

The total surgery obstruction revisited

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Abstract. The total surgery obstruction of a finite n -dimensional Poincaré complex X is an element $s(X)$ of a certain abelian group $\mathbb{S}_n(X)$ with the property that for $n \geq 5$ we have $s(X) = 0$ if and only if X is homotopy equivalent to a closed n -dimensional topological manifold. The definitions of $\mathbb{S}_n(X)$ and $s(X)$ and the property are due to Ranicki in a combination of results of two books and several papers. In this paper we present these definitions and a detailed proof of the main result so that they are in one place and we also add some of the details not explicitly written down in the original sources.

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1. INTRODUCTION

An important problem in the topology of manifolds is deciding whether there is an n -dimensional closed topological manifold in the homotopy type of a given n -dimensional finite Poincaré complex X .

Recall that the “classical surgery theory” alias “Browder-Novikov-Sullivan-Wall-Kirby-Siebenmann theory” provides a method to decide this question in the form of a two-stage obstruction theory, when $n \geq 5$. A result of Spivak provides us with the Spivak normal fibration (SNF) $\nu_X : X \rightarrow \text{BSG}$, which is a spherical fibration, stably unique in some sense. If X is homotopy equivalent to a closed manifold then ν_X reduces to a stable topological block bundle, say $\bar{\nu}_X : X \rightarrow \text{BSTOP}$. The existence of such a reduction is the first obstruction. In terms of classifying spaces, the composition

$$(1.1) \quad H \circ \nu_X : X \rightarrow \text{BSG} \rightarrow \text{B(G/TOP)}$$

has to be homotopic to a constant map. Any reduction $\bar{\nu}_X$ determines a degree one normal map $(f, b) : M \rightarrow X$ from some n -dimensional closed topological manifold M to X with a surgery obstruction, which we call the quadratic signature of (f, b) and denote

$$(1.2) \quad \text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}\bullet}(f, b) \in L_n(\mathbb{Z}\pi_1(X)).$$

The complex X is homotopy equivalent to a closed manifold if and only if there exists a reduction for which $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}\bullet}(f, b) = 0$.

The “algebraic theory of surgery” of Ranicki replaces the above theory with a single obstruction, namely the total surgery obstruction

$$(1.3) \quad s(X) \in \mathbb{S}_n(X)$$

where $\mathbb{S}_n(X)$ is the n -dimensional structure group of X in the sense of the algebraic theory of surgery, which is a certain abelian group associated to X . It is the aim of this paper to discuss the definitions of $\mathbb{S}_n(X)$ and $s(X)$ and explain how they replace the classical theory.

The advantage of the algebraic theory is two-fold. On the one hand it is a single obstruction theory which by itself can be more convenient. On the other hand it turns out that the group $\mathbb{S}_n(X)$ has an L -theoretic definition, in fact it is isomorphic to a homotopy group of the homotopy fiber of a certain assembly map in L -theory. Hence the alternative approach allows us to solve our problem by entirely L -theoretic methods, for example by showing that the assembly map induces an isomorphism on homotopy groups and so $\mathbb{S}_n(X) = 0$. This possibility is in contrast with the classical surgery theory, where the first obstruction (1.1) is not L -theoretic in nature.

However, in practice, often slightly different assembly maps turn out to be more accessible for studying, as is the case for example in the recent papers [2], [3]. Then the theory needs to be modified to accommodate in addition the integer valued Quinn resolution obstruction. In the concluding Section 15 we offer more comments on this generalization and applications as well as examples.

The ingredients in the theory surrounding the total surgery obstruction are:

- The algebraic theory of surgery of Ranicki from [16], [17]. This comprises various sorts of L -groups of chain complexes over various additive categories with chain duality and with various notions of Poincaré duality. The sorts of L -groups are “symmetric”, “quadratic” and “normal”. The last notion is also due to Weiss in [24], [25].
- The classical surgery theory in the topological category from [4], [23]. The algebraic theory is not independent of the classical theory, in the sense that the proof that the algebraic theory answers our problem uses the classical theory.
- Topological transversality in all dimensions and codimensions as provided by Kirby–Siebenmann [8] and Freedman–Quinn [6].
- The surgery obstruction isomorphism, [8, Essay V, Thm. C.1]:

$$\text{sign}_{\mathbb{Z}}^{\mathbf{L}^\bullet} : \pi_n(\mathbf{G}/\mathbf{TOP}) \xrightarrow{\cong} L_n(\mathbb{Z}) \text{ for } n \geq 1.$$

- Geometric normal spaces and geometric normal transversality, both of which were invented by Quinn. However, the whole theory as announced in [14] is not needed. It is replaced by the algebraic normal L -groups from the first item.

We would also like to mention that an alternative approach to the total surgery obstruction has been proposed to us by Michael Weiss. Broadly speaking, in it, the algebraic normal L -groups would be removed at the expense of proving a certain general theorem relating symmetric and quadratic structures. This would enable an alternative definition of the total surgery obstruction. Also a modified proof has been suggested, we remark on this idea throughout the text more.

1.1. The basics of the algebraic theory of surgery. Mishchenko and Ranicki defined for a ring R with involution the symmetric and quadratic L -groups $L^n(R)$ and $L_n(R)$ respectively, as cobordism groups of chain complexes over R with symmetric and quadratic Poincaré structure respectively. The quadratic L -groups are isomorphic to the surgery obstruction groups of Wall [23].

