Münster J. of Math. **6** (2013), 445–482
urn:nbn:de:hbz:6-55309457491

Münster Journal of Mathematics

Completely bounded representations of convolution algebras of locally compact quantum groups

Michael Brannan, Matthew Daws, and Ebrahim Samei

(Communicated by Joachim Cuntz)

Abstract. Given a locally compact quantum group $\mathbb{G}$, we study the structure of completely bounded homomorphisms $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$, and the question of when they are similar to $*$-homomorphisms. By analogy with the cocommutative case (representations of the Fourier algebra $A(G)$), we are led to consider the associated (a priori unbounded) map $\pi^* : L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H)$ given by $\pi^*(\omega) = \pi(\omega^\sharp)^*$. We show that the corepresentation V_π of $L^\infty(\mathbb{G})$ associated to π is invertible if and only if both π and π^* are completely bounded. To prove the "if" part of this claim we show that coefficient operators of such representations give rise to completely bounded multipliers of the dual convolution algebra $L^1(\hat{\mathbb{G}})$. An application of these results is that any (co)isometric corepresentation is automatically unitary. An averaging argument then shows that when $\mathbb{G}$ is amenable, π is similar to a $*$-homomorphism if and only if π^* is completely bounded. For compact Kac algebras, and for certain cases of $A(G)$, we show that any completely bounded homomorphism π is similar to a $*$-homomorphism, without further assumption on π^*. Using free product techniques, we construct new examples of compact quantum groups $\mathbb{G}$ such that $L^1(\mathbb{G})$ admits bounded, but not completely bounded, representations.

1. Introduction

Given a locally compact group G, a *uniformly bounded representation* of G on a Hilbert space H is a weakly continuous homomorphism π_0 from G into the invertible group of the algebra $\mathcal{B}(H)$ of bounded operators on H, with $\|\pi_0\| := \sup_{s\in G} \|\pi_0(s)\| < \infty$. The study of uniformly bounded representations of locally compact groups has been central to the development of harmonic analysis and to understanding the structure of operator algebras associated to locally compact groups (see [11, 15, 30] for example).

In this paper we initiate the study of "uniformly bounded representations" of locally compact quantum groups. To motivate the objects of study in this paper, recall that for a locally compact group G, there is a bijective correspondence between nondegenerate bounded representations $\pi : L^1(G) \to \mathcal{B}(H)$, of

the Banach $*$-algebra $L^1(G)$, and uniformly bounded representations $\pi_0 : G \to \mathcal{B}(H)$. The correspondence between π and π_0 is given in the usual way by integration:

$$\big(\pi(\omega)\alpha|\beta\big) = \int_G \omega(s)\big(\pi_0(s)\alpha|\beta\big)\ ds \qquad (\omega \in L^1(G), \alpha, \beta \in H).$$

Note that $\|\pi\| = \|\pi_0\|$. Further, as $L^1(G) = L^\infty(G)_*$ is a maximal operator space, π is automatically completely bounded with $\|\pi\|_{\mathcal{CB}(L^1(G),\mathcal{B}(H))} = \|\pi\|$. Moreover, from the above correspondence, it is easy to see that a nondegenerate representation $\pi : L^1(G) \to \mathcal{B}(H)$ is (similar to) a $*$-representation of $L^1(G)$ if and only if π_0 is (similar to) a unitary representation.

Thus, if $\mathbb{G}$ is a locally compact quantum group with von Neumann algebra $L^\infty(\mathbb{G})$ and convolution algebra $L^1(\mathbb{G})$, we are led to consider (completely) bounded representations $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$. A representation $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ is called a *$*$-representation* if the restriction of π to the canonical dense $*$-subalgebra $L^1_\sharp(\mathbb{G}) \subseteq L^1(\mathbb{G})$ (with involution $\omega \mapsto \omega^\sharp$), is a $*$-representation. This is equivalent to the fact that $\pi|_{L^1_\sharp(\mathbb{G})} = \pi^*$, where $\pi^* : L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H)$ is the representation obtained from π via the formula $\pi^*(\omega) = \pi(\omega^\sharp)^*$. As usual, we say π is *similar to a $*$-representation* if $T\pi(\cdot)T^{-1}$ is a $*$-representation in the above sense, for some invertible $T \in \mathcal{B}(H)$.

A necessary condition for a representation $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ to be similar to a $*$-representation is that both π and π^* extend to *completely bounded* maps from $L^1(\mathbb{G})$ to $\mathcal{B}(H)$ (see Proposition 3.4). Furthermore, when π and π^* are completely bounded, coefficients of π induce *completely bounded multipliers* ([12, 13, 22]) of $L^1(\hat{\mathbb{G}})$, see Theorem 4.7 and Proposition 4.8. We prove this result by making careful use of the duality theory for locally compact quantum groups to identify, via the underlying Hilbert spaces, dense subspaces of $L^1(\mathbb{G})$ and $L^1(\hat{\mathbb{G}})$.

Furthermore, by restricting ourselves to the completely bounded case, we can use that completely bounded representations $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ correspond to *corepresentations* $V_\pi \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$, that is, operators with $(\Delta \otimes \iota)(V_\pi) = V_{\pi,13}V_{\pi,23}$. We show in Theorem 3.3 that when V_π is invertible in $L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$, then π^* is automatically completely bounded. Using our results on completely bounded multipliers, we can prove the converse to this result—if π and π^* are completely bounded, then V_π is invertible (with an appropriate modification when π is a degenerate homomorphism). From this perspective, our results on completely bounded multipliers hence generalize a classical result of de Canniere and Haagerup [15, Thm. 2.2], showing that coefficient functions of uniformly bounded representations of a group G are always completely bounded multipliers of the Fourier algebra $A(G)$.

Putting all these results together, we believe that invertible corepresentations, equivalently, π such that π^* is also completely bounded, should be thought of as the correct quantum generalization of a uniformly bounded group representation.

An interesting corollary of these results is that a corepresentation U which is not necessarily unitary, but is at least a partial isometry, and is either injective or surjective, must automatically be unitary. This follows since if π is the associated homomorphism, then the condition on U is enough to show that π^* is completely bounded, and then U must be invertible. An application of this is an improvement upon a construction of Kustermans: the "induced corepresentations" constructed in [25] are always unitary, without any "integrability" condition; see Section 4.13.

Another application of our results is to the similarity problem for amenable locally compact quantum groups. When $\mathbb{G}$ is amenable, Theorem 4.7 combined with some averaging techniques with invariant means shows that a representation $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ is similar to a $*$-representation if and only if both π and π^* are completely bounded maps (Theorem 5.1). This generalizes the results of Day and Dixmier [14, 16] on the unitarizability of amenable groups, as well as some known results for compact quantum groups. When $\mathbb{G}$ is a compact Kac algebra (and for certain cases of $A(G)$) we are able to improve these results and show that every completely bounded representation $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ is similar to a $*$-representation, without assumption on π^* (see Theorems 6.2, 8.1, and 8.2).

In [8], the first examples of bounded, but not completely bounded, representations of Fourier algebras were constructed. Using noncommutative Khintchine inequalities from free probability theory, we show that similar constructions hold for certain free products of compact quantum groups; such representations cannot, of course, be similar to $*$-representations. Moreover, for the representations constructed here, we show that there are coefficient operators which fail to induce bounded multipliers of $L^1(\hat{\mathbb{G}})$ (Corollary 7.2). This provides further evidence that *completely* bounded maps are indeed the "correct" maps to study.

The paper is organised as follows. In Section 2, we review some basic facts about locally compact quantum groups which will be needed in this paper, fixing some notation along the way. We also make some remarks on the extended Haagerup tensor product. In Section 3 we show the correspondence between completely bounded representations π on $L^1(\mathbb{G})$ and corepresentations V_π, and show that if π is similar to a $*$-representation, or V_π is invertible, then π^* is also completely bounded. In Section 4 we prove the converse to this result. We first study the duality theory of locally compact quantum groups, trying to find a quantum analogue of the function space $L^1(G) \cap A(G)$. We then show that coefficients of representations π, such that both π and π^* are completely bounded, induce completely bounded multipliers. This result is then applied to prove that V_π is invertible. In Section 5, we consider the similarity problem for locally compact quantum groups, focusing on the amenable case, and refining our results for compact Kac algebras in Section 6. In Section 7, we construct bounded representations of $L^1(\mathbb{G})$ which are not completely bounded, where $\mathbb{G}$ is a certain free product of compact quantum groups. Section 8 deals only with representations of Fourier algebras, and extends some of the results obtained

in [7]. Finally Appendix A looks at weak*-closed operators, and proves a technical result about analytic generators of σ-weakly continuous, one-parameter groups.

Finally, a quick word on notation. We use $\langle \cdot, \cdot \rangle$ to denote the bilinear pairing between a Banach space and its dual, and $(\cdot|\cdot)$ to denote the inner product of a Hilbert space. We use the symbol $\overline{\otimes}$ for the von Neumann algebraic tensor product; for tensor products of Hilbert spaces, or the minimal tensor products of C*-algebras, we write simply $\otimes$. We shall use standard results from the theory of operator spaces, following [17] for example.

2. Locally compact quantum groups

For the convenience of the reader, we give a brief overview of the theory of locally compact quantum groups. Our main references are [27] and [28]. The first reference is a self-contained account of the C*-algebraic approach to locally compact quantum groups, and the second reference discusses the von Neumann algebraic approach. For other readable introductions, see [26] and [41].

A *Hopf–von Neumann algebra* is a pair (M, Δ) where M is a von Neumann algebra and $\Delta : M \to M\overline{\otimes}M$ is a unital normal $*$-homomorphism which is coassociative: $(\iota \otimes \Delta) \circ \Delta = (\Delta \otimes \iota) \circ \Delta$. Then Δ_* induces a completely contractive Banach algebra product on the predual M_*. We shall write the product in M_* by juxtaposition, so

$$\langle x, \omega\omega' \rangle = \langle \Delta(x), \omega \otimes \omega' \rangle \qquad (x \in M, \omega, \omega' \in M_*).$$

Recall the notion of a normal semifinite faithful weight $\varphi : M^+ \to [0, \infty]$ (see [36, Chap. VII] for example). We let

$$\mathfrak{n}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}, \qquad \mathfrak{m}_\varphi = \operatorname{span}\{x^*y \mid x, y \in \mathfrak{n}_\varphi\},$$
$$\mathfrak{m}_\varphi^+ = \{x \in M^+ \mid \varphi(x) < \infty\}.$$

Then $\mathfrak{m}_\varphi$ is a hereditary $*$-subalgebra of M, $\mathfrak{n}_\varphi$ is a left ideal, and $\mathfrak{m}_\varphi^+$ is equal to $M^+ \cap \mathfrak{m}_\varphi$. We can perform the GNS construction for φ, which leads to a Hilbert space H, a linear map $\Lambda : \mathfrak{n}_\varphi \to H$ with dense range, and a unital normal $*$-homomorphism $\pi : M \to \mathcal{B}(H)$ with $\pi(x)\Lambda(y) = \Lambda(xy)$. In future, we shall tend to suppress π in our notation. Tomita–Takesaki theory gives us the modular conjugation J and the modular automorphism group $(\sigma_t)_{t \in \mathbb{R}}$. Recall that there is a (unbounded) positive nonsingular operator ∇ (denoted Δ in [36]) which induces (σ_t) via $\sigma_t(x) = \nabla^{it} x \nabla^{-it}$ for $x \in M$ and $t \in \mathbb{R}$. Finally, M is in standard position on H, so in particular, for each $\omega \in M_*$, there are $\xi, \eta \in H$ with $\omega = \omega_{\xi,\eta}$, where $\langle x, \omega_{\xi,\eta} \rangle = (x\xi|\eta)$ for $\xi, \eta \in H$.

A von Neumann algebraic *locally compact quantum group* is a Hopf–von Neumann algebra (M, Δ), together with faithful normal semifinite weights φ, ψ which are left and right invariant, respectively. This means that

$$\varphi\big((\omega \otimes \iota)\Delta(x)\big) = \varphi(x)\langle 1, \omega \rangle, \qquad \psi\big((\iota \otimes \omega)\Delta(y)\big) = \psi(y)\langle 1, \omega \rangle,$$

for $\omega \in M_*^+, x \in \mathfrak{m}_\varphi^+$ and $y \in \mathfrak{m}_\psi^+$. Using these weights, we can construct an antipode S, which will in general be unbounded. Then S has a decomposition $S = R\tau_{-i/2}$, where R is the unitary antipode, and $(\tau_t)_{t\in\mathbb{R}}$ is the scaling group. The unitary antipode R is a normal $*$-antiautomorphism of M, and $\Delta R = \sigma(R \otimes R)\Delta$, where $\sigma : M\overline{\otimes}M \to M\overline{\otimes}M$ is the tensor swap map. As R is normal, it drops to an isometric linear map $R_* : M_* \to M_*$, which is anti-multiplicative. As usual, we make the canonical choice that $\varphi = \psi \circ R$.

Associated to (M, Δ) is a *reduced* C*-algebraic quantum group (A, Δ). Here A is a C*-subalgebra of M, and $\Delta : A \to M(A \otimes A)$, the multiplier algebra of $A \otimes A$. Here we identify $M(A \otimes A)$ with a C*-subalgebra of $M\overline{\otimes}M$; indeed, given a C*-algebra $B \subseteq \mathcal{B}(H)$, we can always identify the multiplier algebra $M(B)$ with $\{x \in B'' : xb, bx \in B \ (b \in B)\}$.

There is a unitary W, the *fundamental unitary*, acting on $H \otimes H$ (the Hilbert space tensor product) such that $\Delta(x) = W^*(1 \otimes x)W$ for $x \in M$. Then W is multiplicative, in the sense that $W_{12}W_{13}W_{23} = W_{23}W_{12}$. Here, and later, we use the standard leg number notation: $W_{12} = W \otimes I, W_{13} = \Sigma_{23}W_{12}\Sigma_{23}$ and so forth, where here $\Sigma : H \otimes H \to H \otimes H$ is the swap map. The *left-regular representation* of M_* is the map $\lambda : \omega \mapsto (\omega \otimes \iota)W$. This is an injective homomorphism, the norm closure of $\lambda(M_*)$ is a C*-algebra denoted by $\hat{A}$, and the σ-weak closure of $\lambda(M_*)$ is a von Neumann algebra, which we denote by $\hat{M}$. We define a coproduct $\hat{\Delta} : \hat{M} \to \hat{M}\overline{\otimes}\hat{M}$ by $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, where $\hat{W} = \Sigma W \Sigma$. Then we can find invariant weights to turn $(\hat{M}, \hat{\Delta})$ into a locally compact quantum group—the dual quantum group to M. We then have the Pontryagin duality theorem which states that $\hat{\hat{M}} = M$ canonically. There is a nice link between the scaling group of M, and the modular theory of the (left) Haar weight of $\hat{M}$, in that $\tau_t(x) = \hat{\nabla}^{it} x \hat{\nabla}^{-it}$ for $x \in M, t \in \mathbb{R}$.

In this paper we will use the symbol $\mathbb{G}$ to indicate the abstract "object" to be thought of as a locally compact quantum group. Inspired by the classical examples of locally compact groups, we write $L^\infty(\mathbb{G})$ for M, $C_0(\mathbb{G})$ for A, $C^b(\mathbb{G})$ for $M(A)$ (the multiplier algebra of A), $L^1(\mathbb{G})$ for M_*, and $L^2(\mathbb{G})$ for H. We denote the locally compact quantum group dual to $\mathbb{G}$ by $\hat{\mathbb{G}}$, and use the notations $L^\infty(\hat{\mathbb{G}}) = \hat{M}$, $L^1(\hat{\mathbb{G}}) = \hat{M}_*$, and so on. In order to distinguish between objects attached to $\hat{\mathbb{G}}$ from those attached to $\mathbb{G}$, we decorate elements in $L^\infty(\hat{\mathbb{G}})$, $L^1(\hat{\mathbb{G}})$ etc. with "hats". For example, we write $\hat{x} \in L^\infty(\hat{\mathbb{G}})$, and $\hat{\omega} \in L^1(\hat{\mathbb{G}})$.

Since the antipode S of $\mathbb{G}$ is unbounded in general, there is not a natural way to turn $L^1(\mathbb{G})$ into a $*$-algebra. However, $L^1(\mathbb{G})$ always contains a dense subalgebra which has a conjugate-linear involution. We follow [24], see also [28, Sec. 2]. Define $L^1_\sharp(\mathbb{G})$ to be the collection of those $\omega \in L^1(\mathbb{G})$ such that there exists $\omega^\sharp \in L^1(\mathbb{G})$ with $\langle x, \omega^\sharp \rangle = \overline{\langle S(x)^*, \omega \rangle}$ for each $x \in D(S)$. Then $L^1_\sharp(\mathbb{G})$ is a dense subalgebra of $L^1(\mathbb{G})$, and $\omega \mapsto \omega^\sharp$ defines an involution on $L^1_\sharp(\mathbb{G})$. As [24, Prop. 3.1] shows, given $\omega \in L^1(\mathbb{G})$, we have that $\omega \in L^1_\sharp(\mathbb{G})$ if

and only if $\lambda(\omega)^* = \lambda(\omega')$ for some $\omega' \in L^1(\mathbb{G})$. Furthermore, the left regular representation λ becomes a $*$-homomorphism, when restricted to $L^1_\sharp(\mathbb{G})$.

When G is a locally compact group, we write $VN(G)$ for $L^\infty(\hat{G})$ (the von Neumann algebra generated by the left regular representation of $\lambda_G : G \to \mathcal{U}(L^2(G))$), and write $A(G)$ for $L^1(\hat{G})$, and call this the Fourier algebra of G. The Fourier algebra was first studied as a Banach algebra by Eymard [19]. The left regular representation $\hat{\lambda} : A(G) \to C_0(G)$, is a completely contractive $*$-homomorphism, which allows us to identify $A(G)$ with a dense $*$-subalgebra of $C_0(G)$, the identification being given by

$$\hat{\lambda}(\hat{\omega})(t) = \langle \lambda_G(t^{-1}), \hat{\omega} \rangle \qquad (\hat{\omega} \in A(G), t \in G).$$

See [37] for details (note that the above identification differs from the one considered by Eymard, where t is replaced by t^{-1} in [19]). So as to not overburden notation, in many places (especially in Section 8) we will drop the notation $\hat{\lambda}$ from above, and just view $A(G)$ as a concrete $*$-subalgebra of $C_0(G)$.

2.1. **The extended Haagerup tensor product.** Let M and N be von Neumann algebras. The *extended (or weak*) Haagerup tensor product* of M with N is the collection of $x \in M \overline{\otimes} N$ such that

$$x = \sum_i x_i \otimes y_i$$

with σ-weak convergence, and such that $\sum_i x_i x_i^*$ and $\sum_i y_i^* y_i$ converge σ-weakly in M and N respectively. The natural norm is then

$$\|x\|_{eh} = \inf \left\{ \Big\| \sum_i x_i x_i^* \Big\|^{1/2} \Big\| \sum_i y_i^* y_i \Big\|^{1/2} \;\Big|\; x = \sum_i x_i \otimes y_i \right\},$$

and we write $M \otimes_{eh} N$ for the resulting normed space. See [6] or [18] for further details.

