Münster J. of Math. **6** (2013), 483–523
urn:nbn:de:hbz:6-55309457271
Münster Journal of Mathematics
© Münster J. of Math. 2013

Topology of the octonionic flag manifold

Augustin-Liviu Mare and Matthieu Willems

(Communicated by Linus Kramer)

Abstract. The octonionic flag manifold $\mathrm{Fl}(\mathbb{O})$ is the space of all pairs in $\mathbb{O}\mathrm{P}^2 \times \mathbb{O}\mathrm{P}^2$ (where $\mathbb{O}\mathrm{P}^2$ denotes the octonionic projective plane) which satisfy a certain "incidence" relation. It comes equipped with the projections $\pi_1, \pi_2 : \mathrm{Fl}(\mathbb{O}) \to \mathbb{O}\mathrm{P}^2$, which are $\mathbb{O}\mathrm{P}^1$-bundles, as well as with an action of the group Spin(8). The first two results of this paper give Borel type descriptions of the usual, respectively Spin(8)-equivariant cohomology of $\mathrm{Fl}(\mathbb{O})$ in terms of π_1 and π_2 (actually the Euler classes of the tangent spaces to the fibers of π_1, respectively π_2, which are rank 8 vector bundles on $\mathrm{Fl}(\mathbb{O})$). We then obtain a Goresky–Kottwitz–MacPherson type description of the ring $H^*_{\mathrm{Spin}(8)}(\mathrm{Fl}(\mathbb{O}))$. Finally, we consider the Spin(8)-equivariant K-theory ring of $\mathrm{Fl}(\mathbb{O})$ and obtain a Goresky–Kottwitz–MacPherson type description of this ring.

Contents

1. Introduction

Let $\mathbb{O}$ denote the (normed, unital, noncommutative, and nonassociative) algebra of octonions and let $\mathbb{O}\mathrm{P}^2$ be the octonionic projective plane (see for instance [3], [10], and [26]). This space is an important example in incidence geometry. It turns out that there exists a natural identification between the space of lines in $\mathbb{O}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ itself. The octonionic flag manifold $\mathrm{Fl}(\mathbb{O})$ is the space of all pairs $(p, \ell) \in \mathbb{O}\mathrm{P}^2 \times \mathbb{O}\mathrm{P}^2$, where p is a point and ℓ a line, such that p and ℓ are incident (see Definition 2.2 below). Both $\mathbb{O}\mathrm{P}^2$ and $\mathrm{Fl}(\mathbb{O})$ carry natural structures of differentiable manifolds. More precisely, we have the natural identifications

$$\mathbb{O}\mathrm{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9) \quad \text{and} \quad \mathrm{Fl}(\mathbb{O}) = \mathrm{F}_4/\mathrm{Spin}(8), \tag{1.1}$$

where F_4 denotes the compact, connected, simply connected Lie group whose Lie algebra is the (compact) real form of the complex simple Lie algebra of type F_4. We consider the natural $\mathbb{O}\mathrm{P}^1$-bundles $\pi_1, \pi_2 : \mathrm{Fl}(\mathbb{O}) \to \mathbb{O}\mathrm{P}^2$ given by

$$\pi_1(p,\ell) := p \quad \text{and} \quad \pi_2(p,\ell) := \ell.$$

Let also $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the rank 8 vector bundles on $\mathrm{Fl}(\mathbb{O})$ given by

$$\mathcal{E}_1(p,\ell) := T_{(p,\ell)}\pi_1^{-1}(p) \quad \text{and} \quad \mathcal{E}_2(p,\ell) := T_{(p,\ell)}\pi_2^{-1}(\ell). \tag{1.2}$$

We will use their Euler classes, $e(\mathcal{E}_1)$ and $e(\mathcal{E}_2)$, relative to appropriate orientations. They both live in the integral cohomology ring of $\mathrm{Fl}(\mathbb{O})$. Our first result gives a presentation of this ring in terms of generators and relations. Before stating it, we make the following convention, which will be used throughout this paper: if it is not specified, the coefficient ring of a cohomology group is $\mathbb{R}$.

Theorem 1.1. *(a) The group $H^*(\mathrm{Fl}(\mathbb{O});\mathbb{Z})$ is free.*

(b) We can orient the bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ in such a way that the cohomology classes $2e(\mathcal{E}_1)+e(\mathcal{E}_2)$ and $e(\mathcal{E}_1)+2e(\mathcal{E}_2)$ are multiples of 3. Moreover, the ring $H^*(\mathrm{Fl}(\mathbb{O});\mathbb{Z})$ is generated by $\frac{1}{3}(2e(\mathcal{E}_1)+e(\mathcal{E}_2))$ and $\frac{1}{3}(e(\mathcal{E}_1)+2e(\mathcal{E}_2))$, the ideal of relations being generated by

$$S_i\left(\frac{1}{3}(2e(\mathcal{E}_1)+e(\mathcal{E}_2)), \frac{1}{3}(-e(\mathcal{E}_1)+e(\mathcal{E}_2)), -\frac{1}{3}(e(\mathcal{E}_1)+2e(\mathcal{E}_2))\right) = 0,$$

$i = 2, 3$. Here S_2 and S_3 denote the second, respectively third elementary symmetric polynomials in three variables.

The proof of this theorem is given in Section 3. It relies on results of Hsiang, Palais, and Terng, see [18], concerning the rational cohomology ring of isotropy orbits of Riemannian symmetric spaces.

We also study the topology of $\mathrm{Fl}(\mathbb{O})$ from the point of view of the action of the group

$$M := \mathrm{Spin}(8)$$

which is canonically induced by equation (1.1). More precisely, we are interested in the equivariant cohomology ring $H^*_M(\mathrm{Fl}(\mathbb{O}))$. We recall that this ring has a natural structure of $H^*(BM)$-module, which is defined as follows: we pick a point $x_0 \in \mathrm{Fl}(\mathbb{O})$ which is fixed by the M-action and consider the ring homomorphism $P^* : H^*(BM) = H^*_M(\{x_0\}) \to H^*_M(\mathrm{Fl}(\mathbb{O}))$ induced by the constant map $P : \mathrm{Fl}(\mathbb{O}) \to \{x_0\}$. We define

$$a.\alpha := P^*(a)\alpha,$$

for all $a \in H^*(BM)$ and all $\alpha \in H^*_M(\mathrm{Fl}(\mathbb{O}))$. In fact, $H^*_M(\mathrm{Fl}(\mathbb{O}))$ becomes in this way a $H^*(BM)$-algebra. It is a unital algebra, and this provides us with a canonical embedding of $H^*(BM)$ into $H^*_M(\mathrm{Fl}(\mathbb{O}))$; otherwise expressed, we identify $H^*(BM)$ with its image under P^*. As a general observation, the fact that the M-equivariant cohomology group of a space is an $H^*(BM)$-algebra *with unit* will be sometimes used in what follows without being explicitly mentioned.

It is worth noting that, since M is a compact Lie group of rank four, $H^*(BM)$ is a polynomial ring with four generators. More precisely, we have

$$H^*(BM) = H^*(BT)^{W_M},$$

where $T \subset M$ is a maximal torus and W_M the Weyl group of the pair (M, T). This gives

$$H^*(BM) = \mathbb{R}[a_1, a_2, a_3, a_4], \tag{1.3}$$

where a_1 lives in $H^4(BM)$, a_2 and a_3 in $H^8(BM)$, and a_4 in $H^{12}(BM)$ (see [19, Sec. 3.7]). The group $H^*(BM; \mathbb{Z})$ is described in [20]; as it turns out from that description, it contains 2-torsion elements, and this is the reason which prevented us from discussing the M-equivariant cohomology with integer coefficients in this paper.

We will give two descriptions of the equivariant cohomology ring $H^*_M(\mathrm{Fl}(\mathbb{O}))$. We first note that the vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ are M-equivariant and orientable, so we can associate to them the equivariant Euler classes $e_M(\mathcal{E}_1)$ and $e_M(\mathcal{E}_2)$, which are elements of $H^8_M(\mathrm{Fl}(\mathbb{O}))$. We also consider the equivariant Euler classes

$$b_k := e_M(\mathcal{E}_k|_{x_0}). \tag{1.4}$$

$k = 1, 2$. These two elements of $H^8_M(\{x_0\}) = H^8(BM)$ are linearly independent and we have $H^*(BM) = \mathbb{R}[a_1, b_1, b_2, a_4]$ (see Lemma 5.8).

Theorem 1.2. *We can orient the bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ in such a way that, as an $H^*(BM)$-algebra, $H^*_M(\mathrm{Fl}(\mathbb{O}))$ is generated by $e_M(\mathcal{E}_1)$ and $e_M(\mathcal{E}_2)$, the ideal of relations being generated by:*

$$\begin{aligned} S_i(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -(e_M(\mathcal{E}_1) + 2e_M(\mathcal{E}_2))) \\ = S_i(2b_1 + b_2, -b_1 + b_2, -(b_1 + 2b_2)), \end{aligned} \tag{1.5}$$

$i = 2, 3$. *As before, S_2 and S_3 are the elementary symmetric polynomials of degree two, respectively three, in three variables.*

The second result about $H^*_M(\mathrm{Fl}(\mathbb{O}))$ gives a Goresky–Kottwitz–MacPherson type presentation of this ring (cp. [11], where formulae for the equivariant cohomology of certain spaces with actions of tori have been obtained). We will see that the fixed point set of the M-action on $\mathrm{Fl}(\mathbb{O})$ can be identified with the symmetric group Σ_3. We put

$$\tilde{b}_1 := b_1, \quad \tilde{b}_2 := -b_2, \quad \tilde{b}_3 := b_1 - b_2. \tag{1.6}$$

We will show the following:

Theorem 1.3. *(a) The (restriction) map*

$$\imath^* : H^*_M(\mathrm{Fl}(\mathbb{O})) \to H^*_M(\Sigma_3) = \prod_{\sigma\in\Sigma_3} H^*(BM)$$

induced by the inclusion $\imath : \Sigma_3 = \mathrm{Fl}(\mathbb{O})^M \hookrightarrow \mathrm{Fl}(\mathbb{O})$ *is injective.*

(b) The image of $\imath^*$ *consists of all ordered sets* $(f_\sigma) \in \prod_{\sigma\in\Sigma_3} H^*(BM)$ *such that* $f_\sigma - f_{(i,j)\sigma}$ *is a multiple of* $\tilde{b}_i - \tilde{b}_j$*, for all* $\sigma \in \Sigma_3$ *and all* i, j *with* $1 \le i < j \le 3$*. Here* $(1,2)$*,* $(2,3)$*, and* $(1,3)$ *denote the obvious elements (transpositions) of* Σ_3*.*

This is a precise description of $H^*_M(\mathrm{Fl}(\mathbb{O}))$, if we take into account that

$$H^*(BM) = \mathbb{R}[a_1, \tilde{b}_1, \tilde{b}_2, a_4],$$

see Lemma 5.8.

The last two theorems are proved in Sections 4 and 5.3 respectively. Theorem 1.3 relies on a cell decomposition of $\mathrm{Fl}(\mathbb{O})$, which is the analogue of the classical Bruhat decomposition for complex flag manifolds; once we have this, we simply apply a result of Harada, Henriques, and Holm, see [13]. The proof of Theorem 1.2 can be outlined as follows: first, $e_M(\mathcal{E}_1)$ and $e_M(\mathcal{E}_2)$ generate $H^*_M(\mathrm{Fl}(\mathbb{O}))$ as an $H^*(BM)$-algebra, roughly because $H^*_M(\mathrm{Fl}(\mathbb{O}))$ is isomorphic to $H^*(\mathrm{Fl}(\mathbb{O}))\otimes H^*(BM)$ as an $H^*(BM)$-module and $H^*(\mathrm{Fl}(\mathbb{O}))$ is generated as a ring by $e(\mathcal{E}_1)$ and $e(\mathcal{E}_2)$, see Theorem 1.1; secondly, one shows that if f is any polynomial in three variables with coefficients in $H^*(BM)$, then the restriction of the cohomology class $f(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -(e_M(\mathcal{E}_1) + 2e_M(\mathcal{E}_2)))$ to an arbitrary M-fixed point $\sigma \in \Sigma_3$ is equal to $g(2b_1+b_2, -b_1+b_2, -(b_1+2b_2))$, where g is a polynomial obtained from f by permuting the three variables in a certain way: this, along with the injectivity of $\imath^*$, explains the relations (1.5).

The last main result of the paper concerns the M-equivariant K-theory ring of $\mathrm{Fl}(\mathbb{O})$. By the "equivariant K-theory ring" of an M-space we always mean the Grothendieck group of all topological M-equivariant complex vector bundles over that space, with the multiplication induced by the tensor product (for more details, we refer to [28]). To describe this ring for $\mathrm{Fl}(\mathbb{O})$, we need some information about the (complex) representation ring $R[M]$ of M. It is known (see for instance [2]) that the ring $R[\mathrm{Spin}(8)]$ is the polynomial ring generated over $\mathbb{Z}$ by X_1, X_2, X_3, X_4, which are as follows: the canonical representation of $\mathrm{SO}(8)$ on $\mathbb{C}^8$ composed with the covering $\mathrm{Spin}(8) \to \mathrm{SO}(8)$, the two complex half-spin representations, and the complexified adjoint action. Our result is a Goresky–Kottwitz–MacPherson type description of the ring $K_M(\mathrm{Fl}(\mathbb{O}))$.

Theorem 1.4. *The canonical homomorphism*

$$K_M(\mathrm{Fl}(\mathbb{O})) \to K_M(\Sigma_3) = \prod_{\sigma\in\Sigma_3} R[M] = \prod_{\sigma\in\Sigma_3} \mathbb{Z}[X_1, X_2, X_3, X_4]$$

induced by the inclusion $\Sigma_3 = \mathrm{Fl}(\mathbb{O})^M \hookrightarrow \mathrm{Fl}(\mathbb{O})$ *is injective. Its image consists of all* $(f_\sigma) \in \prod_{\sigma\in\Sigma_3} \mathbb{Z}[X_1, X_2, X_3, X_4]$ *such that* $f_\sigma - f_{(i,j)\sigma}$ *is a multiple of*

$X_i - X_j$, *for all* $\sigma \in \Sigma_3$ *and all* i, j *with* $1 \leq i < j \leq 3$. *Here* $(1,2)$, $(2,3)$, *and* $(1,3)$ *have the same meaning as in Theorem 1.3.*

A proof can be found in Subsection 5.9. The main tool is again the theorem of Harada, Henriques, and Holm mentioned above.

Remark 1.5. The omnipresence of the symmetric group Σ_3 in the above descriptions is not surprising if we take into accout that $\mathrm{Fl}(\mathbb{O})$ is diffeomorphic to the homogeneous space $\mathrm{F}_4/\mathrm{Spin}(8)$. It is well known that many geometric properties of $\mathrm{Spin}(8)$ and F_4 involve Σ_3-symmetry. The generic term one uses for such phenomena is "triality", see [1], [2, Chap. 5 and 14], [3], and the references therein. For example, a result that goes back to É. Cartan in the 1920s says that the group of outer automorphisms of $\mathrm{Spin}(8)$ is isomorphic to Σ_3 and it acts on the $\mathrm{Spin}(8)$-modules V_8, S_8^+, and S_8^- by permuting them. These representations of $\mathrm{Spin}(8)$ are also important for us here, as follows. First, they induce the complex representations X_1, X_2, and X_3 which appear in Theorem 1.4. (Interesting enough, X_4, which is the adjoint representation of $\mathrm{Spin}(8)$, has no relevance in Theorem 1.4 and can practically be neglected.) In the same spirit, it will turn out that the elements $\tilde{b}_i - \tilde{b}_j$ of $H^*(B\mathrm{Spin}(8))$ we are using in Theorem 1.3 are the $\mathrm{Spin}(8)$-equivariant Euler classes of V_8, S_8^+, and S_8^-, regarded as $\mathrm{Spin}(8)$-equivariant vector bundles over a point and equipped with appropriate orientations. The vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ in Theorems 1.1 and 1.2 are induced by V_8 and S_8^+, respectively, in a way which is described in Section 2.6. We will see there that in the same way, to S_8^- corresponds a third vector bundle, $\mathcal{E}_3$, which we can use in order to bring even more Σ_3-symmetry into our first two theorems: this is explained in detail in Remarks 3.2 and 4.10.

Remark 1.6. The space $\mathrm{Fl}(\mathbb{O})$ is a generalized real flag manifold. By definition, such a manifold is an orbit of the isotropy representation of a Riemannian symmetric space (for more details, see Appendix B). The cohomology ring of the principal orbits of these representations was computed by Hsiang, Palais, and Terng in [18]. An important class of such manifolds consists of those with uniform multiplicity 2, 4, or 8: these are the principal adjoint orbits of compact Lie groups, the quaternionic flag manifold $\mathrm{Fl}_n(\mathbb{H})$, and $\mathrm{Fl}(\mathbb{O})$ respectively. The descriptions given in [18] show that the cohomology ring of each of these spaces is expressed by a Borel type formula, that is, it is isomorphic to the coinvariant ring of a certain Weyl group, see [5]. The spaces $\mathrm{Fl}_n(\mathbb{H})$ and $\mathrm{Fl}(\mathbb{O})$ admit natural group actions similar to the action of a maximal torus on an adjoint orbit (e.g., for $\mathrm{Fl}(\mathbb{O})$ this group is $\mathrm{Spin}(8)$, see above). The equivariant cohomology and equivariant K-theory of a principal adjoint orbit with the action of a maximal torus is well understood (see for example [21]). A natural goal is to decide whether $\mathrm{Fl}_n(\mathbb{H})$ and $\mathrm{Fl}(\mathbb{O})$ behave like adjoint orbits also in the equivariant setting. Positive answers have been given for $\mathrm{Fl}_n(\mathbb{H})$ from the point of view of equivariant cohomology (see [23]) and equivariant K-theory (see [24]). In this paper we discuss the remaining space, which is $\mathrm{Fl}(\mathbb{O})$.