Let W be the standard $\mathbb{Z}[\mathbb{Z}_2]$ -resolution of $\mathbb{Z}$. An n -dimensional *symmetric structure* on a chain complex C is an n -dimensional cycle

$$\varphi \in W^{\%}(C) := \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C) \cong \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Hom}_R(C^{-*}, C)).$$

Every such cycle can be written out in components $\varphi = (\varphi_i : C^{n-*} \rightarrow C_{*+i})_{i \in \mathbb{N}}$. If $\varphi_0 : C^{n-*} \rightarrow C$ is a chain homotopy equivalence, then the structure is called *Poincaré*. Every n -dimensional cycle $x \in C(X)$ has a corresponding symmetric structure $\varphi(x)$ on $C(\tilde{X})$ over $\mathbb{Z}[\pi_1(X)]$ with $\varphi(x)_0 = - \cap x : C(\tilde{X})^{n-*} \rightarrow C(\tilde{X})$ given by an equivariant version of the familiar Alexander–Whitney diagonal approximation construction. If X is a Poincaré complex with the fundamental class $[X]$, then we obtain the *symmetric signature* of X ,

$$(1.4) \quad \text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet}(X) = [(C(\tilde{X}), \varphi([X]))] \in L^n(\mathbb{Z}[\pi_1(X)]).$$

An n -dimensional *quadratic structure* on a chain complex C is an n -dimensional cycle

$$\psi \in W_{\%}(C) := W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_R C) \cong W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (\mathrm{Hom}_R(C^{-*}, C)).$$

There is a symmetrization map $1 + T : W_{\%}(C) \rightarrow W^{\%}(C)$ which allows us to see quadratic structures as refinements of symmetric structures. A quadratic structure is called *Poincaré* if its symmetrization is Poincaré. Such a quadratic structure is more subtle to obtain from a geometric situation. As explained in Construction 2.16, given an n -dimensional cycle $x \in C(X)$ and a stable map $F : \Sigma^p X_+ \rightarrow \Sigma^p M_+$ there is a quadratic structure $\psi(x)$ over $\mathbb{Z}[\pi_1(X)]$ on $C(\tilde{M})$. A degree one normal map $(f, b) : M \rightarrow X$ between n -dimensional Poincaré complexes induces a map of Thom spaces $\mathrm{Th}(b) : \mathrm{Th}(\nu_M) \rightarrow \mathrm{Th}(\nu_X)$ which in turn, using S -duality,¹ produces a stable map $F : \Sigma^p X_+ \rightarrow \Sigma^p M_+$ for some p . The quadratic construction ψ produces from the fundamental class $[X]$ a quadratic structure on $C(\tilde{M})$. Considering the Umkehr map $f^! : C(\tilde{X}) \rightarrow \Sigma^{-p} C(\Sigma^p \tilde{X}_+) \rightarrow \Sigma^{-p} C(\Sigma^p \tilde{M}_+) \rightarrow C(\tilde{M})$ and the inclusion into the algebraic mapping cone $e : C(\tilde{M}) \rightarrow C(f^!)$ we obtain an n -dimensional quadratic Poincaré complex called the *quadratic signature* of (f, b)

$$(1.5) \quad \mathrm{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}_{\bullet}}(f, b) = [(C(f^!), e_{\%}\psi([X]))] \in L_n(\mathbb{Z}[\pi_1(X)])$$

If (f, b) is a degree one normal map with M an n -dimensional manifold, then the quadratic signature (1.5) coincides with the classical surgery obstruction.

1.2. The structure group $\mathbb{S}_n(X)$. A generalization of the theory in 1.1 is obtained by replacing the ring R with an algebraic bordism category Λ . Such a category contains an underlying additive category with chain duality $\mathbb{A}$. The category Λ specifies a subcategory of the category of structured chain complexes² in $\mathbb{A}$ and a type of Poincaré duality. We obtain cobordism groups of such chain complexes $L^n(\Lambda)$ and $L_n(\Lambda)$ and also spectra $\mathbf{L}^{\bullet}(\Lambda)$ and $\mathbf{L}_{\bullet}(\Lambda)$ whose homotopy groups are these L -groups.

The notion of an additive category with chain duality allows us to consider structured chain complexes over a simplicial complex X , with $\pi = \pi_1(X)$. Informally one can think of such a structured chain complex over X as a compatible collection of structured chain complexes over $\mathbb{Z}$ indexed by simplices of X . “Forgetting” the indexing “assembles” such a complex over X to a complex over $\mathbb{Z}$ and an equivariant version of this process yields a complex over $\mathbb{Z}[\pi]$. The algebraic bordism categories allow us to consider various types of Poincaré duality for structured complexes over X . There is the *local Poincaré duality* where it is required that the structure over each simplex is Poincaré, with the category of all such complexes denoted $\Lambda(\mathbb{Z})_*(X)$. It turns out that³

$$(1.6) \quad L_n(\Lambda(\mathbb{Z})_*(X)) \cong H_n(X; \mathbf{L}_{\bullet}(\mathbb{Z})), \quad L^n(\Lambda(\mathbb{Z})_*(X)) \cong H_n(X; \mathbf{L}^{\bullet}(\mathbb{Z})).$$

¹See Section 3 for more details if needed.

²meaning chain complexes with a symmetric or quadratic structure

³Here $\mathbf{L}_{\bullet}(\mathbb{Z})$ is short for $\mathbf{L}_{\bullet}(\Lambda(\mathbb{Z})_*(\mathrm{pt.}))$.

