x_2, x_3)e_1, (x_i, x_{i+1})e_i, e_2e_4\rho_{12}, e_3e_4 \mid i = 3, \ldots, n - 1\}.$$ 

(ii) The subgroup generated by $Y_4 - \{e_2e_4\rho_{12}\}$ is finite.

(iii) For $\alpha, \beta$ in $Y_4$ the subgroup generated by $\{\alpha, \beta\}$ is finite.

**Proof.** Using the relation $e_i e_j\rho_{ij} e_j e_i = \lambda_{ij}$ for $i, j = 1, \ldots, n$ with $i \neq j$ we obtain $\SAut(F_n) = \{\{\rho_{i(i+1)}, \rho_{01}, e_i e_{i+1}, e_\rho e_1 \mid i = 1, \ldots, n - 1\}\}$. As a next step in the proof, we claim that $\SAut(F_n)$ is generated by the set

$$Y' = \{(x_1, x_2)e_1e_2e_3, (x_2, x_3)e_1, (x_i, x_{i+1})e_3, e_2e_4\rho_{12}, e_3e_4 \mid i = 3, \ldots, n - 1\}.$$ 

The element $e_2e_4 = (x_2, x_3)e_1 e_3 e_4 e_1(x_2, x_3)$ is contained in $\langle Y' \rangle$, therefore

$$\rho_{12} = e_4 e_2 e_4 \rho_{12} \in \langle Y' \rangle.$$ 

From the relation $e_2 e_3 = e_2 e_4 e_3 e_4 \in \langle Y' \rangle$ we see that

$$(x_2, x_3)(x_1, x_2) = (x_2, x_3)e_1 e_2 e_3 e_3 e_2 e_1(x_1, x_2) \in \langle Y' \rangle.$$ 

Using $e_1 e_4 = (x_1, x_2)e_1 e_2 e_3 e_2 e_4 e_3 e_2 e_1(x_1, x_2) \in \langle Y' \rangle$ and $e_3 e_1 = e_1 e_4 e_3 e_4 \in \langle Y' \rangle$ we see that

$$(x_3, x_4)(x_2, x_3) = (x_3, x_4)e_3 e_1 e_1(x_2, x_3) \in \langle Y' \rangle.$$ 

Now we show that the element $e_3(x_1, x_n)$ is contained in $\langle Y' \rangle$. We consider the relation $e_1 e_2 = (x_2, x_3)e_1 e_1 e_3 e_1(x_2, x_3) \in \langle Y' \rangle$, therefore $(x_1, x_2)e_3 = \ldots$. 

\[(x_1, x_2) e_1 e_2 e_3 e_1 e_2 \text{ is contained in } \langle Y' \rangle. \] If \( n \) is odd, then we have
\[
e_3(x_1, x_n) = e_3(x_1, x_2) \left( x_2, x_3 \right) e_3(x_3, x_4) e_3(x_4, x_5) \cdots e_3(x_{n-2}, x_{n-1}) \left( x_{n-1}, x_n \right) e_3(x_n, x_1)
\]
and if \( n \) is even, then
\[
(x_1, x_n) e_3 = (x_1, x_2) e_3(x_2, x_3) \left( x_3, x_4 \right) e_3(x_4, x_5) \cdots e_3(x_{n-2}, x_{n-1}) \left( x_{n-1}, x_n \right) e_3(x_n, x_1)
\]
Using \( e_3(x_1, x_n) \in \langle Y' \rangle \) and \( (x_{n-1}, x_n) e_3 \in Y' \) we obtain \( (x_1, x_n)(x_{n-1}, x_n) \in \langle Y' \rangle \). From the relations
\[
\rho_{(i+1)(i+2)} = (x_i, x_{i+1})(x_{i+1}, x_{i+2}) \rho_{i,i+1} \rho_{i,i+1} \rho_{i+1}(x_{i+1}, x_{i+2})(x_i, x_{i+1}),
\]
\[
e_{i+1}e_{i+2} = (x_i, x_{i+1})(x_{i+1}, x_{i+2}) e_{i+1} e_{i+1} (x_{i+1}, x_{i+2})(x_i, x_{i+1})
\]
we see that \( \rho_{i(i+1)}, e_i e_{i+1} \) are in \( \langle Y' \rangle \) for \( i = 1, \ldots, n - 1 \). Using the relations
\[
\rho_{n1} = (x_{n-1}, x_n)(x_1, x_n) \rho_{(n-1)n} (x_1, x_n)(x_{n-1}, x_n),
\]
\[
e_n e_1 = (x_{n-1}, x_n)(x_1, x_n) e_{n-1} e_n (x_1, x_n)(x_{n-1}, x_n)
\]
we obtain that \( \rho_{n1}, e_n e_1 \in \langle Y' \rangle \) and therefore \( Y' \) is a generating set of \( \text{SAut}(F_n) \).

Now we show that \( (x_i, x_{i+1}) e_3 \) is contained in \( \langle Y'_i \rangle \) for \( i = 4, \ldots, n - 1 \). We have
the relations
\[ e_{5}e_{3} = (x_{4}, x_{5})e_{4}e_{3}e_{4}(x_{4}, x_{5}) \in \langle Y_{4} \rangle, \]
\[ e_{6}e_{3} = (x_{5}, x_{6})e_{5}e_{3}e_{5}(x_{5}, x_{6}) \in \langle Y_{4} \rangle, \]
\[ \ldots \]
\[ e_{n-1}e_{3} = (x_{n-2}, x_{n-1})e_{n-2}e_{3}e_{n-2}(x_{n-2}, x_{n-1}) \in \langle Y_{4} \rangle \]
and we see that \( e_{i}e_{3} \in \langle Y_{4} \rangle \) for \( i = 4, \ldots, n - 1 \) and
\[ (x_{i}, x_{i+1})e_{3} = (x_{i}, x_{i+1})e_{i}e_{i}e_{3} \in \langle Y_{4} \rangle, \]
hence \( \text{SAut}(F_{n}) = \langle Y_{4} \rangle \).

Now we prove the second statement of the proposition. It is easy to verify that the subgroup \( \langle Y' - \{ e_{2}e_{4}\rho_{12} \} \rangle \) of \( \text{SAut}(F_{n}) \) is isomorphic to a subgroup of the semidirect product \( \text{Sym}(n) \ltimes \mathbb{Z}_{2}^{n} \) and therefore finite.