2. The octonionic flag manifold

The goal of this section is to define the flag manifold $\mathrm{Fl}(\mathbb{O})$ and discuss some of its basic properties. For reader's convenience we have included an appendix (see Appendix A) where the complex flag manifold $\mathrm{Fl}_3(\mathbb{C})$ is discussed in a way appropriate to serve us as a model here.

2.1. **$\mathrm{Fl}(\mathbb{O})$ via the Jordan algebra $(\mathfrak{h}_3(\mathbb{O}), \circ)$.** We first recall that, by definition, the space $\mathbb{O}$ has a basis consisting of the elements $e_1 = 1, e_2, \ldots, e_8$; they satisfy certain multiplication rules which make $\mathbb{O}$ into a nonassociative algebra with division (for more details, see [3, Sec. 2]). Let

$$p = x_1 + x_2e_2 + \ldots + x_8e_8$$

be an element of $\mathbb{O}$, where $x_1, x_2, \ldots, x_8 \in \mathbb{R}$. We define its real part,

$$\mathrm{Re}(p) := x_1,$$

its conjugate,

$$\overline{p} := x_1 - x_2e_2 - \ldots - x_8e_8,$$

as well as its norm, $|p|$, given by

$$|p|^2 := p \cdot \overline{p} = x_1^2 + x_2^2 + \ldots + x_8^2.$$

Let us consider

$$\mathfrak{h}_3(\mathbb{O}) := \left\{ \left(\begin{array}{ccc} x_1 & p & q \\ \bar{p} & x_2 & r \\ \bar{q} & \bar{r} & x_3 \end{array} \right) \middle|\, p, q, r \in \mathbb{O},\ x_1, x_2, x_3 \in \mathbb{R} \right\},$$

the space of all 3×3 Hermitian matrices with entries in $\mathbb{O}$.

Definition 2.2. (a) The *octonionic projective plane* $\mathbb{O}\mathrm{P}^2$ is the set of all matrices $a \in \mathfrak{h}_3(\mathbb{O})$ with

$$a^2 = a \quad \text{and} \quad \mathrm{tr}(a) = 1.$$

(b) The *octonionic flag manifold* $\mathrm{Fl}(\mathbb{O})$ is the set of all pairs $(a, b) \in \mathbb{O}\mathrm{P}^2 \times \mathbb{O}\mathrm{P}^2$ with

$$\mathrm{Re}(\mathrm{tr}(ab)) = 0.$$

In the language of incidence geometry, this condition says that the "point" a and the "line" b are "incident" (see for instance [10, Sec. 7.2]).

We equip $\mathfrak{h}_3(\mathbb{O})$ with the $\mathbb{R}$-linear product[1] given by

$$a \circ b := \frac{1}{2}(ab + ba), \tag{2.1}$$

for all $a, b \in \mathfrak{h}_3(\mathbb{O})$.

Definition 2.3. The group F_4 consists of all $\mathbb{R}$-linear transformations g of $\mathfrak{h}_3(\mathbb{O})$ such that

$$g.(a \circ b) = (g.a) \circ (g.b),$$

for all $a, b \in \mathfrak{h}_3(\mathbb{O})$.

[1]The pair $(\mathfrak{h}_3(\mathbb{O}), \circ)$ is actually a Jordan algebra (see [3] and [10]).

The following is a list of properties of the group F_4 which will be needed later. The details can be found for instance in [10], [26], and [2].

- The group F_4 is a compact, connected, simply connected Lie group whose Lie algebra is the compact real form of the complex simple Lie algebra of type F_4.
- For any $a \in \mathfrak{h}_3(\mathbb{O})$ there exist $g \in \mathrm{F}_4$ and $x_1, x_2, x_3 \in \mathbb{R}$ such that
$$x_1 \geq x_2 \geq x_3$$
and
$$g.a = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}.$$
The numbers x_1, x_2, x_3 are uniquely determined by a.
- We have
$$\mathrm{tr}(g.a) = \mathrm{tr}(a), \tag{2.2}$$
for all $g \in \mathrm{F}_4$ and all $a \in \mathfrak{h}_3(\mathbb{O})$.
- We have
$$g.I = I, \tag{2.3}$$
for any $g \in \mathrm{F}_4$. Here I denotes the diagonal matrix $\mathrm{Diag}(1,1,1)$.
- Denote by $\mathfrak{d} \cong \mathbb{R}^3$ the space of all diagonal matrices in $\mathfrak{h}_3(\mathbb{O})$. We have
$$\{g \in \mathrm{F}_4 \mid g.x = x \text{ for all } x \in \mathfrak{d}\} \cong \mathrm{Spin}(8). \tag{2.4}$$
- The space $\mathbb{O}\mathrm{P}^2$ is the F_4-orbit of
$$d_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
The stabilizer of d_1 is isomorphic to the Lie group $\mathrm{Spin}(9)$. Thus, we have the identification
$$\mathbb{O}\mathrm{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9).$$

We also have the following description of $\mathrm{Fl}(\mathbb{O})$.

Proposition 2.4. *The (diagonal) action of* F_4 *on* $\mathrm{Fl}(\mathbb{O})$ *is transitive. If*
$$d_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
then the stabilizer of (d_1, d_2) *is isomorphic to the group* $\mathrm{Spin}(8)$ *given by equation* (2.4). *Thus, we have the identification*
$$\mathrm{Fl}(\mathbb{O}) = \mathrm{F}_4/\mathrm{Spin}(8).$$

Proof. The transitivity of the F_4-action follows from [10, Sec. 7.2 and 7.6]. The second assertion follows from the fact that $g \in \mathrm{F}_4$ fixes $\mathfrak{d}$ pointwise if and only if it fixes d_1 and d_2 (by equation (2.3)). □

Let us now consider the maps $\pi_1, \pi_2 : \mathrm{Fl}(\mathbb{O}) \to \mathbb{O}\mathrm{P}^2$, given by

$$\pi_1(a,b) := a \quad \text{and} \quad \pi_2(a,b) := b,$$

for all $(a,b) \in \mathrm{Fl}(\mathbb{O})$. From the previous considerations we deduce that they are both F_4-equivariant maps.

Proposition 2.5. *The maps π_1 and π_2 are $\mathbb{O}\mathrm{P}^1$-bundles. Here, in analogy with Definition 2.2(a), $\mathbb{O}\mathrm{P}^1$ (the octonionic projective line) is the space of all idempotent elements of $\mathfrak{h}_2(\mathbb{O})$ with trace equal to 1.*

Proof. We show that π_1 is an $\mathbb{O}\mathrm{P}^1$-bundle. Since π_1 is F_4-equivariant, it is sufficient to prove that $\pi_1^{-1}(d_1) = \mathbb{O}\mathrm{P}^1$ (because then, for any $g \in \mathrm{F}_4$ we have $\pi_1^{-1}(g.d_1) = g.\mathbb{O}\mathrm{P}^1$). Indeed, the elements of $\pi_1^{-1}(d_1)$ are of the form (d_1, a), where $a \in \mathbb{O}\mathrm{P}^2$ is such that

$$\mathrm{tr}(ad_1) = 0.$$

The last equation and the fact that $a^2 = a$ imply that

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_2 & r \\ 0 & \bar{r} & x_3 \end{pmatrix}$$

for $x_2, x_3 \in \mathbb{R}$ and $r \in \mathbb{O}$. The set of all such a with $a^2 = a$ and $\mathrm{tr}(a) = 1$ is the subspace $\mathbb{O}\mathrm{P}^1$ of $\{0\} \times \mathfrak{h}_2(\mathbb{O})$ (the latter being canonically embedded in $\mathfrak{h}_3(\mathbb{O})$). This finishes the proof. □

2.6. **Fl($\mathbb{O}$) as a real flag manifold.** Let $\mathfrak{h}_3^0(\mathbb{O})$ be the space of all elements of $\mathfrak{h}_3(\mathbb{O})$ with trace equal to 0. The representation of F_4 on the space $\mathfrak{h}_3(\mathbb{O})$ mentioned in the previous subsection leaves $\mathfrak{h}_3^0$ invariant, see (2.2). The main point of this subsection is that the induced representation of F_4 on $\mathfrak{h}_3^0(\mathbb{O})$ is just the isotropy representation of the (noncompact) Riemannian symmetric space $\mathrm{E}_{6(-26)}/\mathrm{F}_4$. Here $\mathrm{E}_{6(-26)}$ is a certain noncompact real simple Lie group whose Lie algebra $\mathfrak{e}_{6(-26)}$ is a real form of the simple complex Lie algebra of type E_6 (see [16, Table V, Sec. 6, Chap. X]). Appendix C contains more details about this. We extract from there the relevant information, as follows. We have the Cartan decomposition[2]

$$\mathfrak{e}_{6(-26)} = \mathfrak{f}_4 \oplus \mathfrak{h}_3^0(\mathbb{O}) \tag{2.5}$$

where $\mathfrak{f}_4$ is the Lie algebra of F_4 and $\mathfrak{h}_3^0(\mathbb{O})$ the space of all elements of $\mathfrak{h}_3(\mathbb{O})$ with trace equal to 0. We denote by $\mathfrak{d}^0$ the space of all elements of $\mathfrak{d}$ with trace equal to 0. It is a maximal abelian subspace of $\mathfrak{h}_3^0(\mathbb{O})$. Let us also consider the following subspaces of $\mathfrak{h}_3(\mathbb{O})$:

$$\mathfrak{h}_{\gamma_1} := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & \bar{r} & 0 \end{pmatrix} \;\middle|\; r \in \mathbb{O} \right\},$$

[2]This also explains the subscript -26 from $\mathfrak{e}_{6(-26)}$. It is the signature of the Killing form of this Lie algebra. This form is negative definite on $\mathfrak{f}_4$ (of dimension 52) and positive definite on $\mathfrak{h}_3^0(\mathbb{O})$ (of dimension 26).

$$\mathfrak{h}_{\gamma_2} := \left\{ \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & 0 \\ \bar{q} & 0 & 0 \end{pmatrix} \;\middle|\; q \in \mathbb{O} \right\},$$

and

$$\mathfrak{h}_{\gamma_3} := \left\{ \begin{pmatrix} 0 & p & 0 \\ \bar{p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; p \in \mathbb{O} \right\}.$$

We have the obvious decomposition

$$\mathfrak{h}_3^0(\mathbb{O}) = \mathfrak{d}^0 \oplus \mathfrak{h}_{\gamma_1} \oplus \mathfrak{h}_{\gamma_2} \oplus \mathfrak{h}_{\gamma_3}.$$

The spaces $\mathfrak{h}_{\gamma_k}$ are in fact root spaces, in the sense that we have

$$\mathfrak{h}_{\gamma_k} = \{a \in \mathfrak{h}_3(\mathbb{O}) \mid [x,[x,a]] = \gamma_k(x)^2 a \text{ for all } x \in \mathfrak{d}^0\}, \tag{2.6}$$

$k = 1, 2, 3$. Here the bracket $[\ ,\ ]$ is the usual commutator of matrices and $\gamma_1, \gamma_2, \gamma_3 : \mathfrak{d}^0 \to \mathbb{R}$ are described by

$$\begin{aligned} \gamma_1(x_1, x_2, x_3) &:= x_3 - x_2, \\ \gamma_2(x_1, x_2, x_3) &:= x_1 - x_3, \\ \gamma_3(x_1, x_2, x_3) &:= x_1 - x_2, \end{aligned} \tag{2.7}$$

where (x_1, x_2, x_3) stands for $\mathrm{Diag}(x_1, x_2, x_3)$, for any $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 + x_2 + x_3 = 0$ (for more details concerning equation (2.6), see Appendix C). The elements of $\Phi := \{\pm\gamma_1, \pm\gamma_2, \pm\gamma_3\}$ are the roots[3] of $\mathrm{E}_{6(-26)}/\mathrm{F}_4$ with respect to $\mathfrak{d}^0$. We also consider the subsets

$$\Phi^+ = \{\gamma_1, \gamma_2, \gamma_3\} \quad \text{and} \quad \Phi^- = \{-\gamma_1, -\gamma_2, -\gamma_3\}$$

of Φ. They are the positive, respectively negative roots relative to the simple roots γ_1 and γ_2. The following proposition concerns the action of F_4 on $\mathfrak{h}_3^0(\mathbb{O})$ mentioned above.

Proposition 2.7. *Take $x_0 = \mathrm{Diag}(x_1^0, x_2^0, x_3^0) \in \mathfrak{d}^0$ such that x_1^0, x_2^0, and x_3^0 are any two different. Then the F_4-stabilizer of x_0 is the group $\mathrm{Spin}(8)$ in Proposition 2.4. One identifies in this way*

$$\mathrm{Fl}(\mathbb{O}) = \mathrm{F}_4.x_0. \tag{2.8}$$

Proof. An element $g \in \mathrm{F}_4$ leaves x_0 fixed if and only if it leaves the entire $\mathfrak{d}^0$ pointwise fixed (see Proposition B.1). By equation (2.3) this is the same as saying that g leaves $\mathfrak{d}$ pointwise fixed. By equation (2.4), this is equivalent to $g \in \mathrm{Spin}(8)$. □

Consequently $\mathrm{Fl}(\mathbb{O})$ is a real flag manifold (see Appendix B for more on this notion). We deduce from this that the root spaces $\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$, and $\mathfrak{h}_{\gamma_3}$ are $\mathrm{Spin}(8)$-invariant. In fact, the corresponding representations can be described explicitly as follows (see [3, p. 179]):

[3]Strictly speaking, the roots are $\pm\frac{1}{2}(x_3 - x_2), \pm\frac{1}{2}(x_1 - x_3)$, and $\pm\frac{1}{2}(x_1 - x_2)$ (see the end of Appendix C).

- $\mathfrak{h}_{\gamma_1} = V_8$, the standard (matrix) representation of SO(8) on $\mathbb{R}^8$, composed with the covering map $\pi : \mathrm{Spin}(8) \to \mathrm{SO}(8)$
- $\mathfrak{h}_{\gamma_2} = S_8^+$
- $\mathfrak{h}_{\gamma_3} = S_8^-$,

where $S_8^\pm$ are the two real half-spin representations of Spin(8).

The Weyl group of $\mathrm{E}_{6(-26)}/\mathrm{F}_4$ with respect to $\mathfrak{d}^0$ is

$$W := \{n \in \mathrm{F}_4 \mid n.\mathfrak{d}^0 \subset \mathfrak{d}^0\}/\mathrm{Spin}(8). \tag{2.9}$$

The obvious action of this group on $\mathfrak{d}^0$ is faithful. The corresponding group of transformations of $\mathfrak{d}_0$ is generated by the reflections of $\mathfrak{d}^0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ about the lines $\ker \gamma_1, \ker \gamma_2$, and $\ker \gamma_3$ respectively. Thus, W can be identified with the symmetric group Σ_3 which acts on $\mathfrak{d}^0$ by permuting the coordinates x_1, x_2, x_3. Consequently, it also acts on Φ, by

$$(\sigma\gamma)(x) = \gamma(\sigma^{-1}x),$$

for all $\sigma \in \Sigma_3$, $\gamma \in \Phi$, and $x \in \mathfrak{d}^0$.

The tangent space to $\mathrm{Fl}(\mathbb{O})$ (regarded as a submanifold of euclidean space $\mathfrak{h}_3^0(\mathbb{O})$) at the point x_0 introduced in Proposition 2.7 is

$$T_{x_0}\mathrm{Fl}(\mathbb{O}) = \mathfrak{h}_{\gamma_1} \oplus \mathfrak{h}_{\gamma_2} \oplus \mathfrak{h}_{\gamma_3}.$$

Consider the vector bundles $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ on $\mathrm{Fl}(\mathbb{O})$ given by

$$\mathcal{E}_k|_{g.x_0} = g.\mathfrak{h}_{\gamma_k}, \tag{2.10}$$

for any $g \in \mathrm{F}_4$, $k = 1, 2, 3$. These are subbundles of the tangent bundle of $\mathrm{Fl}(\mathbb{O})$. In what follows we will show that $\mathcal{E}_1$ and $\mathcal{E}_2$ defined by equation (2.10) are the same as $\mathcal{E}_1$ and $\mathcal{E}_2$ defined by equation (1.2).

Proposition 2.8. *The vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ defined by equation* (2.10) *satisfy*

$$\mathcal{E}_1|_{g.x_0} = T_{g.x_0}\pi_1^{-1}(\pi_1(g.x_0)) \text{ and } \mathcal{E}_2|_{g.x_0} = T_{g.x_0}\pi_2^{-1}(\pi_2(g.x_0))$$

for all $g \in \mathrm{F}_4$.

Proof. We prove the first equality. By F_4-equivariance, we only need to prove that

$$\mathfrak{h}_{\gamma_1} = T_{(d_1,d_2)}\pi_1^{-1}(d_1).$$

Here we have used that x_0 corresponds to (d_1, d_2) via the isomorphism (2.8). We saw in the proof of Proposition 2.5 that $\pi_1^{-1}(d_1)$ consists of all

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - x_3 & r \\ 0 & \bar{r} & x_3 \end{pmatrix}$$

where $x_3 \in \mathbb{R}$ and $r \in \mathbb{O}$ such that $a^2 = a$. This gives

$$|r|^2 + \left(x_3 - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

This is a full sphere in the 9-dimensional space $\mathbb{O} \times \mathbb{R}$ whose tangent space at $(r, x_3) = (0, 1)$ is described by $x_3 = 0$. Regarded as a subspace of $\mathfrak{h}_3^0(\mathbb{O})$, the latter space is just $\mathfrak{h}_{\gamma_1}$. □

Since $\mathrm{Fl}(\mathbb{O})$ is a real flag manifold, we deduce from Appendix B (especially Theorem B.2) that it has the following natural cell decomposition:

$$\mathrm{Fl}(\mathbb{O}) = \bigsqcup_{\sigma \in \Sigma_3} C_\sigma. \tag{2.11}$$

For each $\sigma \in \Sigma_3$, the cell C_σ is invariant under the action of Spin(8) and we have a Spin(8)-equivariant diffeomorphism

$$C_\sigma \cong \bigoplus \mathfrak{h}_\gamma \tag{2.12}$$

where the sum runs over all $\gamma \in \Phi^+$ such that $\sigma^{-1}\gamma \in \Phi^-$ (see Corollary B.4). The following result will play an important role in our investigation:

Proposition 2.9. *Each C_σ can be identified with $\mathbb{C}^{n(\sigma)}$ for some number $n(\sigma)$. In this way, the canonical maximal torus T of* Spin(8) *(see for example* [2, Chap. 3] *or* [8, Chap. IV, Thm. 3.9]*) acts $\mathbb{C}$-linearly on C_σ.*

Proof. By the decomposition (2.12), it is sufficient to study the action of T on V_8, S_8^+, and S_8^-. The last two representations of Spin(8) are obtained from the first one by (outer) automorphisms of Spin(8) (see [2, Thm. 5.6]). Since any of these automorphisms leave T invariant, it is sufficient to consider the action of T on V_8. Without giving the exact description of T (see the references above), we recall that if $\pi : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ is the canonical double covering, then the elements of $\pi(T)$ are block diagonal 8×8 matrices consisting of four blocks of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta \in \mathbb{R}$. If we identify $\mathbb{R}^8 = \mathbb{C}^4$ via

$$(x_1, x_2, \ldots, x_7, x_8) = (x_1 + ix_2, \ldots, x_7 + ix_8),$$

then the action of any element of T is given by four copies of a map of the form

$$x_1 + ix_2 \mapsto (\cos\theta + i\sin\theta)(x_1 + ix_2)$$

for all $x_1 + ix_2 \in \mathbb{C}$. This map is obviously $\mathbb{C}$-linear (since the multiplication of complex numbers is commutative). □

Finally, we describe the fixed points of the Spin(8)-action on $\mathrm{Fl}(\mathbb{O})$.