Then there is the *global Poincaré duality* where only the assembled structure is required to be Poincaré, with the category of all such complexes denoted $\Lambda(\mathbb{Z}[\pi])$ for the purposes of this introduction.⁴ The assembly gives a functor $A : \Lambda(\mathbb{Z})_*(X) \rightarrow \Lambda(\mathbb{Z}[\pi])$ which induces on the L -groups the assembly maps

$$(1.7) \quad A : H_n(X; \mathbf{L}^\bullet(\mathbb{Z})) \rightarrow L^n(\mathbb{Z}[\pi]), \quad A : H_n(X; \mathbf{L}_\bullet(\mathbb{Z})) \rightarrow L_n(\mathbb{Z}[\pi]).$$

In a familiar situation such chain complexes arise as follows. A triangulated n -dimensional manifold X has a dual cell decomposition, where for each simplex $\sigma \in X$ the dual cell $(D(\sigma), \partial D(\sigma))$ is an $(n - |\sigma|)$ -dimensional submanifold with boundary. The collection of chain complexes $C(D(\sigma), \partial D(\sigma))$ together with corresponding symmetric structures provides the symmetric signature of X over X , which is a locally Poincaré symmetric complex over X

$$(1.8) \quad \text{sign}_X^{\mathbf{L}^\bullet}(X) \in H_n(X; \mathbf{L}^\bullet(\mathbb{Z})), \quad A(\text{sign}_X^{\mathbf{L}^\bullet}(X)) = \text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}^\bullet}(X) \in L^n(\mathbb{Z}[\pi]).$$

A degree one normal map $(f, b) : M \rightarrow X$ from an n -dimensional manifold M to a triangulated n -dimensional manifold X can be made transverse to the dual cells of X . So if we set $M(\sigma) = f^{-1}(D(\sigma))$ then such a degree one normal map (f, b) yields a collection of degree one normal maps of manifolds with boundary $(f(\sigma), b(\sigma)) : (M(\sigma), \partial(M(\sigma))) \rightarrow (D(\sigma), \partial D(\sigma))$. The collection of chain complexes $C(f(\sigma)^\dagger)$ with the corresponding quadratic structures provides the quadratic signature of (f, b) over X which is a locally Poincaré quadratic complex over X

$$(1.9) \quad \text{sign}_X^{\mathbf{L}^\bullet}(f, b) \in H_n(X; \mathbf{L}_\bullet(\mathbb{Z})), \quad A(\text{sign}_X^{\mathbf{L}^\bullet}(f, b)) = \text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}^\bullet}(f, b) \in L_n(\mathbb{Z}[\pi]).$$

The L -groups relevant for our geometric problem are modifications of the above concepts obtained by using certain connective versions.⁵

The cobordism group of n -dimensional quadratic 1-connective complexes that are locally Poincaré turns out to be isomorphic to the homology group $H_*(X, \mathbf{L}_\bullet\langle 1 \rangle)$, where $\mathbf{L}_\bullet\langle 1 \rangle$ denotes the 1-connective quadratic L -theory spectrum. The cobordism group of n -dimensional quadratic 1-connective complexes that are globally Poincaré turns out to be isomorphic to $L_n(\mathbb{Z}[\pi])$. The assembly functor induces an assembly map analogous to (1.7).

The structure group $\mathbb{S}_n(X)$ is the cobordism group of $(n - 1)$ -dimensional quadratic chain complexes over X that are locally Poincaré, locally 1-connective and globally contractible. All these groups fit into the algebraic surgery exact sequence:

$$(1.10) \quad \rightarrow H_n(X, \mathbf{L}_\bullet\langle 1 \rangle) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\partial} \mathbb{S}_n(X) \xrightarrow{I} H_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle) \rightarrow$$

The map I is induced by the inclusion of categories and the map ∂ will be described below.

⁴The notation here is justified by Proposition 6.6 which says that the L -theory of this category is indeed isomorphic to the L -theory of the group ring.

⁵This is a technical point addressed in Section 15.

1.3. The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$. We need to explain how to associate to X an $(n-1)$ -dimensional quadratic chain complex over X that is locally Poincaré, locally 1-connective and globally contractible.

Being Poincaré is by definition a global condition. What local structure does a Poincaré complex have? The answer is the structure of a normal complex.

An n -dimensional normal complex (Y, ν, ρ) is a space Y with a k -dimensional spherical fibration $\nu : Y \rightarrow \text{BSG}(k)$ and a map $\rho : S^{n+k} \rightarrow \text{Th}(\nu)$. There is also a notion of a normal pair, a normal cobordism, and normal cobordism groups. A Poincaré complex X embedded into a large euclidean space has a regular neighborhood, whose boundary produces a model for the SNF ν_X and collapsing the boundary gives a model for the Thom space $\text{Th}(\nu_X)$ with the collapse map ρ_X . In a general normal complex the underlying space Y does not have to be Poincaré. Nevertheless it has a preferred homology class $h(\rho) \cap u(\nu) = [Y] \in C_n(Y)$, where $u(\nu) \in C^k(\text{Th}(\nu))$ is some choice of the Thom class and h denotes the Hurewicz homomorphism. The class $[Y]$ produces a preferred equivalence class of symmetric structures on $C(Y)$.