For the proof of the last statement of the proposition we note that the elements in \( Y_{4} \) have finite order. If \( \{ \alpha, \beta \} \) is a subset of \( Y_{4} - \{ e_{2}e_{4}\rho_{12} \} \), then the statement is obvious. Otherwise we consider the subset \( \{ e_{2}e_{4}\rho_{12}, \alpha \} \) for \( \alpha \in Y_{4} \). If the commutator of \( e_{2}e_{4}\rho_{12} \) and \( \alpha \) is equal to one, then the subgroup \( \{ e_{2}e_{4}\rho_{12}, \alpha \} \) is finite. If \( e_{2}e_{4}\rho_{12} \) does not commute with \( \alpha \), then \( \alpha \in \{(x_{1}, x_{2})e_{1}e_{2}e_{3}, (x_{2}, x_{3})e_{1}, (x_{3}, x_{4})e_{3}, (x_{4}, x_{5})e_{4} \} \). We note that
\[ \text{ord}((x_{1}, x_{2})e_{1}e_{2}e_{3}) = \text{ord}((x_{2}, x_{3})e_{1}) = 2, \]
\[ \text{ord}(e_{2}e_{4}\rho_{12}(x_{1}, x_{2})e_{1}e_{2}e_{3}) = 6 \]
and
\[ \text{ord}(e_{2}e_{4}\rho_{12}(x_{2}, x_{3})e_{1}) = 4. \]

It follows that the subgroups
\[ \{ e_{2}e_{4}\rho_{12}, (x_{1}, x_{2})e_{1}e_{2}e_{3} \} \]
and
\[ \{ e_{2}e_{4}\rho_{12}, (x_{2}, x_{3})e_{1} \} \]
are finite. If \( \alpha \) is equal to \( (x_{3}, x_{4})e_{3} \) or \( (x_{4}, x_{5})e_{4} \), then the dihedral group \( D_{4} := \{ r, s \mid r^{4} = s^{2} = 1, srs = r^{-1} \} \) has an epimorphism onto \( \{ e_{2}e_{4}\rho_{12}, (x_{3}, x_{4})e_{3} \} \) and \( \{ e_{2}e_{4}\rho_{12}, (x_{4}, x_{5})e_{4} \} \) and hence these groups are finite. \( \square \)

4. SOME FACTS ABOUT CAT(0) SPACES

In this section we briefly present the main definitions and properties concerning CAT(0) metric spaces. A detailed description of these spaces and their geometry can be found in [5].

We start by reviewing the concept of geodesic spaces. Let \((X, d)\) be a metric space and \(x, y \in X\). A geodesic joining \(x\) and \(y\) is a map \(c_{xy} : [0, l] \to X\), such that \(c_{xy}(0) = x\), \(c_{xy}(l) = y\) and \(d(c_{xy}(t), c_{xy}(t')) = |t - t'|\) for all \(t, t' \in [0, l]\). The image of \(c_{xy}\), denoted by \([x, y]\), is called a geodesic segment. A metric space \((X, d)\) is said to be a geodesic space if every two points in \(X\) can be joined by a geodesic. We say that \(X\) is uniquely geodesic if for all \(x, y \in X\) there is exactly one geodesic joining \(x\) and \(y\).

A geodesic triangle in \(X\) consists of three points \(p_1, p_2, p_3\) in \(X\) and a choice of three geodesic segments \([p_1, p_2]\), \([p_2, p_3]\), \([p_3, p_1]\). Such a geodesic triangle will be denoted by \(\Delta(p_1, p_2, p_3)\). A triangle \(\Delta(\overline{p_1, p_2, p_3})\) in Euclidian space \(\mathbb{R}^2\) is called a comparison triangle for \(\Delta(p_1, p_2, p_3)\) if it is a geodesic triangle in \(\mathbb{R}^2\) and if \(d(p_i, p_j) = d(\overline{p_i, p_j})\) for \(i, j = 1, 2, 3\). A point \(\overline{x}\) in \([\overline{p_1, p_2}, \overline{p_2, p_3}]\) is called a comparison point for \(x \in [p_1, p_2]\) if \(d(x, p_1) = d(\overline{x}, \overline{p_1})\) and \(d(x, p_2) = d(\overline{x}, \overline{p_2})\). A geodesic triangle in \(X\) is said to satisfy the CAT(0) inequality if for all \(x\) and \(y\) in the geodesic triangle and all comparison points \(\overline{x}\) and \(\overline{y}\), the inequality \(d(x, y) \leq d(\overline{x}, \overline{y})\) holds.

**Definition 4.1.** A metric space \(X\) is called a CAT(0) space if \(X\) is a geodesic space and all of its geodesic triangles satisfy the CAT(0) inequality.

One can easily verify from the definition of a CAT(0) space that these spaces are uniquely geodesic, therefore we may use the notation \([x, y]\) for the geodesic segment between \(x\) and \(y\) in the CAT(0) space \(X\) without ambiguity. A subset \(Y\) of a CAT(0) space \(X\) is called convex if for all \(x\) and \(y\) in \(Y\) the geodesic segment \([x, y]\) is contained in \(Y\). Indeed, convex subspaces of a CAT(0) space are again CAT(0) spaces. The diameter of \(Y\) is defined as \(\text{diam}(Y) = \sup \{d(x, y) \mid x, y \in Y\}\). The subset \(Y\) is called bounded if \(\text{diam}(Y)\) is finite. We also note that the metric on a CAT(0) metric space is convex, meaning that for each pair of geodesics \(c_1 : [0, a_1] \to X\) and \(c_2 : [0, a_2] \to X\) with \(c_1(0) = c_2(0)\) the inequality \(d(c_1(ta_1), c_2(ta_2)) \leq td(c_1(a_1), c_2(a_2))\) holds for all \(t \in [0, 1]\).

The class of CAT(0) spaces is large. Perhaps the easiest examples of CAT(0) spaces besides \(d\)-dimensional Euclidean spaces \(\mathbb{R}^d\) are metric trees and in particular simplicial trees, where each edge of a simplicial tree has length 1.

Let us mention an important property of CAT(0) spaces which will be needed later.

**Proposition 4.2.** [5, II.1.4] Any CAT(0) metric space is contractible, in particular all of its higher singular homology groups are trivial.

Now that we have introduced a class of spaces, we need, as in other mathematical theories, structure preserving maps. For a metric space \((X, d)\) an isometry \(f : X \to X\) is a bijection such that \(d(f(x), f(y)) = d(x, y)\) for all \(x\) and \(y\) in \(X\). The group of all isometries of \(X\) will be denoted by \(\text{Isom}(X)\). One easily checks that the fixed point set of an isometry of a CAT(0) space is closed and convex (or empty).
The following version of the Bruhat-Tits Fixed Point Theorem [5, II.2.8] is crucial for our arguments.

**Proposition 4.3.** Let $G$ be a group acting on a complete CAT(0) space $X$ by isometries. Then the following conditions are equivalent:

(i) The group $G$ has a global fixed point.
(ii) Each orbit of $G$ is bounded.
(iii) The group $G$ has a bounded orbit.

If the group $G$ satisfies one of the conditions above, then $G$ is called bounded on $X$.

The implications (i)⇒(ii) and (ii)⇒(iii) are trivial, and (iii)⇒(i) is proven in [5, II.2.8].

The following corollary is standard consequence of Proposition 4.3.

**Corollary 4.4.** Let $G_1$, $G_2$ be groups, $X$ a complete CAT(0) space and $\phi_1 : G_1 \to \text{Isom}(X)$, $\phi_2 : G_2 \to \text{Isom}(X)$ homomorphisms. If $G_1$ and $G_2$ are bounded on $X$ and if $\phi_1(g_1) \circ \phi_2(g_2) = \phi_2(g_2) \circ \phi_1(g_1)$ holds for all $g_1$ in $G_1$ and $g_2$ in $G_2$, then the map

$\phi_1 \times \phi_2 : G_1 \times G_2 \to \text{Isom}(X)$

$(g_1, g_2) \mapsto \phi(g_1) \circ \phi(g_2)$

is a homomorphism and $G_1 \times G_2$ is bounded on $X$.

5. **Helly’s Theorem for Complete CAT(0) Spaces and Homological Properties of Nerves**

The purpose of this section is to verify one important result of convexity theory, Helly’s Theorem, for the class of finite dimensional CAT(0) spaces.