Proposition 2.10. *The fixed point set of the* Spin(8)*-action on* $\mathrm{Fl}(\mathbb{O}) = \mathrm{F}_4.x_0$ *is*

$$\mathrm{Fl}(\mathbb{O})^{\mathrm{Spin}(8)} = \Sigma_3 x_0.$$

If $T \subset$ Spin(8) *is the canonical maximal torus, then the fixed points of the T-and the* Spin(8)*-action on* $\mathrm{Fl}(\mathbb{O})$ *are the same.*

Proof. We start with the following claim.
Claim. If $a \in \mathfrak{h}_3^0(\mathbb{O})$ is fixed by T, then a is in $\mathfrak{d}^0$.
To prove this we decompose

$$a = a_0 + a_1 + a_2 + a_3,$$

where $a_0 \in \mathfrak{d}^0$ and $a_j \in \mathfrak{h}_{\gamma_j}$, $j = 1, 2, 3$. Since $\mathfrak{d}^0, \mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$, and $\mathfrak{h}_{\gamma_3}$ are Spin(8)-invariant (see above), all four of a_0, a_1, a_2, a_3 are fixed by T. Assume that a is not in $\mathfrak{d}^0$. Then at least one of a_1, a_2, and a_3 is nonzero. Say first that a_1 is nonzero. We have

$$\pi(g) \cdot a_1 = a_1, \tag{2.13}$$

for all $g \in T$. Here $\pi : \mathrm{Spin}(8) \to \mathrm{SO}(8)$ is the canonical double covering and "$\cdot$" is the matrix multiplication. The SO(8)-stabilizer of a_1 is isomorphic to SO(7). Equation (2.13) says that this stabilizer contains the four dimensional torus $\pi(T)$ as a subgroup, which contradicts $\mathrm{rank}(\mathrm{SO}(7)) = 3$. If a_2 (or a_3) is different from 0, the argument we use is similar: the representation of Spin(8) on $\mathfrak{h}_{\gamma_2} = S_8^+$ (respectively on $\mathfrak{h}_{\gamma_3} = S_8^-$) differs from V_8 by an (outer) automorphism of Spin(8).

The claim implies that

$$\mathrm{Fl}(\mathbb{O})^T \subset \mathrm{Fl}(\mathbb{O}) \cap \mathfrak{d}^0 = \Sigma_3 x_0.$$

For the last equality we have used [10, Section 5 (Hauptachsentransformation von $\mathfrak{J}$)] (see also [26, Section 5, Lemma 1]). On the other hand, equation (2.4) implies that

$$\mathrm{Fl}(\mathbb{O}) \cap \mathfrak{d}^0 \subset \mathrm{Fl}(\mathbb{O})^{\mathrm{Spin}(8)}.$$

This finishes the proof. □

3. Cohomology of Fl(𝕆)

Let us consider again the projection maps $\pi_1, \pi_2 : \mathrm{Fl}(\mathbb{O}) \to \mathbb{O}\mathrm{P}^2$ defined by equation (1.2). We would like to describe π_1 and π_2 by using the identification between $\mathrm{Fl}(\mathbb{O})$ and the orbit $\mathrm{F}_4.x_0$ (see Proposition 2.7). To this end, we consider the following two elements of $\mathfrak{d}^0$:

$$d_1^0 := d_1 - \frac{1}{3}I = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \text{ and } d_2^0 := d_2 - \frac{1}{3}I = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

For each of them, the F_4-stabilizer is a copy of Spin(9) which contains the F_4-stabilizer of x_0, see Proposition 2.7. Thus, the F_4-orbits of d_1^0 and d_2^0 are both diffeomorphic to $\mathbb{O}\mathrm{P}^2$. The maps

$$p_1 : \mathrm{F}_4.x_0 \to \mathrm{F}_4.d_1^0 \quad \text{and} \quad p_2 : \mathrm{F}_4.x_0 \to \mathrm{F}_4.d_2^0$$

given by $p_1(g.x_0) = g.d_1^0$ and $p_2(g.x_0) = g.d_2^0$ are well defined. Let us consider the following diagram:

$$\begin{array}{ccc} \mathrm{Fl}(\mathbb{O}) & \xrightarrow{\pi_1} & \mathbb{O}\mathrm{P}^2 \\ \big\downarrow & & \big\downarrow \\ \mathrm{F}_4.x_0 & \xrightarrow{p_1} & \mathrm{F}_4.d_1^0 \end{array}$$

Here, the vertical arrow in the left-hand side is the F_4-equivariant diffeomorphism which maps (d_1, d_2) to x_0 (see Proposition 2.7). The other vertical arrow in the diagram is the diffeomorphism given by

$$x \mapsto x - \frac{1}{3}I,$$

for all $x \in \mathbb{O}\mathrm{P}^2$: it is an F_4-equivariant diffeomorphism too. The diagram is commutative. We also have a similar diagram which involves p_2 and π_2. Thus, if we identify

$$\mathrm{F}_4.x_0 = \mathrm{Fl}(\mathbb{O}),\ \mathrm{F}_4.d_1^0 = \mathbb{O}\mathrm{P}^2,\ \text{and } \mathrm{F}_4.d_2^0 = \mathbb{O}\mathrm{P}^2$$

then we have the following result:

Proposition 3.1. *The maps* $p_1, p_2 : \mathrm{Fl}(\mathbb{O}) \to \mathbb{O}\mathrm{P}^2$ *defined above are* Spin(8)*-equivariant* $\mathbb{O}\mathrm{P}^1$*-bundles. The vector bundles* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *defined by equation* (2.10) *satisfy*

$$\mathcal{E}_1|_{g.x_0} = T_{g.x_0} p_1^{-1}(p_1(g.x_0)) \text{ and } \mathcal{E}_2|_{g.x_0} = T_{g.x_0} p_2^{-1}(p_2(g.x_0))$$

for all $g \in \mathrm{F}_4$.

This proposition is a direct consequence of Propositions 2.5 and 2.8.

We will use the notation

$$X := \mathrm{Fl}(\mathbb{O}) = \mathrm{F}_4.x_0.$$

Let us consider again the functions $\gamma_1, \gamma_2, \gamma_3 : \mathfrak{d}^0 \to \mathbb{R}$ defined in the previous section (actually the restrictions to $\mathfrak{d}^0$ of the functions given by equation (2.7)). Recall that $\{\pm\gamma_1, \pm\gamma_2, \pm\gamma_3\}$ is a root system of type A_2. We choose the simple root system consisting of γ_1 and γ_2; then $\gamma_3 = \gamma_1 + \gamma_2$ is the third positive root.

In what follows we will construct an orientation on each of the bundles $\mathcal{E}_k$, $k = 1, 2, 3$. First, we pick an orientation on $\mathcal{E}_k|_{x_0} = \mathfrak{h}_{\gamma_k}$ (see below). Then, if $g \in \mathrm{F}_4$, we choose the orientation on $\mathcal{E}_k|_{g.x_0} = g.\mathfrak{h}_{\gamma_k}$ in such a way that the map g is orientation preserving (note that this definition does not depend on g, since the stabilizer group $(\mathrm{F}_4)_{x_0} = \mathrm{Spin}(8)$ is connected and each of its elements acts on $\mathfrak{h}_{\gamma_k}$ as a linear orthogonal transformation, see Section 2). Thus, orienting $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ amounts to choosing orientations on $\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$, and $\mathfrak{h}_{\gamma_3}$. We proceed as follows. First we take into account that $\gamma_3 = s_2\gamma_1$, where s_2 denotes the element of the Weyl group W given by the reflection of $\mathfrak{d}^0$ about $\ker\gamma_2$ (see (2.9) for the definition of W). There exists $n_2 \in \mathrm{F}_4$ with $n_2.\mathfrak{d}^0 = \mathfrak{d}^0$ such that s_2 is equal to the coset $[n_2] = n_2\mathrm{Spin}(8)$ in Σ_3. Consequently, we have

$$\gamma_3 = \gamma_1 \circ n_2^{-1}.$$

This implies that n_2 maps $\mathfrak{h}_{\gamma_1}$ to $\mathfrak{h}_{\gamma_3}$. Similarly, there exists $n_1 \in \mathrm{F}_4$ such that

$$\gamma_3 = \gamma_2 \circ n_1^{-1}.$$

Thus, n_1^{-1} maps $\mathfrak{h}_{\gamma_3}$ to $\mathfrak{h}_{\gamma_2}$. We pick and fix an orientation on $\mathfrak{h}_{\gamma_1}$; the orientations we equip $\mathfrak{h}_{\gamma_2}$ and $\mathfrak{h}_{\gamma_3}$ with are such that the maps n_1 and n_2 are orientation preserving.

The main goal of this section is to prove Theorem 1.1. We proceed as follows. First, observe that X is a 24-dimensional manifold. It is known (see for instance [18, Sec. 5]) that the group $H^*(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module such that

$$\text{rank } H^k(X;\mathbb{Z}) = \begin{cases} 0, \text{ if } k \notin \{0,8,16,24\}, \\ 2, \text{ if } k \in \{8,16\}, \\ 1, \text{ if } k \in \{0,24\}. \end{cases} \tag{3.1}$$

A basis of $H^8(X;\mathbb{Z})$ can be constructed as follows. By Proposition 3.1, the subspaces

$$\mathcal{S}_1 := p_1^{-1}(d_1^0) \quad \text{and} \quad \mathcal{S}_2 := p_2^{-1}(d_2^0)$$

of $\mathrm{Fl}(\mathbb{O})$ are diffeomorphic to $\mathbb{O}\mathrm{P}^1$, hence to the sphere S^8. Moreover, the tangent bundle of $\mathcal{S}_1$ is just $\mathcal{E}_1|_{\mathcal{S}_1}$: thus, the orientation of $\mathcal{E}_1$ chosen above induces an orientation on $\mathcal{S}_1$. Similarly we can also orient $\mathcal{S}_2$. The homology classes $[\mathcal{S}_1]$ and $[\mathcal{S}_2]$ carried by $\mathcal{S}_1$ and $\mathcal{S}_2$ are a basis of $H_8(X;\mathbb{Z})$. Thus, the cohomology classes $\beta_1, \beta_2 \in H^8(X;\mathbb{Z})$ determined by

$$(\beta_i, [\mathcal{S}_j]) = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ otherwise,} \end{cases}$$

$1 \leq i, j \leq 2$, are a basis of $H^8(X;\mathbb{Z})$ (here $(\ ,\ ) : H^8(X;\mathbb{Z}) \otimes H_8(X;\mathbb{Z}) \to \mathbb{Z}$ denotes the evaluation pairing).

We take into account that the elements d_1^0 and d_2^0 of $\mathfrak{d}^0$ satisfy $\gamma_1(d_1^0) = 0$ and $\gamma_2(d_2^0) = 0$. The following equations can be deduced from [18, Proof of Thm. 6.12] (see also [22, Proof of Lemma 3.3]):

$$\begin{aligned} e(\mathcal{E}_1) &= 2\beta_1 + \frac{2\langle\gamma_1,\gamma_2\rangle}{\langle\gamma_2,\gamma_2\rangle}\beta_2 = 2\beta_1 - \beta_2 \\ e(\mathcal{E}_2) &= \frac{2\langle\gamma_2,\gamma_1\rangle}{\langle\gamma_1,\gamma_1\rangle}\beta_1 + 2\beta_2 = -\beta_1 + 2\beta_2 \\ e(\mathcal{E}_3) &= e(\mathcal{E}_1) + e(\mathcal{E}_2). \end{aligned}$$

Thus, we have

$$\beta_1 = \frac{1}{3}(2e(\mathcal{E}_1) + e(\mathcal{E}_2)) \quad \text{and} \quad \beta_2 = \frac{1}{3}(e(\mathcal{E}_1) + 2e(\mathcal{E}_2)). \tag{3.2}$$

From equation (2.10) we deduce that the tangent bundle TX can be split as

$$TX = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3.$$

This implies:

$$e(TX) = e(\mathcal{E}_1)e(\mathcal{E}_2)e(\mathcal{E}_3) = e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)).$$

If $[X]$ is the fundamental homology class of X, then

$$(e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)), [X]) = (e(TX), [X]) = \chi(X) = 6,$$

where $\chi(X)$ is the Euler–Poincaré characteristic of X. We know it is equal to 6 by equation (3.1). Consequently, the cohomology class

$$\frac{1}{6}e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1) + e(\mathcal{E}_2)) \tag{3.3}$$

is a basis of $H^{24}(X;\mathbb{Z})$ over $\mathbb{Z}$.

Let us now consider separately the root system $\{\pm\gamma_1, \pm\gamma_2, \pm\gamma_3\}$. The fundamental weights corresponding to the simple roots γ_1, γ_2 are

$$\lambda_1 = \frac{1}{3}(2\gamma_1 + \gamma_2) \quad \text{and} \quad \lambda_2 = \frac{1}{3}(\gamma_1 + 2\gamma_2).$$

We know that there exists a canonical isomorphism between the ring

$$\mathbb{Q}[\lambda_1, \lambda_2]/\langle \text{nonconstant symmetric polynomials in } \lambda_1, \lambda_2 - \lambda_1, -\lambda_2 \rangle$$

and $H^*(\mathrm{Fl}_3(\mathbb{C});\mathbb{Q})$, see [5]. By a theorem of Bernstein, I. M. Gelfand, and S. I. Gelfand, see [4], the Schubert basis of $H^*(\mathrm{Fl}_3(\mathbb{C});\mathbb{Q})$ over $\mathbb{Q}$ is obtained by considering (the coset of)

$$\frac{1}{6}\gamma_1\gamma_2(\gamma_1 + \gamma_2)$$

and applying successively the divided difference operators Δ_{γ_1} and Δ_{γ_2}. Here, the operator Δ_γ corresponding to the root $\gamma \in \{\gamma_1, \gamma_2\}$ is defined by

$$\Delta_\gamma(f) = \frac{f - f \circ s_\gamma}{\gamma}$$

for any $f \in \mathbb{Q}[\lambda_1, \lambda_2]$ (by s_γ we denote the reflection about the line $\ker \gamma$). The Bernstein–Gelfand–Gelfand basis of $H^*(\mathrm{Fl}_3(\mathbb{C});\mathbb{Q})$ mentioned above consists of the cosets of the following polynomials:

$$\frac{1}{3}\gamma_1(\gamma_1 + \gamma_2), \ \frac{1}{3}\gamma_2(\gamma_1 + \gamma_2)$$
$$\lambda_1, \ \lambda_2$$
$$1.$$

The Schubert classes corresponding to λ_1 and $\frac{1}{3}\gamma_1(\gamma_1 + \gamma_2)$ are Poincaré dual to each other, hence we have:

$$\lambda_1 \cdot \frac{1}{3}\gamma_1(\gamma_1 + \gamma_2) = \frac{1}{6}\gamma_1\gamma_2(\gamma_1 + \gamma_2) + f \tag{3.4}$$

where $f \in \mathbb{Q}[\lambda_1, \lambda_2]$ is in the ideal generated by the nonconstant symmetric polynomials in $\lambda_1, \lambda_2 - \lambda_1, -\lambda_2$.

We now return to the cohomology of X. By [18, Thm. 6.12] (see also [22, Sec. 3]), the ring $H^*(X;\mathbb{Q})$ is generated by β_1 and β_2, the ideal of relations being generated by the symmetric polynomials in $\beta_1, \beta_2-\beta_1, -\beta_2$. Equations (3.2) and (3.4) imply that the equality

$$\beta_1 \frac{1}{3} e(\mathcal{E}_1)(e(\mathcal{E}_1)+e(\mathcal{E}_2)) = \frac{1}{6} e(\mathcal{E}_1)e(\mathcal{E}_2)(e(\mathcal{E}_1)+e(\mathcal{E}_2))$$

holds in $H^*(X;\mathbb{Q})$. The right-hand side of the equation is the fundamental cohomology class of X over $\mathbb{Z}$ (see equation (3.3)). Since β_1 is in $H^*(X;\mathbb{Z})$, we deduce that the cohomology class

$$\frac{1}{3} e(\mathcal{E}_1)(e(\mathcal{E}_1)+e(\mathcal{E}_2)) \tag{3.5}$$

belongs to $H^*(X;\mathbb{Z})$, being the Poincaré dual of β_1 in $H^*(X;\mathbb{Z})$. Similarly, the class

$$\frac{1}{3} e(\mathcal{E}_2)(e(\mathcal{E}_1)+e(\mathcal{E}_2)) \tag{3.6}$$

is in $H^*(X;\mathbb{Z})$, being the Poincaré dual of β_2. Consequently, the classes given by (3.5) and (3.6) are a basis of $H^{16}(X;\mathbb{Z})$.