Suppose we are given $(x_i)_{i \in I} \subseteq M$ and $(y_i)_{i \in I} \subseteq N$ with $\sum_i x_i x_i^*$ and $\sum_i y_i^* y_i$ converging σ-weakly. Let $M \subseteq \mathcal{B}(H)$ and $N \subseteq \mathcal{B}(K)$. Then there are bounded maps $X : H \to H \otimes \ell^2(I)$ and $Y : K \to K \otimes \ell^2(I)$ given by

$$X(\xi) = \sum_i x_i^*(\xi) \otimes \delta_i, \qquad Y(\xi) = \sum_i y_i(\xi) \otimes \delta_i \qquad (\xi \in L^2(\mathbb{G})),$$

where $(\delta_i)_{i \in I}$ is the canonical orthonormal basis for $\ell^2(I)$. Indeed, $X^*X = \sum_i x_i x_i^*$ and $Y^*Y = \sum_i y_i^* y_i$. Letting $\Sigma : K \otimes \ell^2(I) \to \ell^2(I) \otimes K$ be the swap map, a simple calculation shows that

$$(X^* \otimes 1)(1 \otimes \Sigma)(1 \otimes Y) = \sum_i x_i \otimes y_i \in M \overline{\otimes} N \subseteq \mathcal{B}(H \otimes K).$$

This argument hence shows that $\sum_i x_i \otimes y_i$ always converges σ-weakly in $M \overline{\otimes} N$, as long as $\sum_{i \in I} x_i x_i^*$ and $\sum_{i \in I} y_i^* y_i$ converge σ-weakly. Finally, note that if $H = K$ then

$$\sum_i x_i y_i = X^* Y \in \mathcal{B}(H),$$

and in particular, if $M = N$, then $\sum_i x_i y_i$ converges σ-weakly in M to X^*Y.

3. Completely bounded homomorphisms and corepresentations

In this section, we investigate the correspondence between corepresentations of $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ (which are not assumed unitary) and completely bounded representations π of $L^1(\mathbb{G})$. We show how various naturally defined variants of π are associated to variants of U. In particular, we show that π is similar to a $*$-representation if and only if U is similar (in the sense of corepresentations) to a unitary. One of our variants of π is defined using the unbounded $*$-map on $L^1_\sharp(\mathbb{G})$. We show that if U is invertible, then all of the variants of π are completely bounded. A principle result, Theorem 4.10 below, is that the converse to this result is also true; and hence we have a bijection between invertible (not necessarily unitary) corepresentations and a certain class of completely bounded representations.

We are interested in completely bounded homomorphisms $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$. We have the standard identifications (see [17, Chap. 7]):

$$\mathcal{CB}(L^1(\mathbb{G}), \mathcal{B}(H)) \cong \big(L^1(\mathbb{G})\widehat{\otimes}\mathcal{T}(H)\big)^* \cong L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H).$$

Here $\widehat{\otimes}$ denotes the operator space projective tensor product, and $\overline{\otimes}$ the von Neumann algebraic tensor product. We use the notation that $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ is associated with $V_\pi \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$, the relationship is that

$$\pi(\omega) = (\omega \otimes \iota)V_\pi \qquad (\omega \in L^1(\mathbb{G})).$$

We use the standard leg numbering notation, see for example [2, 39].

Lemma 3.1. *Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a completely bounded map, associated to $V_\pi \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$. Then π is a homomorphism if and only if $(\Delta \otimes \iota)V_\pi = V_{\pi,13}V_{\pi,23}$. Similarly, π is an anti-homomorphism if and only if $(\Delta \otimes \iota)V_\pi = V_{\pi,23}V_{\pi,13}$.*

Proof. For $\omega_1, \omega_2 \in L^1(\mathbb{G})$,

$$\pi(\omega_1\omega_2) = \big(\Delta_*(\omega_1 \otimes \omega_2) \otimes \iota\big)V_\pi = (\omega_1 \otimes \omega_2 \otimes \iota)\big((\Delta \otimes \iota)V_\pi\big),$$

while

$$\pi(\omega_1)\pi(\omega_2) = (\omega_1 \otimes \iota)V_\pi(\omega_2 \otimes \iota)V_\pi = (\omega_1 \otimes \omega_2 \otimes \iota)\big(V_{\pi,13}V_{\pi,23}\big),$$

and so π is a homomorphism if and only if $(\Delta \otimes \iota)V_\pi = V_{\pi,13}V_{\pi,23}$. The result for anti-homomorphisms is analogous. □

The condition that $(\Delta\otimes\iota)V_\pi = V_{\pi,13}V_{\pi,23}$ says that V_π is a *co-representation* of $L^\infty(\mathbb{G})$. Similarly, we say that V_π is a *co-anti-representation* when $(\Delta \otimes \iota)V_\pi = V_{\pi,23}V_{\pi,13}$.

Following [7], given a locally compact group G and a representation $\pi : A(G) \to \mathcal{B}(H)$, we define $\check{\pi}, \pi^*$ and $\tilde{\pi}$ by

$$\check{\pi}(\omega) = \pi(\check{\omega}), \quad \pi^*(\omega) = \pi(\overline{\omega})^*, \quad \tilde{\pi}(\omega) = \check{\pi}(\overline{\omega})^* = \pi(\overline{\check{\omega}})^* \qquad (\omega \in A(G)),$$

where $\check{\omega}(g) = \omega(g^{-1})$ (viewing $A(G)$ as a subalgebra of $C_0(G)$).

For $\omega \in L^1(\mathbb{G})$, define ω^* by $\langle x, \omega^* \rangle = \overline{\langle x^*, \omega \rangle}$ for $x \in L^\infty(\mathbb{G})$. As Δ is a $*$-homomorphism, it is easy to see that $L^1(\mathbb{G}) \to L^1(\mathbb{G}); \omega \mapsto \omega^*$ is a conjugate linear homomorphism. Recalling the definition of $L^1_\sharp(\mathbb{G})$ from Section 2, given $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$, we define

$$\begin{aligned} \check{\pi} &: L^1_\sharp(\mathbb{G})^* \to \mathcal{B}(H); \quad \omega \mapsto \pi\big((\omega^*)^\sharp\big), \\ \pi^* &: L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H); \quad \omega \mapsto \pi(\omega^\sharp)^*, \\ \tilde{\pi} &: L^1(\mathbb{G}) \to \mathcal{B}(H); \quad \omega \mapsto \pi(\omega^*)^*. \end{aligned}$$

These are defined by analogy with the $A(G)$ case, and the following results show that they are natural choices.

Proposition 3.2. *Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a bounded map. Then π is completely bounded if and only if $\tilde{\pi}$ is completely bounded, and in this case, $V_\pi^* = V_{\tilde{\pi}}$. Furthermore, $\check{\pi}$ extends to a completely bounded map defined on $L^1(\mathbb{G})$ if and only if π^* extends to a completely bounded map defined on $L^1(\mathbb{G})$, and in this case, $V_{\check{\pi}}^* = V_{\pi^*}$.*

Proof. Let $U \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(H)$ and define $\phi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ by $\phi(\omega) = (\omega \otimes \iota)(U)$. Then

$$\phi(\omega^*)^* = \big((\omega^* \otimes \iota)(U)\big)^* = \big((\omega \otimes \iota)(U^*)^*\big)^* = (\omega \otimes \iota)(U^*).$$

It follows immediately that π is completely bounded if and only if $\tilde{\pi}$ is, and that $V_\pi^* = V_{\tilde{\pi}}$. As $\pi^*(\omega^*) = \check{\pi}(\omega)^*$, similarly $\check{\pi}$ extends to completely bounded map if and only if π^* does, and $V_{\check{\pi}}^* = V_{\pi^*}$. □

We wish to show that $*$-representations of $L^1_\sharp(\mathbb{G})$ do give rise to completely bounded π such that also π^* is completely bounded. To do this, we need to investigate how unitary, or more generally, invertible corepresentations interact with the antipode.

Let us recall a characterization of the antipode using ideas very close to the extended Haagerup tensor product. This is shown in [27, Cor. 5.34] in the C*-algebra setting, but the same proofs work in the von Neumann case. Let $x, y \in L^\infty(\mathbb{G})$ be such that there are families $(x_i), (y_i) \subseteq L^\infty(\mathbb{G})$ with $\sum_i x_i x_i^* < \infty$ and $\sum y_i^* y_i < \infty$, and such that

$$x \otimes 1 = \sum_i \Delta(x_i)(1 \otimes y_i), \qquad y \otimes 1 = \sum_i (1 \otimes x_i)\Delta(y_i).$$

Then $x \in D(S)$ and $S(x) = y$. Note that these sums converge σ-weakly, compare Section 2.1. For further details on characterizing the antipode in this way, see [42, Prop. 5.6].

Theorem 3.3. *Let $V_\pi \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(H)$ be an invertible corepresentation, associated to a completely bounded homomorphism $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$. Then $\check{\pi}$ is a completely bounded anti-homomorphism, and $V_\pi^{-1} = V_{\check{\pi}}$.*

Proof. Let $\alpha, \beta \in H$, let (f_i) be an orthonormal basis of H, and set

$$x_i = (\iota \otimes \omega_{f_i, \beta})(V_\pi), \qquad y_i = (\iota \otimes \omega_{\alpha, f_i})(V_\pi^{-1}).$$

As in Section 2.1, it is easy to see that $\sum_i x_i x_i^* = (\iota\otimes\omega_\alpha)(V_\pi V_\pi^*)$ and $\sum_i y_i^* y_i = (\iota\otimes\omega_\beta)((V_\pi^{-1})^* V_\pi^{-1})$, with the sums converging σ-weakly. Now, using that $(\Delta\otimes\iota)(V_\pi) = V_{\pi,13}V_{\pi,23}$,

$$\begin{aligned}\sum_i \Delta(x_i)(1\otimes y_i) &= \sum_i (\iota\otimes\iota\otimes\omega_{f_i,\beta})((\Delta\otimes\iota)(V_\pi))(1\otimes y_i)\\ &= \sum_i (\iota\otimes\iota\otimes\omega_{f_i,\beta})(V_{\pi,13}V_{\pi,23})(1\otimes(\iota\otimes\omega_{\alpha,f_i})(V_\pi^{-1})).\end{aligned}$$

Now, for any Hilbert space K, and $S,T\in\mathcal{B}(K\otimes H)$,

$$\sum_i (\iota\otimes\omega_{f_i,\beta})(S)(\iota\otimes\omega_{\alpha,f_i})(T) = (\iota\otimes\omega_{\alpha,\beta})(ST).$$

Thus

$$\sum_i \Delta(x_i)(1\otimes y_i) = (\iota\otimes\iota\otimes\omega_{\alpha,\beta})(V_{\pi,13}V_{\pi,23}V_{\pi,23}^{-1}) = (\iota\otimes\omega_{\alpha,\beta})(V_\pi)\otimes 1.$$

Similarly, as Δ is a homomorphism, $(\Delta\otimes\iota)(V_\pi^{-1}) = V_{\pi,23}^{-1}V_{\pi,13}^{-1}$, and so

$$\begin{aligned}\sum_i (1\otimes x_i)\Delta(y_i) &= \sum_i (1\otimes(\iota\otimes\omega_{f_i,\beta})(V_\pi))(\iota\otimes\iota\otimes\omega_{\alpha,f_i})(V_{\pi,23}^{-1}V_{\pi,13}^{-1})\\ &= (\iota\otimes\iota\otimes\omega_{\alpha,\beta})(V_{\pi,23}V_{\pi,23}^{-1}V_{\pi,13}^{-1}) = (\iota\otimes\omega_{\alpha,\beta})(V_\pi^{-1})\otimes 1.\end{aligned}$$

Thus $(\iota\otimes\omega_{\alpha,\beta})(V_\pi)\in D(S)$ and $S\big((\iota\otimes\omega_{\alpha,\beta})(V_\pi)\big) = (\iota\otimes\omega_{\alpha,\beta})(V_\pi^{-1})$.

Let $\omega^*\in L^1_\sharp(\mathbb{G})$, so we have that

$$\langle x,(\omega^*)^\sharp\rangle = \overline{\langle S(x)^*,\omega^*\rangle} = \langle S(x),\omega\rangle \qquad (x\in D(S)).$$

It follows that

$$\begin{aligned}(\check\pi(\omega)\alpha|\beta) &= \langle(\iota\otimes\omega_{\alpha,\beta}(V_\pi),(\omega^*)^\sharp\rangle = \langle S((\iota\otimes\omega_{\alpha,\beta}(V_\pi)),\omega\rangle\\ &= \langle(\iota\otimes\omega_{\alpha,\beta})(V_\pi^{-1}),\omega\rangle = \langle V_\pi^{-1},\omega\otimes\omega_{\alpha,\beta}\rangle.\end{aligned}$$

As such ω are dense in $L^1(\mathbb{G})$, it follows that $\check\pi$ is completely bounded, with $V_{\check\pi} = V_\pi^{-1}$, as claimed. □

We now show that any $*$-homomorphism on $L^1_\sharp(\mathbb{G})$ gives rise to a completely bounded homomorphism π such that also π^* is completely bounded. We use a result of Kustermans that any $*$-representation has a "generator"—that is, V_π exists and is unitary. The interaction between a unitary corepresentation and the antipode (which, in some sense, we generalized in the previous proposition) is of course well-known, see [49, Thm. 1.6] for example.

Proposition 3.4. *Let $\pi : L^1_\sharp(\mathbb{G})\to\mathcal{B}(H)$ be a homomorphism which is similar to a $*$-homomorphism. Then π and π^* extend by continuity to completely bounded homomorphisms $L^1(\mathbb{G})\to\mathcal{B}(H)$, and $\check\pi$ and $\tilde\pi$ extend by continuity to completely bounded anti-homomorphisms $L^1(\mathbb{G})\to\mathcal{B}(H)$.*

Proof. We first show this in the case when π is a $*$-homomorphism. Then $\pi(\omega^\sharp) = \pi(\omega)^*$ for each $\omega \in L^1_\sharp(\mathbb{G})$, and so $\pi^* = \pi$. As π is a $*$-homomorphism, it is necessarily contractive (see [35, Chap. I, Prop. 5.2] for example) and so extends by continuity to a homomorphism defined on all of $L^1(\mathbb{G})$.

To show that π extends to a completely bounded homomorphism, we use a nontrivial result of Kustermans. Suppose for the moment that $\pi : L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H)$ is nondegenerate. By [24, Cor. 4.3], there exists a unitary $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ (actually, $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H)) \subseteq L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$) such that $\pi(\omega) = (\omega \otimes \iota)U$ for each $\omega \in L^1_\sharp(\mathbb{G})$. It follows immediately that π does indeed extend to a completely bounded homomorphism from $L^1(\mathbb{G})$, and that thus actually $U = V_\pi$. As U is unitary, so invertible, the previous propositions shows that π^* is also completely bounded.

If π is degenerate, then as π is a $*$-homomorphism, we can orthogonally decompose H as $H_1 \oplus H_2$, where π restricts to H_1, and $\pi(\omega)\xi = 0$ for each $\xi \in H_2$, $\omega \in L^1_\sharp(\mathbb{G})$. Let π_1 be the restriction of π to H_1; so $\pi_1 : L^1(\mathbb{G}) \to \mathcal{B}(H_1)$ is a completely bounded homomorphism. It follows immediately that the same must be true of π and of π^*.

If π is only similar to a $*$-representation, then there exists an invertible $T \in \mathcal{B}(H)$ such that $\theta : L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H)$, $\omega \mapsto T^{-1}\pi(\omega)T$ is a $*$-homomorphism. Then θ extends to a completely bounded homomorphism, and hence so also does $\pi = T\theta(\cdot)T^{-1}$. The same result holds for π^*, as

$$\pi^*(\omega) = \pi(\omega^\sharp)^* = (T^{-1})^*\theta^*(\omega)T^* \qquad (\omega \in L^1_\sharp(\mathbb{G})).$$

Having established that π and π^* extend to completely bounded maps $L^1(\mathbb{G}) \to \mathcal{B}(H)$, it follows from Proposition 3.2 that also $\check{\pi}$ and $\tilde{\pi}$ extend to completely bounded maps $L^1(\mathbb{G}) \to \mathcal{B}(H)$ which are easily seen to be anti-homomorphisms. □

4. Invertible corepresentations

In the previous section, we provided a necessary condition for $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ to be similar to a $*$-representation, namely, that both π and π^* are completely bounded. Furthermore, if π is associated to a corepresentation V_π, and V_π is invertible, then π^* is completely bounded. In this section, we use the duality theory of locally compact quantum groups to show the converse: if π and π^* are completely bounded, then V_π is invertible (together with an appropriate interpretation of this when π is a degenerate homomorphism).

4.1. **Identification of "function" spaces.** In [7], an important component of the argument was to use the space $L^1(G) \cap A(G)$, identified as a function space on G. For a quantum group, we would like to be able to talk about the space $L^1(\mathbb{G}) \cap L^1(\hat{\mathbb{G}})$; to make sense of this, we shall embed dense subspaces of $L^1(\mathbb{G})$ and $L^1(\hat{\mathbb{G}})$ into $L^2(\mathbb{G})$, making use of the fact that we always identify $L^2(\mathbb{G})$ with $L^2(\hat{\mathbb{G}})$. This section is slightly technical—the key result is Proposition 4.4 below, which is used in the following section.

We first introduce the nonstandard, but instructive notation

$$L^1(\mathbb{G}) \cap L^2(\mathbb{G}) = \big\{\omega \in L^1(\mathbb{G}) \mid \exists\, \xi \in L^2(\mathbb{G}), \langle x^*, \omega\rangle = (\xi|\Lambda(x)) \ (x \in \mathfrak{n}_\varphi)\big\}.$$

This space is denoted by $\mathcal{I}$ in [28, Sec. 1.1] and [27, Notation 8.4]. Then $\hat{\varphi}$ is the weight with GNS construction $(L^2(\mathbb{G}), \iota, \hat{\Lambda})$ where $\lambda(L^1(\mathbb{G})\cap L^2(\mathbb{G}))$ forms a σ-strong* core for $\hat{\Lambda}$, and we have that

$$\langle x^*, \omega\rangle = \big(\hat{\Lambda}(\lambda(\omega))\big|\Lambda(x)\big) \qquad (x \in \mathfrak{n}_\varphi, \omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})).$$

Then [27, Result 8.6] shows that $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a left ideal in $L^1(\mathbb{G})$ and

$$\hat{\Lambda}\big(\lambda(\omega\omega_1)\big) = \lambda(\omega)\hat{\Lambda}\big(\lambda(\omega_1)\big) \qquad (\omega \in L^1(\mathbb{G}), \omega_1 \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})).$$

This formula is of course immediate from the fact that $\hat{\Lambda}$ is a GNS map, but this reasoning is circular(!) if one is following [27, Sec. 8]. Further, [27, Result 8.6] can easily be adapted (or just perform the obvious calculation) to show that $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a left $L^\infty(\mathbb{G})$ module, and

$$\hat{\Lambda}\big(\lambda(x\omega_1)\big) = x\hat{\Lambda}\big(\lambda(\omega_1)\big) \qquad (x \in L^\infty(\mathbb{G}), \omega_1 \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})).$$

Continuing to follow [27, Sec. 8], we find that there is a norm continuous, one-parameter group $(\rho_t)_{t\in\mathbb{R}}$ of isometries on $L^1(\mathbb{G})$, such that ρ_t is an algebra homomorphism for each t, and with $\hat{\sigma}_t(\lambda(\omega)) = \lambda(\rho_t(\omega))$ for $\omega \in L^1(\mathbb{G})$ (where $(\hat{\sigma}_t)_t$ is the modular automorphism group associated to the dual left Haar weight $\hat{\varphi}$). As observed before [28, Prop. 2.8], each ρ_t maps $L^1_\sharp(\mathbb{G})$ into itself, and $\rho_t(\omega^\sharp) = \rho_t(\omega)^\sharp$ for $\omega \in L^1_\sharp(\mathbb{G})$. Finally, for $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, also $\rho_t(\omega) \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ and $\hat{\Lambda}(\lambda(\rho_t(\omega))) = \hat{\nabla}^{it}\hat{\Lambda}(\lambda(\omega))$.