**Theorem 5.1** (Helly’s classical Theorem, [12]). Let $S$ be a finite family of nonempty closed convex subspaces of $\mathbb{R}^d$. If the intersection of $(d+1)$-elements of $S$ is always nonempty, then $\bigcap S$ is nonempty.

There exist numerous different versions of this theorem for CAT(0) spaces in the literature, e.g. for finite families of nonempty convex open resp. closed subspaces, see [4, 3.2], [9, 2], [10, 3.2] and [13, 5.3].

We will study the proof by Debrunner for this result [9]. Debrunner formulated and proved Helly’s Theorem for a family of convex open subspaces of $\mathbb{R}^d$. As we will see in this section, the same line of arguments as in [9] also works for a family of convex open subspaces of a $d$-dimensional CAT(0) space. Farb observed in [10] that Helly’s Theorem for open convex subspaces of a $d$-dimensional CAT(0) space implies the version for closed convex subspaces. Here we include a complete proof for this version.

We require the following definition. For a topological space $X$ we consider the reduced singular homology groups $\tilde{H}_q(X)$ for $q \in \mathbb{Z}$.
**Definition 5.2.** A topological space $X$ is said to be *acyclic* if $\tilde{H}_q(X) = 0$ for all $q \in \mathbb{Z}$.

In particular, if $X$ is acyclic then $X$ is nonempty and connected. For example, every contractible space is acyclic. Hence nonempty CAT(0) spaces are acyclic, see Proposition 4.2.

The proof by Debrunner is based on the following proposition.

**Proposition 5.3.** [9, Lemma $A_m$] Let $X$ be a topological space and $\mathcal{S}$, with $|\mathcal{S}| \geq 2$, a finite family of open nonempty subspaces such that $\cap \mathcal{T}$ is acyclic for all $\mathcal{T} \in \mathcal{S}$ with $|\mathcal{T}| = 1, \ldots, |\mathcal{S}| - 1$.

(i) If $\cap \mathcal{S}$ is empty, then $\tilde{H}_{|\mathcal{S}|-2}(\cup \mathcal{S}) \neq 0$.

(ii) If $\cap \mathcal{S}$ is nonempty, then $\tilde{H}_*(\cup \mathcal{S}) \cong \tilde{H}_{* - |\mathcal{S}|+1}(\cap \mathcal{S})$. In particular, $\cup \mathcal{S}$ is acyclic if and only if $\cap \mathcal{S}$ is acyclic.

**Proof.** We prove both statements by induction on $m := |\mathcal{S}|$. Suppose that $\mathcal{S} = \{X_1, X_2\}$. If $\cap \mathcal{S}$ is empty, then $\cup \mathcal{S}$ is not connected and we have $\tilde{H}_0(\cup \mathcal{S}) \neq 0$. If $\cap \mathcal{S}$ is nonempty, then we consider the reduced Mayer-Vietoris sequence for the pair $(X_1, X_2)$.

\[ \ldots \rightarrow \tilde{H}_q(X_1) \oplus \tilde{H}_q(X_2) \rightarrow \tilde{H}_q(X_1 \cup X_2) \rightarrow \tilde{H}_{q-1}(X_1 \cap X_2) \rightarrow \tilde{H}_{q-1}(X_1) \oplus \tilde{H}_{q-1}(X_2) \rightarrow \ldots \]

We know that $X_1$ and $X_2$ are acyclic, so we obtain

\[ \ldots \rightarrow 0 \rightarrow \tilde{H}_q(X_1 \cup X_2) \rightarrow \tilde{H}_{q-1}(X_1 \cap X_2) \rightarrow 0 \rightarrow \ldots \]

and therefore

\[ \tilde{H}_q(X_1 \cup X_2) \cong \tilde{H}_{q-1}(X_1 \cap X_2). \]

Now assume that $m > 2$. Let $\mathcal{S} = \{X_1, \ldots, X_m\}$ be a family of open subspaces such that the intersection of each $r$ members of this family is acyclic whenever $r = 1, \ldots, m - 1$. We define $U_1 := X_1 \cup \ldots \cup X_{m-1}$, $U_2 := X_m$ and consider the reduced Mayer-Vietoris sequence for the pair $(U_1, U_2)$.

\[ \ldots \rightarrow \tilde{H}_q(U_1) \oplus \tilde{H}_q(U_2) \rightarrow \tilde{H}_q(U_1 \cup U_2) \rightarrow \tilde{H}_{q-1}(U_1 \cap U_2) \rightarrow \tilde{H}_{q-1}(U_1) \oplus \tilde{H}_{q-1}(U_2) \rightarrow \ldots \]

The subspace $U_2$ is acyclic by assumption, and $U_1$ is acyclic by part (ii) of the induction hypothesis. We have

\[ \ldots \rightarrow 0 \rightarrow \tilde{H}_q(U_1 \cup U_2) \rightarrow \tilde{H}_{q-1}(U_1 \cap U_2) \rightarrow 0 \rightarrow \ldots \]

and therefore

\[ \tilde{H}_q(U_1 \cup U_2) \cong \tilde{H}_{q-1}(U_1 \cap U_2). \]

Now we define $\mathcal{S}' = \{X_1 \cap X_m, X_2 \cap X_m, \ldots, X_{m-1} \cap X_m\}$. This is a finite family of open subspaces such that the intersection of each $r$ members of this family is acyclic whenever $r = 1, \ldots, m - 2$.

If $\cap \mathcal{S} = \cap \mathcal{S}'$ is empty, then we have

\[ \tilde{H}_{m-2}(\bigcup \mathcal{S}) = \tilde{H}_{m-2}(U_1 \cup U_2) \]

\[ \cong \tilde{H}_{m-3}(U_1 \cap U_2) \]

\[ = \tilde{H}_{m-3}(\bigcup \mathcal{S}') \]

\[ \neq 0. \quad (\text{Ind. hyp. (i)}) \]
If $\cap S = \cap S'$ is nonempty, then
\[
\tilde{H}_q(\bigcup S) = \tilde{H}_q(U_1 \cup U_2) \\
\cong \tilde{H}_{q-1}(U_1 \cap U_2) \\
= \tilde{H}_{q-1}(S') \\
\cong \tilde{H}_{q-1-(m-1)+1}(\bigcap S') \\
= \tilde{H}_{q-m+1}(\bigcap S). \\
\cap S' = \cap S
\]

Proposition 5.3 gives the following topological version of Helly’s Theorem.

**Theorem 5.4.** (compare [9, Thm. 2]) Suppose that $X$ is a topological space, $d$ a natural number and that $S$ is a finite family of open nonempty subspaces with the properties

(i) $\tilde{H}_q(\bigcup T) = 0$ for all $q \geq d$ and all $T \subseteq S$,
(ii) $\bigcap T$ is acyclic for $T \subseteq S$ with $|T| = 1, \ldots, d + 1$.

Then $\bigcap S$ is acyclic.

**Proof.** Assume that there exist families satisfying the hypotheses but not the conclusion. Let $\{X_1, \ldots, X_m\}$ be such a family of minimal order. Using (ii) we have $m \geq d+2$. This family satisfies hypothesis of Proposition 5.3 by minimality of $m$.

If $X_1 \cap \ldots \cap X_m$ is empty, then by Proposition 5.3 (i) we have $\tilde{H}_{m-2}(\bigcup S) \neq 0$. This contradicts (i).