There exists a notion of an n -dimensional normal algebraic complex (C, θ) over any additive category with chain duality $\mathbb{A}$. At this stage we only say that the normal structure θ contains a symmetric structure, and should be seen as a certain refinement of that symmetric structure.⁶ Again one can consider normal complexes in an algebraic bordism category Λ , specifying an interesting subcategory and the type of Poincaré duality on the underlying symmetric structure. The cobordism groups are denoted $NL^\bullet(\Lambda)$ and there are also associated spectra $\mathbf{NL}^\bullet(\Lambda)$. For a ring R we have the cobordism groups $NL^n(R)$ of n -dimensional normal complexes over R with no Poincaré duality in this case! A geometric normal complex (Y, ν, ρ) gives rise to a normal algebraic complex, called the normal signature

$$(1.11) \quad \text{sign}_{\mathbb{Z}[\pi_1(Y)]}^{\mathbf{NL}^\bullet}(Y) \in NL^n(\mathbb{Z}[\pi_1(Y)]).$$

whose symmetric substructure is the one associated to its fundamental class $[Y]$.

So how is a Poincaré complex X locally normal? For a simplex $\sigma \in X$ consider the dual cell $(D(\sigma), \partial D(\sigma))$, which is a pair of spaces, not necessarily Poincaré. The SNF ν_X can be restricted to $(D(\sigma), \partial D(\sigma))$, remaining a spherical fibration, say $(\nu_X(\sigma), \nu_X(\partial\sigma))$. A certain trick⁷ is needed to obtain a map

$$\rho(\sigma) : (D^{n+k-|\sigma|}, S^{n-1+k-|\sigma|}) \rightarrow (\text{Th}(\nu_X(\sigma)), \text{Th}(\nu_X(\partial\sigma)))$$

providing us with a normal complex “with boundary”. The collection of these gives rise to a compatible collection of normal algebraic complexes over $\mathbb{Z}$ and we obtain an n -dimensional normal algebraic complex over X whose symmetric substructure is globally Poincaré. As such it can be viewed as a normal complex in two distinct algebraic bordism categories.

⁶The details are presented in Section 3.

⁷presented in Section 11

First there is the category $\widehat{\Lambda}(\mathbb{Z})\langle 1/2 \rangle(X)$, which is a $1/2$ -connective version of all normal complexes over X with no Poincaré duality. Here we obtain the $1/2$ -connective normal signature of X over X ⁸

$$(1.12) \quad \text{sign}_X^{\mathbf{NL}^\bullet}(X) \in NL^n(\widehat{\Lambda}(\mathbb{Z})\langle 1/2 \rangle(X)) \cong H_n(X, \mathbf{NL}^\bullet\langle 1/2 \rangle).$$

Similarly as before the assembly of (1.12) becomes (1.11).

Then there is the category $\Lambda(\mathbb{Z})\langle 1/2 \rangle(X)$, which is a $1/2$ -connective version of all normal complexes over X with global Poincaré duality. The cobordism group $NL^n(\Lambda(\mathbb{Z})\langle 1/2 \rangle(X))$ is called the $1/2$ -connective visible symmetric group and denoted $VL^n(X)$. We obtain the visible signature of X over X

$$(1.13) \quad \text{sign}_X^{\mathbf{VL}^\bullet}(X) \in VL^n(X).$$

The forgetful functor $\Lambda(\mathbb{Z})\langle 1/2 \rangle(X) \rightarrow \widehat{\Lambda}(\mathbb{Z})\langle 1/2 \rangle(X)$ induces a map on the NL -groups which sends (1.13) to (1.12).

But we are after an $(n-1)$ -dimensional quadratic complex. To obtain it we need in addition the concept of a boundary of a structured chain complex. Consider an n -dimensional symmetric complex (C, φ) over any additive category with chain duality $\mathbb{A}$. Its boundary $(\partial C, \partial \varphi)$ is an $(n-1)$ -dimensional symmetric complex in $\mathbb{A}$ whose underlying chain complex is defined as $\partial C = \Sigma^{-1}C(\varphi_0)$. The $(n-1)$ -dimensional symmetric structure $\partial \varphi$ is inherited from φ . It becomes Poincaré in $\mathbb{A}$, meaning $\mathcal{C}((\partial \varphi)_0)$ is contractible in $\mathbb{A}$, by a formal argument.⁹ The boundary $(\partial C, \partial \varphi)$ can be viewed as measuring how the complex (C, φ) itself is Poincaré in $\mathbb{A}$. It is shown in Proposition 3.17 that an n -dimensional symmetric complex (C, φ) which is a part of a normal complex (C, θ) comes with a quadratic refinement $\partial \psi$ of the symmetric structure $\partial \varphi$ on the boundary.¹⁰

From this description it follows that the boundary produces the following two maps:

$$(1.14) \quad \partial : L_n(\mathbb{Z}[\pi]) \rightarrow \mathbb{S}_n(X) \text{ and } \partial : VL^n(X) \rightarrow \mathbb{S}_n(X).$$

The total surgery obstruction $s(X)$ is defined as the $(n-1)$ -dimensional quadratic complex over X , obtained as the boundary of the visible signature

$$(1.15) \quad s(X) = \partial \text{sign}_X^{\mathbf{VL}^\bullet}(X) \in \mathbb{S}_n(X).$$

It is locally Poincaré, because by the above discussion any boundary of a complex over X is locally Poincaré. It is also globally contractible since X is Poincaré and hence the boundary of the assembled structure is contractible. The connectivity assumption is also fulfilled.