Lemma 4.2. *The collection of $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ with $\omega \in D(\rho_i)$, $\rho_i(\omega) \in L^1_\sharp(\mathbb{G})$, and $\rho_i(\omega)^\sharp \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is dense in $L^1(\mathbb{G})$. Furthermore, the resulting collection of vectors $\hat{\Lambda}(\lambda(\omega))$ is dense in $L^2(\mathbb{G})$.*

Proof. We use a "smearing argument", compare (for example) [26, Prop. 5.21]. Let $\omega \in L^1_\sharp(\mathbb{G})\cap(L^1(\mathbb{G})\cap L^2(\mathbb{G}))$. Such ω are dense in $L^1(\mathbb{G})$ by [28, Lemma 2.5]. For $n \in \mathbb{N}$ and $z \in \mathbb{C}$ define

$$\omega(n, z) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t + z)^2)\rho_t(\omega)\ dt.$$

Then $\omega(n, 0)$ is analytic for ρ and $\rho_z(\omega(n, 0)) = \omega(n, z)$. As ρ is norm continuous, it follows that $\omega(n, 0) \to \omega$ in norm as $n \to \infty$.

For $x \in D(S)$, using that $\rho_t(\omega)^\sharp = \rho_t(\omega^\sharp)$,

$$\begin{aligned}\langle S(x)^*, \omega(n, z)\rangle &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t + z)^2)\langle S(x)^*, \rho_t(\omega)\rangle\ dt \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t + z)^2)\overline{\langle x, \rho_t(\omega^\sharp)\rangle}\ dt = \overline{\langle x, (\omega^\sharp)(n, \overline{z})\rangle}.\end{aligned}$$

Hence $\omega(n, z) \in L^1_\sharp(\mathbb{G})$ and $\omega(n, z)^\sharp = (\omega^\sharp)(n, \overline{z})$.

For $x \in \mathfrak{n}_\varphi$,

$$\begin{aligned}\langle x^*, (\omega^\sharp)(n,z)\rangle &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t+z)^2)\langle x^*, \rho_t(\omega^\sharp)\rangle \; dt \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t+z)^2)\big(\hat{\Lambda}(\lambda(\rho_t(\omega^\sharp)))\big|\Lambda(x)\big) \; dt = \big(\xi\big|\Lambda(x)\big),\end{aligned}$$

where

$$\xi = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2(t+z)^2)\hat{\nabla}^{it}\hat{\Lambda}(\lambda(\omega^\sharp)) \; dt.$$

In particular, $(\omega^\sharp)(n,z) \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$.

It follows that $\{\omega(n,0) : n \in \mathbb{N}, \omega \in L^1_\sharp(\mathbb{G}) \cap L^2(\mathbb{G})\}$ is dense in $L^1(\mathbb{G})$, analytic for ρ, satisfies that $\rho_z(\omega(n,0)) \in L^1_\sharp(\mathbb{G})$ for all z, and satisfies that $\rho_z(\omega(n,0))^\sharp \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. Furthermore, a similar calculation to that above yields that

$$\hat{\Lambda}(\lambda(\omega(n,0))) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} \exp(-n^2t^2)\hat{\nabla}^{it}\hat{\Lambda}(\lambda(\omega)) \; dt.$$

This converges to $\hat{\Lambda}(\lambda(\omega))$ in norm, as $n \to \infty$. In particular, as ω varies, the collection of vectors $\hat{\Lambda}(\lambda(\omega(n,0)))$ is dense in $L^2(\mathbb{G})$. □

For the following, let $\hat{T}$ be the Tomita map, which is the closure of $\hat{\Lambda}(\mathfrak{n}_{\hat{\varphi}} \cap \mathfrak{n}^*_{\hat{\varphi}}) \to L^2(\mathbb{G}), \hat{\Lambda}(x) \mapsto \hat{\Lambda}(x^*)$. For further details, see [36, Chap. 1] (where this map is denoted by S, a notation which we avoid, as it clashes with the antipode).

Proposition 4.3. *Let $X = \{\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G}) \mid \hat{\Lambda}(\lambda(\omega)) \in D(\hat{T}^*)\}$. Then X is dense in $L^1(\mathbb{G})$, and $\hat{\Lambda}(\lambda(X))$ is dense in $L^2(\mathbb{G})$. For $x \in D(S)^*$ and $\omega \in X$, we have that $x\omega \in X$, and $\hat{T}^*\hat{\Lambda}(\lambda(x\omega)) = S(x^*)\hat{T}^*\hat{\Lambda}(\lambda(\omega))$.*

Proof. Let $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ be given by Lemma 4.2. As $\lambda(\rho_t(\omega)) = \hat{\sigma}_t(\lambda(\omega))$, an analytic continuation argument shows that $\lambda(\omega) \in D(\hat{\sigma}_i)$ with $\hat{\sigma}_i(\lambda(\omega)) = \lambda(\rho_i(\omega))$. As also $\rho_i(\omega) \in L^1_\sharp(\mathbb{G})$ and $\rho_i(\omega)^\sharp \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, we have that

$$\begin{aligned}\hat{\Lambda}\big(\lambda(\rho_i(\omega)^\sharp)\big) &= \hat{\Lambda}\big(\lambda(\rho_i(\omega))^*\big) = \hat{T}\hat{\Lambda}\big(\lambda(\rho_i(\omega))\big) = \hat{T}\hat{\Lambda}\big(\sigma_i(\lambda(\omega))\big) \\ &= \hat{T}\hat{\nabla}^{-1}\hat{\Lambda}\big(\lambda(\omega)\big) = \hat{T}^*\hat{\Lambda}\big(\lambda(\omega)\big).\end{aligned}$$

As such ω are dense in $L^1(\mathbb{G})$, it follows that X is dense. As the collection of vectors $\hat{\Lambda}(\lambda(\omega))$ is dense in $L^2(\mathbb{G})$, it also follows that $\hat{\Lambda}(\lambda(X))$ is dense.

Now, $D(S)^* = D(\tau_{i/2})$. For $x \in L^\infty(\mathbb{G})$ and $t \in \mathbb{R}$, τ_t is implemented as $\tau_t(x) = \hat{\nabla}^{it}x\hat{\nabla}^{-it}$. Thus, that $x \in D(\tau_{i/2})$ means that $\hat{\nabla}^{-1/2}x\hat{\nabla}^{1/2}$ extends to a bounded operator, namely $\tau_{i/2}(x)$. Thus also $\hat{J}\hat{\nabla}^{-1/2}x\hat{\nabla}^{1/2}\hat{J} \subseteq S(x^*)$. As $\hat{T}^* = \hat{J}\hat{\nabla}^{-1/2} = \hat{\nabla}^{1/2}\hat{J}$, we conclude that $\hat{T}^*x\hat{T}^* \subseteq S(x^*)$. Thus, for $x \in S(x)^*$ and $\omega \in X$,

$$S(x^*)\hat{T}^*\hat{\Lambda}(\lambda(\omega)) = \hat{T}^*x\hat{T}^*\hat{T}^*\hat{\Lambda}(\lambda(\omega)) = \hat{T}^*x\hat{\Lambda}(\lambda(\omega)) = \hat{T}^*\hat{\Lambda}(\lambda(x\omega)),$$

as required. Here we used that $\hat{T}^* = (\hat{T}^*)^{-1}$, see [36, Lemma 1.5, Chap. 1]. □

Let us make one final definition:

$$L^1_\sharp(\mathbb{G}) \cap L^2_\sharp(\mathbb{G}) = \{\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G}) \mid \omega \in L^1_\sharp(\mathbb{G}) \text{ and } \omega^\sharp \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})\}.$$

Then [28, Prop. 2.6] shows that $L^1_\sharp(\mathbb{G}) \cap L^2_\sharp(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, and $\hat{\Lambda}(\lambda(L^1_\sharp(\mathbb{G}) \cap L^2_\sharp(\mathbb{G})))$ is dense in $L^2(\mathbb{G})$.

Proposition 4.4. *Let $\omega_1 \in L^1_\sharp(\mathbb{G}) \cap L^2_\sharp(\mathbb{G})$, and let $\omega_2 \in X$ be as in the previous proposition. Let $x \in L^\infty(\mathbb{G})$, set $\xi = \hat{\Lambda}(\lambda(\omega_1))$ and $\eta = \hat{T}^*\hat{\Lambda}(\lambda(\omega_2))$, and set $\hat{\omega} = \hat{\omega}_{x\xi,\eta}$. Then $\hat{\omega} \in L^1(\hat{\mathbb{G}}) \cap L^2(\mathbb{G})$ and $\Lambda(\hat{\lambda}(\hat{\omega})) = \hat{\Lambda}(\lambda((x\omega_1)\omega_2))$. When we take $x = 1$, as ω_1, ω_2 vary, such $\hat{\omega}$ are dense in $L^1(\hat{\mathbb{G}})$.*

Proof. Let $\hat{x} \in \mathfrak{n}_{\hat{\varphi}}$. As $\lambda(x\omega_1) \in \mathfrak{n}_{\hat{\varphi}}$ and $\mathfrak{n}_{\hat{\varphi}}$ is a left ideal, both $\lambda(x\omega_1)^*\hat{x} \in \mathfrak{n}_{\hat{\varphi}}$ and $\hat{x}^*\lambda(x\omega_1) \in \mathfrak{n}_{\hat{\varphi}}$, so that $\hat{x}^*\lambda(x\omega_1) \in \mathfrak{n}_{\hat{\varphi}} \cap \mathfrak{n}_{\hat{\varphi}}^*$. Thus $\hat{\Lambda}(\hat{x}^*\lambda(x\omega_1)) \in D(\hat{T})$ and $\hat{T}\hat{\Lambda}(\hat{x}^*\lambda(x\omega_1)) = \hat{\Lambda}(\lambda(x\omega_1)^*\hat{x})$. Thus

$$\begin{aligned}
\langle \hat{x}^*, \hat{\omega} \rangle &= (\hat{x}^* x\xi | \eta) = \big(\hat{x}^* x \hat{\Lambda}(\lambda(\omega_1)) \big| \hat{T}^*\hat{\Lambda}(\lambda(\omega_2))\big) \\
&= \big(\hat{x}^*\hat{\Lambda}(\lambda(x\omega_1)) \big| \hat{T}^*\hat{\Lambda}(\lambda(\omega_2))\big) = \big(\hat{\Lambda}(\lambda(\omega_2)) \big| \hat{T}\hat{\Lambda}(\hat{x}^*\lambda(x\omega_1))\big) \\
&= \big(\hat{\Lambda}(\lambda(\omega_2)) \big| \hat{\Lambda}(\lambda(x\omega_1)^*\hat{x})\big) = \hat{\varphi}\big(\hat{x}^*\lambda(x\omega_1)\lambda(\omega_2)\big) \\
&= \big(\hat{\Lambda}(\lambda((x\omega_1)\omega_2)) \big| \hat{\Lambda}(\hat{x})\big).
\end{aligned}$$

As $\hat{x} \in \mathfrak{n}_{\hat{\varphi}}$ was arbitrary, this shows that $\hat{\omega} \in L^1(\hat{\mathbb{G}}) \cap L^2(\mathbb{G})$ with $\Lambda(\hat{\lambda}(\hat{\omega})) = \hat{\Lambda}(\lambda((x\omega_1)\omega_2))$ as claimed. As above, the collection of allowed ξ is dense in $L^2(\mathbb{G})$, and by Proposition 4.3, the collection of allowed η is also dense in $L^2(\mathbb{G})$. Hence certainly the collection of thus constructed $\hat{\omega}$ will be dense in $L^1(\hat{\mathbb{G}})$. □

4.5. **Coefficients of representations.** Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a completely bounded (anti-)homomorphism with associated co(-anti-)representation V_π. Given $\alpha, \beta \in H$, a *coefficient* of π is the operator

$$T^\pi_{\alpha,\beta} = (\iota \otimes \omega_{\alpha,\beta})V_\pi \in L^\infty(\mathbb{G}). \tag{1}$$

Equivalently, $T^\pi_{\alpha,\beta}$ is determined by the dual pairing

$$\langle T^\pi_{\alpha,\beta}, \omega \rangle = \big(\pi(\omega)\alpha \big| \beta\big) \qquad (\omega \in L^1(\mathbb{G})).$$

Note that the latter definition of $T^\pi_{\alpha,\beta}$ makes sense even if π is bounded, but not necessarily completely bounded—a situation we will address in Section 7.

In this subsection and the next one, we study the analytic structure of coefficients of completely bounded representations $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$. We show that if π^* is also completely bounded, then coefficients of π give rise to completely bounded multipliers of $L^1(\hat{\mathbb{G}})$, a tool which will allow us to prove the main theorems of this section (Theorems 4.10 and 4.12) below.

Proposition 4.6. *Suppose that both π and π^* extend to completely bounded maps $L^1(\mathbb{G}) \to \mathcal{B}(H)$. For every $\alpha, \beta \in H$, we have that $T^{\pi^*}_{\alpha,\beta} \in D(S)$ with $S(T^{\pi^*}_{\alpha,\beta})^* = T^\pi_{\beta,\alpha}$.*

Proof. Let $\omega \in L^1_\sharp(\mathbb{G})$, so that

$$\langle T^\pi_{\beta,\alpha}, \omega^\sharp \rangle = \big(\pi(\omega^\sharp)\beta \big| \alpha\big) = \overline{\big(\pi(\omega^\sharp)^* \alpha \big| \beta\big)} = \overline{(\pi^*(\omega)\alpha|\beta)} = \overline{\langle T^{\pi^*}_{\alpha,\beta}, \omega\rangle}.$$

Comparing this with the definition of $\sharp$, we might hope that $T^\pi_{\beta,\alpha} \in D(S)$ with $S(T^\pi_{\beta,\alpha})^* = T^{\pi^*}_{\alpha,\beta}$. This is indeed true, but a little care is needed, see Proposition A.1 in the appendix. As $S \circ * \circ S = *$, we also see that $T^{\pi^*}_{\alpha,\beta} \in D(S)$ with $S(T^{\pi^*}_{\alpha,\beta})^* = T^\pi_{\beta,\alpha}$, as claimed. □

Given a coefficient $T^\pi_{\alpha,\beta}$, arguing as in Section 2.1 and as in the proof of Theorem 3.3, we see that

$$\begin{aligned}\Delta(T^\pi_{\alpha,\beta}) &= (\iota \otimes \iota \otimes \omega_{\alpha,\beta})(V_{\pi,13}V_{\pi,23}) = \sum_i (\iota \otimes \omega_{f_i,\beta})(V_\pi) \otimes (\iota \otimes \omega_{\alpha,f_i})(V_\pi) \\ &= \sum_i T^\pi_{f_i,\beta} \otimes T^\pi_{\alpha,f_i},\end{aligned}$$

the sum converging σ-weakly. This observation, combined with the previous proposition, shows that the hypotheses of the next theorem are not so outlandish; compare with the proof of Theorem 4.10 below.

Theorem 4.7. *Let $\mathbb{G}$ be a locally compact quantum group, let $x \in L^\infty(\mathbb{G})$, and suppose that $\sigma\Delta(x) = \sum_i b_i \otimes a_i \in L^\infty(\mathbb{G}) \otimes_{eh} L^\infty(\mathbb{G})$. Suppose furthermore that each $b_i \in D(S)^*$, and $\sum_i S(b_i^*)^* S(b_i^*) < \infty$. Then, for $\xi, \eta \in L^2(\mathbb{G})$,*

$$x\hat{\lambda}(\hat{\omega}_{\xi,\eta}) = \hat{\lambda}\Big(\sum_i \hat{\omega}_{a_i\xi, S(b_i^*)\eta}\Big),$$

the sum converging absolutely in $L^1(\hat{\mathbb{G}})$.

Proof. Let $\omega_1, \omega_2 \in L^1(\mathbb{G})$ be given by Proposition 4.4. Let $\xi = \hat{\Lambda}(\lambda(\omega_1)), \eta = \hat{T}^*\hat{\Lambda}(\lambda(\omega_2))$; recall that such choices are both dense in $L^2(\mathbb{G})$. Then $\hat{\omega}_{\xi,\eta} \in L^1(\hat{\mathbb{G}}) \cap L^2(\mathbb{G})$ and $\Lambda(\hat{\lambda}(\hat{\omega}_{\xi,\eta})) = \hat{\Lambda}(\lambda(\omega_1\omega_2))$.

For each i, let $\xi_i = a_i(\xi)$ and $\eta_i = S(b_i^*)(\eta)$. Then

$$\sum_i \|\xi_i\|^2 = \sum_i \langle a_i^* a_i, \omega_{\xi,\xi}\rangle \leq \Big\| \sum_i a_i^* a_i \Big\| \|\xi\|^2.$$

Similarly,

$$\sum_i \|\eta_i\|^2 \leq \Big\| \sum_i S(b_i^*)^* S(b_i^*) \Big\| \|\eta\|^2.$$

It follows that $\sum_i \hat{\omega}_{\xi_i,\eta_i}$ converges absolutely in $L^1(\hat{\mathbb{G}})$.

By Proposition 4.3, $\eta_i = \hat{T}^*\hat{\Lambda}(\lambda(b_i\omega_2))$, and also $\xi_i = a_i\hat{\Lambda}(\lambda(\omega_1))$. By Proposition 4.4,

$$\Lambda(\hat{\lambda}(\hat{\omega}_{\xi_i,\eta_i})) = \hat{\Lambda}\big(\lambda\big((a_i\omega_1)(b_i\omega_2)\big)\big).$$

Thus, for $y \in \mathfrak{n}_\varphi$,

$$\begin{aligned}\sum_i \big(\Lambda(\hat\lambda(\hat\omega_{\xi_i,\eta_i}))\big|\Lambda(y)\big) &= \sum_i \big(\hat\Lambda\big(\lambda\big((a_i\omega_1)(b_i\omega_2)\big)\big)\big|\Lambda(y)\big)\\ &= \sum_i \langle y^*, (a_i\omega_1)(b_i\omega_2)\rangle = \sum_i \langle \Delta(y^*)(a_i\otimes b_i), \omega_1\otimes\omega_2\rangle\\ &= \langle \Delta(y^*x), \omega_1\otimes\omega_2\rangle = \langle y^*, x(\omega_1\omega_2)\rangle\\ &= \big(\hat\Lambda(\lambda(x(\omega_1\omega_2)))\big|\Lambda(y)\big).\end{aligned}$$

Thus,

$$\Lambda\big(x\hat\lambda(\hat\omega_{\xi,\eta})\big) = x\hat\Lambda(\lambda(\omega_1\omega_2)) = \hat\Lambda(\lambda(x(\omega_1\omega_2))) = \sum_i \Lambda(\hat\lambda(\omega_{\xi_i,\eta_i})),$$

which completes the proof, as Λ injects, and such ξ, η are dense. □

We remark that we could weaken the hypothesis that $\sigma\Delta(x) \in L^\infty(\mathbb{G}) \otimes_{eh} L^\infty(\mathbb{G})$ to just requiring that $\Delta(x) = \sum_i a_i \otimes b_i$, the sum converging σ-weakly in $L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, and with $\sum_i a_i^* a_i < \infty$ and $\sum_i S(b_i^*)^* S(b_i^*) < \infty$.