If $X_1 \cap \ldots \cap X_m$ is nonempty, then there exists $q \geq 0$ with $\tilde{H}_q(X_1 \cap \ldots \cap X_m) \neq 0$. By Proposition 5.3 (ii) follows that $\tilde{H}_{q+m-1}(X_1 \cup \ldots \cup X_m) \neq 0$. We have $q + m - 1 \geq d$. This contradicts (i). □

Using Theorem 5.4 we can easily prove the following version of Helly’s Theorem for a family of convex open subspaces of a $d$-dimensional CAT(0) space. Recall that here by dimension we mean the covering dimension of a metric space. In contrast to that, the compact dimension of a space $X$ is defined as

\[
\text{cdim}(X) = \max \{\dim(Y) \mid Y \subseteq X \text{ compact}\}.
\]

We note that for a metric space $X$ we have clearly

\[
\text{cdim}(X) \leq \dim(X),
\]

since $\dim(Y) \leq \dim(X)$ holds for all compact subsets $Y \subseteq X$.

Before we turn to the proof we need the following result.

**Proposition 5.5.** Let $X$ be a $d$-dimensional CAT(0) space. Then the reduced singular homology groups are $\tilde{H}_q(U) = 0$ for all open subspaces $U \subseteq X$ and all $q \geq d$. 

Proof. For $d = 0$ there is nothing to prove. Assume that $d \geq 1$. As shown by Kleiner in [13, Thm. A] one has for any CAT(0) space $X$

$$\text{cdim}(X) = \max \{k \mid H_k(U, V) \neq 0 \text{ for some open pair } (U, V) \text{ in } X\}.$$ 

We have $d = \dim(X) \geq \text{cdim}(X)$, therefore we obtain $\widetilde{H}_q(U, V) = 0$ for all open pairs $(U, V)$ in $X$ and all $q > d$. Let $U \subseteq X$ be an open subspace. We consider the long exact sequence for the pair $(X, U)$:

$$\ldots \rightarrow \widetilde{H}_{n+1}(X) \rightarrow \widetilde{H}_{n+1}(X, U) \rightarrow \widetilde{H}_n(U) \rightarrow \widetilde{H}_n(X) \rightarrow \ldots$$

The CAT(0) space $X$ is contractible, therefore $\widetilde{H}_q(X) = 0$ and we have

$$\widetilde{H}_{n+1}(X, U) \cong \widetilde{H}_n(U).$$

Using $\widetilde{H}_q(X, U) = 0$ for all $q > d$ we obtain $\widetilde{H}_q(U) = 0$ for all $q \geq d$. □

**Theorem 5.6** (Helly’s Theorem for open convex subspaces of a CAT(0) space). Let $X$ be a $d$-dimensional complete CAT(0) space and $\mathcal{S}$ a finite family of nonempty open convex subspaces. If the intersection of each $(d+1)$-elements of $\mathcal{S}$ is nonempty, then $\bigcap \mathcal{S}$ is nonempty.

**Proof.** The CAT(0) space $X$ has covering dimension $d$, therefore by Proposition 5.5, we have $\widetilde{H}_q(\bigcup T) = 0$ for all $T \subseteq \mathcal{S}$ and $q \geq d$. Since the intersection of convex sets in $T \subseteq \mathcal{S}$ with $|T| = 1, \ldots, d+1$ is nonempty and convex, $\bigcap T$ is by Proposition 4.2 contractible and hence acyclic. By Theorem 5.4 it follows that $\bigcap \mathcal{S}$ is acyclic, in particular nonempty. □

For our application we require a variation of Helly’s Theorem for closed convex subspaces of a $d$-dimensional CAT(0) space and we include a complete proof of this result here. Let us outline the structure of our proof: first we replace each of the closed convex subspaces by a bounded closed convex subspace. For this new family we then construct a swelling consisting of open convex bounded subspaces. Applying Helly’s Theorem 5.6 to this family we obtain a nonempty intersection of it and hence the intersection of a family we started with is also nonempty.

We need the following definition.

**Definition 5.7.** Let $X$ be a topological space. A swelling of a family $(A_i)_{i \in I}$ with $A_i \subseteq X$ is a family $(B_i)_{i \in I}$ with $B_i \subseteq X$, such that $A_i \subseteq B_i$ for every $i \in I$ and for every finite subset $J \subseteq I$ we have

$$\bigcap_{j \in J} A_j = \emptyset \quad \text{if and only if} \quad \bigcap_{j \in J} B_j = \emptyset.$$

Let us first recall an important property of a CAT(0) space which we will need in the proof of the next proposition. By definition, a family of subsets $(A_i)_{i \in I}$ of a metric space is said to have the finite intersection property if the intersection of each finite subfamily is nonempty. Monod proved in [15, Thm. 14] that a family consisting of bounded closed convex subsets of a complete CAT(0) space with the finite intersection property has a nonempty intersection.
Proposition 5.8. Let $X$ be a complete CAT(0) space and let $A, B \subseteq X$ be nonempty closed convex subsets with $A \cap B = \emptyset$ and $A$ bounded, then
\[ d(A, B) := \inf \{ d(a, b) \mid a \in A, b \in B \} > 0. \]

Proof. We assume that $d(A, B) = 0$. Then there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $A$ and a sequence $(b_n)_{n \in \mathbb{N}}$ in $B$ such that $\lim_{n} d(a_n, b_n) = 0$. Let for all $n \in \mathbb{N}$, $A_n \subseteq A$ be the closed convex hull of the set $\{ a_k \mid k \geq n \}$ and consider the family $(A_n)_{n \in \mathbb{N}}$. Since $A$ is bounded, this family consists of bounded closed convex subsets and has the finite intersection property. Therefore the intersection of $\{ A_n \mid n \in \mathbb{N} \}$ is nonempty. Let $x$ be in $\bigcap_{n \in \mathbb{N}} A_n \subseteq A$. Further we define
\[ B_n := \{ y \in X \mid d(y, B) \leq d(a_n, b_n) \}. \]
The set $B_n$ is a closed convex set and $A_n \subseteq B_n$ for $n \in \mathbb{N}$. Therefore $x$ is in $\bigcap_{n \in \mathbb{N}} B_n$. Using $d(x, B) \leq d(a_n, b_n)$ for all $n \in \mathbb{N}$ we obtain $d(x, B) = 0$. The set $B$ is closed, hence $x \in B$ and therefore $x \in A \cap B$. A contradiction. \hfill \Box

Using Proposition 5.8 we can now construct a swelling of a finite family of closed bounded convex subsets of a complete CAT(0) space which consists of open bounded convex subsets.

Proposition 5.9. Let $X$ be a complete CAT(0) space and $\mathcal{F} = \{ F_1, \ldots, F_k \}$ a finite family of nonempty closed bounded convex subsets. Then there exists a swelling $\mathcal{U} = \{ U_1, \ldots, U_k \}$ of $\mathcal{F}$ consisting of nonempty open bounded convex subsets.

Proof. We define
\[ \mathcal{F}_1 := \{ \bigcap \mathcal{G} \mid \mathcal{G} \in \mathcal{F}, \bigcap \mathcal{G} \neq \emptyset, \bigcap \mathcal{G} \cap F_1 = \emptyset \}. \]
By Proposition 5.8, $\min \{ 1, d(F_1, S_i) \mid S_i \in \mathcal{F}_1 \} = \epsilon_1 > 0$. The family
\[ \mathcal{V}_1 = \{ V_1, F_2, \ldots, F_k \} \]
where $V_1 := \{ x \in X \mid d(x, F_1) \leq \frac{\epsilon_1}{2} \}$ is a swelling of $\mathcal{F}$ which consists of nonempty closed bounded convex subsets. More precisely, the subset $V_1$ is convex because the subset $F_1$ and the CAT(0) metric are convex.