To complete the proof, it only remains to show that the cohomology classes given by (3.5) and (3.6) can be expressed as polynomials with integer coefficients in β_1 and β_2. Indeed, by using (3.2), we can see that

$$\frac{1}{3} e(\mathcal{E}_1)(e(\mathcal{E}_1)+e(\mathcal{E}_2)) = \beta_1^2$$

and

$$\frac{1}{3} e(\mathcal{E}_2)(e(\mathcal{E}_1)+e(\mathcal{E}_2)) = \beta_2^2.$$

Here we have used the relation

$$\beta_1^2+\beta_2^2-\beta_1\beta_2 = 0,$$

which follows from the fact that the second symmetric polynomial in $\beta_1, \beta_2 - \beta_1, -\beta_2$ is equal to 0.

Remark 3.2. Denoting by $\mathcal{E}_3^-$ the vector bundle $\mathcal{E}_3$ equipped with the orientation which is opposite to the one we have defined above, we can rephrase Theorem 1.1 by saying that the ring $H^*(\mathrm{Fl}(\mathbb{O});\mathbb{Z})$ is generated by

$$x_1 := \frac{1}{3}(e(\mathcal{E}_3^-)-e(\mathcal{E}_2)),\ x_2 := \frac{1}{3}(e(\mathcal{E}_1)-e(\mathcal{E}_3^-)),\ x_3 := \frac{1}{3}(e(\mathcal{E}_2)-e(\mathcal{E}_1)),$$

subject to the relations given by the vanishing of the symmetric polynomials in x_1, x_2, and x_3. See also Appendix A and Remark 1.6.

Remark 3.3. The result stated in Theorem 1.1 is not entirely new: a similar description has been obtained for example in [29, Thm. 2.3] (cp. also [6, Lemma 20.4]). The novelty of Theorem 1.1 is that it gives geometric descriptions of the generators of the cohomology ring.

4. Equivariant cohomology of $\mathrm{Fl}(\mathbb{O})$: generators and relations

In this section we will prove Theorem 1.2. As before, we denote

$$M := \mathrm{Spin}(8) \quad \text{and} \quad X := \mathrm{Fl}_3(\mathbb{O}) = \mathrm{F}_4.x_0.$$

We first recall that the vector spaces $\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$, and $\mathfrak{h}_{\gamma_3}$, as well as the vector bundles $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ have been endowed with orientations in Section 3. The following result is an immediate consequence of Theorem 1.1 (see also Section 3):

Lemma 4.1. *The ring $H^*(X)$ is generated by $e(\mathcal{E}_1)$ and $e(\mathcal{E}_2)$, subject to the relations*

$$S_i(2e(\mathcal{E}_1) + e(\mathcal{E}_2), -e(\mathcal{E}_1) + e(\mathcal{E}_2), -e(\mathcal{E}_1) - 2e(\mathcal{E}_2)) = 0,$$

$i = 2, 3$. *Here S_i denotes the i-th fundamental symmetric polynomial in three variables.*

We are actually interested here in the equivariant cohomology ring $H^*_M(X)$. We recall that, by definition, we have $H^*_M(X) = H^*(EM \times_M X)$, where EM is the total space of the classifying principal bundle $EM \to BM$ of M. As explained in the introduction, $H^*_M(X)$ has a canonical structure of $H^*(BM)$-module. In the case at hand, this module turns out to be free, of rank equal to $\dim H^*(M)$: we say that the M-action on X is *equivariantly formal.* This follows readily from the fact that $H^{\mathrm{odd}}(X) = \{0\}$, see Lemma 4.1, and some standard results in equivariant cohomology, see for example [12, Lemma C.24 and Prop. C.26]. The result stated in the following proposition is a direct consequence of equivariant formality (see for example [15, Prop. 4.4]).

Proposition 4.2. *The graded ring homomorphism $\jmath^* : H^*_M(X) \to H^*(X)$ induced by the canonical inclusion $\jmath : X \to EM \times_M X$ is surjective. Its kernel is*

$$\ker \jmath^* = \langle H^+(BM).H^*_M(X)\rangle,$$

where $H^+(BM)$ denotes the space of all elements of $H^*(BM)$ of strictly positive degree and $\langle H^+(BM).H^*_M(X)\rangle$ is the $\mathbb{R}$-span of all elements of the form $a.\alpha$, with $a \in H^+(BM)$ and $\alpha \in H^*_M(X)$.

Our first goal is to prove that the equations (1.5) hold true. The elements b_1, b_2 of $H^*(BM)$ involved there can actually be expressed as

$$b_k = e_M(\mathfrak{h}_{\gamma_k}),$$

$k = 1, 2$ (see Proposition 2.8). Let us also define

$$b_3 := e_M(\mathfrak{h}_{\gamma_3}). \tag{4.1}$$

The following notation is standard: if $\alpha \in H^*_M(X)$ and $x \in X^M$, then the restriction of α to x is

$$\alpha|_x := i_x^*(\alpha),$$

where $i_x : \{x\} \to X$ is the inclusion map (note that $\alpha|_x \in H^*_M(\{x\}) = H^*(BM)$). The following lemma will be needed later. It is worthwhile recalling at this point that $H^*(BM)$ is identified via P^* with a subspace of $H^*_M(X)$, see Section 1.

Lemma 4.3. *We have*

$$e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2) - e_M(\mathcal{E}_3) = b_1 + b_2 - b_3. \tag{4.2}$$

Proof. For any $k \in \{1, 2, 3\}$ we have $\jmath^*(e_M(\mathcal{E}_k)) = e(\mathcal{E}_k)$ (since, by definition, $e_M(\mathcal{E}_k)$ is the Euler class of a vector bundle over $EM \times_M X$ whose pullback via $\jmath$ is $\mathcal{E}_k$). We deduce that

$$\jmath^*(e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2) - e_M(\mathcal{E}_3)) = e(\mathcal{E}_1) + e(\mathcal{E}_2) - e(\mathcal{E}_3) = 0.$$

From Proposition 4.2 and the fact that $H^k(X) = \{0\}$ for all $1 \le k \le 7$ (see equation (3.1)) we deduce that

$$e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2) - e_M(\mathcal{E}_3) \in P^*(H^*(BM)).$$

The composition $P \circ i_{x_0}$ is the identity function of $\{x_0\}$. Thus, it is now sufficient to note that

$$\begin{aligned}(e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2) - e_M(\mathcal{E}_3))|_{x_0} &= e_M(\mathcal{E}_1|_{x_0}) + e_M(\mathcal{E}_2|_{x_0}) - e_M(\mathcal{E}_3|_{x_0}) \\ &= b_1 + b_2 - b_3,\end{aligned}$$

where we have used equation (2.10). □

The following localization result will also be used here. It will be proved in Subsection 5.3. We recall, see Lemma 2.10, that the fixed points of the M-action on X are given by

$$X^M = \Sigma_3 x_0.$$

Lemma 4.4. *The restriction map*

$$H^*_M(X) \to H^*_M(X^M)$$

is injective.

The strategy we will use in order to justify (1.5) is by showing for each equation that the two sides are equal when restricted to any point in X^M. We recall that X^M is equal to the W-orbit of x_0, where W acts on $\mathfrak{d}^0$ as the reflection group of the root system $\{\pm\gamma_1, \pm\gamma_2, \pm\gamma_3\}$. Since $\{\gamma_1, \gamma_2\}$ is a simple root system, the reflections $s_1 := s_{\gamma_1}$ and $s_2 := s_{\gamma_2}$ generate W. Moreover, we have

$$W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}, \tag{4.3}$$

where $s_1 s_2 s_1 = s_2 s_1 s_2$.

Lemma 4.5. *The restrictions of $e_M(\mathcal{E}_1)$, $e_M(\mathcal{E}_2)$, and $e_M(\mathcal{E}_3)$ to X^M are as follows:*

σ	1	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
$e_M(\mathcal{E}_1)\vert_{\sigma x_0}$	b_1	$-b_1$	b_3	b_2	$-b_3$	$-b_2$
$e_M(\mathcal{E}_2)\vert_{\sigma x_0}$	b_2	b_3	$-b_2$	$-b_3$	b_1	$-b_1$
$e_M(\mathcal{E}_3)\vert_{\sigma x_0}$	b_3	b_2	b_1	$-b_1$	$-b_2$	$-b_3$

TABLE 1

Proof. We have

$$e_M(\mathcal{E}_1)|_{s_1x_0} = e_M\left(\mathcal{E}_1|_{s_1x_0}\right).$$

By definition, $\mathcal{E}_1|_{x_0} = \mathfrak{h}_{\gamma_1}$. The points x_0 and s_1x_0 are antipodal points of the eight dimensional sphere $\mathcal{S}_1 = p_1^{-1}(d_1^0)$, which is embedded in X (see Section 3). By Proposition 3.1, the tangent bundle of $\mathcal{S}_1$ is just the restriction of $\mathcal{E}_1$ to $\mathcal{S}_1$. The orientation of $\mathcal{E}_1$ induces an orientation of the sphere $\mathcal{S}_1$. The space $\mathcal{E}_1|_{s_1x_0}$ is the same as $\mathcal{E}_1|_{x_0} = \mathfrak{h}_{\gamma_1}$, but with the reversed orientation. Consequently,

$$e_M(\mathcal{E}_1)|_{s_1x_0} = -e_M(\mathfrak{h}_{\gamma_1}) = -b_1.$$

Let us now determine

$$e_M(\mathcal{E}_1)|_{s_2x_0} = e_M\left(\mathcal{E}_1|_{s_2x_0}\right).$$

Like in Section 3, we consider again $n_2 \in \mathrm{F}_4$ such that s_2 is the coset $[n_2] = n_2\mathrm{Spin}(8)$ in the Weyl group $W = \Sigma_3$. By definition, since $s_2x_0 = n_2.x_0$, we have

$$\mathcal{E}_1|_{s_2x_0} = n_2.\mathfrak{h}_{\gamma_1}.$$

Moreover, n_2 is an orientation preserving map from $\mathfrak{h}_{\gamma_1}$ to $\mathcal{E}_1|_{s_2x_0}$ (from the way we have oriented $\mathcal{E}_1$ in Section 3). On the other hand, we saw in Section 3 that n_2 maps $\mathfrak{h}_{\gamma_1}$ to $\mathfrak{h}_{\gamma_3}$ by preserving the orientation. We deduce that

$$e_M\left(\mathcal{E}_1|_{s_2x_0}\right) = e_M(\mathfrak{h}_{\gamma_3}) = b_3.$$

We determine now

$$e_M(\mathcal{E}_1)|_{s_1s_2x_0} = e_M\left(\mathcal{E}_1|_{s_1s_2x_0}\right).$$

Take $n_1 \in \mathrm{F}_4$ such that $s_1 = [n_1] = n_1\mathrm{Spin}(8)$ in Σ_3. We have

$$s_1s_2x_0 = s_1^{-1}s_2x_0 = n_1^{-1}.(n_2.x_0).$$

Thus, $\mathcal{E}_1|_{s_1s_2x_0}$ is obtained from $\mathfrak{h}_{\gamma_1}$ by applying first n_2 (and obtaining $\mathfrak{h}_{\gamma_3}$), followed by n_1^{-1} (which gives $\mathfrak{h}_{\gamma_2}$). Consequently,

$$e_M\left(\mathcal{E}_1|_{s_1s_2x_0}\right) = e_M(\mathfrak{h}_{\gamma_2}) = b_2.$$

All other restriction formulae can be proved similarly. □

The following lemma expresses $e_M(\mathcal{E}_3)$ in terms of $e_M(\mathcal{E}_1)$ and $e_M(\mathcal{E}_2)$.

Lemma 4.6. *We have*

$$b_3 = b_1 + b_2$$

and

$$e_M(\mathcal{E}_3) = e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2).$$

Proof. We take equation (4.2) and restrict both sides to s_1x_0. The left-hand side changes according to Lemma 4.5. The right-hand side doesn't change. Indeed, for any $k \in \{1,2,3\}$ we have

$$P^*(b_k)|_{s_1x_0} = i^*_{s_1x_0}(P^*(b_k)) = i^*_{s_1x_0}(P^*(e_M(\mathfrak{h}_{\gamma_k}))) = (P \circ i_{s_1x_0})^*(e_M(\mathfrak{h}_{\gamma_k})),$$

which is the same as the M-equivariant Euler class of the pullback of $\mathfrak{h}_{\gamma_k}$ via the map $P \circ i_{s_1x_0} : \{s_1x_0\} \to \{x_0\}$; this is equal to b_k. Equation (4.2) implies

$$-b_1 + b_3 - b_2 = b_1 + b_2 - b_3,$$

which, in turn, implies the desired equations. □

We are now ready to show that the relations given by equation (1.5) hold true. For each of them we restrict the left-hand side to x_0, s_1x_0, $s_2x_0, \ldots$, $s_1s_2s_1x_0$ and use Lemmata 4.5 and 4.6; each time we do this, we obtain $S_2(2b_1+b_2, -b_1+b_2, -b_1-2b_2)$, respectively $S_3(2b_1+b_2, -b_1+b_2, -b_1-2b_2)$. Indeed, let S be one of the (symmetric) polynomials S_2 and S_3. We have as follows:

$$\begin{aligned}
S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))|_{s_1x_0}&\\
= S(-2b_1 + b_3, b_1 + b_3, b_1 - 2b_3)&\\
= S(-b_1 + b_2, 2b_1 + b_2, -b_1 - 2b_2)&\\
= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).&
\end{aligned}$$

$$\begin{aligned}
S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))|_{s_2x_0}&\\
= S(2b_3 - b_2, -b_3 - b_2, -b_3 + 2b_2)&\\
= S(2b_1 + b_2, -b_1 - 2b_2, -b_1 + b_2)&\\
= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).&
\end{aligned}$$

$$\begin{aligned}
S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))|_{s_1s_2x_0}&\\
= S(2b_2 - b_3, -b_2 - b_3, -b_2 + 2b_3)&\\
= S(-b_1 + b_2, -b_1 - 2b_2, 2b_1 + b_2)&\\
= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).&
\end{aligned}$$

$$\begin{aligned}
S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))|_{s_2s_1x_0}&\\
= S(-2b_3 + b_1, b_3 + b_1, b_3 - 2b_1)&\\
= S(-b_1 - 2b_2, 2b_1 + b_2, -b_1 + b_2)&\\
= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).&
\end{aligned}$$

$$\begin{aligned}
S(2e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) + e_M(\mathcal{E}_2), -e_M(\mathcal{E}_1) - 2e_M(\mathcal{E}_2))|_{s_1s_2s_1x_0}&\\
= S(-2b_2 - b_1, b_2 - b_1, b_2 + 2b_1)&\\
= S(-b_1 - 2b_2, -b_1 + b_2, 2b_1 + b_2)&\\
= S(2b_1 + b_2, -b_1 + b_2, -b_1 - 2b_2).&
\end{aligned}$$

Our second goal is to show that $e_M(\mathcal{E}_1)$ and $e_M(\mathcal{E}_2)$ generate $H^*_M(X)$ as an $H^*(BM)$-algebra. To this end we first recall that the action of M on X is equivariantly formal. From equation (3.1) we deduce that there exists a basis $\bar{\alpha}_0, \dots, \bar{\alpha}_5$ of $H^*_M(X)$ over $H^*(BM)$, such that each $\bar{\alpha}_k$ is a homogeneous element of degree given by

$$\deg \bar{\alpha}_k = \begin{cases} 0, & \text{if } k = 0, \\ 8, & \text{if } k \in \{1, 2\}, \\ 16, & \text{if } k \in \{3, 4\}, \\ 24, & \text{if } k = 5. \end{cases}$$

We need the following lemma.

Lemma 4.7. *There exists a basis $\{\tilde{\alpha}_k \mid k = 0, \dots, 5\}$ of $H^*_M(X)$ as an $H^*(BM)$-module such that:*

(i) if $k \in \{0, \dots, 5\}$, then both $\tilde{\alpha}_k$ and

$$\alpha_k := \jmath^*(\tilde{\alpha}_k) \in H^*(X)$$

are homogeneous of degree given by

$$\deg \tilde{\alpha}_k = \deg \alpha_k = \deg \bar{\alpha}_k,$$

(ii) the set $\{\alpha_k \mid k = 0, \dots, 5\}$ is a basis of $H^*(X)$ over $\mathbb{R}$,

(iii) we have

$$\tilde{\alpha}_1 = e_M(\mathcal{E}_1), \ \tilde{\alpha}_2 = e_M(\mathcal{E}_2),$$

and

$$\alpha_1 = e(\mathcal{E}_1), \ \alpha_2 = e(\mathcal{E}_2).$$

Proof. We set

$$\tilde{\alpha}_k := \begin{cases} \bar{\alpha}_k, & \text{if } k \neq 1, 2, \\ e_M(\mathcal{E}_1), & \text{if } k = 1, \\ e_M(\mathcal{E}_2), & \text{if } k = 2. \end{cases}$$

It is sufficient to show that

$$\bar{\alpha}_1 = r_{11} e_M(\mathcal{E}_1) + r_{21} e_M(\mathcal{E}_2) + a_1$$
$$\bar{\alpha}_2 = r_{12} e_M(\mathcal{E}_1) + r_{22} e_M(\mathcal{E}_2) + a_2$$

where $r_{11}, r_{21}, r_{12}, r_{22}$ are real numbers such that the matrix $(r_{ij})_{1 \le i,j \le 2}$ is nonsingular and a_1, a_2 are in $H^*(BM)$. Indeed, we have

$$\jmath^*(e_M(\mathcal{E}_1)) = e(\mathcal{E}_1) \quad \text{and} \quad \jmath^*(e_M(\mathcal{E}_2)) = e(\mathcal{E}_2).$$

The cohomology classes $e(\mathcal{E}_1)$ and $e(\mathcal{E}_2)$ are a basis of $H^8(X)$ (see Section 3). Also $\jmath^*(\bar{\alpha}_1)$ and $\jmath^*(\bar{\alpha}_2)$ are a basis of $H^8(X)$ (because $\ker \jmath^* = \langle H^+(BM).H^*_M(X)\rangle$). Thus, we can write

$$\jmath^*(\bar{\alpha}_1) = r_{11} \jmath^*(e_M(\mathcal{E}_1)) + r_{21} \jmath^*(e_M(\mathcal{E}_2))$$
$$\jmath^*(\bar{\alpha}_2) = r_{12} \jmath^*(e_M(\mathcal{E}_1)) + r_{22} \jmath^*(e_M(\mathcal{E}_2))$$

for some numbers $r_{11}, r_{21}, r_{12}, r_{22}$ such that the matrix $(r_{ij})_{1\le i,j\le 2}$ is nonsingular. Consequently, the differences $\bar{\alpha}_1 - r_{11}e_M(\mathcal{E}_1) - r_{21}e_M(\mathcal{E}_2)$ and $\bar{\alpha}_2 - r_{12}e_M(\mathcal{E}_1) - r_{22}e_M(\mathcal{E}_2)$ are linear combinations with coefficients in $H^+(BM)$ of $\bar{\alpha}_0, \dots, \bar{\alpha}_5$. By dimension reasons, both of them must live in $H^+(BM)$. This finishes the proof. □

Let us now consider the isomorphism of $H^*(BM)$-modules

$$\Psi : H^*_M(X) \to H^*(X) \otimes H^*(BM)$$

given by $\Psi(\tilde{\alpha}_k) := \alpha_k$, for all $k = 0, \dots, 5$.