⁸The connectivity condition turns out to be fulfilled, see Section 9 for explanation if needed.

⁹The choice of terminology is explained below Definition 2.25.

¹⁰The normal structure on C provides a second stable symmetric structure on ∂C in addition to $\partial \varphi$ and the two structures stably coincide. Such a situation yields a quadratic structure.

Main Theorem. [19, Thm. 17.4]

Let X be a finite Poincaré complex of formal dimension $n \geq 5$. Then X is homotopy equivalent to a closed n -dimensional topological manifold if and only if

$$0 = s(X) \in \mathbb{S}_n(X).$$

The proof is based on:

Main Technical Theorem.

Let X be a finite Poincaré complex of formal dimension $n \geq 5$ and denote by $t(X) = I(s(X)) \in H_{n-1}(X, \mathbf{L}_\bullet\langle 1 \rangle)$. Then we have

- (I) $t(X) = 0$ if and only if there exists a topological block bundle reduction of the SNF $\nu_X : X \rightarrow \text{BSG}$.
 (II) If $t(X) = 0$ then we have

$$\begin{aligned} \partial^{-1}s(X) &= \{ -\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}_\bullet}(f, b) \in L_n(\mathbb{Z}[\pi_1(X)]) \mid \\ &\quad (f, b) : M \rightarrow X \text{ degree one normal map, } M \text{ manifold} \}. \end{aligned}$$

Proof of Main Theorem assuming Main Technical Theorem.

If X is homotopy equivalent to a manifold then $t(X) = 0$ and by (II) the set $\partial^{-1}s(X)$ contains 0, hence $s(X) = 0$.

If $s(X) = 0$ then $t(X) = 0$ and hence by (I) the SNF of X has a topological block bundle reduction. Also $\partial^{-1}s(X)$ must contain 0 and hence by (II) there exists a degree one normal map with the target X and with the surgery obstruction 0. $\square$

Remark 1.4. The condition (I) might be puzzling for the following reason. As recalled earlier, the classical surgery gives an obstruction to the reduction of the SNF in the group $[X, \text{B}(\text{G}/\text{TOP})] = H^1(X; \text{G}/\text{TOP})$. It is important to note that here the Ω^∞ -space structure used on G/TOP corresponds to the Whitney sum and hence not the one that is compatible with the well-known homotopy equivalence $\text{G}/\text{TOP} \simeq \mathbf{L}\langle 1 \rangle_0$. On the other hand $t(X) \in H_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle)$. We note that the claim of (I) is NOT that the two groups are isomorphic, it merely says that one obstruction is zero if and only if the other is zero.

1.5. Informal discussion of the proof of Main Technical Theorem.

Part (I): The crucial result is the relation between the quadratic, symmetric, and normal L -groups of a ring R via the long exact sequence

$$(1.16) \quad \dots \longrightarrow L_n(R) \xrightarrow{1+T} L^n(R) \xrightarrow{J} NL^n(R) \xrightarrow{\partial} L_{n-1}(R) \longrightarrow \dots$$

Here the maps $1+T$ and ∂ were already discussed. The map J exists because a symmetric Poincaré structure on a chain complex yields a preferred normal structure, reflecting the observation that a Poincaré complex has the SNF and hence gives a geometric normal complex. Using suitable connective versions there is a related homotopy fibration sequence of spectra (here the implicit ring is $\mathbb{Z}$)

$$(1.17) \quad \mathbf{L}_\bullet\langle 1 \rangle \rightarrow \mathbf{L}^\bullet\langle 0 \rangle \rightarrow \mathbf{NL}^\bullet\langle 1/2 \rangle \rightarrow \Sigma \mathbf{L}_\bullet\langle 1 \rangle.$$

The exactness of (1.16) and fibration property of (1.17) are not easily observed. It is a result of [24, 25] for which we offer some explanation in Section 3. The sequence (1.17) induces a long exact sequence in homology

$$(1.18) \quad \cdots \rightarrow H_n(X; \mathbf{L}^\bullet\langle 0 \rangle) \xrightarrow{J} H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle) \xrightarrow{\partial} H_{n-1}(X; \mathbf{L}^\bullet\langle 1 \rangle) \rightarrow \cdots$$

Another tool is the S -duality from stable homotopy theory, which gives

$$(1.19) \quad H_n(X; \mathbf{E}) \cong H^k(\mathrm{Th}(\nu_X); \mathbf{E}) \text{ with } \mathbf{E} = \mathbf{L}^\bullet\langle 1 \rangle, \mathbf{L}^\bullet\langle 0 \rangle, \text{ or } \mathbf{NL}^\bullet\langle 1/2 \rangle.$$

and transforms the exact sequence (1.18) into an exact sequence in cohomology of the Thom space $\mathrm{Th}(\nu_X)$.

The proof of the theorem is organized within the following commutative braid:

We observe that

$$(1.20) \quad t(X) = \partial \operatorname{sign}_X^{\mathbf{NL}^\bullet}(X) \in H_{n-1}(X; \mathbf{L}^\bullet\langle 1 \rangle)$$

with the normal signature over X from (1.12).