Now we assume that for $i \in \{ 1, \ldots, j \}$ the family $\mathcal{V}_j := \{ V_1, \ldots, V_j, F_{j+1}, \ldots, F_k \}$ is defined and is a swelling of $\mathcal{F}$ which consists of nonempty closed bounded subsets. We define
\[ \mathcal{F}_{j+1} := \{ \bigcap \mathcal{G} \mid \mathcal{G} \in \mathcal{V}_j, \bigcap \mathcal{G} \neq \emptyset, \bigcap \mathcal{G} \cap F_{j+1} = \emptyset \}. \]
By Proposition 5.8, $\min \{ 1, d(F_{j+1}, S_i) \mid S_i \in \mathcal{F}_{j+1} \} = \epsilon_{j+1} > 0$. The family
\[ \mathcal{V}_{j+1} = \{ V_1, \ldots, V_{j+1}, F_{j+2}, \ldots, F_k \} \]
where $V_{j+1} := \{ x \in X \mid d(x, F_{j+1}) \leq \frac{\epsilon_{j+1}}{2} \}$ is a swelling of $\mathcal{F}$ which consists of nonempty closed bounded convex subsets. Thus, we can assume that $\mathcal{V}_k = \{ V_1, \ldots, V_k \}$ with $V_i = \{ x \in X \mid d(x, F_i) \leq \epsilon_i \}$ is defined and is a swelling of $\mathcal{F}$. The family $\mathcal{U} := \{ U_1, \ldots, U_k \}$ with $U_i = \{ x \in X \mid d(x, F_i) < \frac{\epsilon_i}{2} \}$ is a swelling of $\mathcal{F}$ which consists of nonempty bounded open convex subsets. \hfill \Box
We are now ready to prove Helly’s Theorem for a finite family of closed convex subspaces of a CAT(0) space.

**Theorem 5.10** (Helly’s Theorem for closed convex subspaces of a CAT(0) space). Let $X$ be a $d$-dimensional complete CAT(0) space and $\mathcal{S}$ a finite family of nonempty closed convex subspaces. If the intersection of each $(d+1)$-elements of $\mathcal{S}$ is nonempty, then $\bigcap \mathcal{S}$ is nonempty.

**Proof.** For each subset $\mathcal{T}$ of $\mathcal{S}$ of order equal to $d+1$ we choose an element $p$ in $\bigcap \mathcal{T}$. Let the union of these elements be the set $\mathcal{P}$. This set is finite and we define

$$\mathcal{S}' = \{\text{conv} \{\mathcal{P} \cap S\} \mid S \in \mathcal{S}\},$$

where $\text{conv} \{\mathcal{P} \cap S\}$ is the closure of the convex hull of $\{\mathcal{P} \cap S\}$. The set $\mathcal{S}'$ consists of nonempty, closed bounded convex subspaces and the intersection of each $(d+1)$-elements of $\mathcal{S}'$ is nonempty. By Proposition 5.9 there exists a swelling $\mathcal{U}$ of $\mathcal{S}'$ which consists of nonempty open convex subspaces. By Helly’s Theorem 5.6 it follows that $\bigcap \mathcal{U}$ is nonempty and therefore $\bigcap \mathcal{S}'$ is nonempty. We have $\emptyset \neq \bigcap \mathcal{S}' \subseteq \bigcap \mathcal{S}$. This completes the proof. \qed

Our main technique in the proofs of Theorems A and B is based on the following crucial corollary. Indeed, it was Farb who discovered the connection between Helly’s Theorem and the combinatorics of generating sets for a large class of groups.

**5.1. Farb’s Fixed Point Criterion.** Let $G$ be a group, $Y$ a finite generating set of $G$ and $X$ a complete $d$-dimensional CAT(0) space. Let $\Phi : G \to \text{Isom}(X)$ be a homomorphism. If each $(d+1)$-element subset of $Y$ has a fixed point in $X$, then $G$ has a fixed point in $X$.

**Proof.** Recall that $\text{Fix}(y) \subseteq X$, the fixed point set of $y \in Y$, is a closed convex subset. Let $y_1$ and $y_2$ be in $Y$, then $\text{Fix}(y_1) \cap \text{Fix}(y_2)$ is equal to $\text{Fix}((y_1, y_2))$ and therefore the statement immediately follows from Helly’s Theorem for closed convex subspaces of a CAT(0) space, 5.10. \qed

6. SOME FACTS ABOUT SIMPLICIAL COMPLEXES AND NERVES

In the previous section we presented an important tool concerning global fixed point properties for isometric actions of groups, namely Farb’s Fixed Point Criterion 5.1. Using this criterion it remains to find a ‘nice’ generating set $Y$ of $\text{Aut}(F_n)$ such that each of its $(d+1)$-element subsets has a fixed point. The purpose of this section is to present techniques to show that an infinite subgroup which is generated by $(d+1)$-elements of $Y$ has a fixed point. The methods presented here are based on certain simplicial complexes.

Let us recall some basic properties of abstract simplicial complexes, see [5] for details. A *simplicial complex* $\Delta$ with a nonempty vertex set $\mathcal{V}$ is a collection of finite subsets of $\mathcal{V}$, called simplices, such that every one element subset of $\mathcal{V}$ is a simplex and $\Delta$ is closed under taking subsets. Let $A \in \Delta$ be a simplex. The cardinality $r$ of $A$ is called the *rank* of $A$ and $r-1$ is called the *dimension*
of $A$. The *dimension* of $\Delta$ is defined as: $\dim(\Delta) := \sup \{ \dim(A) \mid A \in \Delta \}$. As usual, we denote by $|\Delta|$ the geometric realization of the simplicial complex $\Delta$.

In the following we need one basic construction that allows us to produce new simplicial complexes from old ones.

**Definition 6.1.** Let $K_1, K_2$ be simplicial complexes with vertex sets $V_1, V_2$. The *join* $K_1 \ast K_2$ of $K_1$ and $K_2$ is a simplicial complex with vertex set equal to the union $V_1 \cup V_2$ and $A \subseteq V_1 \cup V_2$ is a simplex in $K_1 \ast K_2$ if and only if $A = A_1 \cup A_2$ where $A_1$ is a simplex in $K_1$ and $A_2$ is a simplex in $K_2$.

For example, the join of the standard $n$-simplex, with vertex set $\{0, 1, \ldots, n\}$, denoted by $\Delta_n$, and the standard $m$-simplex $\Delta_m$, with vertex set $\{n+1, \ldots, n+m+1\}$ is an $(n+m+1)$-simplex with a vertex set $\{0, 1, \ldots, n+m+1\}$, see for example Figure 1.

![Figure 1](image1.png)

**Figure 1**

If the geometric realization of a simplicial complex $K_i$ is homeomorphic to a sphere of dimension $d_i$ for $i = 1, 2$, then the geometric realization of $K_1 \ast K_2$ is homeomorphic to a sphere of dimension $d_1 + d_2 + 1$. In particular, we have:

$$|\partial \Delta_n \ast \partial \Delta_m| \cong S^{n+m-1}$$

for $n, m > 0$, where $\partial \Delta_n$ resp. $\partial \Delta_m$ is a boundary of $\Delta_n$ resp. $\Delta_m$, see for example Figure 2.