From now on we identify the $H^*(BM)$-algebra $H^*_M(X)$ with $H^*(X) \otimes H^*(BM)$ equipped with the product $\circ$. The latter is defined by the fact that it is $H^*(BM)$-bilinear and it satisfies the condition

$$\alpha_k \circ \alpha_\ell := \Psi(\tilde{\alpha}_k \tilde{\alpha}_\ell),$$

for all $k, \ell \in \{0, \dots, 5\}$. We stress that

$$H^*_M(X) = (H^*(X) \otimes \mathbb{R}[a_1, a_2, a_3, a_4], \circ) \tag{4.4}$$

as $\mathbb{R}[a_1, a_2, a_3, a_4]$-algebras, see equation (1.3). The usual grading of $H^*(X)$ together with

$$\deg a_1 = 4,\ \deg a_2 = \deg a_3 = 8,\ \deg a_4 = 12,$$

induces a grading on $H^*(X) \otimes H^*(BM)$. The following two properties of the product $\circ$ will be used later. If $\alpha, \beta \in H^*(X)$ are homogeneous elements, then we have:

(i) $\alpha \circ \beta$ is a homogeneous element of $H^*(X) \otimes H^*(BM)$ of degree given by $\deg(\alpha \circ \beta) = \deg \alpha + \deg \beta$,
(ii) $\alpha \circ \beta = \alpha\beta +$(a linear combination of multiples of $H^+(BM)$).

Point (i) follows from the fact that the map Ψ is degree preserving. To justify point (ii) it is sufficient to take $\alpha = \alpha_k$ and $\beta = \alpha_\ell$, where $k, \ell \in \{0, \dots, 5\}$; we use the fact that the following diagram is commutative:

$$\begin{array}{ccc} H^*_M(X) & \xrightarrow{\quad\Psi\quad} & H^*(X) \otimes H^*(BM) \\ & {\scriptstyle \jmath^*}\searrow \quad \swarrow & \\ & H^*(X). & \end{array}$$

Here the arrow in the right-hand side is the canonical projection.

Lemma 4.8. *The classes*

$$\epsilon_1 := e_M(\mathcal{E}_1) \quad \textit{and} \quad \epsilon_2 := e_M(\mathcal{E}_2)$$

generate $H^*_M(X)$ as an $H^*(BM)$-algebra. Equivalently, in terms of the identification (4.4)*, the classes*

$$\epsilon_1 = e(\mathcal{E}_1) \quad \textit{and} \quad \epsilon_2 = e(\mathcal{E}_2)$$

generate $(H^*(X) \otimes H^*(BM), \circ)$ as an $H^*(BM)$-algebra.

Proof. It is sufficient to prove that for any $k \in \{0, \ldots, 5\}$, α_k can be written as a polynomial expression in ϵ_1 and ϵ_2 with coefficients in $H^*(BM)$, the product being $\circ$. We prove this by induction on k. The claim is obvious for $k = 0$, as α_0 is just a number (element of $H^0(X)$). Let us now make the induction step: take $k \in \{0, \ldots, 5\}$, $k \geq 1$. We know that ϵ_1 and ϵ_2 generate $H^*(X)$ (see Lemma 4.1). Thus, we have

$$\alpha_k = f(\epsilon_1, \epsilon_2),$$

where f is a polynomial in two variables and the product in the right hand side is the usual (cup) product. Let $f^\circ(\epsilon_1, \epsilon_2)$ be the element of $H^*(X) \otimes H^*(BM)$ obtained by evaluating f in terms of the product $\circ$. By property (ii) of $\circ$, $\alpha_k - f^\circ(\epsilon_1, \epsilon_2)$ is a linear combination of terms of the form $a.\alpha_\ell$, where $a \in H^+(BM)$ and $\ell \in \{0, \ldots, 5\}$ with $\deg \alpha_\ell < \deg \alpha_k$. The last condition implies $\ell < k$: we only need to use the induction hypothesis. □

The following lemma will finish the proof of Theorem 1.2.

Lemma 4.9. *The ideal of relations in $H^*_M(X)$ with respect to ϵ_1 and ϵ_2 is generated by* (1.5)*.*

Proof. Let us consider the polynomials $g_2, g_3 \in \mathbb{R}[x_1, x_2]$ given by

$$g_i = S_i(2x_1 + x_2, -x_1 + x_2, -(x_1 + 2x_2)), \tag{4.5}$$

$i = 2, 3$. We prove that if $f(x_1, x_2) \in H^*(BM) \otimes \mathbb{R}[x_1, x_2]$ such that

$$f^\circ(\epsilon_1, \epsilon_2) = 0, \tag{4.6}$$

then f is in the ideal generated by the polynomials

$$f_i(x_1, x_2) := g_i(x_1, x_2) - g_i(b_1, b_2),$$

$i = 2, 3$ (here $f^\circ(\epsilon_1, \epsilon_2)$ is the element of $H^*(X) \otimes H^*(BM)$ obtained by evaluating $f(x_1, x_2)$ on ϵ_1, ϵ_2 in the ring $(H^*(X) \otimes H^*(BM), \circ)$). We prove this claim by induction on $\deg f$: throughout this proof, *degree* will always be considered only with respect to x_1 and x_2. If $\deg f = 0$ then the claim is obvious. Let us now perform the induction step. We consider a nonconstant polynomial $f(x_1, x_2)$ as above, satisfying equation (4.6). Let $h(x_1, x_2)$ be the component of $f(x_1, x_2)$ of highest degree (with respect to x_1, x_2). From the fact that $f^\circ(\epsilon_1, \epsilon_2) = 0$ and property (ii) of $\circ$ we deduce that

$$h(\epsilon_1, \epsilon_2) = 0,$$

the product involved in the left-hand side being the usual (cup) product. By Lemma 4.1, $h(x_1, x_2)$ is a combination with coefficients in $H^*(BM) \otimes \mathbb{R}[x_1, x_2]$ of $g_2(x_1, x_2)$ and $g_3(x_1, x_2)$. We come back to equation (4.6) and replace h by the expression mentioned in the previous sentence, where we complete each occurrence of g_i to f_i (by adding and subtracting the necessary quantity). The cancellations which we obtain allow us to obtain another condition of type (4.6), this time with a polynomial f of degree strictly smaller than the previous one. Finally, we use the induction hypothesis. □

Remark 4.10. Like in Remark 3.2, we denote by $\mathcal{E}_3^-$ the vector bundle $\mathcal{E}_3$ with the reversed orientation. Set

$$\tilde{x}_1 := \frac{1}{3}(e_M(\mathcal{E}_3^-) - e_M(\mathcal{E}_2)),\ \tilde{x}_2 := \frac{1}{3}(e_M(\mathcal{E}_1) - e_M(\mathcal{E}_3^-)),$$
$$\tilde{x}_3 := \frac{1}{3}(e_M(\mathcal{E}_2) - e_M(\mathcal{E}_1)),$$
$$u_1 := \frac{1}{3}(-b_3 - b_2),\ u_2 := \frac{1}{3}(b_1 + b_3),\ u_3 := \frac{1}{3}(b_2 - b_1).$$

Theorem 1.2 can be rephrased by saying that $H^*_M(\mathrm{Fl}(\mathbb{O}))$ is generated as an $H^*(BM)$-algebra by $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, subject to the following relations:

$$S_i(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = S_i(u_1, u_2, u_3),\ i = 2, 3.$$

A similar description holds for $H^*_T(\mathrm{Fl}_3(\mathbb{C}))$, see Appendix A.

Remark 4.11. Another presentation of the ring $H^*_M(X) = H^*_{\mathrm{Spin}(8)}(\mathrm{F}_4/\mathrm{Spin}(8))$ can be deduced from [17, Cor. 5.10], since F_4 and Spin(8) have the same rank.

5. Presentations of Goresky–Kottwitz–MacPherson type

5.1. **A theorem of Harada, Henriques, and Holm.** Motivated by the well-known result of Goresky, Kottwitz, and MacPherson, see [11], concerning the equivariant cohomology of complex projective varieties that are acted on by tori, Harada, Henriques, and Holm considered in [13] actions of arbitrary topological groups along with equivariant cohomology theories associated to them. They obtained descriptions of the corresponding (cohomology) rings for spaces equipped with a certain stratification. We will confine ourselves here to state a weaker version of their main result, which is strictly what we need in order to prove Theorems 1.3 and 1.4. If M is an arbitrary compact connected Lie group and X a space acted on by M, we denote by $E^*_M(X)$ the M-equivariant cohomology ring with real coefficients or the M-equivariant complex topological K-theory ring of X: the result is valid for both $H^*_M(\cdot)$ and $K^*_M(\cdot)$.

Theorem 5.2 ([13]). *Let*

$$X = \bigsqcup_{k=1}^{s} C_k$$

be a finite CW complex whose open cells C_k, $1 \le k \le s$, satisfy the following properties:

(i) C_k is an even dimensional real vector space equipped with an M-linear action with a unique fixed point, say p_k, which is identified with 0.

(ii) We can decompose

$$C_k = \bigoplus_{1 \le \ell \le k} C_{k\ell}, \tag{5.1}$$

where $C_{k\ell}$ are vector subspaces (possibly equal to $\{0\}$) of C_k; the boundary $\partial_X(C_{k\ell})$ of $C_{k\ell}$ in X consists of only one point, which is fixed by the M-action (in the case where $C_{k\ell} = \{0\}$, the fixed point is p_k).

(iii) For any $k \in \{1, \dots, s\}$, the equivariant Euler classes $e_M(C_{k\ell})$, where $1 \le \ell \le k$ such that $C_{k\ell} \neq \{0\}$, are relatively prime elements of $E_M^*(\mathrm{pt.})$ (here we regard $C_{k\ell}$ as a vector bundle over a point).

Then the map $\imath^* : E_M^*(X) \to E_M^*(X^M)$ induced by the inclusion of the M-fixed point set X^M into X is injective. Moreover, the image of $\imath^*$ consists of all

$$(f_k) \in E_M^*(X^M) = \prod_{k=1}^{s} E_M^*(\mathrm{pt.})$$

such that $f_k - f_\ell$ is divisible by $e_M(C_{k\ell})$ for all $1 \le \ell < k \le s$ with $C_{k\ell} \neq \{0\}$.

5.3. **The CW complex structure of Fl($\mathbb{O}$).** We aim to apply Theorem 5.2 to the special case of $X := \mathrm{Fl}(\mathbb{O})$ and $M := \mathrm{Spin}(8)$. In this section we are concerned with assumptions (i) and (ii) in that theorem. More precisely, recall that

$$X := \mathrm{Fl}(\mathbb{O}) = \mathrm{F}_4.x_0,$$

where $x_0 = \mathrm{Diag}(x_1^0, x_2^0, x_3^0)$, with $x_1^0, x_2^0, x_3^0 \in \mathbb{R}$, any two distinct, such that $x_1^0 + x_2^0 + x_3^0 = 0$; this time we also assume that $x_2^0 < x_3^0 < x_1^0$, i.e., $\gamma_1(x_0) > 0$ and $\gamma_2(x_0) > 0$. We use the CW decomposition of Fl($\mathbb{O}$) described by equation (2.11). That is, we choose $C_k := C_\sigma$, for $\sigma \in \Sigma_3$. The splitting (5.1) is the one described by equation (2.12). Assumption (i) in Theorem 5.2 follows from Proposition 2.10 and the fact that $C_\sigma \cap \Sigma_3 x_0 = \{\sigma x_0\}$. For assumption (ii), we will need the explicit embedding of C_σ in X, as given in Theorem B.2 (b). That is, we consider the root spaces $\mathfrak{g}_\gamma \subset \mathfrak{e}_{6(-26)}$, where $\gamma \in \Phi$, as well as the diffeomorphism $\sum \mathfrak{g}_\gamma \to X$, $x \mapsto \exp(x)(\sigma x_0)$, where the sum in the domain runs over all $\gamma \in \Phi^+$ such that $\sigma^{-1}\gamma \in \Phi^-$. Assumption (ii) follows readily from the following lemma.

Lemma 5.4. *If σ and γ are as above, then the boundary of $\exp(\mathfrak{g}_\gamma)(\sigma x_0)$ in X is $\{s_\gamma \sigma x_0\}$.*

Proof. Let us consider again $\Phi = \{\pm\gamma_1, \pm\gamma_2, \pm\gamma_3\}$, which is a root system of type A_2. The corresponding Weyl group W, see (2.9), is isomorphic to Σ_3. It contains the reflections s_{γ_i} about $\ker \gamma_i$, $i = 1, 2, 3$. Each of those is a transformation of $\mathfrak{d}^0$ which permutes the three coordinates of any vector in the following way:

$$s_{\gamma_1} = (2,3),\ s_{\gamma_2} = (1,3),\ s_{\gamma_3} = (1,2). \tag{5.2}$$

Here, as usual, (i, j) denotes the i, j transposition in Σ_3. In fact, W is generated by $s_1 := s_{\gamma_1}$ and $s_2 := s_{\gamma_2}$. The precise description of W is given by equation (4.3). We will need the following table, which gives for every $\sigma \in \Sigma_3$ the set of all $\gamma \in \Phi^+ = \{\gamma_1, \gamma_2, \gamma_3\}$ such that $\sigma^{-1}\gamma \in \Phi^-$.
Here we have used the formulae: $s_k(\gamma_k) = -\gamma_k$ for $k = 1, 2$, $s_1(\gamma_2) = s_2(\gamma_1) = \gamma_3$, $s_1(\gamma_3) = \gamma_2$, and $s_2(\gamma_3) = \gamma_1$.

σ	γ
s_1	γ_1
s_2	γ_2
s_2s_1	$\gamma_2,\ \gamma_3$
s_1s_2	$\gamma_1,\ \gamma_3$
$s_1s_2s_1$	$\gamma_1,\ \gamma_2,\ \gamma_3$

Table 2

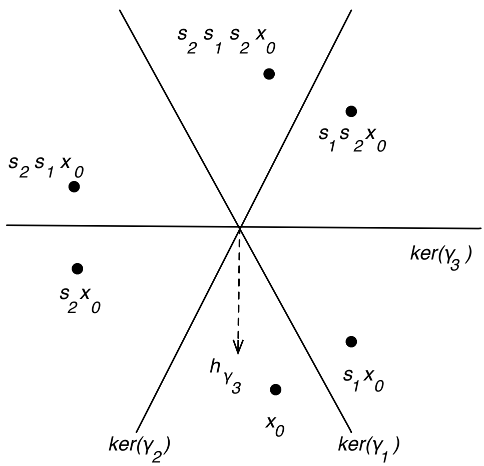

Figure 1.

Let us discuss in detail the following two situations.
Case 1. $(\sigma, \gamma) = (s_1, \gamma_1)$. We need to show that the boundary of $\exp(\mathfrak{g}_{\gamma_1})(s_1x_0)$ is x_0. To this end, we note that $\exp(\mathfrak{g}_{\gamma_1})(s_1x_0)$ is a Schubert cell (see Appendix B). Thus, by [9, Sec. 4] (especially equation (4.10)), its closure consists of the cell itself together with the 0 dimensional cell $\{x_0\}$.
Case 2. $(\sigma, \gamma) = (s_1s_2s_1, \gamma_3) = (s_{\gamma_3}, \gamma_3)$. We now show that the boundary of $\exp(\mathfrak{g}_{\gamma_3})(s_3x_0)$ is $\{x_0\}$. To simplify notations, we set

$$G := \mathrm{E}_{6(-26)},\ K := \mathrm{F}_4,\ \mathfrak{g} := \mathfrak{e}_{6(-26)},\ \mathfrak{k} := \mathfrak{f}_4,\ \mathfrak{s} := \mathfrak{h}_3^0(\mathbb{O}),\ \text{and } \gamma_3 := \gamma.$$

As usual, we denote $M = \mathrm{Spin}(8)$. We also denote by N and A the connected Lie subgroups of G of Lie algebras $\mathfrak{g}_{\gamma_1} + \mathfrak{g}_{\gamma_2} + \mathfrak{g}_{\gamma_3}$, respectively $\mathfrak{a}$ (the notations above have been used in the general case in Appendix B). We will use the rank-one reduction procedure, as described in [16, Chap. IX, Sec. 2]. Let us denote by $\mathfrak{g}^{\gamma}$ the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\gamma}$ and $\mathfrak{g}_{-\gamma}$. Take $h_\gamma \in \mathfrak{a}$ determined by $\langle h_\gamma, h\rangle = \gamma(h)$, for all $h \in \mathfrak{a}$ (here $\langle\ ,\ \rangle$ is the Killing form of $\mathfrak{g}$).