We can interpret this result in terms of *left multipliers* or *centralizers* on $L^1(\hat{\mathbb{G}})$. We shall follow [13] (see also [22], but be aware of differing language). A left multiplier of $L^1(\hat{\mathbb{G}})$ is a linear map $L : L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$ with $L(\hat\omega_1\hat\omega_2) = L(\hat\omega_1)\hat\omega_2$ for each $\hat\omega_1, \hat\omega_2 \in L^1(\hat{\mathbb{G}})$. If additionally L is completely bounded, then an equivalent condition is that the adjoint $L^* : L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}})$ satisfies $\hat\Delta L^* = (L^* \otimes \iota)\hat\Delta$.

Given data as in the above theorem, we may define a σ-weakly continuous linear map $L^* : L^\infty(\hat{\mathbb{G}}) \to \mathcal{B}(L^2(\mathbb{G}))$ by

$$L^*(\hat x) = \sum_i S(b_i^*)^* \hat x a_i \qquad (\hat x \in L^\infty(\hat{\mathbb{G}})). \tag{2}$$

Then clearly the preadjoint $L : \mathcal{B}(L^2(\mathbb{G}))_* \to L^1(\hat{\mathbb{G}})$ satisfies $\hat\lambda(L\omega_{\xi,\eta}) = x\hat\lambda(\hat\omega_{\xi,\eta})$ for all $\xi, \eta \in L^2(\mathbb{G})$. Thus L factors through the quotient map $\mathcal{B}(L^2(\mathbb{G}))_* \to L^1(\hat{\mathbb{G}})$ and induces a map on $L^1(\hat{\mathbb{G}})$, still denoted by L. Hence we can regard L^* as a map on $L^\infty(\hat{\mathbb{G}})$. We record the following for use later.

Proposition 4.8. *With notation as above, $L^* \in \mathcal{CB}(L^\infty(\hat{\mathbb{G}}))$,*

$$\|L^*\|_{cb} \le \Big\| \sum_i S(b_i^*)^* S(b_i^*) \Big\|^{1/2} \Big\| \sum_i a_i^* a_i \Big\|^{1/2},$$

and $(L^* \otimes \iota)(\hat W) = (1 \otimes x)\hat W$.

Proof. The norm estimate is a standard calculation from the definition of L^* given by (2). The last assertion follows from [13, Prop. 2.4]; indeed, this is just a calculation using the definition of $\hat\lambda$ and that $x\hat\lambda(\hat\omega_{\xi,\eta}) = \hat\lambda(L\hat\omega_{\xi,\eta})$. □

We remark that it follows from [13] that $x \in C^b(\mathbb{G})$ (compare with Theorem 4.12 below) and that $x \in D(S)$.

4.9. **When we get an invertible corepresentation.** We will work in a little generality, and deal with possibly degenerate homomorphisms. Given a homomorphism $\pi : A \to \mathcal{B}(H)$ of a Banach algebra A on a Hilbert space H, the *essential space of* π is H_e, the closed linear span of $\{\pi(a)\xi : a \in A, \xi \in H\}$.

The following shows that if π and π^* are completely bounded, then V_π is an invertible operator (suitably interpreted in the degenerate case), and thus we have a converse to Theorem 3.3.

Theorem 4.10. *Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a completely bounded representation such that π^* extends to a completely bounded representation. Letting H_e be the essential space of π, we have that $V_{\tilde{\pi}}V_\pi = V_\pi V_{\tilde{\pi}}$ is a (not necessarily orthogonal) projection of $L^2(\mathbb{G}) \otimes H$ onto $L^2(\mathbb{G}) \otimes H_e$. Furthermore, both V_π and $V_{\tilde{\pi}}$ have ranges equal to $L^2(\mathbb{G}) \otimes H_e$. Hence V_π and $V_{\tilde{\pi}}$ restrict to operators on $L^2(\mathbb{G}) \otimes H_e$, and are mutual inverses in $\mathcal{B}(L^2(\mathbb{G}) \otimes H_e)$.*

Proof. Fix $\alpha, \beta \in H$. By Proposition 3.2, we know that $V_{\tilde{\pi}} = V_\pi^*$, and thus $T^{\tilde{\pi}}_{\alpha,\beta} = (T^{\pi}_{\beta,\alpha})^*$. As in the discussion before Theorem 4.7,

$$\sigma\Delta(T^{\tilde{\pi}}_{\alpha,\beta}) = \sigma\Delta(T^{\pi}_{\beta,\alpha})^* = \sum_i (T^{\pi}_{\beta,f_i})^* \otimes (T^{\pi}_{f_i,\alpha})^* = \sum_i T^{\tilde{\pi}}_{f_i,\beta} \otimes T^{\tilde{\pi}}_{\alpha,f_i}.$$

As π^* is completely bounded, by Proposition 4.6,

$$(T^{\tilde{\pi}}_{f_i,\beta})^* = T^{\pi}_{\beta,f_i} \in D(S) \quad \text{with} \quad S\big((T^{\tilde{\pi}}_{f_i,\beta})^*\big) = S\big(T^{\pi}_{\beta,f_i}\big) = (T^{\pi^*}_{f_i,\beta})^* = T^{\check{\pi}}_{\beta,f_i},$$

where the last equality uses Proposition 3.2. Moreover,

$$\begin{aligned}\sum_i S\big((T^{\tilde{\pi}}_{f_i,\beta})^*\big)^* S\big((T^{\tilde{\pi}}_{f_i,\beta})^*\big) &= \sum_i T^{\pi^*}_{f_i,\beta}(T^{\pi^*}_{f_i,\beta})^* = \sum_i (\iota \otimes \omega_{f_i,\beta})(V_{\pi^*})\big((\iota \otimes \omega_{\beta,f_i})(V_{\pi^*})\big)^* \\ &= \sum_i (\iota \otimes \omega_{f_i,\beta})(V_{\pi^*})(\iota \otimes \omega_{f_i,\beta})(V_{\pi^*}^*) = (\iota \otimes \omega_\beta)(V_{\pi^*}V_{\pi^*}^*),\end{aligned}$$

and similarly $\sum_i (T^{\tilde{\pi}}_{\alpha,f_i})^* T^{\tilde{\pi}}_{\alpha,f_i} = (\iota \otimes \omega_\alpha)(V_\pi V_\pi^*)$. We can hence apply Theorem 4.7 and Proposition 4.8 to conclude that if we define

$$L^* : L^\infty(\hat{\mathbb{G}}) \to \mathcal{B}(L^2(\mathbb{G})); \quad \hat{x} \mapsto \sum_i T^{\pi^*}_{f_i,\beta} \hat{x} T^{\tilde{\pi}}_{\alpha,f_i}, \tag{3}$$

then $L^* \in \mathcal{CB}(L^\infty(\hat{\mathbb{G}}))$, and $(L^* \otimes \iota)(\hat{W}) = (1 \otimes T^{\tilde{\pi}}_{\alpha,\beta})\hat{W}$. As $\hat{W} = \sigma W^* \sigma$, this shows that $(\iota \otimes L^*)(W^*)W = T^{\tilde{\pi}}_{\alpha,\beta} \otimes 1$. That is,

$$T^{\tilde{\pi}}_{\alpha,\beta} \otimes 1 = \sum_i (1 \otimes T^{\pi^*}_{f_i,\beta}) W^* (1 \otimes T^{\tilde{\pi}}_{\alpha,f_i}) W = \sum_i (1 \otimes T^{\pi^*}_{f_i,\beta}) \Delta(T^{\tilde{\pi}}_{\alpha,f_i}).$$

So, let $\omega_1, \omega_2 \in L^1(\mathbb{G})$, and consider

$$\begin{aligned}\sum_i \langle (1 \otimes T^{\pi^*}_{f_i,\beta})\Delta(T^{\tilde{\pi}}_{\alpha,f_i}), \omega_1 \otimes \omega_2\rangle &= \sum_i \langle T^{\tilde{\pi}}_{\alpha,f_i}, \omega_1\big(\omega_2 T^{\pi^*}_{f_i,\beta}\big)\rangle \\ &= \sum_i \big(\tilde{\pi}\big(\omega_1\big(\omega_2 T^{\pi^*}_{f_i,\beta}\big)\big)\alpha\big|f_i\big) = \sum_i \big(\tilde{\pi}\big(\omega_2 T^{\pi^*}_{f_i,\beta}\big)\tilde{\pi}(\omega_1)\alpha\big|f_i\big) \\ &= \sum_i \langle T^{\pi^*}_{f_i,\beta} T^{\tilde{\pi}}_{\tilde{\pi}(\omega_1)\alpha,f_i}, \omega_2\rangle = \langle (\iota \otimes \omega_{\tilde{\pi}(\omega_1)\alpha,\beta})(V_{\pi^*}V_{\tilde{\pi}}), \omega_2\rangle.\end{aligned}$$

However, we know that this is equal to $\langle T^{\tilde{\pi}}_{\alpha,\beta} \otimes 1, \omega_1 \otimes \omega_2\rangle$. So we conclude that

$$(\iota \otimes \omega_{\tilde{\pi}(\omega)\alpha,\beta})(V_{\pi^*}V_{\tilde{\pi}}) = \langle T^{\tilde{\pi}}_{\alpha,\beta}, \omega\rangle 1 = \big(\tilde{\pi}(\omega)\alpha\big|\beta\big)1 \qquad (\omega \in L^1(\mathbb{G}), \alpha, \beta \in H).$$

Letting $\tilde{H}_e$ be the essential space of $\tilde{\pi}$, we conclude that $V_{\pi^*}V_{\tilde{\pi}} = 1$ on $L^2(\mathbb{G}) \otimes \tilde{H}_e \subseteq L^2(\mathbb{G}) \otimes H$.

As $L^1_\sharp(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, we see that the essential space for π^* also equals $\tilde{H}_e$. Letting (e_i) be an orthonormal basis for $L^2(\mathbb{G})$, a simple calculation shows that

$$V_{\pi^*}(\xi \otimes \gamma) = \sum_i e_i \otimes \pi^*(\omega_{\xi,e_i})(\gamma) \in L^2(\mathbb{G}) \otimes \tilde{H}_e \qquad (\xi \in L^2(\mathbb{G}), \gamma \in H).$$

So V_{π^*} has range contained in $L^2(\mathbb{G}) \otimes \tilde{H}_e$. It follows that $V_{\pi^*}V_{\tilde{\pi}}$ is actually a (not necessarily orthogonal) projection from $L^2(\mathbb{G}) \otimes H$ onto $L^2(\mathbb{G}) \otimes \tilde{H}_e$. Furthermore, the range of V_{π^*} must actually be $L^2(\mathbb{G}) \otimes \tilde{H}_e$.

Now set $\phi = \pi^*$, so ϕ is also a completely bounded homomorphism $L^1(\mathbb{G}) \to \mathcal{B}(H)$. Thus the same argument now applied to ϕ shows that $V_{\phi^*}V_{\tilde{\phi}} = V_\pi V_{\pi^*}^* = V_\pi V_{\tilde{\pi}}$ is a (not necessarily orthogonal) projection from $L^2(\mathbb{G}) \otimes H$ onto $L^2(\mathbb{G}) \otimes H_e$ where H_e is the essential space of $\tilde{\phi}$, which agrees with the essential space for $\phi^* = \pi$. We also conclude that the range of $V_{\phi^*} = V_\pi$ is $L^2(\mathbb{G}) \otimes H_e$.

Following [28, Sec. 4], the *opposite quantum group* to $\mathbb{G}$ is $\mathbb{G}^{\mathrm{op}}$, where $L^\infty(\mathbb{G}^{\mathrm{op}}) = L^\infty(\mathbb{G})$ and $\Delta^{\mathrm{op}} = \sigma\Delta$. That is, we reverse the multiplication in $L^1(\mathbb{G})$ to get $L^1(\mathbb{G}^{\mathrm{op}})$. Then $R^{\mathrm{op}} = R$ and $\tau_t^{\mathrm{op}} = \tau_{-t}$ for each t. Thus $S^{\mathrm{op}} = R^{\mathrm{op}}\tau^{\mathrm{op}}_{-i/2} = R\tau_{i/2} = S^{-1} = * \circ S \circ *$. For $x \in D(S^{\mathrm{op}}) = D(S)^*$, and $\omega \in L^1_\sharp(\mathbb{G}^{\mathrm{op}})$,

$$\langle x, \omega^{\sharp,\mathrm{op}}\rangle = \overline{\langle S^{\mathrm{op}}(x)^*, \omega\rangle} = \overline{\langle S(x^*), \omega\rangle} = \langle S(x^*)^*, \omega^*\rangle.$$

It follows that $\omega^* \in L^1_\sharp(\mathbb{G})$, as for each $y \in D(S)$, setting $x = y^* \in D(S)^*$, we see that

$$\overline{\langle S(y)^*, \omega^*\rangle} = \langle S(x^*), \omega\rangle = \overline{\langle x, \omega^{\sharp,\mathrm{op}}\rangle} = \langle y, (\omega^{\sharp,\mathrm{op}})^*\rangle.$$

So $(\omega^*)^\sharp = (\omega^{\sharp,\mathrm{op}})^*$, and reversing this argument shows that $L^1_\sharp(\mathbb{G}) = L^1_\sharp(\mathbb{G}^{\mathrm{op}})^*$.

Thus, if we set $\phi = \tilde{\pi} : L^1(\mathbb{G}^{\mathrm{op}}) \to \mathcal{B}(H)$ then ϕ is a completely bounded representation, and $\tilde{\phi} = \pi$, so that the essential space of ϕ is H_e. For $\omega \in L^1_\sharp(\mathbb{G}^{\mathrm{op}})$, we have that $\omega^* \in L^1_\sharp(\mathbb{G})$, and $\phi^*(\omega) = \phi(\omega^{\sharp,\mathrm{op}})^* = \tilde{\pi}(((\omega^*)^\sharp)^*)^* = \pi((\omega^*)^\sharp) = \check{\pi}(\omega)$. Hence, as $\check{\pi}$ is completely bounded, also ϕ^* is completely bounded; and so $\phi^* = \check{\pi}$ and similarly $\check{\phi} = \pi^*$. Applying the previous argument to ϕ, we

conclude that $V_{\phi^*}V_{\check{\phi}} = V_{\check{\pi}}V_{\pi}$ is a (not necessarily orthogonal) projection onto $L^2(\mathbb{G}) \otimes H_e$. We also see that the range of $V_{\phi^*} = V_{\check{\pi}}$ is $L^2(\mathbb{G}) \otimes H_e$.

Finally, we also have that $V_{\phi}V_{\check{\phi}} = V_{\check{\pi}}V_{\pi^*}$ is a (not necessarily orthogonal) projection onto $L^2(\mathbb{G}) \otimes \tilde{H}_e$. It follows that

$$\ker\big(V_{\pi}V_{\check{\pi}}\big) = \mathrm{Im}(V_{\check{\pi}}^*V_{\pi}^*)^{\perp} = \mathrm{Im}(V_{\pi^*}V_{\check{\pi}})^{\perp} = \big(L^2(\mathbb{G}) \otimes \tilde{H}_e\big)^{\perp},$$

and analogously, also $\ker(V_{\check{\pi}}V_{\pi}) = \mathrm{Im}(V_{\check{\pi}}V_{\pi^*})^{\perp} = (L^2(\mathbb{G}) \otimes \tilde{H}_e)^{\perp}$. So we conclude that $\ker(V_{\pi}V_{\check{\pi}}) = \ker(V_{\check{\pi}}V_{\pi})$. So $V_{\pi}V_{\check{\pi}}$ and $V_{\check{\pi}}V_{\pi}$ are projections with the same range and kernel, and hence are equal. □

The above theorem says that if both π and π^* are completely bounded and nondegenerate, then V_{π} is an invertible corepresentation of $L^{\infty}(\mathbb{G})$, and, informally,

$$(S \otimes \iota)(V_{\pi}) = V_{\pi}^{-1}.$$

This is well-known in, for example, the theory of algebraic compact quantum groups (compare with [39, Prop. 3.1.7(iii)]). It is interesting that our arguments seem to require a lot of structure—for example, by using the duality theory for locally compact quantum groups.

From Proposition 4.8 and the proof of Theorem 4.10, we obtain the following corollary, showing that the coefficients $T^{\check{\pi}}_{\alpha,\beta}$ considered in Theorem 4.10 naturally induce completely bounded multipliers on $L^1(\hat{\mathbb{G}})$.

Corollary 4.11. *Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a completely bounded representation such that π^* extends to a completely bounded representation. Then for any $\alpha, \beta \in H$, the coefficient $T^{\check{\pi}}_{\alpha,\beta}$ represents a completely bounded left multiplier $L : L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$, determined by*

$$\hat{\lambda}(L\hat{\omega}) = T^{\check{\pi}}_{\alpha,\beta}\hat{\lambda}(\hat{\omega}) \qquad (\hat{\omega} \in L^1(\hat{\mathbb{G}})).$$

Moreover, $\|L\|_{cb} \leq \|\pi\|_{cb}\|\pi^*\|_{cb}\|\alpha\|\|\beta\|$.

Proof. The fact that L is a completely bounded left multiplier was already observed in the proof of Theorem 4.10. Moreover, from Proposition 4.8 and the proof of Theorem 4.10, we deduce that

$$\begin{aligned}\|L\|_{cb} = \|L^*\|_{cb} &\leq \|(\iota \otimes \omega_{\beta})(V_{\pi^*}V_{\pi^*}^*)\|^{1/2}\|(\iota \otimes \omega_{\alpha})(V_{\check{\pi}}^*V_{\check{\pi}})\|^{1/2} \\ &\leq \|\pi\|_{cb}\|\pi^*\|_{cb}\|\alpha\|\|\beta\|.\end{aligned}$$

□

When $\mathbb{G}$ is a locally compact group, the above result is classical and is due to de Canniere and Haagerup [15, Thm. 2.2]. (See also [7, Thm. 3] for when $\mathbb{G}$ is the dual of a locally compact group.)

Let G be a locally compact group, $\pi : L^1(G) \to \mathcal{B}(H)$ a bounded representation, and $\pi_0 : G \to \mathcal{B}(H)$ the uniformly bounded representation associated to π. The correspondence $\pi \leftrightarrow \pi_0$ given in Section 1 implies that each coefficient operator $T^{\pi}_{\alpha,\beta} \in L^{\infty}(G)$ is actually (a.e.) equal to the continuous function $\varphi^{\pi}_{\alpha,\beta} \in C^b(G)$, $s \mapsto \big(\pi_0(s)\alpha|\beta\big)$.

Let $\mathcal{B}_0(H)$ be the compact operators on H, so that $M(\mathcal{B}_0(H)) = \mathcal{B}(H)$. We hence get the *strict topology* on $\mathcal{B}(H)$, where a bounded net (T_α) is strictly-null if and only if the nets $(T_\alpha x)$ and (xT_α) are norm-null, for each $x \in \mathcal{B}_0(H)$. One consequence of the above correspondence is that the corepresentation $V_\pi \in L^\infty(G)\overline{\otimes}\mathcal{B}(H)$ is actually a member of $C^b_{str}(G, B(H))$, the space of bounded continuous maps $G \to \mathcal{B}(H)$, where $B(H)$ is given the strict topology. Indeed, V_π corresponds to the map $G \to \mathcal{B}(H); s \mapsto \pi_0(s)$.

When $\mathbb{G}$ is a locally compact quantum group, the natural replacement for $C^b_{str}(G, \mathcal{B}(H))$ is the multiplier algebra $M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$. This follows, as $M(C_0(G) \otimes \mathcal{B}_0(H))$ is isomorphic to $C^b_{str}(G, \mathcal{B}(H))$. Indeed, given $F \in C^b_{str}(G, \mathcal{B}(H))$ and $f \in C_0(G, \mathcal{B}_0(H)) = C_0(G) \otimes \mathcal{B}_0(H)$, clearly the pointwise products Ff, fF are members of $C_0(G, \mathcal{B}_0(H))$, and so F is a multiplier, and a partition of unity argument shows that every multiplier arises in this way.