![Figure 2](image2.png)

**Figure 2**

In the following we want to represent a family of subspaces of a topological space by a combinatorial structure. For this reason we need the following definition.

**Definition 6.2.** Let $X$ be a set and $\mathcal{S}$ a family of subsets of $X$. The *nerve* $\mathcal{N}(\mathcal{S})$ is the simplicial complex whose vertex set is $\mathcal{S}$ and whose nonempty simplices are all finite subsets $\{S_1, \ldots, S_k\} \subseteq \mathcal{S}$ with $S_1 \cap \ldots \cap S_k \neq \emptyset$.
For example, let $S = \{S_0, \ldots, S_k\}$ be a set of nonempty closed convex subspaces of a $d$-dimensional complete CAT(0) space. We consider $\mathcal{N}(S)$ as a subcomplex of the standard $k$-simplex $\Delta_k$. If the nerve $\mathcal{N}(S)$ contains the full $d$-skeleton of $\Delta_k$, then by Helly’s Theorem 5.10 it follows that $|\mathcal{N}(S)| \geq |\Delta_k|$. The main use of the nerve is the following proposition, due to McCord. Recall that a cover of a topological space is said to be point-finite if every point of this space is contained in only finitely many sets of this cover.

**Proposition 6.3.** [14, 2] Let $Y$ be a topological space and $\mathcal{U}$ be a point-finite open cover of $Y$ such that the intersection of any finite subcollection of $\mathcal{U}$ is either empty or contractible. Then $H_*(\mathcal{N}(\mathcal{U})) \cong H_*(Y)$.

The next important result is Theorem 6.5, whose proof relies on Proposition 6.3 and the following result.

**Proposition 6.4.** [4, 3.3] Let $X$ be a complete CAT(0) space and let $S_1, \ldots, S_l$ be subsets of $\text{Isom}(X)$ such that $[S_i, S_j] = 1$ holds for all $1 \leq i < j \leq l$. Let $\mathcal{F}_i = \{\text{Fix}(s) \mid s \in S_i\}$ and $\mathcal{N}_i = \mathcal{N}(\mathcal{F}_i)$. Put $\mathcal{N} = \mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l)$. Then we have

$$\mathcal{N} = \mathcal{N}_1 \ast \ldots \ast \mathcal{N}_l.$$

**Proof.** We first note that the vertex sets of $\mathcal{N}$ and $\mathcal{N}_1 \ast \ldots \ast \mathcal{N}_l$ are equal.

Let $A = \{\text{Fix}(s_1), \ldots, \text{Fix}(s_k)\}$ be a simplex in $\mathcal{N}$. We write the set $A$ as $A = A_1 \cup \ldots \cup A_l$ with $A_i \subseteq \mathcal{F}_i$ for $i$ in $\{1, \ldots, l\}$. The intersection $\bigcap A_i$ is nonempty for all $i$ in $\{1, \ldots, l\}$ and therefore the set $A_i$ is a simplex in $\mathcal{N}_i$ for every $i$ in $\{1, \ldots, l\}$. It follows that the subset $A$ is a union of simplices in $\mathcal{N}_i$ and therefore a simplex in $\mathcal{N}_1 \ast \ldots \ast \mathcal{N}_l$. We have shown that $\mathcal{N} \subseteq \mathcal{N}_1 \ast \ldots \ast \mathcal{N}_l$.

Now we prove the other inclusion. Let $B$ be a simplex in $\mathcal{N}_1 \ast \ldots \ast \mathcal{N}_l$. We know that $B = B_1 \cup \ldots \cup B_l$ where $B_i$ is a simplex in $\mathcal{N}_i$ for $i$ in $\{1, \ldots, l\}$. Now we have to show that $\bigcap B$ is nonempty. We consider the set $S_i' := \{s \in S_i \mid \text{Fix}(s) \in B_i\}$ and the group which is generated by $S_i'$ for $i$ in $\{1, \ldots, l\}$. The subset $B_i$ is a simplex in $\mathcal{N}_i$ and therefore the fixed point set $\text{Fix}(\langle S_i' \rangle)$ is nonempty for all $i$ in $\{1, \ldots, l\}$. Next we note that $[\langle S_i' \rangle, \langle S_j' \rangle] = 1$ for $i \neq j$. It follows from Corollary 4.4 that

$$\bigcap_{i=1}^l \text{Fix}(\langle S_i' \rangle) = \text{Fix}(\bigcup_{i=1}^l (\langle S_i' \rangle)) \neq \emptyset.$$

In particular the set $\bigcap B$ is nonempty and therefore $B$ is a simplex in $\mathcal{N}$. This completes the proof. \qed

**Theorem 6.5.** [4, 3.4] Let $k_1, \ldots, k_l$ be in $\mathbb{N}_{>0}$ and let $X$ be a $d$-dimensional complete CAT(0) space with $0 < d < k_1 + \ldots + k_l$. Let $S_1, \ldots, S_l$ be subsets of $\text{Isom}(X)$ such that $[S_i, S_j] = 1$ holds for all $1 \leq i < j \leq l$. If each $k_i$-element subset of $S_i$ has a fixed point in $X$ for all $i \in \{1, \ldots, l\}$, then for some $j \in \{1, \ldots, l\}$ every finite subset of $S_j$ has a fixed point.
Proof. Assume this is false, i.e. for each $i \in \{1, \ldots, l\}$ let $k_i' \geq k_i$ be minimal such that there exists a $(k_i' + 1)$-element subset

$$T_i = \left\{ s_{i,1}, \ldots, s_{i,k_i'+1} \right\} \subseteq S_i$$

with empty fixed point set. By minimality of $k_i'$, we know that each $k_i'$-element subset of $T_i$ has a fixed point. Therefore the nerve of

$$\mathcal{F}_i = \left\{ \text{Fix}(s_{i,1}), \ldots, \text{Fix}(s_{i,k_i'+1}) \right\}$$

is the boundary of a $k_i'$-simplex. It follows from Proposition 6.4 that

$$\mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_i) \cong \partial \Delta_{k_i'} \ast \ldots \ast \partial \Delta_{k_l'}.$$ 

The geometric realization of this nerve is homeomorphic to a sphere, hence

$$|\mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_i)| \cong S^{k_i'+\ldots+k_l'-1}.$$ 

Therefore the singular homology groups of the above spaces are isomorphic, i.e.

$$H_\ast(|\mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_i)|) \cong H_\ast(S^{k_i'+\ldots+k_l'-1}).$$

Now we replace $\{\mathcal{F}_1, \ldots, \mathcal{F}_l\}$ by a family consisting of a bounded convex closed subsets, as in the proof of Theorem 5.10 and then by a swelling $\{\mathcal{F}_1', \ldots, \mathcal{F}_l'\}$ consisting of bounded convex open subsets, see Proposition 5.9. We have

$$\mathcal{N}(\mathcal{F}_1' \cup \ldots \cup \mathcal{F}_l') \cong \mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l).$$

Using Proposition 6.3 we obtain

$$H_{k_i'+\ldots+k_l'-1}(\mathcal{F}_1' \cup \ldots \cup \mathcal{F}_l') \cong H_{k_i'+\ldots+k_l'-1}(\mathcal{N}(\mathcal{F}_1' \cup \ldots \cup \mathcal{F}_l'))$$

$$\cong H_{k_i'+\ldots+k_l'-1}(|\mathcal{N}(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l)|)$$

$$\cong H_{k_i'+\ldots+k_l'-1}(S^{k_1'+\ldots+k_l'-1}).$$

Because the CAT(0) space $X$ is $d$-dimensional, we have by Proposition 5.5 that the singular homology groups $H_q(\mathcal{F}_1' \cup \ldots \cup \mathcal{F}_l') = 0$ for all $q \geq d$. We have the inequality $k_1' + \ldots + k_l' - 1 \geq d$, in particular $H_{k_i'+\ldots+k_l'-1}(\mathcal{F}_1' \cup \ldots \cup \mathcal{F}_l') \cong 0$. This contradicts

$$H_{k_i'+\ldots+k_l'-1}(S^{k_1'+\ldots+k_l'-1}) \cong \mathbb{Z}.$$

The following consequence of Theorem 6.5 is a crucial tool for proving global fixed point results for infinite subgroups.