We have the Cartan decomposition

$$\mathfrak{g}^\gamma = \mathfrak{k}^\gamma \oplus \mathfrak{s}^\gamma,$$

where $\mathfrak{k}^\gamma = \mathfrak{k} \cap \mathfrak{g}^\gamma$ and $\mathfrak{s}^\gamma = \mathfrak{s} \cap \mathfrak{g}^\gamma$ (see also equation (2.5)). The space $\mathbb{R}h_\gamma$ is maximal abelian in $\mathfrak{s}^\gamma$. Let G^γ, K^γ, and A^γ denote the connected Lie subgroups of G of Lie algebras $\mathfrak{g}^\gamma, \mathfrak{k}^\gamma$, respectively $\mathbb{R}h_\gamma$. Then we have $K^\gamma = K \cap G^\gamma$ and $A^\gamma = A \cap G^\gamma$. Moreover, if M^γ denotes the centralizer of h_γ in K^γ, then we have $M^\gamma = M \cap G^\gamma$. The connected Lie subgroup of G^γ of Lie algebra $\mathfrak{g}_\gamma$ is $N^\gamma = G^\gamma \cap N$. The Iwasawa decomposition of G^γ is

$$G^\gamma = K^\gamma A^\gamma N^\gamma.$$

Without loss of generality we may assume that $x_0 = h_\gamma$: since the last two vectors are in the same Weyl chamber (see Figure 1), their K-orbits are G-equivariantly diffeomorphic. Consequently, we have $s_{\gamma_3} x_0 = -h_\gamma$. The orbit $X^\gamma := K^\gamma.h_\gamma$ is contained in $X = K.h_\gamma$ (for both orbits, the group action is the Adjoint one). In fact, the inclusion is G^γ-equivariant. Indeed, the action of G on X is induced by the identification $X = G/MAN$. Consequently, the subgroup G^γ of G acts on X and the orbit of the coset of e is $G^\gamma/(MAN \cap G^\gamma) = G^\gamma/(M^\gamma A^\gamma N^\gamma) = X^\gamma$ (here we have used that the map $K \times A \times N \to G$, $(k, a, n) \mapsto kan$, for all $(k, a, n) \in K \times A \times N$ is a diffeomorphism). The Schubert cell decomposition of X^γ described in Theorem B.2(b) is

$$X^\gamma = \exp(\mathfrak{g}_\gamma)(-h_\gamma) \bigsqcup \{h_\gamma\}.$$

Thus, the cell $\exp(\mathfrak{g}_\gamma)(-h_\gamma)$ is dense in X^γ. We deduce that the closure of $\exp(\mathfrak{g}_\gamma)(-h_\gamma)$ in X is equal to X^γ. This finishes the proof.

The other cases follow immediately from the two above. For instance, to show that the boundary of $\exp(\mathfrak{g}_{\gamma_1})(s_1 s_2 x_0)$ is $s_1(s_1 s_2 x_0) = s_2 x_0$ we use Case 2. Indeed, we replace x_0 by $s_2 x_0$ and s_1, s_2 by s_2, s_3 respectively (reflections about the walls of the Weyl chamber which contains $s_2 x_0$). □

5.5. **The root structure of Spin(8).** It remains to verify assumption (iii) in Theorem 5.2 for the cell decomposition (2.11) and the splittings (2.12) in two situations: equivariant cohomology and equivariant K-theory. This will be done by calculating explicitly the corresponding Euler classes. We need the following description of the roots, weights, and of the representation ring of Spin(8). The details can be found for instance in [8, Chap. V, Sec. 6 and Chap. VI, Sec. 6] or [2, Chap. 4]. The Lie algebra of Spin(8) is the space $\mathfrak{so}(8)$ of all skew-symmetric 8×8 matrices whose entries are real numbers. Recall that by T we have denoted the canonical maximal torus of Spin(8) (see Proposition 2.9). Its Lie algebra, call it $\mathfrak{t}$, consists of all matrices of the form

$$\begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \theta_4 \\ & & & -\theta_4 & 0 \end{pmatrix},$$

where $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R}$. For any $j \in \{1, 2, 3, 4\}$ we denote by L^j the linear function on $\mathfrak{t}$ which assigns to each matrix of the form above the number θ_j. The set $\{L^1, L^2, L^3, L^4\}$ is a basis of the dual space $\mathfrak{t}^*$.

- The set of roots of Spin(8) with respect to $\mathfrak{t}$ is

$$\Phi_{\mathrm{Spin}(8)} = \{\pm L^i \pm L^j \mid 1 \le i < j \le 4\}.$$

- A simple root system is

$$\Pi = \{L^1 - L^2, L^2 - L^3, L^3 - L^4, L^3 + L^4\}.$$

The corresponding set of positive roots is

$$\Phi^+_{\mathrm{Spin}(8)} = \{L^i \pm L^j \mid 1 \le i < j \le 4\}.$$

- The corresponding fundamental weights are:

(5.3)
$$\rho_1 = L^1,\ \rho_2 = L^1 + L^2,\ \rho_3 = \frac{L^1 + L^2 + L^3 - L^4}{2},\ \rho_4 = \frac{L^1 + L^2 + L^3 + L^4}{2}.$$

Since Spin(8) is simply connected, these weights are a basis of the lattice $\mathfrak{t}^*_{\mathbb{Z}}$ of integral forms. We will also use the presentation

$$\mathfrak{t}^*_{\mathbb{Z}} = \oplus_{1 \le i \le 5} \mathbb{Z}\omega^i / (2\omega^5 - \omega^1 - \omega^2 - \omega^3 - \omega^4),$$

where we have denoted as follows:

$$\omega^1 := L^1,\ \omega^2 := L^2,\ \omega^3 := L^3,\ \omega^4 := L^4,\ \omega^5 := \frac{L^1 + L^2 + L^3 + L^4}{2}.$$

As usual, to any integral form $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ corresponds the character $e^\lambda \in R[T]$. In this way, if we denote $y_j := e^{\omega^j}$, $1 \le j \le 5$, we obtain the following presentation:

$$R[T] \cong \mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}]/(y_5^2 - y_1 y_2 y_3 y_4).$$

- The canonical action of the Weyl group $W_{\mathrm{Spin}(8)} = N_{\mathrm{Spin}(8)}(T)/T$ on $\mathfrak{t}^*$ is faithful. The linear automorphisms of $\mathfrak{t}^*$ induced in this way are those η with the property that for any $1 \le i \le 4$, there exists $1 \le j \le 4$ such that $\eta(L^i) = \pm L^j$, the number of "$-$" signs being even.
- The representation ring of Spin(8) is $R[\mathrm{Spin}(8)] = \mathbb{Z}[X_1, X_2, X_3, X_4]$ where

$$X_1 = V_8 \otimes \mathbb{C},\ X_2 = S_8^+ \otimes \mathbb{C},\ X_3 = S_8^- \otimes \mathbb{C},$$

and X_4 is the complexified adjoint representation of Spin(8) (recall that V_8 is induced by the standard representation of SO(8) on $\mathbb{R}^8$ via the covering map Spin(8) $\to$ SO(8) and $S_8^\pm$ are the real half-spin representations of Spin(8)). Their weights are as follows (see [2, Prop. 4.2]):

(i) For X_1:

(5.4)
$$\pm L^1, \pm L^2, \pm L^3, \text{ and } \pm L_4.$$

(ii) For X_2:

$$\frac{\pm L^1 \pm L^2 \pm L^3 \pm L^4}{2}, \tag{5.5}$$

where the number of "$-$" signs is even.

(iii) For X_3:

$$\frac{\pm L^1 \pm L^2 \pm L^3 \pm L^4}{2}, \tag{5.6}$$

where the number of "$-$" signs is odd.

(iv) For X_4: all roots of Spin(8) relative to T.

The (complex) dimension of each weight space is equal to 1. Consequently, the restriction/inclusion map $R[\mathrm{Spin}(8)] = R[T]^{W_{\mathrm{Spin}(8)}} \to R[T]$ is given by

$$\begin{aligned}
X_1 =& y_1 + y_1^{-1} + y_2 + y_2^{-1} + y_3 + y_3^{-1} + y_4 + y_4^{-1} \\
X_2 =& y_5 + y_5y_1^{-1}y_2^{-1} + y_5y_1^{-1}y_3^{-1} + y_5y_1^{-1}y_4^{-1} + y_5y_2^{-1}y_3^{-1} + y_5y_2^{-1}y_4^{-1} \\
& + y_5y_3^{-1}y_4^{-1} + y_5y_1^{-1}y_2^{-1}y_3^{-1}y_4^{-1} \\
X_3 =& y_5y_1^{-1} + y_5y_2^{-1} + y_5y_3^{-1} + y_5y_4^{-1} + y_5y_1^{-1}y_2^{-1}y_3^{-1} + y_5y_1^{-1}y_2^{-1}y_4^{-1} \\
& + y_5y_1^{-1}y_3^{-1}y_4^{-1} + y_5y_2^{-1}y_3^{-1}y_4^{-1} \\
X_4 =& \sum_{1\le i<j\le 4} y_i^{\pm 1}y_j^{\pm 1}.
\end{aligned}$$

Recall that the T-actions on V_8 and $S_8^{\pm}$ are $\mathbb{C}$-linear relative to certain complex linear structures on these spaces, see Proposition 2.9. We would like now to calculate the weights of each of these three T-modules. For V_8 they are L_1, L_2, L_3, and L_4. The Spin(8)-module S_8^+ differs from V_8 by a group automorphism of Spin(8), see [2, Chap. 14]. This is just one of the outer automorphisms that arise from the many symmetries of the Dynkin diagram of Spin(8). It leaves T invariant and the induced automorphism of $\mathfrak{t}$ is the reflection $s_{\omega_5-\omega_4}$ through $\ker(\omega_5 - \omega_4)$ (equip $\mathfrak{t}$ with the inner product which makes $(\theta_1, \theta_2, \theta_3, \theta_4)$ into an orthonormal coordinate system). Thus, the weights of S_8^+ are:

$$\begin{aligned}
s_{\omega_5-\omega_4}(L_1) &= \omega_1 + \omega_4 - \omega_5 = \rho_1 - \rho_3 \\
s_{\omega_5-\omega_4}(L_2) &= \omega_2 + \omega_4 - \omega_5 = -\rho_1 + \rho_2 - \rho_3 \\
s_{\omega_5-\omega_4}(L_3) &= \omega_3 + \omega_4 - \omega_5 = -\rho_2 + \rho_4 \\
s_{\omega_5-\omega_4}(L_4) &= \omega_5 = \rho_4.
\end{aligned}$$

A similar reasoning holds for S_8^-, the automorphism of $\mathfrak{t}$ being this time s_{ω_5}. The resulting weights are:

$$\begin{aligned} s_{\omega_5}(L_1) &= \omega_1 - \omega_5 = \rho_1 - \rho_4 \\ s_{\omega_5}(L_2) &= \omega_2 - \omega_5 = -\rho_1 + \rho_2 - \rho_4 \\ s_{\omega_5}(L_3) &= \omega_3 - \omega_5 = \rho_3 - \rho_2 \\ s_{\omega_5}(L_4) &= \omega_4 - \omega_5 = -\rho_3. \end{aligned}$$

5.6. **Equivariant cohomology of Fl($\mathbb{O}$).** We can now calculate the Euler classes $e_M(C_{k\ell})$ mentioned in Theorem 5.2 for $M = \mathrm{Spin}(8)$ and $X = \mathrm{Fl}(\mathbb{O})$. They are the Spin(8)-equivariant Euler classes of V_8 and $S_8^{\pm}$. If V is any of these representations, then we can split $V = \bigoplus_{i=1}^4 \ell_i$, where ℓ_i are 1-dimensional T-invariant complex vector subspaces, see Proposition 2.9. Consequently,

$$e_T(V) = c_4^T(\bigoplus_{i=1}^{4} \ell_i) = c_1^T(\ell_1)\cdots c_1^T(\ell_4),$$

where c_4^T and c_1^T denote the T-equivariant Chern classes. We know that the 1-dimensional complex representations of T are labeled by the character group $\mathrm{Hom}(T, S^1)$, and the map $\mathrm{Hom}(T, S^1) \to H^2(BT;\mathbb{Z})$ given by $L \mapsto c_1^T(L)$ is a group isomorphism (see for example [19, Chap. 20, Sec. 11]). In turn, $\mathrm{Hom}(T, S^1)$ is isomorphic to the lattice of integral forms on $\mathfrak{t}$. The formulae obtained at the end of Subsection 5.5 thus give readily descriptions of $e_T(V_8)$ and $e_T(S_8^{\pm})$ as elements of $H_T^*(\mathrm{pt.}) = H^*(BT) = S(\mathfrak{t}^*)$. On the other hand, there is a canonical inclusion $H^*(B\mathrm{Spin}(8))) \hookrightarrow H^*(BT)$, which maps Spin(8)-equivariant to T-equivariant Euler classes. We deduce:

$$e_{\mathrm{Spin}(8)}(V_8) = \rho_1(-\rho_1+\rho_2)(-\rho_2+\rho_3+\rho_4)(-\rho_3+\rho_4) \tag{5.7}$$

$$e_{\mathrm{Spin}(8)}(S_8^+) = (\rho_1-\rho_3)(-\rho_1+\rho_2-\rho_3)(-\rho_2+\rho_4)\rho_4 \tag{5.8}$$

$$e_{\mathrm{Spin}(8)}(S_8^-) = -(\rho_1-\rho_4)(-\rho_1+\rho_2-\rho_4)(\rho_3-\rho_2)\rho_3. \tag{5.9}$$

Since $H^*_{\mathrm{Spin}(8)}(\mathrm{pt.}) = H^*(B\mathrm{Spin}(8)) = \mathbb{R}[\rho_1,\rho_2,\rho_3,\rho_4]^{W_{\mathrm{Spin}(8)}}$, we have shown:

Lemma 5.7. *The equivariant Euler classes $e_{\mathrm{Spin}(8)}(V_8)$, $e_{\mathrm{Spin}(8)}(S_8^-)$, and $e_{\mathrm{Spin}(8)}(S_8^+)$ are pairwise relatively prime elements of $H^*_{\mathrm{Spin}(8)}(\mathrm{pt.})$.*

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. From Theorem 5.2 we deduce that the map $\imath^* : H_M^*(\mathrm{Fl}(\mathbb{O})) \to H_M^*(\Sigma_3 x_0) = \prod_{\sigma\in\Sigma_3} H^*(BM)$ is injective. Moreover, its image consists of those $(f_\sigma)_{\sigma\in\Sigma_3}$ with the following property:

(P1) $f_\sigma - f_{s_\gamma\sigma}$ is divisible by $e_M(\mathfrak{h}_\gamma)$ for any $\sigma \in \Sigma_3$ and any $\gamma \in \Phi^+$ such that $\sigma^{-1}\gamma \in \Phi^-$.

Condition (P1) is equivalent to:

(P2) $f_\sigma - f_{s_\gamma\sigma}$ is divisible by $e_M(\mathfrak{h}_\gamma)$ for any $\sigma \in \Sigma_3$ and any $\gamma \in \Phi^+$.

Indeed, (P2) implies (P1). Also (P1) implies (P2): assume that (P1) holds true and take $\sigma \in \Sigma_3$ and $\gamma \in \Phi^+$ such that $\sigma^{-1}\gamma \in \Phi^+$; then we have $s_\gamma(s_\gamma\sigma) = \sigma$ and also $(s_\gamma\sigma)^{-1}\gamma = -\sigma^{-1}\gamma$, which is in Φ^-; thus, by (P1), the difference $f_{s_\gamma\sigma} - f_\sigma$ is divisible by $e_M(\mathfrak{h}_\gamma)$.

Finally, recall that $\tilde{b}_i - \tilde{b}_j = e_M(\mathfrak{h}_{\gamma_k})$, where $1 \le i < j \le 3$ and $\{k\} = \{1,2,3\} \setminus \{i,j\}$ (see equations (1.4), (1.6), (2.10), and (4.1), as well as Proposition 2.8 and Lemma 4.6). We also take into account equation (5.2). □

The next lemma is relevant for the observation made right after Theorem 1.3. First recall from the introduction that $H^*(BM) = \mathbb{R}[a_1, a_2, a_3, a_4]$, where $a_1 \in H^4(BM)$, $a_2, a_3 \in H^8(BM)$, and $a_4 \in H^{12}(BM)$.

Lemma 5.8. *The elements a_1^2, $\tilde{b}_1$, and $\tilde{b}_2$ of $H^8(BM)$ are linearly independent. Consequently, we have*

$$H^*(BM) = \mathbb{R}[a_1, \tilde{b}_1, \tilde{b}_2, a_4].$$

Proof. As before, we regard $H^*(BM)$ as a subspace of the polynomial ring $H^*(BT) = \mathbb{R}[\rho_1, \rho_2, \rho_3, \rho_4]$. One can see that $\tilde{b}_1$ and $\tilde{b}_2$ are, up to a possible negative sign, just the Euler classes $e_{\mathrm{Spin}(8)}(V_8)$ and $e_{\mathrm{Spin}(8)}(S_8^+)$, respectively. Concretely, these are given by equations (5.7) and (5.8), respectively: observe that those two polynomials are linearly independent. Assume now that there exists a linear combination of them which is equal to a_1^2. Recall that a_1, regarded as a (polynomial) function on $\mathfrak{t}$, is nothing but the norm squared of a vector. In particular, the only zero of a_1 is for $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$. On the other hand, the polynomials given by (5.7) and (5.8) vanish whenever $\rho_1 = \rho_2 = \rho_3 = 0$ and ρ_4 is arbitrary. This is a contradiction. □

5.9. **Equivariant K-theory of Fl($\mathbb{O}$).** Our aim here is to prove Theorem 1.4. Unlike in the previous section, we will apply the Harada–Henriques–Holm theorem for the T-action, rather than the Spin(8)-action. We then take into account that $K_{\mathrm{Spin}(8)}(\mathrm{Fl}(\mathbb{O})) = K_T(\mathrm{Fl}(\mathbb{O}))^{W_{\mathrm{Spin}(8)}}$.