The following gives a summary of the results of this section. The argument about multiplier algebras should be compared with [49, Sec. 4], which in turn is inspired by [2, p. 441].

Theorem 4.12. *Let $\mathbb{G}$ be a locally compact quantum group, and let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a nondegenerate, completely bounded representation. Then the following are equivalent:*

(1) $\check{\pi}$ extends to a completely bounded map $L^1(\mathbb{G}) \to \mathcal{B}(H)$;
(2) V_π is invertible.

In this case, $V_\pi^{-1} = V_{\check{\pi}}$, and $V_\pi \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$.

Proof. The equivalences have been established by Theorem 3.3 and Theorem 4.10. Suppose now that V_π^{-1} exists. Recall that the coproduct on $L^\infty(\mathbb{G})$ is implemented by the fundamental unitary by $\Delta(x) = W^*(1 \otimes x)W$ for $x \in L^\infty(\mathbb{G})$. Furthermore, $W \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(L^2(\mathbb{G})))$. Then $V_{\pi,13} = (\Delta \otimes \iota)(V_\pi)V_{\pi,23}^{-1} = W_{12}^* V_{\pi,23} W_{12} V_{\pi,23}^{-1}$ which is a member of $M\big(C_0(\mathbb{G}) \otimes B_0(L^2(\mathbb{G})) \otimes \mathcal{B}_0(H)\big)$. Thus $V_\pi \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$ as claimed. □

4.13. **Application to (co)isometric corepresentations.** A corepresentation $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ is a *coisometric corepresentation* if U is a coisometry, namely $UU^* = 1$. As Δ is a $*$-homomorphism, we see that U^* is a co-anti-representation.

Proposition 4.14. *Let $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ be a coisometric corepresentation, associated to $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$. Then $\check{\pi}$ is a completely bounded representation, with $V_{\check{\pi}} = U^*$.*

Proof. This follows almost exactly as the proof of Theorem 3.3. Let $\alpha, \beta \in H$, let (f_i) be an orthonormal basis of H, and set

$$x_i = (\iota \otimes \omega_{f_i,\beta})(U), \quad y_i = (\iota \otimes \omega_{\alpha,f_i})(U^*).$$

Then as before, we see that

$$\sum_i \Delta(x_i)(1 \otimes y_i) = (\iota \otimes \iota \otimes \omega_{\alpha,\beta})(U_{13}U_{23}U_{23}^*) = (\iota \otimes \omega_{\alpha,\beta})(U) \otimes 1,$$

and similarly

$$\sum_i (1 \otimes x_i)\Delta(y_i) = (\iota \otimes \iota \otimes \omega_{\alpha,\beta})(U_{23}U_{23}^*U_{13}^*) = (\iota \otimes \omega_{\alpha,\beta})(U^*) \otimes 1.$$

Hence $(\iota \otimes \omega_{\alpha,\beta})(U) \in D(S)$ with $S((\iota \otimes \omega_{\alpha,\beta})(U)) = (\iota \otimes \omega_{\alpha,\beta})(U^*)$. As before, it now follows that $\check{\pi}$ is completely bounded with $V_{\check{\pi}} = U^*$. □

The following is now immediate from Theorem 4.10.

Corollary 4.15. *Let $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ be a coisometric corepresentation. Then U is unitary.*

Corollary 4.16. *Let $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ be an isometric corepresentation. Then U is unitary.*

Proof. As in the proof of Theorem 4.10, following [28, Sec. 4], the *opposite quantum group* to $\mathbb{G}$ is $\mathbb{G}^{\rm op}$, where $L^\infty(\mathbb{G}^{\rm op}) = L^\infty(\mathbb{G})$ and $\Delta^{\rm op} = \sigma\Delta$. It follows that corepresentations of $\mathbb{G}$ are co-anti-representations of $\mathbb{G}^{\rm op}$, and vice versa. The proof follows from the observation that U^* is thus a coisometric corepresentation of $\mathbb{G}^{\rm op}$. □

An application of this result is to the theory of induced corepresentations in the sense of Kustermans, [25]. In this paper, a theory is developed allowing one to "induce" a corepresentation from a "smaller" quantum group to a larger one. However, in general the resulting corepresentation is only a coisometry, and not unitary (see [25, Notation 5.3]), and under a further "integrability" condition, and with an elaborate argument, it is shown that this corepresentation is unitary, [25, Prop. 7.5]. Our result shows that a further condition is not necessarily, and the induced corepresentation is always unitary.

5. The similarly problem

We recall from [4, Def. 3.2] (for example) that $\mathbb{G}$ is *amenable* if there is a state m on $L^\infty(\mathbb{G})$ such that $\langle m, (\iota \otimes \omega)\Delta(x)\rangle = \langle m, x\rangle\langle 1, \omega\rangle$ for $x \in L^\infty(\mathbb{G}), \omega \in L^1(\mathbb{G})$. Using the unitary antipode, we see that we could have equivalently used the obvious "left" variant of this definition instead. We call m a *right invariant state*.

Based on the results of the previous section, we are now a position to state the "similarity problem" for locally compact quantum groups: Suppose $\mathbb{G}$ has the property that every completely bounded homomorphism $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$, with π^* also extending to a completely bounded homomorphism, is similar to a $*$-homomorphism. Then is $\mathbb{G}$ necessarily amenable? By Theorem 4.12, this is equivalent to asking if having all invertible corepresentations being similar to unitary corepresentations forces $\mathbb{G}$ to be amenable? When $\mathbb{G}$ is commutative, this reduces to the standard conjecture that a unitarizable

locally compact group G is amenable. When $\mathbb{G}$ is cocommutative, [7] proves this conjecture in the affirmative—there is of course the stronger conjecture that actually, no condition on π^* is needed, and we provide more evidence for this in Section 8 below.

In the following, we shall show how to use an invariant state to "average" an invertible corepresentation to a unitary corepresentation. This is well-known in the compact quantum group case (see, for example, [47, Thm. 5.2]). The noncompact case is similar, but we provide the details, as we are unaware of a good reference.

Theorem 5.1. *Let $\mathbb{G}$ be an amenable locally compact quantum group, and let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a nondegenerate homomorphism. The following are equivalent:*

(1) π is similar to a $*$-homomorphism;
(2) both π and $\check{\pi}$ extend to completely bounded (anti-)homomorphisms $L^1(\mathbb{G}) \to \mathcal{B}(H)$.

Proof. That (1) implies (2) is Proposition 3.4.

Now suppose that (2) holds, and that π is nondegenerate, so by Theorem 4.10, V_π is invertible. Let $m \in L^\infty(\mathbb{G})^*$ be a right invariant state. Set

$$T = (m \otimes \iota)(V_\pi^* V_\pi) \in \mathcal{B}(H).$$

This means (by definition) that $\langle T, \omega\rangle = \langle m, (\iota \otimes \omega)(V_\pi^* V_\pi)\rangle$ for $\omega \in \mathcal{B}(H)_*$.

As V_π is invertible, so is the positive operator $V_\pi^* V_\pi$, and so, for some $\epsilon > 0$, we have that $V_\pi^* V_\pi \geq \epsilon 1$. It follows that for any $\omega \in \mathcal{B}(H)_*^+$,

$$\langle T, \omega\rangle = \langle m, (\iota \otimes \omega)(V_\pi^* V_\pi)\rangle \geq \langle m, (\iota \otimes \omega)(\epsilon 1)\rangle = \epsilon\langle \omega, 1\rangle.$$

Hence $T \geq \epsilon 1$, and so we may define

$$V = (1 \otimes T^{1/2}) V_\pi (1 \otimes T^{-1/2}) \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(H).$$

Then, as Δ is a unital $*$-homomorphism,

$$\begin{aligned}(\Delta \otimes \iota)(V) &= (1 \otimes 1 \otimes T^{1/2}) V_{\pi,13} V_{\pi,23} (1 \otimes 1 \otimes T^{-1/2}) \\ &= (1 \otimes 1 \otimes T^{1/2}) V_{\pi,13} (1 \otimes 1 \otimes T^{-1/2})(1 \otimes 1 \otimes T^{1/2}) V_{\pi,23} (1 \otimes 1 \otimes T^{-1/2}) \\ &= V_{13} V_{23}.\end{aligned}$$

So V is a corepresentation, say inducing a homomorphism $\phi : L^1(\mathbb{G}) \to \mathcal{B}(H)$. For $\omega \in L^1(\mathbb{G})$, we have that $\phi(\omega) = (\omega \otimes \iota)(V) = T^{1/2}\pi(\omega)T^{-1/2}$, and so ϕ is similar to π.

We next show that V is unitary. For $\omega_1 \in L^1(\mathbb{G})$ and $\omega_2 \in \mathcal{B}(H)_*$, we may define $\psi \in \mathcal{B}(H)_*$ by

$$\langle R, \psi\rangle = \langle V_\pi^*(1 \otimes R) V_\pi, \omega_1 \otimes \omega_2\rangle \qquad (R \in \mathcal{B}(H)).$$

Then we see that

$$\begin{aligned}(\iota \otimes \psi)(V_\pi^* V_\pi) &= (\iota \otimes \omega_1 \otimes \omega_2)\big(V_{\pi,23}^* V_{\pi,13}^* V_{\pi,13} V_{\pi,23}\big) \\ &= (\iota \otimes \omega_1 \otimes \omega_2)\big((\Delta \otimes \iota)(V_\pi^* V_\pi)\big).\end{aligned}$$

Thus

$$\begin{aligned}\langle T,\psi\rangle &= \langle m,(\iota\otimes\psi)(V_\pi^*V_\pi)\rangle = \langle m,(\iota\otimes\omega_1\otimes\omega_2)\big((\Delta\otimes\iota)(V_\pi^*V_\pi)\big)\rangle\\ &= \langle m,(\iota\otimes\omega_1)\Delta((\iota\otimes\omega_2)(V_\pi^*V_\pi))\rangle\\ &= \langle m,(\iota\otimes\omega_2)(V_\pi^*V_\pi)\rangle\langle 1,\omega_1\rangle = \langle 1\otimes T,\omega_1\otimes\omega_2\rangle.\end{aligned}$$

As ω_1,ω_2 were arbitrary, we conclude that $V_\pi^*(1\otimes T)V_\pi = 1\otimes T$. Thus

$$\begin{aligned}V^*V &= (1\otimes T^{-1/2})V_\pi^*(1\otimes T^{1/2})(1\otimes T^{1/2})V_\pi(1\otimes T^{-1/2})\\ &= (1\otimes T^{-1/2})(1\otimes T)(1\otimes T^{-1/2}) = 1,\end{aligned}$$

as required.

Arguing as in Theorem 3.3, it is now easy to see that ϕ is a $*$-homomorphism, because V is unitary. □

We can also deal with the case of degenerate representations of $L^1(\mathbb{G})$. The proof of [7, Prop. 6] actually shows the following:

Lemma 5.2. *Let A be a $*$-algebra, and let $\pi : A \to \mathcal{B}(H)$ be a homomorphism. Let $\pi_e : A \to \mathcal{B}(H_e)$ be the subrepresentation given by restricting to the essential space H_e. Suppose there is a (not necessarily orthogonal) projection Q from H to H_e such that $\pi(a)Q = \pi(a)$ for all $a \in A$. Then π is similar to $*$-representation if and only if π_e is similar to a $*$-representation.*

Theorem 5.3. *Let $\mathbb{G}$ be an amenable locally compact quantum group, and let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a homomorphism. The following are equivalent:*

(1) π is similar to a $*$-representation;
(2) π and $\check{\pi}$ extend to a completely bounded representation, respectively anti-representation, $L^1(\mathbb{G}) \to \mathcal{B}(H)$.

Proof. That (1) implies (2) is again Proposition 3.4. If (2) holds, then Theorem 4.10 shows that V_π restricts to an invertible operator on $L^2(\mathbb{G})\otimes H_e$; that is, V_{π_e} is invertible. So Theorem 5.1 shows that π_e is similar to a $*$-representation. By Lemma 5.2 it suffices to construct a projection $Q : H \to H_e$ with $\pi(\omega)Q = \pi(\omega)$ for $\omega \in L^1(\mathbb{G})$.

By Theorem 4.10, $P = V_\pi V_{\check{\pi}}$ is a projection of $L^2(\mathbb{G})\otimes H$ onto $L^2(\mathbb{G})\otimes H_e$. The proof actually shows that $\ker P = \ker V_\pi V_{\check{\pi}} = (L^2(\mathbb{G})\otimes\tilde{H}_e)^\perp = L^2(\mathbb{G})\otimes\tilde{H}_e^\perp$. Thus $L^2(\mathbb{G})\otimes H_e \cap L^2(\mathbb{G})\otimes\tilde{H}_e^\perp = \{0\}$, and so $H_e\cap\tilde{H}_e^\perp = \{0\}$. Similarly, $L^2(\mathbb{G})\otimes H_e + L^2(\mathbb{G})\otimes\tilde{H}_e^\perp = L^2(\mathbb{G})\otimes H$, and so $H_e + \tilde{H}_e^\perp = H$. It follows that there is a projection Q of H onto H_e with $\ker Q = \tilde{H}_e^\perp$.

Now, $\alpha\in\tilde{H}_e^\perp$ if and only if

$$0 = (\alpha|\tilde{\pi}(\omega)\beta) = (\alpha|\pi(\omega^*)^*\beta) = (\pi(\omega^*)\alpha|\beta)\qquad(\omega\in L^1(\mathbb{G}),\beta\in H).$$

It follows that for $\omega\in L^1(\mathbb{G}),\alpha\in\ker Q$, we have that $\pi(\omega)\alpha = 0 = \pi(\omega)Q\alpha$. Clearly $\pi(\omega)Q = \pi(\omega)$ on H_e, and thus it follows that $\pi(\omega)Q = \pi(\omega)$ on H, as required. □

6. Special cases

In this section, we take an approach which is closer in spirit to [7]. A cost is that we shall have to assume that $\hat{\mathbb{G}}$ is coamenable. Recall that $\hat{\mathbb{G}}$ is *coamenable* if $L^1(\hat{\mathbb{G}})$ has a bounded approximate identity. This is equivalent to a large number of other conditions, see [4, Thm. 3.1]. In particular, the proof of this reference shows that $L^1(\hat{\mathbb{G}})$ has a contractive approximate identity in this case. It is conjectured that the coamenability of $\hat{\mathbb{G}}$ is equivalent to $\mathbb{G}$ being amenable, but except when $\mathbb{G}$ is discrete (see [40]) it is only known that $\hat{\mathbb{G}}$ coamenable implies that $\mathbb{G}$ is amenable. A benefit to the approach of this section is that in special cases (as for $\hat{\mathbb{G}}$ being a SIN group in [7]) we can show that if $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ is completely bounded, then π is similar to a $*$-homomorphism, without assumption about $\check{\pi}$.

Theorem 6.1. *Let $\mathbb{G}$ be a locally compact quantum group such that $\hat{\mathbb{G}}$ is coamenable. Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be a completely bounded homomorphism, such that $\check{\pi}$ extends a completely bounded map $L^1(\mathbb{G}) \to \mathcal{B}(H)$. Then π is similar to a $*$-homomorphism.*

Proof. Let $\alpha, \beta \in H$ and $\xi, \eta \in L^2(\mathbb{G})$. From Corollary 4.11, $T^{\tilde{\pi}}_{\beta,\alpha}\hat{\lambda}(\hat{\omega}_{\xi,\eta}) \in \lambda(L^1(\hat{\mathbb{G}}))$. Moreover, an application of Theorem 4.7 to $T^{\tilde{\pi}}_{\beta,\alpha}$ shows that

$$T^{\tilde{\pi}}_{\beta,\alpha}\hat{\lambda}(\hat{\omega}_{\xi,\eta}) = \sum_i \hat{\lambda}\big(\hat{\omega}_{a_i\xi,S(b_i^*)\eta}\big),$$

where, if (f_i) is an orthonormal basis for H,

$$a_i = T^{\tilde{\pi}}_{\beta,f_i}, \quad S(b_i^*) = S\big((T^{\tilde{\pi}}_{f_i,\alpha})^*\big) = T^{\check{\pi}}_{\alpha,f_i}.$$

In particular,

$$\begin{aligned}\sum_i \|\hat{\omega}_{a_i\xi,S(b_i^*)\eta}\| &\le \Big(\sum_i \|a_i\xi\|^2\Big)^{1/2}\Big(\sum_i \|S(b_i^*)\eta\|^2\Big)^{1/2}\\ &\le \Big\|\sum_i a_i^*a_i\Big\|^{1/2}\Big\|\sum_i S(b_i^*)^*S(b_i^*)\Big\|^{1/2}\|\xi\|\|\eta\|\\ &\le \|\xi\|\|\eta\|\|\alpha\|\|\beta\|\|V_{\tilde{\pi}}\|\|V_{\check{\pi}}\| = \|\xi\|\|\eta\|\|\alpha\|\|\beta\|\|\pi\|_{cb}\|\check{\pi}\|_{cb}.\end{aligned}$$

Let $(\hat{\omega}_k)$ be a contractive approximate identity for $L^1(\hat{\mathbb{G}})$. It follows that for each k, there exists $\hat{\omega}'_k \in L^1(\hat{\mathbb{G}})$ with

$$(T^{\pi}_{\alpha,\beta})^*\hat{\lambda}(\hat{\omega}_k) = T^{\tilde{\pi}}_{\beta,\alpha}\hat{\lambda}(\hat{\omega}_k) = \hat{\lambda}(\hat{\omega}'_k) \quad \text{with} \quad \|\hat{\omega}'_k\| \le \|\alpha\|\|\beta\|\|\pi\|_{cb}\|\check{\pi}\|_{cb}.$$

As $\hat{\lambda}(L^1(\hat{\mathbb{G}}))$ is σ-weakly dense in $L^\infty(\mathbb{G})$, it follows that for $\omega \in L^1(\mathbb{G})$,

$$\begin{aligned}\overline{\langle T^{\pi}_{\alpha,\beta},\omega\rangle} = \langle (T^{\pi}_{\alpha,\beta})^*,\omega^*\rangle &= \lim_k \langle (T^{\pi}_{\alpha,\beta})^*\hat{\lambda}(\hat{\omega}_k),\omega^*\rangle\\ &= \lim_k \langle \hat{\lambda}(\hat{\omega}'_k),\omega^*\rangle = \lim_k \langle \lambda(\omega)^*,\hat{\omega}'_k\rangle.\end{aligned}$$

The final equality follows by a simple calculation, see [12, Lemma 8.10]. Thus

$$|(\pi(\omega)\alpha|\beta)| = |\langle T^{\pi}_{\alpha,\beta},\omega\rangle| \le \|\alpha\|\|\beta\|\|\pi\|_{cb}\|\check{\pi}\|_{cb}\|\lambda(\omega)\|.$$

As $\lambda(L^1(\mathbb{G}))$ is dense in $C_0(\hat{\mathbb{G}})$, by continuity, there is a homomorphism $\phi : C_0(\hat{\mathbb{G}}) \to \mathcal{B}(H)$ with

$$\|\phi\| \leq \|\pi\|_{cb}\|\check{\pi}\|_{cb}, \quad \phi \circ \lambda = \pi.$$

As $\hat{\mathbb{G}}$ is coamenable, it follows that $\mathbb{G}$ is amenable (see [4, Thm. 3.2]) and that hence $C_0(\hat{\mathbb{G}})$ is a nuclear C*-algebra (see [4, Thm. 3.3]). The similarly problem has an affirmative answer for nuclear C*-algebras (see [9, Thm. 4.1]) so ϕ is similar to a $*$-homomorphism. As λ is a $*$-homomorphism on $L^1_\sharp(\mathbb{G})$, it follows that π (restricted to $L^1_\sharp(\mathbb{G})$) is also similar to a $*$-homomorphism.