**Corollary 6.6.** [4, 3.6] Let $k$ and $l$ be in $\mathbb{N}_{>0}$ and let $X$ be a complete $d$-dimensional CAT(0) space, with $d < k \cdot l$. Let $S$ be a subset of $\text{Isom}(X)$ and let $S_1, \ldots, S_l$ be conjugates of $S$ such that $[S_i, S_j] = 1$ for $i \neq j$. If each $k$-element subset of $S$ has a fixed point in $X$, then each finite subset of $S$ has a fixed point in $X$.

**Proof.** This is clear from Theorem 6.5 since the fixed point sets of the sets $S_i$ are conjugate.

7. Proof of Theorem A

Now we have all the ingredients to prove Theorem A.

**Theorem A.** If $n \geq 4$ and $d < \min \left\{ k \left\lfloor \frac{n}{k+2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\}$, then $\text{Aut}(F_n)$ has property $\mathcal{F}A_d$. In particular, if $n \geq 4$ and $d < 2 \left\lfloor \frac{n}{4} \right\rfloor - 1$, then $\text{Aut}(F_n)$ has property $\mathcal{F}A_d$.

**Proof.** Let $X$ be a $d$-dimensional complete CAT(0) space and

$$\Phi : \text{Aut}(F_n) \rightarrow \text{Isom}(X)$$

an action of $\text{Aut}(F_n)$ on $X$. By Proposition 2.2 the group $\text{Aut}(F_n)$ is generated by the set

$$Y_2 := \{ (x_1, x_2)e_1e_2, (x_2, x_3)e_1, (x_i, x_{i+1}), e_2\rho_{12}, e_n \mid i = 3, \ldots, n - 1 \}.$$

Let us outline the structure of the proof: combining the Bruhat-Tits Fixed Point Theorem 4.3 with Corollaries 4.4 and 6.6 we show the following: if $k \leq d + 1$ and $d < \min \left\{ k \left\lfloor \frac{n}{k+2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\}$, then each $k$-element subset of $Y_2$ has a fixed point. Then by Farb’s Fixed Point Criterion 5.1 the action $\Phi$ has a global fixed point.

As seen in Proposition 2.2 each element in $Y_2$ is an involution and the order of the product of two elements is finite. Let us consider the Coxeter group

$$W = \langle Y_2 \mid (fg)^{\text{ord}(fg)} = 1, f, g \in Y_2 \rangle$$

whose Coxeter diagram looks as follows.

![Figure 3](image)

In particular, we obtain an epimorphism $\pi : W \rightarrow \text{Aut}(F_n)$ and an action

$$\Phi \circ \pi : W \rightarrow \text{Aut}(F_n) \rightarrow \text{Isom}(X).$$

It is obvious that if a subgroup of $W$ has a fixed point, then the image of this subgroup under $\pi$, a subgroup in $\text{Aut}(F_n)$, also has a fixed point. For $k = 2$ we know by Proposition 2.2 that each pair of the generating set $Y_2$ of $\text{Aut}(F_n)$ generates a finite subgroup, therefore by the Bruhat-Tits Fixed Point Theorem 4.3 we obtain that each 2-element subset of $Y_2$ has a fixed point. Now we assume that each $k$-element subset of $Y_2$ has a fixed point. Let $Y'$ be a $(k + 1)$-element subset of $Y_2$.
If \( e_2 \rho_{12} \) is not in \( Y' \), then it follows by Proposition 2.2 that \( \langle Y' \rangle \) is a finite subgroup of \( \text{Aut}(F_n) \) and this subgroup has by Bruhat-Tits Fixed Point Theorem 4.3 a fixed point.

If \( e_2 \rho_{12} \) is in \( Y' \), we consider the Coxeter diagram of \( \langle Y' \rangle \subseteq W \). If it is not connected, then it follows from hypothesis and from Corollary 4.4 that \( \langle Y' \rangle \) has a fixed point. If the Coxeter diagram of \( \langle Y' \rangle \subseteq W \) is connected, then we have the following cases:

1. \( Y' = \{e_2 \rho_{12}, (x_1, x_2)e_1e_2, (x_2, x_3)e_1, (x_3, x_4), \ldots, (x_k, x_{k+1})\} \)
2. \( Y' = \{e_2 \rho_{12}, (x_2, x_3)e_1, (x_3, x_4), \ldots, (x_{k+1}, x_{k+2})\} \).

The involution \( e_n \) is not in \( Y' \): assume that \( e_n \) is contained in \( Y' \), then \( Y' \) consists of at least \( n \) elements. Therefore we must have \( k + 1 \geq n \) which contradicts our assumption that \( k + 1 \leq d + 1 < 2\left\lfloor \frac{n}{4} \right\rfloor + 1 \).

If \( Y' = \{e_2 \rho_{12}, (x_1, x_2)e_1e_2, (x_2, x_3)e_1, (x_3, x_4), \ldots, (x_k, x_{k+1})\} \), then we define the permutations
\[
\tau_i := (x_1, x_{(k+1)-(i-1)+1})(x_2, x_{(k+1)-(i-1)+2})\ldots(x_{k+1}, x_{(k+1)-(i-1)+k+1})
\]
and the sets
\[
S_i := \tau_i Y' \tau_i^{-1}
\]
for \( i \in \{1, \ldots, \left\lfloor \frac{n}{k+1} \right\rfloor \} \). The sets \( S_1, \ldots, S_{\left\lfloor \frac{n}{k+1} \right\rfloor} \) have the property that \([S_i, S_j] = 1\) for \( i \neq j \) as they act nontrivially only on disjoint subsets of \( X \). By the assumption each \( k \)-element subset of \( Y' \) has a fixed point and it follows from Corollary 6.6 that for \( d < k \left\lfloor \frac{n}{k+1} \right\rfloor \) the set \( Y' \) has a fixed point.

If \( Y' \) is equal to \( \{e_2 \rho_{12}, (x_2, x_3)e_1, (x_3, x_4), \ldots, (x_{k+1}, x_{k+2})\} \), then we define the permutations
\[
\sigma_i := (x_1, x_{(k+2)-(i-1)+1})(x_2, x_{(k+2)-(i-1)+2})\ldots(x_{k+2}, x_{(k+2)-(i-1)+k+2})
\]
and the sets
\[
T_i := \sigma_i Y' \sigma_i^{-1}
\]
for \( i \in \{1, \ldots, \left\lfloor \frac{n}{k+2} \right\rfloor \} \). With similar arguments as above it follows that for \( d < k \left\lfloor \frac{n}{k+2} \right\rfloor \) the set \( Y' \) has a fixed point.