The first step in the proof is made by calculating the K-theoretical T-equivariant Euler classes of C_σ, see equation (2.11). By (2.12), this amounts to calculating $e_T^K(V_8), e_T^K(S_8^-)$, and $e_T^K(S_8^+)$, where the superscript K stands for K-theory. Recall that the Euler class of a direct sum is the product of the Euler classes of the summands; also, if a torus acts on $\mathbb{C}$ with weight λ, then the resulting equivariant K-theoretical Euler class is $1 - e^{-\lambda}$ (see e.g. [7, Note, p. 35]). From the expressions of the weights which we have obtained at the end of Subsection 5.5 we obtain:

$$e_T^K(V_8) = (1 - y_1^{-1})(1 - y_2^{-1})(1 - y_3^{-1})(1 - y_4^{-1})$$
$$e_T^K(S_8^+) = (1 - y_1^{-1}y_4^{-1}y_5)(1 - y_2^{-1}y_4^{-1}y_5)(1 - y_3^{-1}y_4^{-1}y_5)(1 - y_5^{-1})$$
$$e_T^K(S_8^-) = (1 - y_1^{-1}y_5)(1 - y_2^{-1}y_5)(1 - y_3^{-1}y_5)(1 - y_4^{-1}y_5).$$

We immediately deduce:

Lemma 5.10. *The K-theoretical equivariant Euler classes $e_T^K(V_8), e_T^K(S_8^+)$, and $e_T^K(S_8^-)$ are pairwise relatively prime elements of $R[T]$.*

Thus, the T-action on $\mathrm{Fl}(\mathbb{O})$ satisfies the hypotheses of Theorem 5.2 for $K_T^*(\cdot)$ (also recall that, by Proposition 2.10, the T-fixed point set is $\Sigma_3 x_0$). We deduce that $K_T^1(\mathrm{Fl}(\mathbb{O})) = \{0\}$, as well as the following result, which concerns $K_T^0(\mathrm{Fl}(\mathbb{O}))$.

Proposition 5.11. *The ring homomorphism $\imath_T^* : K_T(\mathrm{Fl}(\mathbb{O}))) \to K_T(\Sigma_3 x_0) = \prod_{\sigma\in\Sigma_3} R[T]$ induced by the inclusion map $\imath : \Sigma_3 x_0 \to \mathrm{Fl}(\mathbb{O})$ is injective. Its image consists of all $(f_\sigma) \in \prod_{\sigma\in\Sigma_3} \mathbb{Z}[y_1^{\pm1}, y_2^{\pm1}, y_3^{\pm1}, y_4^{\pm1}, y_5^{\pm1}]/(y_5^2 - y_1y_2y_3y_4)$ with the property that for any $\sigma \in \Sigma_3$ we have:*

(i) $f_{(2,3)\sigma} - f_\sigma$ *is divisible by* $(1-y_1^{-1})(1-y_2^{-1})(1-y_3^{-1})(1-y_4^{-1})$

(ii) $f_{(1,3)\sigma} - f_\sigma$ *is divisible by* $(1-y_1^{-1}y_4^{-1}y_5)(1-y_2^{-1}y_4^{-1}y_5)(1-y_3^{-1}y_4^{-1}y_5)(1-y_5^{-1})$

(iii) $f_{(1,2)\sigma} - f_\sigma$ *is divisible by* $(1-y_1^{-1}y_5)(1-y_2^{-1}y_5)(1-y_3^{-1}y_5)(1-y_4^{-1}y_5)$.

The divisibility referred to above is in the ring $R[T] = \mathbb{Z}[y_1^{\pm1}, y_2^{\pm1}, y_3^{\pm1}, y_4^{\pm1}, y_5^{\pm1}]$ $/(y_5^2 - y_1y_2y_3y_4)$.

We are now ready to accomplish the main goal of the subsection:

Proof of Theorem 1.4. Let $\imath_M^* : K_{\mathrm{Spin}(8)}(\mathrm{Fl}(\mathbb{O})) \to \prod_{\sigma\in\Sigma_3} R[\mathrm{Spin}(8)]$ be the ring homomorphism induced by the inclusion $\imath : \Sigma_3 x_0 \to \mathrm{Fl}(\mathbb{O})$. The map $\imath_M^*$ is obviously $W_{\mathrm{Spin}(8)}$-equivariant, where the action of $W_{\mathrm{Spin}(8)}$ on $\prod_{\sigma\in\Sigma_3} R[\mathrm{Spin}(8)]$ is the diagonal one. We have the ring isomorphisms

(5.10)
$$R[\mathrm{Spin}(8)] \cong R[T]^{W_{\mathrm{Spin}(8)}} \quad \text{and} \quad K_{\mathrm{Spin}(8)}(\mathrm{Fl}(\mathbb{O})) \cong K_T(\mathrm{Fl}(\mathbb{O}))^{W_{\mathrm{Spin}(8)}},$$

where for the last one we invoke [14, Cor. 4.10(ii)]. Since $\imath_T^*$ is $W_{\mathrm{Spin}(8)}$-equivariant, we may identify the two pairs of spaces related by the isomorphisms (5.10) and assume that $\imath_M^*$ is just the restriction of $\imath_T^*$ to the space $K_T(\mathrm{Fl}(\mathbb{O}))^{W_{\mathrm{Spin}(8)}}$. Consequently, the image of $\imath_M^*$ is the intersection of the image of $\imath_T^*$ with $\prod_{\sigma\in\Sigma_3} R[\mathrm{Spin}(8)]$. It thus consists of all (f_σ) in the latter direct product which satisfy the divisibility properties (i), (ii), and (iii) in Proposition 5.11. Recall now that $R[\mathrm{Spin}(8)] = \mathbb{Z}[X_1, X_2, X_3, X_4]$. From the explicit formulae for X_1, X_2, and X_3 given in Subsection 5.5 we deduce by direct calculation:

$$X_2 - X_3 = y_5(1-y_1^{-1})(1-y_2^{-1})(1-y_3^{-1})(1-y_4^{-1})$$
$$X_1 - X_3 = y_4(1-y_1^{-1}y_4^{-1}y_5)(1-y_2^{-1}y_4^{-1}y_5)(1-y_3^{-1}y_4^{-1}y_5)(1-y_5^{-1})$$
$$X_1 - X_2 = -y_5^{-1}(1-y_1^{-1}y_5)(1-y_2^{-1}y_5)(1-y_3^{-1}y_5)(1-y_4^{-1}y_5).$$

Consequently, for $(f_\sigma) \in \prod_{\sigma\in\Sigma_3} \mathbb{Z}[X_1, X_2, X_3, X_4]$, conditions (i), (ii), and (iii) in Proposition 5.11 are equivalent to:

(i') $f_{(2,3)\sigma} - f_\sigma$ is divisible by $X_2 - X_3$,

(ii') $f_{(1,3)\sigma} - f_\sigma$ is divisible by $X_1 - X_3$,

(iii') $f_{(1,2)\sigma} - f_\sigma$ is divisible by $X_1 - X_2$.

This finishes the proof. □

APPENDIX A. THE COMPLEX FLAG MANIFOLD $\mathrm{Fl}_3(\mathbb{C})$

This section is a recollection of well-known facts concerning the complex flag manifold $\mathrm{Fl}_3(\mathbb{C})$. The focus is of course on those aspects whose counterparts in the realm of octonions we deal with in this paper. Our aim here is to smooth the passage from complex numbers to octonions.

Originally, $\mathrm{Fl}_3(\mathbb{C})$ is the set of all nested sequences

$$V_1 \subset V_2 \subset \mathbb{C}^3,$$

where V_1 and V_2 are complex vector subspaces of $\mathbb{C}^3$ such that $\dim V_1 = 1$ and $\dim V_2 = 2$. Alternatively, let us equip $\mathbb{C}^3$ with the standard Hermitian inner product: then $\mathrm{Fl}_3(\mathbb{C})$ is the set of pairs (L_1, L_2), where L_1 and L_2 are 1-dimensional complex vector subspaces of $\mathbb{C}^3$ with L_1 orthogonal to L_2.

Let us now consider the space

$$\mathfrak{h}_3(\mathbb{C}) = \{a \in \mathrm{Mat}^{3\times 3}(\mathbb{C}) \mid a = a^*\}.$$

We have the following presentations.

Proposition A.1. *a) There is a natural identification between the complex projective plane $\mathbb{C}\mathrm{P}^2$ and the set of all matrices $a \in \mathfrak{h}_3(\mathbb{C})$ with*

$$a^2 = a \text{ and } \mathrm{tr}(a) = 1.$$

b) There is a natural identification between the flag manifold $\mathrm{Fl}_3(\mathbb{C})$ and the set of all pairs $(a_1, a_2) \in \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$ with the property that

$$\mathrm{Re}(\mathrm{tr}(a_1 a_2)) = 0.$$

The identifications are as follows.

For $\mathbb{C}\mathrm{P}^2$. A 1-dimensional complex vector subspace V of $\mathbb{C}^3$ is identified with the element of $\mathfrak{h}_3(\mathbb{C})$ which has eigenvalues $1, 0, 0$ and 1-eigenspace equal to V (the 0-eigenspace is implicitly $V^\perp$). Moreover, an element a of $\mathfrak{h}_3(\mathbb{C})$ has eigenvalues $(1, 0, 0)$ if and only if $a^2 = a$ and $\mathrm{tr}(a) = 1$.

For $\mathrm{Fl}_3(\mathbb{C})$. Take L_1, L_2 two 1-dimensional complex vector subspaces of $\mathbb{C}^3$ and a_1, a_2 the Hermitian matrices with eigenvalues $(1, 0, 0)$ and 1-eigenspaces L_1, respectively L_2. The main point is that L_1 is perpendicular to L_2 if and only if $\mathrm{Re}(\mathrm{tr}(a_1 a_2)) = 0$. Indeed, let us choose an orthonormal basis v_1, v_2, v_3 of $\mathbb{C}^3$, where $L_2 = \mathbb{C}v_1$. Then we have:

$$\begin{aligned}\mathrm{tr}(a_1 a_2) &= \langle a_1 a_2(v_1), v_1\rangle + \langle a_1 a_2(v_2), v_2\rangle + \langle a_1 a_2(v_3), v_3\rangle \\ &= \langle a_1(v_1), v_1\rangle = \langle a_1^2(v_1), v_1\rangle = \langle a_1(v_1), a_1^*(v_1)\rangle = \langle a_1(v_1), a_1(v_1)\rangle.\end{aligned}$$

Thus, $\mathrm{Re}(\mathrm{tr}(a_1 a_2)) = 0$ if and and only if $a_1(v_1) = 0$. On the other hand, L_1 is perpendicular to L_2 if and only if L_2 is contained in the 0-eigenspace of a_1, that is, $a_1(v_1) = 0$.

There are three natural projections $\mathrm{Fl}_3(\mathbb{C}) \to \mathbb{C}\mathrm{P}^2$: the first maps an arbitrary pair (L_1, L_2) to L_1, the second to L_2, and the third to the orthogonal complement of $L_1 \oplus L_2$ in $\mathbb{C}^3$. These are all three $\mathbb{C}\mathrm{P}^1$-bundles. By taking the tangent space to the fiber at any point, one obtains three subbundles of the tangent bundle of $\mathrm{Fl}_3(\mathbb{C})$, which we denote by $\mathcal{E}_1^c, \mathcal{E}_2^c$, and $\mathcal{E}_3^c$. These are complex line bundles over $\mathrm{Fl}_3(\mathbb{C})$. On the other hand, $\mathrm{Fl}_3(\mathbb{C})$ has also three tautological bundles, call them $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$. The following identifications are natural:

$$\mathcal{E}_1^c = \mathcal{L}_2 \otimes \mathcal{L}_3^*, \quad \mathcal{E}_2^c = \mathcal{L}_3 \otimes \mathcal{L}_1^*, \quad \mathcal{E}_3^c = \mathcal{L}_2 \otimes \mathcal{L}_1^*.$$

They induce the following relationship concerning the first Chern classes:

$$c_1(\mathcal{E}_1^c) = c_1(\mathcal{L}_2) - c_1(\mathcal{L}_3),\ c_1(\mathcal{E}_2^c) = c_1(\mathcal{L}_3) - c_1(\mathcal{L}_1),\ c_1(\mathcal{E}_3^c) = c_1(\mathcal{L}_2) - c_1(\mathcal{L}_1).$$

Consequently, we have

$$c_1(\mathcal{E}_3^c) = c_1(\mathcal{E}_1^c) + c_1(\mathcal{E}_2^c).$$

But $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ is a trivial vector bundle over $\mathrm{Fl}_3(\mathbb{C})$, hence

$$c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) + c_1(\mathcal{L}_3) = 0.$$

We are led to:

$$\begin{aligned} c_1(\mathcal{L}_1) &= -\frac{1}{3}(c_1(\mathcal{E}_1^c) + 2c_1(\mathcal{E}_2^c)) \\ c_1(\mathcal{L}_2) &= \frac{1}{3}(2c_1(\mathcal{E}_1^c) + c_1(\mathcal{E}_2^c)) \\ c_1(\mathcal{L}_3) &= \frac{1}{3}(-c_1(\mathcal{E}_1^c) + c_1(\mathcal{E}_2^c)). \end{aligned} \tag{A.1}$$

A theorem of Borel, see [6], says that the ring $H^*(\mathrm{Fl}_3(\mathbb{C}); \mathbb{Z})$ is generated by $x_1 := c_1(\mathcal{L}_1), x_2 := c_1(\mathcal{L}_2)$, and $x_3 := c_1(\mathcal{L}_3)$, subject to the relations given by the vanishing of all symmetric polynomials in x_1, x_2, and x_3. We deduce:

Proposition A.2. *The ring $H^*(\mathrm{Fl}_3(\mathbb{C}); \mathbb{Z})$ is generated by $\frac{1}{3}(2c_1(\mathcal{E}_1^c) + c_1(\mathcal{E}_2^c))$ and $\frac{1}{3}(c_1(\mathcal{E}_1^c) + 2c_1(\mathcal{E}_2^c))$, subject to the relations*

$$S_i\left(\frac{1}{3}(2c_1(\mathcal{E}_1^c) + c_1(\mathcal{E}_2^c)), \frac{1}{3}(-c_1(\mathcal{E}_1^c) + c_1(\mathcal{E}_2^c)), -\frac{1}{3}(c_1(\mathcal{E}_1^c) + 2c_1(\mathcal{E}_2^c))\right) = 0,$$

$i = 2, 3$.

Let us now denote by T the 3-torus which consists of all diagonal matrices of the form $\mathrm{Diag}(z_1, z_2, z_3)$, $z_i \in \mathbb{C}$, $|z_i| = 1$. The natural splitting $T = S^1 \times S^1 \times S^1$ induces $BT = BS^1 \times BS^1 \times BS^1$, hence $H^*(BT) = \mathbb{R}[u_1] \otimes \mathbb{R}[u_2] \otimes \mathbb{R}[u_3] = \mathbb{R}[u_1, u_2, u_3]$. There is a canonical action of T on $\mathrm{Fl}_3(\mathbb{C})$. The corresponding T-equivariant cohomology ring $H_T^*(\mathrm{Fl}_3(\mathbb{C}))$ can also be described by a Borel type formula. Concretely, as an $H^*(BT)$-algebra, $H_T^*(\mathrm{Fl}_3(\mathbb{C}))$ is generated by the T-equivariant first Chern classes of $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$, which are:

$$\tilde{x}_1 := c_1^T(\mathcal{L}_1), \quad \tilde{x}_2 := c_1^T(\mathcal{L}_2), \quad \tilde{x}_3 := c_1^T(\mathcal{L}_3).$$

The ideal of relations is generated by:

$$S_j(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = S_j(u_1, u_2, u_3),$$

$j = 1, 2, 3$ (the details are spelled out for instance in [23, Proof of Thm. 1.1]). This time, instead of (A.1) we have:

$$c_1^T(\mathcal{L}_1) = -\frac{1}{3}(c_1^T(\mathcal{E}_1^c) + 2c_1^T(\mathcal{E}_2^c)) + \frac{1}{3}(u_1 + u_2 + u_3)$$
$$c_1^T(\mathcal{L}_2) = \frac{1}{3}(2c_1^T(\mathcal{E}_1^c) + c_1^T(\mathcal{E}_2^c)) + \frac{1}{3}(u_1 + u_2 + u_3)$$
$$c_1^T(\mathcal{L}_3) = \frac{1}{3}(-c_1^T(\mathcal{E}_1^c) + c_1^T(\mathcal{E}_2^c)) + \frac{1}{3}(u_1 + u_2 + u_3).$$

Set

$$b_1 := u_2 - u_3 \text{ and } b_2 := u_3 - u_1.$$

We have proved:

Proposition A.3. *As an $H^*(BT)$-algebra, $H_T^*(\mathrm{Fl}_3(\mathbb{C}))$ is generated by $c_1^T(\mathcal{E}_1^c)$ and $c_1^T(\mathcal{E}_2^c)$, subject to the relations:*

$$S_i(2c_1^T(\mathcal{E}_1^c) + c_1^T(\mathcal{E}_2^c), -c_1^T(\mathcal{E}_1^c) + c_1^T(\mathcal{E}_2^c), -(c_1^T(\mathcal{E}_1^c) + 2c_1^T(\mathcal{E}_2^c))) = S_i(2b_1 + b_2, -b_1 + b_2, -(b_1 + 2b_2)),$$

$i = 2, 3$.

The T-fixed points in $\mathrm{Fl}_3(\mathbb{C})$ are all pairs of type $(\mathbb{C}e_i, \mathbb{C}e_j)$, $1 \le i, j \le 3$, $i \neq j$. Here e_i is the i-th coordinate vector in $\mathbb{C}^3$. Thus, we have a natural identification $\mathrm{Fl}_3(\mathbb{C})^T \cong \Sigma_3$. One can show that the restriction map $H_T^*(\mathrm{Fl}_3(\mathbb{C})) \to H_T^*(\Sigma_3)$ is injective and its image consists of all $(f_\sigma) \in \prod_{\sigma \in \Sigma_3} \mathbb{R}[u_1, u_2, u_3]$ such that $f_\sigma - f_{(i,j)\sigma}$ is divisible by $u_i - u_j$, for all $\sigma \in \Sigma_3$ and $1 \le i < j \le 3$: this is the standard Goresky–Kottwitz–MacPherson type description of $H_T^*(\mathrm{Fl}_3(\mathbb{C}))$, which can be immediately deduced from the original work [11].