The above reference to [9] needed that ϕ (that is, π) is nondegenerate. But as $C_0(\hat{\mathbb{G}})$ has a bounded approximate identity, we can simply follow [7, Prop. 6] to deal with the case when π is degenerate. □

The above proof is interesting because in special cases, it allows us to remove the hypothesis that $\check{\pi}$ is completely bounded (or even bounded on all of $L^1(\mathbb{G})$). This idea was used for SIN groups in [7], the key point being that if G is SIN, then the Plancherel weight on $VN(G)$ can be approximated by *tracial* states in $A(G)$. It seems unlikely that many genuinely "quantum" groups will have this property, and so we shall restrict attention to the compact case, where we have many nontrivial examples.

Recall that a locally compact quantum group $\mathbb{G}$ is *compact* if $C_0(\mathbb{G})$ is unital. Here, the existence of Haar weights follows from simple axioms, see [48]. We shall be interested in the case when the Haar state is tracial—by [48, Thm. 2.5] this is equivalent to $\mathbb{G}$ being a compact Kac algebra. See [3, 46, 45] for some genuinely "quantum" examples of compact Kac algebras.

Theorem 6.2. *Let $\mathbb{G}$ be a compact Kac algebra, and let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be completely bounded homomorphism. Then π is similar to a $*$-homomorphism.*

Proof. As the dual $\hat{\mathbb{G}}$ is discrete, $L^1(\hat{\mathbb{G}})$ is unital. Indeed, if $\Lambda : L^\infty(\mathbb{G}) \to L^2(\mathbb{G})$ is the GNS map, then it is not hard to show that the state $\hat{\omega}_{\Lambda(1)}$ is the unit of $L^1(\hat{\mathbb{G}})$. Following the proof of Theorem 6.1, we wish to show that $T^{\tilde{\pi}}_{\beta,\alpha}\hat{\lambda}(\hat{\omega}_{\Lambda(1)}) = \hat{\lambda}(\hat{\omega})$ for some $\hat{\omega}$ whose norm is controlled. As $S = R$ in this case, we really have to show that

$$\begin{aligned}\sum_i \Big\| R\big((T^{\tilde{\pi}}_{f_i,\alpha})^*\big)\Lambda(1)\Big\|^2 &= \sum_i \varphi\Big(R\big((T^{\tilde{\pi}}_{f_i,\alpha})^*\big)^* R\big((T^{\tilde{\pi}}_{f_i,\alpha})^*\big)\Big) \\ &= \sum_i \varphi\Big((T^{\tilde{\pi}}_{f_i,\alpha})^* T^{\tilde{\pi}}_{f_i,\alpha}\Big) < \infty.\end{aligned}$$

where φ is the (normal, tracial) Haar state on $L^\infty(\mathbb{G})$, which satisfies $\varphi \circ R = \varphi$. However, as φ is a trace, this sum is

$$\begin{aligned}\sum_i \varphi\Big(T^{\tilde{\pi}}_{f_i,\alpha}(T^{\tilde{\pi}}_{f_i,\alpha})^*\Big) &= \sum_i \varphi\Big((T^{\pi}_{\alpha,f_i})^* T^{\pi}_{\alpha,f_i}\Big) \\ &= \varphi\big((\iota \otimes \omega_\alpha)(V_\pi^* V_\pi)\big) \leq \|\pi\|^2_{cb}\|\alpha\|^2.\end{aligned}$$

Then we can just continue as in the proof of Theorem 6.1. □

The above proof could be adapted to a general Kac algebra provided that we can find a bounded approximate identity $(\hat{\omega}_{\xi_k,\eta_k})$ for $L^1(\hat{\mathbb{G}})$ such that, for each k, we have that ω_{η_k} is tracial (by scaling η_k and ξ_k, we can suppose that $\|\eta_k\| = 1$ for each k, but we also need at least that $\sup_k \|\xi_k\| < \infty$). In particular, we can only apply this to $\hat{G}$ if G is a SIN group, by using the proof of [38, Prop. 3.2] (and so we have reproved [7, Thm. 20]). However, this argument will not extend to other classes of groups.

For a general compact quantum group, we can consider the Hopf $*$-algebra $\mathcal{A} \subseteq C_0(\mathbb{G})$ formed from the matrix coefficients of irreducible representations. On this algebra, we have reasonably explicit formulae for the antipode S and the Haar weight φ (see [39, Sec. 3.2] for example). A little calculation shows that if, in reasonable generality, we have something like

$$\sum_i \varphi\big(S(b_i^*)^* S(b_i^*)\big) = \sum_i \varphi(b_i b_i^*),$$

then already φ is a trace. So to extend the above result to general compact quantum groups (if it is true at all!) probably needs a new idea.

7. The importance of complete boundedness

Up to this point, we have restricted our attention to *completely bounded* representations $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$, and derived some interesting structural results for such representations. A natural question which arises is: what can be said about bounded representations π which are *not* completely bounded, if such representations even exist? In [8] Choi and the 3rd named author answered the existence part of this question by constructing a bounded, but not completely bounded, representation $\pi : A(G) \to \mathcal{B}(H)$, for any group G which contains $\mathbb{F}_2$ as a closed subgroup. In particular, for $G = \mathbb{F}_2$, every completely bounded representation of $A(G)$ is similar to a $*$-representation ([7, Thm. 20], see also Theorem 6.2 above) but there are bounded representations of $A(G)$ not similar to $*$-representations.

In this section, we will consider free products of compact quantum groups, and Khintchine inequalities for reduced free products, to construct examples of noncommutative/noncocommutative compact quantum groups $\mathbb{G}$ such that $L^1(\mathbb{G})$ admits bounded representations $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ which are not similar to $*$-representations. To be precise, we will show the following.

Theorem 7.1. *Let $\mathbb{G}$ be a nontrivial compact quantum group, and let $\widetilde{\mathbb{G}} = *_{i\in\mathbb{N}}\mathbb{G}_i$ be the compact quantum group free product of countably many copies of $\mathbb{G}$ (or let $\widetilde{\mathbb{G}} = \mathbb{T} * \mathbb{G}$). Then there exists a bounded representation $\pi : L^1(\widetilde{\mathbb{G}}) \to \mathcal{B}(H)$ which is not completely bounded. In particular, π is not similar to a $*$-representation.*

For completely bounded representations $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ (with π^* also completely bounded), a key result in the preceding sections was Corollary 4.11, which showed that each coefficient operator $T^{\bar{\pi}}_{\alpha,\beta} \in L^\infty(\mathbb{G})$ $(\alpha, \beta \in H)$ induces a completely bounded multiplier of the dual convolution algebra $L^1(\hat{\mathbb{G}})$.

The following corollary shows that for noncompletely bounded representations, there is no hope of generalizing Corollary 4.11.

Corollary 7.2. *Let $\widetilde{\mathbb{G}}$ and $\pi : L^1(\widetilde{\mathbb{G}}) \to \mathcal{B}(H)$ be as in the statement and proof of Theorem 7.1. Then there exist vectors $\alpha, \beta \in H$ with the property that the map $\hat{\lambda}(\hat{\omega}) \mapsto T^{\tilde{\pi}}_{\alpha,\beta}\hat{\lambda}(\hat{\omega})$ $(\hat{\omega} \in L^1(\widehat{\widetilde{\mathbb{G}}}))$ does not induce a bounded left multiplier of $L^1(\widehat{\widetilde{\mathbb{G}}})$.*

Proof of Corollary 7.2. We use an argument similar to the proof of [15, Thm. 2.3]. Suppose, to get a contradiction, that each coefficient $T^{\tilde{\pi}}_{\alpha,\beta}$ induces a multiplier of $L^1(\widehat{\widetilde{\mathbb{G}}})$, as in Corollary 4.11. Then for each $\alpha, \beta \in H$, as $L^1(\widehat{\widetilde{\mathbb{G}}})$ is unital, there exists a unique $\hat{\omega}^{\tilde{\pi}}_{\alpha,\beta} \in L^1(\widehat{\widetilde{\mathbb{G}}})$ such that $T^{\tilde{\pi}}_{\alpha,\beta} = \hat{\lambda}(\hat{\omega}^{\tilde{\pi}}_{\alpha,\beta})$. Define a sesquilinear map $Q : H \times H \to L^1(\widehat{\widetilde{\mathbb{G}}})$ by setting $Q(\alpha, \beta) = \hat{\omega}^{\tilde{\pi}}_{\alpha,\beta}$. We claim that Q is separately continuous. To see this, fix $\beta \in H$ and suppose $\lim_{n\to\infty} \|\alpha - \alpha_n\| = 0$ and $\lim_{n\to\infty} Q(\alpha_n, \beta) = \hat{\omega} \in L^1(\widehat{\widetilde{\mathbb{G}}})$. Then for any $\omega \in L^1(\widetilde{\mathbb{G}})$,

$$\begin{aligned}\langle \hat{\lambda}(\hat{\omega}), \omega\rangle &= \lim_{n\to\infty} \langle \hat{\lambda}(Q(\alpha_n, \beta)), \omega\rangle = \lim_{n\to\infty} \langle T^{\tilde{\pi}}_{\alpha_n,\beta}, \omega\rangle = \lim_{n\to\infty} \big(\tilde{\pi}(\omega)\alpha_n \big| \beta\big) \\ &= \big(\tilde{\pi}(\omega)\alpha \big| \beta\big) = \langle T^{\tilde{\pi}}_{\alpha,\beta}, \omega\rangle = \langle \hat{\lambda}(Q(\alpha, \beta)), \omega\rangle.\end{aligned}$$

Therefore $\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(Q(\alpha, \beta))$, giving $\hat{\omega} = Q(\alpha, \beta)$, and the closed graph theorem implies that Q is continuous in the first variable. The same argument applies to the second variable, proving the claim.

Since any separately continuous sesquilinear map is bounded (see for example [33, Thm. 2.17]), there is a constant $D > 0$ such that $\|\hat{\omega}^{\tilde{\pi}}_{\alpha,\beta}\|_{L^1(\widehat{\widetilde{\mathbb{G}}})} \leq D\|\alpha\|\|\beta\|$ for all $\alpha, \beta \in H$. As a consequence, we have

$$\begin{aligned}|\big(\tilde{\pi}(\omega)\alpha|\beta\big)| &= |\langle \hat{\lambda}(\hat{\omega}^{\tilde{\pi}}_{\alpha,\beta}), \omega\rangle| = |\langle \lambda(\omega^*)^*, \hat{\omega}^{\tilde{\pi}}_{\alpha,\beta}\rangle| \\ &\leq \|\hat{\omega}^{\tilde{\pi}}_{\alpha,\beta}\|\|\lambda(\omega^*)^*\| \leq D\|\alpha\|\|\beta\|\|\lambda(\omega^*)\| \qquad (\omega \in L^1(\widetilde{\mathbb{G}})),\end{aligned}$$

showing that $\tilde{\pi}$ extends to the bounded anti-representation $\sigma : C_0(\widehat{\widetilde{\mathbb{G}}}) \to \mathcal{B}(H)$ determined by $\sigma \circ \lambda = \tilde{\pi}$. Since $\widetilde{\mathbb{G}}$ is compact, $C_0(\widehat{\widetilde{\mathbb{G}}})$ is nuclear, and it follows that σ is automatically completely bounded. But then $\|\pi\|_{cb} = \|\tilde{\pi}\|_{cb} = \|\sigma \circ \lambda\|_{cb} \leq \|\sigma\|_{cb} < \infty$, contradicting the fact that π is not completely bounded. □

7.3. **Reduced free products.** Before proceeding with the proof of Theorem 7.1, let us recall the construction of the reduced free product of a collection of C*-algebras equipped with distinguished states. For further details see [1], [44] or the introduction to [31]. Let I be an index set, and for each $i \in I$ let A_i be a C*-algebra (assumed unital), and let φ_i be a state on A_i with faithful GNS construction (π_i, H_i, ξ_i). Let D_I be the collection of all "words" in I, that is, tuples $(i_1, \cdots, i_n)$ with $i_j \neq i_{j+1}$ for each j; we allow the empty word $\varnothing$. Set

$H_i^0 = \xi_i^\perp \subseteq H_i$, and let H be the Hilbert space direct sum of $H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0$ as (i_j) varies through D_I; we interpret the space for $\varnothing \in D_I$ as being a copy of $\mathbb{C}$. Thus H is some sort of generalized Fock space construction. Let $\Omega \in H$ be the vector $1 \in \mathbb{C}$.

For each $i \in I$ there is an obvious isomorphism

$$\begin{aligned} H &\cong (\mathbb{C} \oplus H_i^0) \otimes \Big(\bigoplus \big\{ H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0 \mid (i_j) \in D_I, i_1 \neq i \big\} \Big) \\ &\cong H_i \otimes \Big(\bigoplus \big\{ H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0 \mid (i_j) \in D_I, i_1 \neq i \big\} \Big). \end{aligned}$$

Let U_i be the unitary implementing this isomorphism, and define $\hat{\pi}_i : A_i \to \mathcal{B}(H); a \mapsto U_i^*(\pi_i(a) \otimes 1)U_i$, a faithful $*$-representation of A_i on H. A description of U_i is given in [1], but let us just note that if $a_i \in A_i$ with $\varphi_i(a_i) = 0$, then $\hat{\pi}_i(a_i)\Omega = a_i\xi_i \in H_i^0$.

Let $\mathcal{A}$ be the algebraic free product of the (A_i), amalgamated over units. Then $\pi = *_{i\in I}\hat{\pi}_i$ is a faithful $*$-representation of $\mathcal{A}$ on $\mathcal{B}(H)$. By definition, the reduced free product $*_{i\in A}A_i$ is the closure of $\pi(\mathcal{A})$. The vector state induced by Ω is the free product of the states, denoted $*_{i\in I}\varphi_i$. Notice that (π, H, Ω) is the GNS construction for $*_{i\in I}\varphi_i$. Denote by A the full free product, which is the completion of $\mathcal{A}$ under the norm given by considering all $*$-representations of A_i on a common Hilbert space.

If each A_i were a von Neumann algebra, with normal states φ_i, then the (reduced) von Neumann algebraic free product is simply the weak operator closure of $*_{i\in I}A_i$ in $\mathcal{B}(H)$.

In [45] Wang studied free products of compact quantum groups (see also [39, Sec. 6.3]). If for each i we have a compact quantum group $\mathbb{G}_i$, then we form $\mathbb{G} = *_{i\in I}\mathbb{G}_i$ as follows. Let $C_0(\mathbb{G})$ be the reduced free product of $(C_0(\mathbb{G}_i), \varphi_i)$, where φ_i is the Haar state. For each i let $\iota_i : C_0(\mathbb{G}_i) \to A$ be the inclusion, and let Δ_i be the coproduct on $C_0(\mathbb{G}_i)$. Set $\rho_i = (\iota_i \otimes \iota_i)\Delta_i : C_0(\mathbb{G}_i) \to A \otimes A$. By the universal property of A, there is a $*$-homomorphism $\Delta : A \to A \otimes A$ such that $\Delta\iota_i = \rho_i$ for each i. It is easy to see that Δ is coassociative, and by using the corepresentation theory of compact quantum groups, one can verify the density conditions to show that (A, Δ) is a compact quantum group.

Then [45, Thm. 3.8] shows that the Haar state on (A, Δ) is just the free product of the Haar states on each $C_0(\mathbb{G}_i)$. It follows that the reduced version of A is just $C_0(\mathbb{G})$, and thus Δ drops to a coproduct on $C_0(\mathbb{G})$.

We require the following noncommutative Khintchine inequality describing the operator space structure of the linear span of a family of centered freely independent operators. See [21, Prop. 7.4] or the introduction to [31]. In fact, we shall only use the $k = 1$ case, which is attributed to Voiculescu [43].

Theorem 7.4. *Let I be an index set, and for each $i \in I$, let A_i be a von Neumann algebra with faithful normal state φ_i. Let $(A, \varphi) = (*_{i\in I}A_i, *_{i\in I}\varphi_i)$ be their von Neumann algebraic free product. Then for any $x_i \in A_i$ with $\varphi_i(x_i) = 0$, and any finitely supported family $\{a_i\}_{i\in I} \in M_k(\mathbb{C})$ ($k \in \mathbb{N}$), we have*

$$\max\Big\{\max_{i\in I}\|a_i\otimes x_i\|_{M_k(\mathbb{C})\otimes A},\ \Big\|\sum_{i\in I}a_i^*a_i\varphi_i(x_i^*x_i)\Big\|_{M_k(\mathbb{C})}^{1/2},\ \Big\|\sum_{i\in I}a_ia_i^*\varphi_i(x_ix_i^*)\Big\|_{M_k(\mathbb{C})}^{1/2}\Big\}$$
$$\leq \Big\|\sum_{i\in I}a_i\otimes x_i\Big\|_{M_k(\mathbb{C})\otimes A}$$
$$\leq 3\max\Big\{\max_{i\in I}\|a_i\otimes x_i\|_{M_k(\mathbb{C})\otimes A},\ \Big\|\sum_{i\in I}a_i^*a_i\varphi_i(x_i^*x_i)\Big\|_{M_k(\mathbb{C})}^{1/2},\ \Big\|\sum_{i\in I}a_ia_i^*\varphi_i(x_ix_i^*)\Big\|_{M_k(\mathbb{C})}^{1/2}\Big\}.$$

Now let $\mathbb{G}$ be a nontrivial compact quantum group, set $\mathbb{G}_i=\mathbb{G}$ for each $i\in\mathbb{N}$, and let $\widetilde{\mathbb{G}}=*_{i\in\mathbb{N}}\mathbb{G}_i$ be the free product. (We will show how to modify the following arguments to address the case of $\widetilde{\mathbb{G}}=\mathbb{T}*\mathbb{G}$ in Remark 7.9.) Fix once and for all a nontrivial irreducible unitary representation $U\in L^\infty(\mathbb{G})\otimes M_d(\mathbb{C})$ and for each $i\in\mathbb{N}$, let $U^i\in L^\infty(\widetilde{\mathbb{G}})\otimes M_d(\mathbb{C})$ be the irreducible unitary representation of $\widetilde{\mathbb{G}}$ induced by the inclusion $L^\infty(\mathbb{G}_i)\hookrightarrow L^\infty(\widetilde{\mathbb{G}})$. Note that the U^i's are pairwise inequivalent representations. Finally, let $L_i^\infty=\{(\iota\otimes\rho)U^i:\rho\in M_d(\mathbb{C})^*\}$ be the coefficient space of U^i and let X be the weak*-closed linear span of all the L_i^∞'s.