Assuming the above, the proof proceeds as follows. If ν_X has a reduction, then it has an associated degree one normal map $(f, b) : M \rightarrow X$ which can be made transverse to the dual cells of X . For each σ the preimage $(M(\sigma), \partial M(\sigma))$ of the dual cell $(D(\sigma), \partial D(\sigma))$ is an $(n - |\sigma|)$ -dimensional submanifold with boundary and generalizing (1.8) we obtain

$$(1.21) \quad \operatorname{sign}_X^{\mathbf{L}^\bullet}(M) \in L^n(\Lambda(\mathbb{Z})\langle 0 \rangle_*(X)) \cong H_n(X; \mathbf{L}^\bullet\langle 0 \rangle)$$

The mapping cylinder of the degree one normal map (f, b) becomes a normal cobordism between M and X and, as such, it produces a normal algebraic cobordism between $J(\operatorname{sign}_X^{\mathbf{L}^\bullet}(M))$ and $\operatorname{sign}_X^{\mathbf{NL}^\bullet}(X)$. In other words the symmetric signature $\operatorname{sign}_X^{\mathbf{L}^\bullet}(M)$ is a lift of the normal signature $\operatorname{sign}_X^{\mathbf{NL}^\bullet}(X)$ from (1.12) and it follows from the exact sequence (1.18) that $t(X)$ vanishes.

The crucial concept used in the proof of the other direction is that of an orientation of a spherical fibration with respect to a ring spectrum, such as $\mathbf{L}^\bullet\langle 0 \rangle$ and $\mathbf{NL}^\bullet\langle 1/2 \rangle$. For such a ring spectrum an $\mathbf{E}$ -orientation of the SNF ν_X is an element in $H^k(\mathrm{Th}(\nu_X); \mathbf{E})$ with a certain property. By the S -duality

(1.19) it corresponds to a homology class in $H_n(X, \mathbf{E})$. It turns out that the SNF ν_X has a canonical $\mathbf{NL}^\bullet\langle 1/2 \rangle$ -orientation which corresponds to the normal signature (1.12) in this way. Similarly if there is a reduction of ν_X with a degree one normal map $(f, b) : M \rightarrow X$, then it gives an $\mathbf{L}^\bullet\langle 0 \rangle$ -orientation of ν_X which corresponds to the symmetric signature (1.21) of M over X .

Theorem 13.16 says that a spherical fibration has a topological block bundle reduction if and only if its canonical $\mathbf{NL}^\bullet\langle 1/2 \rangle$ -orientation has an $\mathbf{L}^\bullet\langle 0 \rangle$ -lift. The proof is by analyzing classifying spaces for spherical fibrations with orientations, a certain diagram (Proposition 13.13) is shown to be a homotopy pull-back. Here is used the fact that the surgery obstruction map $\pi_n(\mathbf{G}/\mathbf{TOP}) \rightarrow L_n(\mathbb{Z})$ is an isomorphism for $n > 1$.

Part (II): To show the inclusion of the right hand side one needs to study the quadratic signatures over X of degree one normal maps $(f, b) : M \rightarrow X$ with M an n -dimensional closed manifold and X an n -dimensional Poincaré complex. That means studying the local structure of such maps which boils down to studying quadratic signatures of degree one normal maps $(g, c) : N \rightarrow Y$ where Y is only a normal complex. In this case one obtains a non-Poincaré quadratic complex whose boundary can be related to the quadratic boundary of the normal complex Y as shown in Proposition 14.2. Passing to complexes over X one obtains a quadratic complex over X , still denoted $\text{sign}_X^{\mathbf{L}^\bullet}(f, b)$ although it is not locally Poincaré, whose boundary is described in Proposition 14.5 establishing the required inclusion.

To study the other inclusion a choice is made of a degree one normal map $(f_0, b_0) : M_0 \rightarrow X$. Recall that all degree one normal maps with the target X are organized in the cobordism set of the normal invariants $\mathcal{N}(X)$. One considers the surgery obstruction map relative to (f_0, b_0)

$$(1.22) \quad \text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet}(-, -) - \text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet}(f_0, b_0) : \mathcal{N}(X) \rightarrow L_n(\mathbb{Z}[\pi_1(X)]).$$

The signature $\text{sign}_X^{\mathbf{L}^\bullet}$ over X relative to (f_0, b_0) produces a map from the normal invariants $\mathcal{N}(X)$ to the homology group $H_n(X; \mathbf{L}_\bullet\langle 1 \rangle)$. The main technical result is now Proposition 14.18 which states that this map provides us with an identification of (1.22) with the assembly map (1.7) for X . In particular it says that $\text{sign}_X^{\mathbf{L}^\bullet}$ relative to (f_0, b_0) produces a bijection. Via the standard identification $\mathcal{N}(X) \cong [X; \mathbf{G}/\mathbf{TOP}]$ and the bijection $[X, \mathbf{G}/\mathbf{TOP}] \cong H^0(X; \mathbf{L}_\bullet\langle 1 \rangle)$ (using the Kirby–Siebenmann isomorphism again) this boils down to identifying $\text{sign}_X^{\mathbf{L}^\bullet}$ with the Poincaré duality with respect to the spectrum $\mathbf{L}_\bullet\langle 1 \rangle$. Here, similarly as in part (I), a relationship between the signatures and orientations with respect to the L -theory spectra plays a prominent role (Proposition 14.30 and Lemma 14.32).

Alternative approach. The alternative approach of M. Weiss to the definition of the total surgery obstruction would run as follows. The main ingredient is Proposition 4.5 in his paper [26] which says that the symmetrization map $\mathbb{S}_n(X) \rightarrow \mathbb{S}^n(X)$ from the structure group to its “symmetric version” is an isomorphism. The group $\mathbb{S}^n(X)$ is defined as the symmetric L -group of the same

underlying algebraic bordism category as $\mathbb{S}_n(X)$. The proof of this proposition is based on the fact shown in [24, 25] that the hyperquadratic structures on a fixed chain complex, which measure the difference between the symmetric and quadratic structures on that chain complex, behave “homologically” in some sense.