A similar description holds for the T-equivariant K-theory ring of $\mathrm{Fl}(\mathbb{O})$. Namely, let us first identify $R[T] = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$. Then the restriction map $K_T(\mathrm{Fl}_3(\mathbb{C})) \to K_T(\Sigma_3)$ is injective and its image consists of all $(f_\sigma) \in \prod_{\sigma \in \Sigma_3} \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$ with the property that $f_\sigma - f_{(i,j)\sigma}$ is divisible by $1 - t_i t_j^{-1}$, for all $\sigma \in \Sigma_3$ and $1 \le i < j \le 3$. This is a direct application of [25, Thm. 1.7] (cp. also the appendix of [27]).

Appendix B. Real flag manifolds and their cell decomposition

In this section we will present some general notions and results concerning real flag manifolds. The main reference is [9] (the background material can be found for instance in [16, Chap. IX]).

Let G be a real connected semisimple Lie group and denote by $\mathfrak{g}$ its Lie algebra. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$$

be a Cartan decomposition: this means that the Killing form of $\mathfrak{g}$ is strictly negative definite on $\mathfrak{k}$ and strictly positive definite on $\mathfrak{s}$. The corresponding Cartan involution is $\theta : \mathfrak{g} \to \mathfrak{g}$,

$$\theta(x + y) = x - y,$$

for all $x \in \mathfrak{k}$ and $y \in \mathfrak{s}$. We pick a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$ and consider the following root space decomposition:

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\gamma \in \Phi} \mathfrak{g}_\gamma. \tag{B.1}$$

Here $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and Φ the set of roots, which are functions $\gamma : \mathfrak{a} \to \mathbb{R}$ such that the root space

$$\mathfrak{g}_\gamma := \{x \in \mathfrak{g} \mid [h, x] = \gamma(h)x \text{ for all } h \in \mathfrak{a}\}$$

is nonzero. The set Φ is a root system in the dual space $\mathfrak{a}^*$. Let us pick a system of simple roots and denote by Φ^+ the corresponding set of positive roots. We set

$$\mathfrak{n} := \sum_{\gamma \in \Phi^+} \mathfrak{g}_\gamma$$

and obtain the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

If K, A, N are the connected Lie subgroups of G of Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ respectively, then we have the following Iwasawa decomposition of G:

$$G = KAN.$$

Let us also denote by M the centralizer of $\mathfrak{a}$ in K and by W the Weyl group, which is

$$W = \{k \in K \mid \mathrm{Ad}_G k(\mathfrak{a}) \subset \mathfrak{a}\}/M.$$

It turns out that, via the adjoint representation of G, the group K leaves $\mathfrak{s}$ invariant. The orbits of the resulting representation are called *real flag manifolds.* We need the following result:

Proposition B.1. *Take $x_0 \in \mathfrak{a}$ such that $\gamma(x_0) \neq 0$ for all $\gamma \in \Phi$. Then the stabilizer of x_0 in K is equal to M.*

Proof. By [9, Prop. 1.2] the stabilizer K_{x_0} of x_0 satisfies

$$K_{x_0} = MK_{x_0}^0,$$

where $K_{x_0}^0$ denotes the identity component of K_{x_0}. The Lie algebra of K_{x_0} is the commutator of x_0 in $\mathfrak{k}$. From the root decomposition (B.1), this is the same as $\mathfrak{m}$. Thus, we have $K_{x_0}^0 \subset M$ and consequently $K_{x_0} = M$. □

Consequently, we can identify

$$X := \mathrm{Ad}_G(K)x_0 = K/M.$$

From this we can see that there is a canonical embedding of the Weyl group W in X.

The natural action of K on X extends to an action of G. This arises from the identification

$$X = K/M = KAN/MAN = G/MAN,$$

where we take into account that MAN is a subgroup of G. The cell decomposition of X we will describe in the following theorem uses the embedding $W \subset X$ and also the action of G on X. The proof can be found in [9, Sec. 3].

Theorem B.2 ([9])**.** *(a) We have*

$$X = \bigsqcup_{w \in W} Nw. \tag{B.2}$$

(b) Fix $w \in W$. The map $\sum \mathfrak{g}_\gamma \to Nw$, $x \mapsto \exp(x)w$ is a diffeomorphism. The sum in the domain runs over all $\gamma \in \Phi^+$ such that $w^{-1}\gamma \in \Phi^-$.

(c) The decomposition (B.2) *makes X into a CW complex.*

The cells Nw, $w \in W$, are usually referred to as *Schubert cells.*

Let us now consider the following root space decomposition of $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{a} + \sum_{\gamma \in \Phi^+} \mathfrak{s}_\gamma, \tag{B.3}$$

where

$$\mathfrak{s}_\gamma = (\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma}) \cap \mathfrak{s} = \{x \in \mathfrak{s} \mid [h,[h,x]] = \gamma(h)^2 x \text{ for all } h \in \mathfrak{a}\}.$$

We can easily see that both $\mathfrak{g}_\gamma$ and $\mathfrak{s}_\gamma$ are M-invariant, where M acts via the Adjoint representation. The following result seems to be known. Since we didn't find it clearly stated and proved in the literature, we included a proof of it.

Proposition B.3. *If $\gamma \in \Phi^+$, then the map $\Theta : \mathfrak{g}_\gamma \to \mathfrak{s}_\gamma$, given by $\Theta(x) = x - \theta x$, for all $x \in \mathfrak{g}_\gamma$, is an M-equivariant linear isomorphism.*

Proof. First, since $\theta(\mathfrak{g}_\gamma) = \mathfrak{g}_{-\gamma}$, Θ is well defined, in the sense that it maps $\mathfrak{g}_\gamma$ to $\mathfrak{s}_\gamma$. The map is also injective: $x - \theta x = 0$ and $x \in \mathfrak{g}_\gamma$ implies $x = 0$. The map is also surjective: if $y \in \mathfrak{s}_\gamma$, then we can write it as $y = y_1 + y_2$, with $y_1 \in \mathfrak{g}_\gamma$ and $y_2 \in \mathfrak{g}_{-\gamma}$; since $y \in \mathfrak{s}$, we have $\theta(y) = -y$, which implies $y_2 = -\theta(y_1)$, thus $y = y_1 - \theta(y_1) = \Theta(y_1)$. The M-equivariance of Θ follows from the M-equivariance of θ. □

We now take into account that the map described in Theorem B.2(b) is M-equivariant, where M acts on the domain by the Adjoint representation and on the codomain via the G-action on X. We deduce:

Corollary B.4. *Fix $w \in W$. We have an M-equivariant diffeomorphism between the Schubert cell Nw and the space $\sum \mathfrak{s}_\gamma$, where the sum runs over all $\gamma \in \Phi^+$ such that $w^{-1}\gamma \in \Phi^-$.*

Appendix C. The symmetric space $\mathrm{E}_{6(-26)}/\mathrm{F}_4$

In this section we will outline the construction of the (noncompact) symmetric space mentioned in the title. We will try to make more clear several aspects mentioned in Subsection 2.6. For instance, we will prove that the root spaces $\mathfrak{s}_\gamma$ in the decomposition described by equation (B.3) are the $\mathfrak{h}_\gamma$

described in Subsection 2.6. This is an important fact, because it allows us to deduce the presentation of C_σ given by equation (2.12) from Theorem B.2(b) and Proposition B.3. The main reference of this section is the article [10] by Freudenthal.

Recall that by Definition 2.3, the group of all linear transformations of $\mathfrak{h}_3(\mathbb{O})$ which preserve the product $\circ$ is F_4. We define the determinant function on $\mathfrak{h}_3(\mathbb{O})$ as follows:

$$\det(a) = \frac{1}{3}\mathrm{tr}(a \circ a \circ a) - \frac{1}{2}\mathrm{tr}(a \circ a)\mathrm{tr}a + \frac{1}{6}(\mathrm{tr}a)^3,$$

for all $a \in \mathfrak{h}_3(\mathbb{O})$. Let us consider the group of all linear transformations of $\mathfrak{h}_3(\mathbb{O})$ which leave the determinant invariant. It turns out that this group is just $\mathrm{E}_{6(-26)}$ (see Subsection 2.6 for the definition of this group). From the formula of the determinant above we deduce easily that $\mathrm{E}_{6(-26)}$ contains F_4. Less obvious is that the latter group is a maximal compact subgroup of the former. The Lie algebra $\mathfrak{f}_4$ consists of all linear transformations of $\mathfrak{h}_3(\mathbb{O})$ of the form

$$\tilde{b} : \mathfrak{h}_3(\mathbb{O}) \to \mathfrak{h}_3(\mathbb{O}), \ \tilde{b}(y) = [b, y],$$

where b is a 3×3 matrix with entries in $\mathbb{O}$ such that $b = -b^*$ (that is, b is skew-Hermitian). Here and everywhere else in this section $[\ ,\ ]$ denotes the usual matrix commutator. To any $a \in \mathfrak{h}_3^0(\mathbb{O})$ we attach the $\mathbb{R}$-linear transformation $\hat{a}$ of $\mathfrak{h}_3(\mathbb{O})$ given by

$$\hat{a} : \mathfrak{h}_3(\mathbb{O}) \to \mathfrak{h}_3(\mathbb{O}), \ \hat{a}(y) = a \circ y, \text{ for all } y \in \mathfrak{h}_3(\mathbb{O}).$$

The Cartan decomposition of $\mathfrak{e}_{6(-26)}$ corresponding to $\mathfrak{f}_4$ is described in the following proposition (see [10], end of Sec. 8.1.1).

Proposition C.1. *If c is in the Lie algebra $\mathfrak{e}_{6(-26)}$, then there exists $a \in \mathfrak{h}_3^0(\mathbb{O})$ and b a 3×3 skew-Hermitian matrix with entries in $\mathbb{O}$ such that $c = \tilde{b} + \hat{a}$. The matrices a and b are uniquely determined by c.*

We see from here that $\mathfrak{e}_{6(-26)} = \mathfrak{f}_4 \oplus \mathfrak{h}_3^0(\mathbb{O})$ is a Cartan decomposition, as already mentioned in Subsection 2.6.

Note that the elements of $\mathfrak{e}_{6(-26)}$ are linear endomorphisms of $\mathfrak{h}_3(\mathbb{O})$. We denote the Lie bracket by $[\ ,\ ]_*$: it is given by the commutator of the endomorphisms. We need the following lemma:

Lemma C.2. *If $a, x \in \mathfrak{h}_3^0(\mathbb{O})$, then:*

(i) $[\hat{x}, \hat{a}]_* = \frac{1}{4}\widetilde{[x, a]}$

(ii) $[\hat{x}, [\hat{x}, \hat{a}]_*]_* = \frac{1}{4}\widehat{[x, [x, a]]}$.

Proof. (i) For any $y \in \mathfrak{h}_3(\mathbb{O})$ we have

$$[\hat{x}, \hat{a}]_*(y) = \hat{x}(\hat{a}(y)) - \hat{a}(\hat{x}(y)) = x \circ (a \circ y) - a \circ (x \circ y) = \frac{1}{4}[[x, a], y].$$

(ii) For any $y \in \mathfrak{h}_3(\mathbb{O})$ we have

$$4[\hat{x}, [\hat{x}, \hat{a}]_*]_*(y) = [\hat{x}, \widetilde{[x,a]}]_*(y) = \hat{x}(\widetilde{[x,a]}(y)) - \widetilde{[x,a]}(\hat{x}(y))$$
$$= x \circ ([[x,a],y]) - [[x,a], x \circ y] = [x,[x,a]] \circ y. \qquad \square$$

Let us now identify $\mathfrak{h}_3^0(\mathbb{O})$ with the subspace $\{\hat{x} \mid x \in \mathfrak{h}_3^0(\mathbb{O})\}$ of $\mathfrak{e}_{6(-26)}$. From equation (ii) above we deduce that $\mathfrak{d}^0$ is a maximal abelian subspace of $\mathfrak{h}_3^0(\mathbb{O})$. By definition, a vector $a \in \mathfrak{h}_3^0(\mathbb{O})$ is a root vector with respect to a root γ if

$$[\hat{x}, [\hat{x}, \hat{a}]_*]_* = \gamma(\hat{x})^2 \hat{a},$$

for all $x \in \mathfrak{d}^0$. Again from equation (ii) we deduce that the roots of the symmetric space $E_{6(-26)}/F_4$ with respect to $\mathfrak{d}^0$ are the functions $\frac{1}{2}(x_3 - x_2)$, $\frac{1}{2}(x_1 - x_3)$, and $\frac{1}{2}(x_1 - x_2)$ along with their negatives (see also equation (2.7)). The corresponding root spaces are the spaces $\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$, and $\mathfrak{h}_{\gamma_3}$ described in Section 2.6.

Acknowledgments

We would like to thank the referee for several valuable suggestions.

References

[1] J. F. Adams, Spin(8), triality, F_4, and all that, *Superspace and Supergravity*, 435–445, ed. S. Hawking and M. Rocek, Cambridge Univ. Press, 1981. MR0625301 (83d:83004)

[2] J. F. Adams, *Lectures on exceptional Lie groups*, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, IL, 1996. MR1428422 (98b:22001)

[3] J. C. Baez, The octonions, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 145–205. MR1886087 (2003f:17003)

[4] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, Schubert cells, and the cohomology of the spaces G/P, Uspehi Mat. Nauk **28** (1973), no. 3(171), 3–26. MR0429933 (55 #2941)

[5] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. (2) **57** (1953), 115–207. MR0051508 (14,490e)

[6] A. Borel, Sur l'homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math. **76** (1954), 273–342. MR0064056 (16,219b)

[7] R. Bott, *Lectures on $K(X)$*, Mathematics Lecture Note Series W. A. Benjamin, Inc., New York, 1969. MR0258020 (41 #2667)

[8] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, 98, Springer, New York, 1985. MR0781344 (86i:22023)

[9] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups, Compositio Math. **49** (1983), no. 3, 309–398. MR0707179 (85e:58150)

[10] H. Freudenthal, Oktaven, Ausnahmegruppen und Oktavengeometrie, Geom. Dedicata **19** (1985), no. 1, 7–63. MR0797151 (86k:17018b)

[11] M. Goresky, R. Kottwitz, and R. MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, Invent. Math. **131** (1998), no. 1, 25–83. MR1489894 (99c:55009)

[12] V. Guillemin, V. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, 98, Amer. Math. Soc., Providence, RI, 2002. MR1929136 (2003m:53149)

[13] M. Harada, A. Henriques, and T. S. Holm, Computation of generalized equivariant cohomologies of Kac-Moody flag varieties, Adv. Math. **197** (2005), no. 1, 198–221. MR2166181 (2006h:53086)

[14] M. Harada, G. D. Landweber, and R. Sjamaar, Divided differences and the Weyl character formula in equivariant K-theory, Math. Res. Lett. **17** (2010), no. 3, 507–527. MR2653685 (2011g:19004)

[15] J.-C. Hausmann and T. Holm, Conjugation spaces and edges of compatible torus actions, *Geometric aspects of analysis and mechanics*, 179–198, Progr. Math., 292 Birkhäuser/Springer, New York. MR2809472 (2012h:53186)

[16] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, corrected reprint of the 1978 original, Graduate Studies in Mathematics, 34, American Mathematical Society, Providence, RI, 2001. MR1834454 (2002b:53081)

[17] T. S. Holm and R. Sjamaar, Torsion and abelianization in equivariant cohomology, Transform. Groups **13** (2008), no. 3-4, 585–615. MR2452608 (2009j:57041)

[18] W.-Y. Hsiang, R. S. Palais, and C.-L. Terng, The topology of isoparametric submanifolds, J. Differential Geom. **27** (1988), no. 3, 423–460. MR0940113 (89m:53104)

[19] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29, Cambridge Univ. Press, Cambridge, 1990. MR1066460 (92h:20002)

[20] A. Kono, On the integral cohomology of BSpin(n), J. Math. Kyoto Univ. **26** (1986), no. 3, 333–337. MR0857221 (88b:55018)

[21] B. Kostant and S. Kumar, T-equivariant K-theory of generalized flag varieties, J. Differential Geom. **32** (1990), no. 2, 549–603. MR1072919 (92c:19006)

[22] A.-L. Mare, Equivariant cohomology of real flag manifolds, Differential Geom. Appl. **24** (2006), no. 3, 223–229. MR2216937 (2007b:53103)

[23] A.-L. Mare, Equivariant cohomology of quaternionic flag manifolds, J. Algebra **319** (2008), no. 7, 2830–2844. MR2397410 (2009h:57053)

[24] A.-L. Mare and M. Willems, Equivariant K-theory of quaternionic flag manifolds, J. K-Theory **4** (2009), no. 3, 537–557. MR2570953 (2010k:19009)

[25] J. McLeod, The Kunneth formula in equivariant K-theory, *Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978)*, 316–333, Lecture Notes in Math., 741, Springer, Berlin, 1979. MR0557175 (80m:55007)

[26] S. Murakami, Exceptional simple Lie groups and related topics in recent differential geometry, *Differential geometry and topology (Tianjin, 1986–87)*, 183–221, Lecture Notes in Math., 1369 Springer, Berlin. MR1001187 (90g:22009)

[27] I. Rosu, Equivariant K-theory and equivariant cohomology. With an appendix by Allen Knutson and Rosu. Math. Z. **243** (2003), no. 3, 423–448. MR1970011 (2004f:19011)

[28] G. Segal, Equivariant K-theory, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 129–151. MR0234452 (38 #2769)

[29] M. Yokotani, The integral cohomology rings of $F_4/\mathrm{Spin}(n)$ and $E_6/\mathrm{Spin}(m)$, Math. J. Okayama Univ. **34** (1992), 181–193 (1994). MR1272620 (95c:57061)

Received June 14, 2011; accepted March 23, 2013

Augustin-Liviu Mare
Department of Mathematics and Statistics
University of Regina
Regina SK, S4S 0A2 Canada
E-mail: mareal@math.uregina.ca
URL: http://www.math.uregina.ca/~mareal

Matthieu Willems
Département d'informatique
Université du Québec à Montréal
Montréal QC, H3C 3P8 Canada
E-mail: matthieu.willems@polytechnique.org