Lemma 7.5. *There is a constant $C>0$ such that*

$$\|x\Omega\|_{L^2(\widetilde{\mathbb{G}})}\leq\|x\|_{L^\infty(\widetilde{\mathbb{G}})}\leq C\|x\Omega\|_{L^2(\widetilde{\mathbb{G}})}\qquad(x\in X).$$

Proof. The lower bound is immediate. To get the upper bound, fix $x\in X$ and write x as an L^2-convergent series $x=\sum_{i\in\mathbb{N}}x_i\in X$ where each x_i is a member of L_i^∞. As U is nontrivial, the Haar state annihilates all coefficients of U, and so we can apply the $k=1$ version of Theorem 7.4 to see that

$$\|x\|_{L^\infty(\widetilde{\mathbb{G}})}\leq 3\max\Big\{\max_i\|x_i\|_{L^\infty(\widetilde{\mathbb{G}})},\ \Big(\sum_{i\in\mathbb{N}}\varphi_i(x_i^*x_i)\Big)^{1/2},\ \Big(\sum_{i\in\mathbb{N}}\varphi_i(x_ix_i^*)\Big)^{1/2}\Big\}$$
$$=3\max\Big\{\max_i\|x_i\|_{L_i^\infty},\ \|x\Omega\|_{L^2(\widetilde{\mathbb{G}})},\ \Big(\sum_{i\in\mathbb{N}}\|x_i^*\xi_i\|^2_{L^2(\mathbb{G}_i)}\Big)^{1/2}\Big\}.$$

Since the spaces L_i^∞ are all finite dimensional and isometrically isomorphic, there exist constants $C_1,C_2>0$ such that

$$\|x_i\|_{L_i^\infty}\leq C_1\|x_i\xi_i\|_{L^2(\mathbb{G}_i)}\quad\text{and}\quad\|x_i^*\xi_i\|_{L^2(\mathbb{G}_i)}\leq C_2\|x_i\xi_i\|_{L^2(\mathbb{G}_i)}\qquad(i\in\mathbb{N}).$$

The lemma now follows by taking $C=3\max\{C_1,C_2\}$. □

Lemma 7.6. *The pre-annihilator ${}^\perp X$ is a closed two-sided ideal in $L^1(\widetilde{\mathbb{G}})$.*

Proof. By linearity, it is enough to show that if $\omega \in {}^{\perp}X$ and $x = (\iota \otimes \rho)U^i$ for some $i \in \mathbb{N}$ and $\rho \in M_d(\mathbb{C})^*$, then $\langle x, \omega\omega' \rangle = 0 = \langle x, \omega'\omega \rangle$ for $\omega' \in L^1(\widetilde{\mathbb{G}})$. However,

$$\langle x, \omega\omega' \rangle = \langle \Delta(x), \omega \otimes \omega' \rangle = \langle U^i_{13} U^i_{23}, \omega \otimes \omega' \otimes \rho \rangle = \langle U^i, \omega \otimes (\omega' \otimes \iota)(U^i)\rho \rangle = 0,$$

as $\rho' = (\omega' \otimes \iota)(U^i)\rho \in M_d(\mathbb{C})^*$ and $(\iota \otimes \rho')(U^i) \in X$. Similarly, $\langle x, \omega'\omega \rangle = 0$. □

Consider now the space $\ell^2(\mathbb{N}, M_d)$, which is a Banach algebra under pointwise operations.

Proposition 7.7. *The map*

$$\phi : L^1(\widetilde{\mathbb{G}}) \to \ell^2(\mathbb{N}, M_d); \quad \omega \mapsto \big((\omega \otimes \iota)(U^i)\big)_{i \in \mathbb{N}}$$

is well-defined, bounded, and is an algebra homomorphism. Furthermore, ϕ drops to give a isomorphism between $L^1(\widetilde{\mathbb{G}})/{}^{\perp}X$ and $\ell^2(\mathbb{N}, M_d)$.

Proof. The dual space of $\ell^2(\mathbb{N}, M_d)$ is $\ell^2(\mathbb{N}, M_d^*)$. Let $\rho = (\rho_i) \in \ell^2(\mathbb{N}, M_d^*)$, and let $x_i = (\iota \otimes \rho_i)(U^i) \in L^\infty_i$. For $n \in \mathbb{N}$, let $x = \sum_{i=1}^n x_i \in X$. Observe that $x\Omega = \sum_{i=1}^n x_i\Omega = \sum_{i=1}^n x_i\xi_i$ as $\varphi_i(x_i) = 0$ for each i. As the $x_i\xi_i$ are pairwise orthogonal in the Fock space H, it follows that

$$\|x\Omega\|^2 = \sum_{i=1}^n \|x_i\xi_i\|^2 \leq \sum_{i=1}^n \|x_i\|^2 \leq \sum_{i=1}^n \|\rho_i\|^2 \leq \|\rho\|^2.$$

Letting C be the constant from Lemma 7.5, we see that

$$\begin{aligned}\Big|\sum_{i=1}^n \langle \rho_i, (\omega \otimes \iota)(U^i) \rangle\Big| = \Big|\sum_{i=1}^n \langle x_i, \omega \rangle\Big| &= \big|\langle x, \omega \rangle\big| \\ &\leq \|\omega\|\|x\| \leq C\|\omega\|\|x\Omega\| \leq C\|\omega\|\|\rho\|.\end{aligned}$$

As n and ρ were arbitrary, it follows that ϕ is well-defined, and $\|\phi\| \leq C$.

For each i, as U^i is a corepresentation, the map $L^1(\widetilde{\mathbb{G}}) \to M_d(\mathbb{C}); \omega \mapsto (\omega \otimes \iota)(U^i)$ is a homomorphism. It follows that ϕ is also a homomorphism. Notice that $\phi(\omega) = 0$ if and only if $(\omega \otimes \iota)(U^i) = 0$ for each i, if and only if $\omega \in {}^{\perp}L^\infty_i$ for each i, if and only if $\omega \in {}^{\perp}X$. So ϕ drops to give an injective map $L^1(\widetilde{\mathbb{G}})/{}^{\perp}X \to \ell^2(\mathbb{N}, M_d)$.

Notice that $\phi^* : \ell^2(\mathbb{N}, M_d^*) \to ({}^{\perp}X)^{\perp} = X$ is the map $(\rho_i) \mapsto \sum_i (\iota \otimes \rho_i)(U^i)$. As U is irreducible, the map $M_d^* \to L^\infty(\mathbb{G}), \rho \mapsto (\iota \otimes \rho)(U)$ is injective, and so also the map $M_d^* \to L^2(\mathbb{G}), \rho \mapsto \Lambda((\iota \otimes \rho)(U))$ is injective. Thus there is a constant C' such that $\|\rho\| \leq C'\|(\iota \otimes \rho)(U)\xi_i\|$ for all $\rho \in M_d^*, i \in \mathbb{N}$. Then, for $(\rho_i) \in \ell^2(\mathbb{N}, M_d^*)$,

$$\begin{aligned}\|(\rho_i)\|^2 \leq C'^2 \sum_i \|(\iota \otimes \rho_i)(U^i)\xi_i\|^2 &= C'^2 \Big\| \sum_i (\iota \otimes \rho_i)(U^i)\Omega \Big\|^2 \\ &\leq C'^2 \Big\| \sum_i (\iota \otimes \rho_i)(U^i) \Big\|^2 = C'^2 \big\|\phi^*((\rho_i))\big\|^2.\end{aligned}$$

Thus ϕ^* is bounded below, which implies that ϕ is a surjection, and so the proof is complete. □

7.8. **Proof of the main theorem.** The final part of the construction is to find a bounded-below homomorphism $\theta : \ell^2(\mathbb{N}, M_d) \to \mathcal{B}(H)$ for a suitable Hilbert space H. Given $\xi, \eta \in H$, denote by $\theta_{\xi,\eta}$ the rank-one operator $H \to H; \gamma \mapsto (\gamma|\eta)\xi$. Recall from [8, Sec. 3.1] that if we denote by (δ_i) the canonical orthonormal basis of $\ell^2(\mathbb{N}_0)$ or $\ell^2(\mathbb{N})$, and define $\theta_0 : \ell^2(\mathbb{N}) \to \mathcal{B}(\ell^2(\mathbb{N}_0))$ by $\theta_0(\delta_i) = \theta_{\delta_i, \delta_i+\delta_0}$, then θ_0 extends by linearity and continuity to a bounded below algebra homomorphism.

Set $H = \mathbb{C}^d \otimes \ell^2(\mathbb{N}_0)$, and regard $\ell^2(\mathbb{N}, M_d)$ as $M_d \otimes \ell^2(\mathbb{N})$. Now define $\theta = (\iota \otimes \theta_0) : \ell^2(\mathbb{N}, M_d) \to M_d \otimes \mathcal{B}(\ell^2(\mathbb{N}_0)) \cong \mathcal{B}(H)$, an algebra homomorphism. As M_d is finite-dimensional, clearly θ is bounded, and bounded below. We can now prove the main result of this section.

Proof of Theorem 7.1. Form X as above, and the algebra isomorphism $\phi : L^1(\widetilde{\mathbb{G}})/^{\perp}X \to \ell^2(\mathbb{N}, M_d)$. Let $q : L^1(\widetilde{\mathbb{G}}) \to L^1(\widetilde{\mathbb{G}})/^{\perp}X$ be the quotient map. Let $\pi = \theta \circ \phi \circ q$, so that π is a bounded representation of $L^1(\widetilde{\mathbb{G}})$.

Towards a contradiction, suppose that π is completely bounded, and hence associated to a corepresentation $V_\pi \in L^\infty(\widetilde{\mathbb{G}}) \overline{\otimes} \mathcal{B}(H)$. Set

$$U = \sum_{i \in \mathbb{N}} U^i \otimes \theta_{\delta_i, \delta_i} \in L^\infty(\widetilde{\mathbb{G}}) \overline{\otimes} M_d \overline{\otimes} \mathcal{B}(\ell^2(\mathbb{N}_0)) \cong L^\infty(\widetilde{\mathbb{G}}) \overline{\otimes} \mathcal{B}(H),$$

where the sum converges σ-weakly (think of U as being a block diagonal matrix, with diagonal entries U^i, which are all unitary). For $\omega \in L^1(\widetilde{\mathbb{G}})$, $\xi \in \mathbb{C}^d$ and $k \in \mathbb{N}_0$, we have that

$$\pi(\omega)(\xi \otimes \delta_k) = \delta_{k,0}(\omega \otimes \iota)(U^i)(\xi) \otimes \delta_i + \delta_{k,i}(\omega \otimes \iota)(U^k)(\xi) \otimes \delta_k.$$

Consequently,

$$\begin{aligned}(\omega \otimes \iota)(V_\pi - U)(\xi \otimes \delta_k) &= \pi(\omega)(\xi \otimes \delta_k) - (\omega \otimes \iota)(U^k)(\xi) \otimes \delta_k \\ &= \delta_{k,0}(\omega \otimes \iota)(U^i)(\xi) \otimes \delta_i.\end{aligned}$$

If we now set

$$x_i = (\iota \otimes \omega_{\delta_0, \delta_i})(V_\pi - U) \in L^\infty(\widetilde{\mathbb{G}}) \overline{\otimes} M_d,$$

then $\sum_i x_i^* x_i$ converges σ-weakly to $(\iota \otimes \omega_{\delta_0,\delta_0})((V_\pi - U)^*(V_\pi - U))$. However, for $\omega \in L^1(\widetilde{\mathbb{G}})$ and $\xi \in \mathbb{C}^d$, also

$$\sum_i (\omega \otimes \iota)(x_i)(\xi) \otimes \delta_i = \sum_i (\omega \otimes \iota)(V_\pi - U)(\xi \otimes \delta_0) = \sum_i (\omega \otimes \iota)(U^i)(\xi) \otimes \delta_i.$$

It follows that $x_i = U^i$ for all i, and hence each x_i is unitary, which contradicts $\sum_i x_i^* x_i$ converging. We conclude that π is not completely bounded, and hence cannot be similar to a $*$-representation. □

Let us make a remark about π^*. As each U^i is a unitary corepresentation, we have that $(\omega^\sharp \otimes \iota)(U^i) = (\omega \otimes \iota)(U^i)^*$ for each $\omega \in L^1_\sharp(\widetilde{\mathbb{G}})$. Thus if we give

$\ell^2(\mathbb{N}, M_d)$ the pointwise $*$ operation, it follows that $\phi \circ q : L^1(\widetilde{\mathbb{G}}) \to \ell^2(\mathbb{N}, M_d)$ is a $*$-homomorphism. Thus

$$\pi^*(\omega) = \pi(\omega^\sharp)^* = \theta\big(\phi(q(\omega))^*\big)^* \qquad (\omega \in L^1_\sharp(\widetilde{\mathbb{G}})).$$

It follows that π^* extends to a bounded homomorphism on $L^1(\widetilde{\mathbb{G}})$. It is easy to compute explicitly what $\theta(a^*)^*$ is, for each $a \in \ell^2(\mathbb{N}, M_d)$, and then adapting the argument in the previous proof will show that π^* is also not completely bounded.

Remark 7.9. We now briefly address the case where $\widetilde{\mathbb{G}} = \mathbb{T} * \mathbb{G}$ in Theorem 7.1. Let z denote the canonical Haar unitary generator of $L^\infty(\mathbb{T})$, and let $U \in L^\infty(\mathbb{G}) \otimes M_d(\mathbb{C})$ be our fixed nontrivial irreducible unitary representation of $\mathbb{G}$. For each $i \in \mathbb{N}$, consider the tensor product unitary representation of $\widetilde{\mathbb{G}}$ given by $V^i = z^i \boxtimes U \boxtimes z^{-i} \in L^\infty(\widetilde{\mathbb{G}}) \otimes M_d(\mathbb{C})$ (where z^i is viewed as a one-dimensional representation of $\mathbb{T}$). It follows from [45] that the representations $\{V^i\}_{i\in\mathbb{N}}$ are pairwise inequivalent and irreducible. Our claim is that the above proof for $\widetilde{\mathbb{G}} = *_{i\in\mathbb{N}}\mathbb{G}_i$ goes through unchanged with the family $\{U^i\}_{i\in\mathbb{N}}$ replaced by $\{V^i\}_{i\in\mathbb{N}}$.

To see this, observe that the only facts that we used about $\widetilde{\mathbb{G}}$ and $\{U^i\}_{i\in\mathbb{N}}$ are: (1) each representation U^i is d-dimensional, (2) the elements of the coefficient spaces L^∞_i associated to U^i are centered with respect to the Haar state, and (3) the von Neumann algebras $\mathcal{A}_i$ generated by L^∞_i are pairwise freely independent in $(L^\infty(\widetilde{\mathbb{G}}), \varphi)$. For the family $\{V^i\}_{i\in\mathbb{N}}$, conditions (1) and (2) are automatically satisfied. To see that condition (3) is also satisfied, note that if $\mathcal{A}_0 \subseteq L^\infty(\mathbb{G}) \hookrightarrow L^\infty(\widetilde{\mathbb{G}})$ denotes the von Neumann algebra generated by the coefficient space of U, then $\mathcal{A}_i = z^i\mathcal{A}_0 z^{-i}$ for each $i \in \mathbb{N}$. Using this fact, condition (3) is easily seen to be a simple consequence of the definition of free independence (see [29, Def. 5.3], for example) and the fact that z is $*$-free from $\mathcal{A}_0$.

8. For the Fourier algebra

In this section, we collect some further special cases for the Fourier algebra. These add further weight to the conjecture that every completely bounded representation $\pi : A(G) \to \mathcal{B}(H)$ is similar to a $*$-representation. The main results of this section are as follows.

Theorem 8.1. *Let G be an amenable locally compact group, and let $\pi : A(G) \to \mathcal{B}(H)$ be a completely* contractive *representation. Then π is a $*$-representation.*

Theorem 8.2. *Let G be an amenable locally compact group which contains an open SIN subgroup $K \subseteq G$. Then every completely bounded representation $\pi : A(G) \to B(H)$ is similar to a $*$-representation.*

The proofs of the above results rely on the fact that when G is an amenable locally compact group, $A(G)$ is always a 1-operator amenable completely contractive Banach algebra, see [32].

Definition 8.3. Let A be completely contractive Banach algebra, and denote by $m : A\widehat{\otimes}A \to A$ the multiplication map. A net $\{d_\alpha\}$ in $A\widehat{\otimes}A$ is called a *bounded approximate diagonal* for A if $\{d_\alpha\}$ is bounded and for every $a \in A$,

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad \text{and} \quad am(d_\alpha) \to 0,$$

where $A\widehat{\otimes}A$ has the usual A-bimodule structure,

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

An element $M \in (A\widehat{\otimes}A)^{**}$ is called a *virtual diagonal* provided that

$$a \cdot M = M \cdot a, \quad am^{**}(M) = a \qquad (a \in A).$$

Ruan proved in [32] that the a has a bounded approximate diagonal in $A\widehat{\otimes}A$ if and only if it has a virtual diagonal in $(A\widehat{\otimes}A)^{**}$ (and, in turn, these are equivalent to A being *operator amenable*).

Now let $L > 0$. A is called L-*operator amenable* if it has a virtual diagonal $M \in (A\widehat{\otimes}A)^{**}$ such that $\|M\| \leq L$. The *operator amenability constant* of A is $N := \inf\{L : A \text{ is L-operator amenable}\}$. By Alaoglu's theorem, the operator amenability constant is attained, that is, A has a N-virtual diagonal in $(A\widehat{\otimes}A)^{**}$. Since any virtual diagonal has norm at least 1, the operator amenability constant is at least 1.

Proof of Theorem 8.1: Since G is amenable, $A(G)$ has a contractive approximate identity. Hence it follows from [7, Rem. 7] (see also Lemma 5.2) that π is a $*$-representation if and only if π_e is a $*$-representation, where $\pi_e = \pi|_{H_e}$ is the essential part of π. So, without loss of generality, we can assume that π is nondegenerate.

Put $A = \overline{\pi(A(G))}$. Since $A(G)$ is 1-operator amenable and π is a complete contraction, it follows that A is also 1-operator amenable. It follows from [5, Thm. 7.4.18(ii)] that A is a (commutative) C*-algebra. (Note that the proof of [5, Thm. 7.4.18(ii)] is for *unital* operator algebras but it can be easily modified to apply to our case as well). We now follow the proof of [7, Cor. 9]. Indeed, the adjoint operation on a commutative C*-algebra (in particular, on A) is a complete isometry. Recall that π^* is defined by

$$\pi^*(u) = \pi(\overline{u})^* \qquad (u \in A(G)).$$

Now, the map $u \mapsto \overline{u}$ is an anti-linear complete isometry, and thus $\|\pi^*\|_{cb} = \|\pi\|_{cb}$. Hence by [7, Thm. 8], there is an invertible operator $T \in \mathcal{B}(H)$ and a $*$-homomorphism $\sigma : C_0(G) \to \mathcal{B}(H)$ such that

$$\pi(u) = T^{-1}\sigma(u)T \qquad (u \in A(G)).$$

Moreover, T can be chosen so that

$$1 \leq \|T\|\|T^{-1}\| \leq \|\pi\|_{cb}^2\|\pi^*\|_{cb}^2 \leq 1.$$

So $\|T\|\|T^{-1}\| = 1$, and so, by replacing T with $T/\|T\|$, we can assume that $\|T\| = \|T^{-1}\| = 1$ which means that T is unitary. Hence

$$\pi(\cdot) = T^{-1}\sigma(\cdot)T = T^*\sigma(\cdot)T$$

is a $*$-homomorphism, as claimed. □

We now turn to the proof of Theorem 8.2. For a locally compact group G and an open subgroup K of G, let $G = \dot{\bigcup}_{x\in I} xK$ denote the decomposition of G to distinct left cosets of K (i.e. $xK \cap yK = \varnothing$ if $x \neq y$). For every element $u \in A(G)$ and $x \in I$, write

$$u_x = u\chi_{xK},$$

where χ_{xK} is the characteristic function of the coset xK. Since K is open, each χ_{xK} is a norm-one idempotent in the Fourier–Stieltjes algebra $B(G) = C^*(G)^*$ [19, Prop. 2.31]. Since $A(G)$ is a closed ideal in $B(G)$, we conclude that $u_x \in A(G)$. We let

$$A(xK) = A(G)\chi_{xK}.$$

Since K is open, the canonical embedding of the Fourier algebra $A(K)$ into $A(G)$ (that is, extending functions by zero outside of K) is completely isometric, allowing us to identify $A(K)$ with its image $A(eK)$ unambiguously. In what follows, we shall consider the translation operators $L_x : A(K) \to A(xK)$ defined by

$$(L_x u)(y) = u(x^{-1}y), \qquad (u \in A(K)).$$

Since left translation on $A(G)$ is completely isometric, $L_x : A(K) \to A(xK)$ is always a completely isometric algebra isomorphism. Finally, for any homomorphism $\pi : A(G) \to \mathcal{B}(H)$, let π_x $(x \in I)$ denote the restriction of π to the ideal $A(xK)$.