An n -dimensional GPC X provides us with a symmetric algebraic complex over X , which is globally Poincaré, but possibly not locally Poincaré. The application of the boundary defines an element in $\mathbb{S}^n(X)$. The symmetrization isomorphism then provides us with an element in $\mathbb{S}_n(X)$ which coincides with $s(X)$ as we define it. From our point of view this definition has a drawback that it does not give us a concrete description of the quadratic refinement, it only tells us it exists. However, for the proof of the Main Technical Theorem we need a good understanding of the relationship of the quadratic structure with the local normal structure on X . The definition we give in Section 12 provides us with such knowledge.

As mentioned before M. Weiss suggested an alternative way to formulating and proving the Main Technical Theorem as well. We elaborate more on this in Section 15. Now we only want to mention that it would prove less (see Theorem 13.16) and also the details of this approach have not yet been worked out.

The purpose of the paper. As the title suggests this article revisits the existing theory which was developed over decades by Andrew Ranicki, with contributions also due to Michael Weiss. On one hand it is meant as a guide to the theory. We decided to write such a guide when we were learning the theory. It turned out that results of various sources needed to be combined and we felt that it might be a good idea to have them in one place. The sources are [11, 15, 18, 19], and also [24, 25]. On the other hand, we found certain statements which were correct, but without proofs, which we were able to supply. These are:

- The fact that the quadratic boundary of a certain (normal, Poincaré) geometric pair associated to a degree one normal map from a manifold to a Poincaré space agrees with the surgery obstruction of that map is proved in our Example 3.27. The claim was stated in [18, p. 622] without proof. The proposition preceding the claim suggests the main idea of the proof, but we felt that writing it down is needed.
- The construction of the normal signature $\text{sign}_X^{\text{NL}\bullet}(X)$ in Section 11 for an n -dimensional geometric Poincaré complex X . This was claimed to exist in [19, Ex. 9.12] (see also [22, Errata for p. 103]), for X any n -dimensional geometric normal complex. We provide details of this construction when X is Poincaré, which is enough for our purposes.
- In the proof of Theorem 13.16 a certain map has to be identified with the surgery obstruction map. The identification was claimed in [15, p. 291] without details. Theorem 13.16 is also essentially equivalent

to [19, Prop. 16.1], which has a sketch proof and is referenced back to [15] for further details.

- The relation between the quadratic complex associated to a degree one normal map from a manifold to a normal complex and the quadratic boundary of the normal complex itself as described in Proposition 14.2. We also provide the proof of Proposition 14.4 which is a relative version of Proposition 14.2 and it is also an ingredient in the proof of Proposition 14.5 which gives information about the quadratic signature over X of a degree one normal map from a manifold to a Poincaré complex X . Proposition 14.2 was stated as [18, Prop. 7.3.4], but only contained a sketch proof. Proposition 14.5 is used in the proof of [19, Thm. 17.4].

Over time we have also heard from several other mathematicians in the area the need for such clarifications. We believe that with this paper we provide an answer to these questions and that the proof of the main theorem as presented here is complete. We also hope that our all-in-one-package paper makes the presentation of the whole theory surrounding the total surgery obstruction more accessible. We would be grateful for comments from an interested reader should there still be unclear parts.

It should be noted however, that we do not bring new technology to the proof, nor do we state any new theorems. Our supplying of the proofs as listed above is in the spirit of the two main sources [15] and [19].

Structure. The reader will recognize that our table of contents closely follows part I and the first two sections of part II of the book [19]. We find most of the book a very good and readable source. So in the background parts of this paper we confine ourselves to survey-like treatment. In the parts where we felt the need for clarification, in particular the proof of the main theorem, we provide the details.

The reader of this article should be familiar with the classical surgery theory and at least basics of the algebraic surgery theory. Sections 2 to 10 contain a summary of the results from part I of [19] which are needed to explain the theory around the main problem, sometimes equipped with informal comments. The reader familiar with these results can skip those sections and start reading Section 11, where the proof of the main theorem really begins. In case the reader is familiar with everything except normal complexes, he may consult in addition Section 3.

Literature. Besides the above mentioned sources some background can be found in [16], [17], [20], [21].

Note. Parts of this work will be used in the PhD thesis of Philipp Kühl.

2. ALGEBRAIC COMPLEXES

In this section we briefly recall the basic concepts of algebraic surgery. The details can be found in [16, 17] and [19, Chap. 1].

Throughout the paper $\mathbb{A}$ denotes an additive category and $\mathbb{B}(\mathbb{A})$ denotes the category of bounded chain complexes in $\mathbb{A}$. The total complex of a double chain complex can be used to extend a contravariant functor $T : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$ to a contravariant functor $T : \mathbb{B}(\mathbb{A}) \rightarrow \mathbb{B}(\mathbb{A})$ as explained in detail in [19, p. 26].

Definition 2.1. A *chain duality* on an additive category $\mathbb{A}$ is a pair (T, e) where

- T is a contravariant functor $T : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$,
- e is a natural transformation $e : T^2 \rightarrow (\text{id} : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A}))$ such that
 - $e_M : T^2(M) \rightarrow M$ is a chain equivalence,
 - $e_{T(M)} \circ T(e_M) = \text{id}$.