So far we have shown that if \( n \geq 4 \) and \( d < \min \left\{ k \left\lfloor \frac{n}{k+2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\} \), then each \( (d + 1) \)-element subset of \( Y_2' \) has a fixed point. By Farb’s Fixed Point Criterion 5.1 it follows that \( \text{Aut}(F_n) \) has a global fixed point. An easy calculation shows:
\[
2 \left\lfloor \frac{n}{4} \right\rfloor - 1 \leq \min \left\{ k \left\lfloor \frac{n}{k+2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\}.
\]
This completes the proof. \( \square \)

Note that, as an immediate corollary of Theorem A, we obtain a similar result for \( \text{GL}_n(\mathbb{Z}) \).

**Corollary 7.1.** If \( n \geq 4 \) and \( d < \min \left\{ k \left\lfloor \frac{n}{k+2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\} \), then \( \text{GL}_n(\mathbb{Z}) \) has property \( \text{FA}_d \). In particular, if \( n \geq 4 \) and \( d < 2\left\lfloor \frac{n}{4} \right\rfloor - 1 \), then \( \text{GL}_n(\mathbb{Z}) \) has property \( \text{FA}_d \).
8. Proof of Theorem B

Using the result of Theorem A, we prove

**Theorem B.** If \( n \geq 5 \) and \( d < \min \left\{ k \left\lfloor \frac{n-1}{k+2} \right\rfloor \mid k = 2, \ldots, d+1 \right\} \), then \( \text{SAut}(F_n) \) has property \( \text{FA}_d \). In particular, if \( n \geq 5 \) and \( d < 2 \left\lfloor \frac{n-1}{4} \right\rfloor - 1 \), then \( \text{SAut}(F_n) \) has property \( \text{FA}_d \).

**Proof.** Let \( X \) be a \( d \)-dimensional complete CAT(0) space and

\[
\Phi : \text{SAut}(F_n) \to \text{Isom}(X)
\]

an action of \( \text{SAut}(F_n) \) on \( X \). By Proposition 3.1 the group \( \text{SAut}(F_n) \) is generated by the set

\[
Y_4 := \{(x_1, x_2)e_1e_2e_3, (x_2, x_3)e_1, (x_i, x_{i+1})e_i, e_2e_4\rho_{12}, e_3e_4 \mid i = 3, \ldots, n-1 \}.
\]

If \( n \leq 8 \), then \( d < 2 \) and the conclusion of Theorem B follows from Proposition 3.1 and Farb’s Fixed Point Criterion 5.1. We hence may assume that \( n \geq 9 \).

We show again the following: if \( k \leq d + 1 \) and \( d < \min \left\{ k \left\lfloor \frac{n-1}{k+2} \right\rfloor \mid k = 2, \ldots, d+1 \right\} \), then each \( k \)-element subset of \( Y_4 \) has a fixed point.

Let us consider the Coxeter-like diagram for the set \( Y_4 \). We draw a graph with \( Y_4 \) as vertex set, joining vertices \( f \) and \( g \) by an edge iff \([f, g] \neq 1\).

![Figure 4](image)

If \( k \) is equal to 2, then we know by Proposition 3.1 and the Bruhat-Tits Fixed Point Theorem 4.3 that each 2-element subset of \( Y_4 \) has a fixed point.

Now we assume that each \( k \)-element subset of \( Y_4 \) has a fixed point. Let \( Y' \) be a \((k+1)\)-element subset of \( Y_4 \). If \( e_2e_4\rho_{12} \) is not in \( Y' \), then it follows from Proposition 3.1 that \( \langle Y' \rangle \) is a finite subgroup of \( \text{SAut}(F_n) \) and this subgroup has by the Bruhat-Tits Fixed Point Theorem 4.3 a fixed point. If \( e_2e_4\rho_{12} \) is in \( Y' \) then we have the following cases:

(1) If there exists a nonempty proper subset $Y''$ of $Y'$ with the property
$$[Y'', Y' - Y''] = 1,$$
then it follows by the assumption and by Corollary 4.4 that $Y'$ has a fixed point.

(2) Otherwise we consider the determinant homomorphism
$$\det : \text{Aut}(F_{n-1}) \to \text{GL}_{n-1}(\mathbb{Z}) \to \mathbb{Z}_2$$
and we define
$$\Psi : \text{Aut}(F_{n-1}) \to \text{SAut}(F_n)$$
as follows
$$f \mapsto f'$$
$$f'(x_k) = \begin{cases} 
  f(x_k) & \text{if } k = 1, \ldots, n-1, \\
  x_k^{\det(f)} & \text{if } k = n.
\end{cases}$$

The homomorphism $\Psi$ is injective and $Y'$ is contained in $\text{im}(\Psi)$ because the element $(x_{n-1}, x_n)e_{n-1}$ is not contained in $Y'$. More precisely, assume that $(x_{n-1}, x_n)e_{n-1} \in Y'$, therefore $k + 1 \geq n - 3$ which contradicts our assumption.

By Theorem A the group $\text{im}(\Psi) \subseteq \text{SAut}(F_n)$ has a global fixed point and therefore $Y'$ has a fixed point.

Again, by Farb’s Fixed Point Criterion 5.1 it follows that $\text{SAut}(F_n)$ has a global fixed point. An easy calculation shows:
$$2 \left\lfloor \frac{n - 1}{4} \right\rfloor - 1 \leq \min \left\{ k \left\lfloor \frac{n - 1}{k + 2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\}.$$ 
This finishes the proof. \[\square\]

Note that, as an immediate corollary of Theorem B we obtain a similar result for $\text{SL}_n(\mathbb{Z})$.

**Corollary 8.1.** If $n \geq 5$ and $d < \min \left\{ k \left\lfloor \frac{n - 1}{k + 2} \right\rfloor \mid k = 2, \ldots, d + 1 \right\}$, then $\text{SL}_n(\mathbb{Z})$ has property $\text{FA}_d$. In particular, if $n \geq 5$ and $d < 2 \left\lfloor \frac{n - 1}{4} \right\rfloor - 1$, then $\text{SL}_n(\mathbb{Z})$ has property $\text{FA}_d$.

**Remark 8.2.**

(i) Bridson proved in personal communication a slightly better bound for $\text{Aut}(F_n)$ for property $\text{FA}_d$, namely the bound $\left\lfloor \frac{2n}{3} \right\rfloor$.

(ii) There exists an upper bound on the dimension $d$ such that $\text{Aut}(F_n)$ can have property $\text{FA}_d$. Consider the symmetric space $P_n(\mathbb{R})$ of positive definite real $n \times n$ matrices. This space is a complete $\text{CAT}(0)$ space of dimension $\frac{1}{2}n(n + 1)$. The group $\text{GL}_n(\mathbb{R})$ acts by isometries on this space via $X \mapsto AXA^t$, where $A \in \text{GL}_n(\mathbb{R})$, $X \in P_n(\mathbb{R})$ and $t$ denotes the transposition. Therefore we have
$$\text{Aut}(F_n) \to \text{GL}_n(\mathbb{R}) \to \text{Isom}(P_n(\mathbb{R}))$$
and this action is fixed point free.
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References


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