Proof of Theorem 8.2: Since G is amenable, $A(G)$ has a contractive approximate identity, and we may again assume that π is nondegenerate.

As above, we have that $G = \dot{\bigcup}_{x\in I} xK$. For every $x \in I$, the mapping $\pi_x \circ L_x : A(K) \to \mathcal{B}(H)$ defines a completely bounded homomorphism of $A(K)$ on H, and so by [7, Thm. 20], $\pi_x \circ L_x$ extends continuously to a bounded representation $\sigma_x : C_0(K) \to \mathcal{B}(H)$ with

$$\|\sigma_x\| \leq \|\pi\|_{cb}^2.$$

Let $\{f_\alpha\}_{\alpha\in\Delta} \subset A(K)$ be a contractive approximate identity for $C_0(K)$. For every $u \in A(G)$, we have

$$\lim_\alpha \sigma_x(f_\alpha)\pi(u) = \lim_\alpha \sigma_x(f_\alpha(L_{x^{-1}}u_x)) = \sigma_x(L_{x^{-1}}u_x) = \pi(u_x).$$

This, together with the fact that π is nondegenerate, implies that E_x, the limit of the net $\{\sigma_x(f_\alpha)\}$ in the strong operator topology of $\mathcal{B}(H)$, exists. Moreover, for every $x \in I$, E_x is an idempotent and for every $x \neq y$,

$$E_x E_y = 0. \tag{4}$$

Now let $A = \overline{\pi(A(G))} \subseteq \mathcal{B}(H)$. Since A is the closure of the range of a completely bounded homomorphism of the operator amenable Banach algebra $A(G)$, it is itself operator amenable. Hence, by combining [5, Prop. 7.4.12(ii)] and [20, Cor. 4.8], it follows that

$$A'' = \overline{A}^{\text{s.o.t.}},$$

where $\overline{A}^{\text{s.o.t.}}$ refers to the closure of A in the strong operator topology of $\mathcal{B}(H)$. In particular, A'' is commutative and for every $x \in I$,

$$E_x \in A''.$$

Therefore, again by combining [5, Prop. 7.4.12(ii)] and [20, Lemma 4.4], there is an invertible operator $T \in \mathcal{B}(H)$ such that $T^{-1}E_xT$ is an orthogonal projection for all $x \in I$. Let

$$P_x = T^{-1}E_xT \quad \text{and} \quad \rho_x(\cdot) = T^{-1}\sigma_x(\cdot)T \qquad (x \in I).$$

For every $x, y \in I$ with $x \neq y$, it follows from (4) that $P_xP_y = 0$. This implies that, for every $u, v \in C_0(K)$,

$$\rho_x(u)^*\rho_y(v) = \rho_x(u)^*P_x^*P_y\rho_y(v) = 0, \tag{5}$$

and

$$\rho_x(u)\rho_y(v)^* = \rho_x(u)P_xP_y^*\rho_y(v)^* = 0. \tag{6}$$

Now let $u \in A(G)$. For every $x, y \in I$ with $x \neq y$, it follows from (5) that

$$\big(T\pi(u_x)T^{-1}\big)^*\big(T\pi(u_y)T^{-1}\big) = \rho_x(L_{x^{-1}}u_x)^*\rho_y(L_{y^{-1}}u_y) = 0.$$

Similarly (6) implies that

$$\big(T\pi(u_x)T^{-1}\big)\big(T\pi(u_y)T^{-1}\big)^* = 0.$$

Hence $\{(T^{-1}\pi(u_x)T)^*(T^{-1}\pi(u_x)T)\}_{x\in I}$ is a family of pairwise orthogonal positive operators on H, and therefore

$$\begin{aligned}
\|T\pi(u)T^{-1}\|^2 &= \|(T\pi(u)T^{-1})^*(T\pi(u)T^{-1})\| \\
&= \Big\| \sum_{x\in I} (T\pi(u_x)T^{-1})^*(T\pi(u_x)T^{-1}) \Big\| \\
&= \sup_{x\in I} \|(T\pi(u_x)T^{-1})^*(T\pi(u_x)T^{-1})\| \leq \|T\|^2\|T^{-1}\|^2 \sup_{x\in I}\|\pi(u_x)\|^2 \\
&\leq \|T\|^2\|T^{-1}\|^2\|\pi\|_{cb}^4\|u\|_\infty^2.
\end{aligned}$$

Thus $T\pi(\cdot)T^{-1}$ extends continuously to a bounded homomorphism of $C_0(G)$ into $\mathcal{B}(H)$, and hence, so does π. The result now follows. □

One class of amenable locally compact groups having open SIN subgroups are the amenable, totally disconnected groups. In particular, if we assume further that they are noncompact, then they give us examples of non-SIN groups satisfying the assumption of Theorem 8.2. These groups, for instance, include Fell groups (see [34, Sec. 2] for more details).

Appendix A. Duality for closed operators

In this appendix, we will prove the following result.

Proposition A.1. *Let $\mathbb{G}$ be a locally compact quantum group, and let $x, y \in L^\infty(\mathbb{G})$ be such that $\langle x, \omega^\sharp\rangle = \overline{\langle y, \omega\rangle}$ for all $\omega \in L^1_\sharp(\mathbb{G})$. Then $x \in D(S)$ with $S(x)^* = y$.*

Firstly, note that as $S = R\tau_{-i/2}$, we have that $\omega \in L^1_\sharp(\mathbb{G})$ if and only if there is $\omega' \in L^1(\mathbb{G})$ with

$$\langle x, \omega' \rangle = \langle \tau_{-i/2}(x), \omega^* \rangle \qquad (x \in D(\tau_{-i/2})).$$

Indeed, if this holds, then for $x \in D(S)$, we have that

$$\langle x, \omega' \circ R \rangle = \langle \tau_{-i/2}(R(x)), \omega^* \rangle = \overline{\langle S(x)^*, \omega \rangle},$$

so $\omega^\sharp = \omega' \circ R$. The converse follows similarly.

Set $D = L^1_\sharp(\mathbb{G})^* = \{\omega^* \mid \omega \in L^1_\sharp(\mathbb{G})\}$ and for $\omega \in D$ let $\omega' = (\omega^*)^\sharp$. It follows that the proposition above is equivalent to the following.

Proposition A.2. *Let $\mathbb{G}$ be a locally compact quantum group, and let $x, y \in L^\infty(\mathbb{G})$ be such that $\langle x, \omega' \rangle = \langle y, \omega \rangle$ for all $\omega \in D$. Then $x \in D(\tau_{-i/2})$ with $y = \tau_{-i/2}(x)$.*

Perhaps the "standard" proof of the proposition would be to "smear" x and y by the one-parameter group (τ_t); compare [26, Prop. 4.22] or the proof of [27, Prop. 5.26], for example. Instead, we wish to indicate how to prove the result by using closed operators.

Given a topological vector space E and a subspace $D(T) \subseteq E$, a linear map $T : D(T) \to E$ is *closed and densely defined* if $D(T)$ is dense in E, and the graph $\mathcal{G}(T) = \{(x, T(x)) \mid x \in D(T)\}$ is closed in $E \times E$. Define also $\mathcal{G}'(T) = \{(T(x), x) \mid x \in E\}$, which is closed if and only if $\mathcal{G}(T)$ is. Let E be a Banach space—we shall consider the case when E has the norm topology, and when E^* has the weak*-topology.

Let $T : D(T) \to E$ be a linear map. Set $D(T^*) = \{\mu \in E^* \mid \exists \lambda \in E^*,\ \langle \lambda, x \rangle = \langle \mu, T(x) \rangle\ (x \in D(T))\}$, and with a slight abuse of notation, set $\mathcal{G}(T^*)$ to be the collection of such (μ, λ). Notice that:

- $\mathcal{G}'(-T^*) = \mathcal{G}(T)^\perp$, and so $\mathcal{G}(T^*)$ is weak*-closed;
- $D(T)$ is dense if and only if $(\lambda, 0) \in \mathcal{G}(T)^\perp \implies \lambda = 0$, which is equivalent to $\mathcal{G}(T^*)$ being the graph of an operator (that is, the choice of λ being unique);
- $\overline{\mathcal{G}(T)} = {}^\perp(\mathcal{G}(T)^\perp) = {}^\perp\mathcal{G}'(-T^*)$ and so $D(T^*)$ is weak*-dense, because $\mathcal{G}(T)$ is the graph of an operator.

It follows that T^* is a closed, densely defined operator, for the weak*-topology. We can similarly reverse the argument, and start with T^*, and define a closed, densely defined operator T by setting $D(T) = \{x \in E \mid \exists y \in E,\ \langle \mu, y \rangle = \langle T^*(\mu), x \rangle\ (\mu \in D(T^*))\}$ and $T(x) = y$. Furthermore, these two constructions are mutual inverses, so if we start with T^*, form T, and then form the adjoint of T, we recover T^*. We are not aware of an entirely satisfactory reference for this construction, but see [23, Sec. 5.5, Chap. III].

Proof of Proposition A.2. We apply the preceding discussion to $\tau_{-i/2}$ acting on $L^\infty(\mathbb{G})$. As $\tau_{-i/2}$ is the analytic generator of the σ-weakly continuous group (τ_t), it follows from [10] that $\tau_{-i/2}$ is σ-weakly closed and densely defined. Let $\tau_{*,-i/2}$ be the pre-adjoint, which from the above is hence a (norm) closed,

densely defined operator on $L^1(\mathbb{G})$. It follows that $D = D(\tau_{*,-i/2})$ and that $\omega' = \tau_{*,-i/2}(\omega)$ for $\omega \in D$. The hypothesis of the proposition now simply states that $x \in D((\tau_{*,-i/2})^*)$ and $y = (\tau_{*,-i/2})^*(x)$. The claim now follows from the fact that $(\tau_{*,-i/2})^* = \tau_{i/2}$. □

We remark that, for example, this argument gives a very easy proof that $L^1_\sharp(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, without the need for a "smearing" argument (compare with the discussion before Proposition 3.1 in [24]).

Acknowledgments

This paper was initiated at the Canadian Abstract Harmonic Analysis Symposium 2010, and the first and second authors wish to thank the third author, Yemon Choi, and the staff at the University of Saskatchewan for their hospitality. We gratefully acknowledge the financial support of PIMS, NSERC and the University of Saskatchewan. The third author was partially supported by NSERC under grant no. 366066-09.

References

[1] D. Avitzour, Free products of C*-algebras, Trans. Amer. Math. Soc. **271** (1982), no. 2, 423–435. MR0654842 (83h:46070)

[2] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 425–488. MR1235438 (94e:46127)

[3] T. Banica and R. Speicher, Liberation of orthogonal Lie groups, Adv. Math. **222** (2009), no. 4, 1461–1501. MR2554941 (2010j:46125)

[4] E. Bédos and L. Tuset, Amenability and co-amenability for locally compact quantum groups, Internat. J. Math. **14** (2003), no. 8, 865–884. MR2013149 (2004k:46129)

[5] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, 30, Oxford Univ. Press, Oxford, 2004. MR2111973 (2006a:46070)

[6] D. P. Blecher and R. R. Smith, The dual of the Haagerup tensor product, J. London Math. Soc. (2) **45** (1992), no. 1, 126–144. MR1157556 (93h:46078)

[7] M. Brannan and E. Samei, The similarity problem for Fourier algebras and corepresentations of group von Neumann algebras, J. Funct. Anal. **259** (2010), no. 8, 2073–2097. MR2671122 (2011j:22009)

[8] Y. Choi and E. Samei, Quotients of Fourier algebras, and representations which are not completely bounded, Proc. Amer. Math. Soc. **141** (2013), no. 7, 2379–2388. MR3043019

[9] E. Christensen, On nonselfadjoint representations of C*-algebras, Amer. J. Math. **103** (1981), no. 5, 817–833. MR0630768 (82k:46085)

[10] I. Ciorănescu and L. Zsidó, Analytic generators for one-parameter groups, Tôhoku Math. J. (2) **28** (1976), no. 3, 327–362. MR0430867 (55 #3872)

[11] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. **96** (1989), no. 3, 507–549. MR0996553 (90h:22008)

[12] M. Daws, Multipliers, self-induced and dual Banach algebras, Dissertationes Math. (Rozprawy Mat.) **470** (2010), 62 pp. MR2681109 (2011i:46048)

[13] M. Daws, Multipliers of locally compact quantum groups via Hilbert C*-modules, J. Lond. Math. Soc. (2) **84** (2011), no. 2, 385–407. MR2835336 (2012h:46123)

[14] M. M. Day, Means for the bounded functions and ergodicity of the bounded representations of semi-groups, Trans. Amer. Math. Soc. **69** (1950), 276–291. MR0044031 (13,357e)
[15] J. De Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, Amer. J. Math. **107** (1985), no. 2, 455–500. MR0784292 (86m:43002)
[16] J. Dixmier, Les moyennes invariantes dans les semi-groups et leurs applications, Acta Sci. Math. Szeged **12** (1950), Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars A, 213–227. MR0037470 (12,267a)
[17] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, 23, Oxford Univ. Press, New York, 2000. MR1793753 (2002a:46082)
[18] E. G. Effros and Z.-J. Ruan, Operator space tensor products and Hopf convolution algebras, J. Operator Theory **50** (2003), no. 1, 131–156. MR2015023 (2004j:46078)
[19] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France **92** (1964), 181–236. MR0228628 (37 #4208)
[20] J. A. Gifford, Operator algebras with a reduction property, J. Aust. Math. Soc. **80** (2006), no. 3, 297–315. MR2236039 (2007b:46087)
[21] M. Junge, Embedding of the operator space OH and the logarithmic 'little Grothendieck inequality', Invent. Math. **161** (2005), no. 2, 225–286. MR2180450 (2006i:47130)
[22] M. Junge, M. Neufang, and Z.-J. Ruan, A representation theorem for locally compact quantum groups, Internat. J. Math. **20** (2009), no. 3, 377–400. MR2500076 (2010c:46128)
[23] T. Kato, *Perturbation theory for linear operators*, Second edition. Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1976. MR0407617 (53 #11389)
[24] J. Kustermans, Locally compact quantum groups in the universal setting, Internat. J. Math. **12** (2001), no. 3, 289–338. MR1841517 (2002m:46108)
[25] J. Kustermans, Induced corepresentations of locally compact quantum groups, J. Funct. Anal. **194** (2002), no. 2, 410–459. MR1934609 (2004c:46106)
[26] J. Kustermans, Locally compact quantum groups, *Quantum independent increment processes. I*, 99–180, Lecture Notes in Math., 1865 Springer, Berlin. MR2132094 (2006f:46073)
[27] J. Kustermans and S. Vaes, Locally compact quantum groups, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 6, 837–934. MR1832993 (2002f:46108)
[28] J. Kustermans and S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting, Math. Scand. **92** (2003), no. 1, 68–92. MR1951446 (2003k:46081)
[29] A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, 335, Cambridge Univ. Press, Cambridge, 2006. MR2266879 (2008k:46198)
[30] T. Pytlik and R. Szwarc, An analytic family of uniformly bounded representations of free groups, Acta Math. **157** (1986), no. 3-4, 287–309. MR0857676 (88e:22014)
[31] É. Ricard and Q. Xu, Khintchine type inequalities for reduced free products and applications, J. Reine Angew. Math. **599** (2006), 27–59. MR2279097 (2009h:46110)
[32] Z.-J. Ruan, The operator amenability of $A(G)$, Amer. J. Math. **117** (1995), no. 6, 1449–1474. MR1363075 (96m:43001)
[33] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973. MR0365062 (51 #1315)
[34] V. Runde and N. Spronk, Operator amenability of Fourier-Stieltjes algebras. II, Bull. Lond. Math. Soc. **39** (2007), no. 2, 194–202. MR2323448 (2008i:46043)
[35] M. Takesaki, *Theory of operator algebras. I*, Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, 2002. MR1873025 (2002m:46083)
[36] M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, 125, Springer, Berlin, 2003. MR1943006 (2004g:46079)

[37] M. Takesaki and N. Tatsuuma, Duality and subgroups. II, J. Functional Analysis **11** (1972), 184–190. MR0384995 (52 #5865)
[38] K. F. Taylor, The type structure of the regular representation of a locally compact group, Math. Ann. **222** (1976), no. 3, 211–224. MR0425007 (54 #12965)
[39] T. Timmermann, *An invitation to quantum groups and duality*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR2397671 (2009f:46079)
[40] R. Tomatsu, Amenable discrete quantum groups, J. Math. Soc. Japan **58** (2006), no. 4, 949–964. MR2276175 (2008g:46122)
[41] S. Vaes, *Locally compact quantum groups*, PhD. thesis, Katholieke Universiteit Leuven, 2001. Available from `http://wis.kuleuven.be/analyse/stefaan/`
[42] S. Vaes and A. Van Daele, Hopf C*-algebras, Proc. London Math. Soc. (3) **82** (2001), no. 2, 337–384. MR1806875 (2002f:46139)
[43] D. Voiculescu, A strengthened asymptotic freeness result for random matrices with applications to free entropy, Internat. Math. Res. Notices **1998**, no. 1, 41–63. MR1601878 (2000d:46080)
[44] D. Voiculescu, Symmetries of some reduced free product C*-algebras, *Operator algebras and their connections with topology and ergodic theory*, Buşteni, 1983, 556–588. Lecture Notes in Math. **1132**, Springer, Berlin, 1985. MR0799593 (87d:46075)
[45] S. Wang, Free products of compact quantum groups, Comm. Math. Phys. **167** (1995), no. 3, 671–692. MR1316765 (95k:46104)
[46] S. Wang, Quantum symmetry groups of finite spaces, Comm. Math. Phys. **195** (1998), no. 1, 195–211. MR1637425 (99h:58014)
[47] S. L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys. **111** (1987), no. 4, 613–665. MR0901157 (88m:46079)
[48] S. L. Woronowicz, Compact quantum groups, *Symétries quantiques (Les Houches, 1995)*, 845–884, North-Holland, Amsterdam. MR1616348 (99m:46164)
[49] S. L. Woronowicz, From multiplicative unitaries to quantum groups, Internat. J. Math. **7** (1996), no. 1, 127–149. MR1369908 (96k:46136)

Received October 29, 2012; accepted March 20, 2013

Michael Brannan
Department of Mathematics,
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
E-mail: `mbrannan@illinois.edu`

Mathew Daws
School of Mathematics, University of Leeds
Leeds LS2 9JT UK
E-mail: `matt.daws@cantab.net`

Ebrahim Samei
Department of Mathematics and Statistics,
McLean Hall, University of Saskatchewan,
106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6
E-mail: `samei@math.usask.ca`