The extension $T : \mathbb{B}(\mathbb{A}) \rightarrow \mathbb{B}(\mathbb{A})$ mentioned before the definition defines the dual $T(C)$ for a chain complex $C \in \mathbb{B}(\mathbb{A})$. A chain duality $T : \mathbb{A} \rightarrow \mathbb{B}(\mathbb{A})$ can be used to define a tensor product of two objects M, N in $\mathbb{A}$ over $\mathbb{A}$ as

$$(2.1) \quad M \otimes_{\mathbb{A}} N = \text{Hom}_{\mathbb{A}}(T(M), N),$$

which is a priori just a chain complex of abelian groups. This definition generalizes for chain complexes C and D in $\mathbb{B}(\mathbb{A})$:

$$C \otimes_{\mathbb{A}} D := \text{Hom}_{\mathbb{A}}(T(C), D).$$

Example 2.2. Let R be a ring with involution $r \mapsto \bar{r}$, e.g. if $R = \mathbb{Z}[\pi]$ is the group ring of a group π , then we have an involution given by $\bar{g} = g^{-1}$ for $g \in \pi$.

The category $\mathbb{A}(R)$ of finitely generated free left R -modules possesses a chain duality by $T(M) = \text{Hom}_R(M, R)$. The involution can be used to turn an a priori right R -module $T(M)$ into a left R -module. The dual $T(C)$ of a bounded chain complex C over R is $\text{Hom}_R(C, R)$.

Chain duality is important because it enables us to define various concepts of Poincaré duality as we will see. Although the chain dual $T(M)$ in the above example is concentrated in dimension 0, this is not necessarily the case in general. In Section 5 we will see examples where this generality is important.

Notation 2.3. Let W and $\widehat{W}$ be the canonical free $\mathbb{Z}[\mathbb{Z}_2]$ -resolution and the free periodic $\mathbb{Z}[\mathbb{Z}_2]$ -resolution of $\mathbb{Z}$ respectively:

$$W := \quad \dots \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \longrightarrow 0$$

$$\widehat{W} := \quad \dots \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-T} \dots$$

The chain duality T can be used to define an involution $T_{C,C}$ on $C \otimes_{\mathbb{A}} C$ which makes it into a $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex, see [19, p. 29].

Definition 2.4. We have the following chain complexes of abelian groups:

$$\begin{aligned} W_{\%}(C) &:= W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C), \\ W^{\%}(C) &:= \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C), \\ \widehat{W}^{\%}(C) &:= \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes_{\mathbb{A}} C). \end{aligned}$$

Notation 2.5. Let $f: C \rightarrow D$ be a chain map in $\mathbb{B}(\mathbb{A})$. Then the map of $\mathbb{Z}[\mathbb{Z}_2]$ -chain complexes $f \otimes f: C \otimes_{\mathbb{A}} C \rightarrow D \otimes_{\mathbb{A}} D$ induces chain maps

$$f_{\%}: W_{\%}(C) \rightarrow W_{\%}(D) \quad f^{\%}: W^{\%}(C) \rightarrow W^{\%}(D) \quad \widehat{f}^{\%}: \widehat{W}^{\%}(C) \rightarrow \widehat{W}^{\%}(D).$$

Definition 2.6. Let C be a chain complex in $\mathbb{B}(\mathbb{A})$. An n -dimensional *symmetric structure* on C is an n -dimensional cycle $\varphi \in W^{\%}(C)_n$. Similarly an n -dimensional *quadratic structure* on C is an n -dimensional cycle $\psi \in W_{\%}(C)_n$, and an n -dimensional *hyperquadratic structure* on C is an n -dimensional cycle $\theta \in \widehat{W}^{\%}(C)_n$.

Note that the dimension n refers only to the degree of the element φ , ψ , or θ and does not mean that the chain complex C has to be concentrated between degrees 0 and n .

Notation 2.7. On chain complexes we use the operations of *suspension* defined by $(\Sigma C)_n = C_{n-1}$ and *desuspension* defined by $(\Sigma^{-1}C)_n = C_{n+1}$. If X is a well-based topological space we can consider the reduced suspension ΣX . For the singular chain complexes $C(X)$ and $C(\Sigma X)$ and there is a natural chain homotopy equivalence which we denote $\Sigma: C(X) \rightarrow \Sigma^{-1}C(\Sigma X)$, see [16, Sec. 1] if needed. Sometimes we use the same symbol for the associated map of degree one of chain complexes $\Sigma: C(X) \rightarrow C(\Sigma X)$.

Remark 2.8. The structures on a chain complex C from Definition 2.6 can also be described in terms of their components. Abbreviating $C^{m-*} = \Sigma^m TC$, an element $\varphi \in W^{\%}(C)_n$ is a collection of maps $\{\varphi_s: C^{n+s-*} \rightarrow C | s \in \mathbb{N}\}$, an element $\psi \in W_{\%}(C)_n$ is a collection of maps $\{\psi_s: C^{n-s-*} \rightarrow C | s \in \mathbb{N}\}$, and an element $\theta \in \widehat{W}^{\%}(C)_n$ is a collection of maps $\{\theta_s: C^{n+s-*} \rightarrow C | s \in \mathbb{Z}\}$, all of them satisfying certain identities, see [19, p. 30]. In the symmetric case these identities describe each φ_s as a chain homotopy between φ_{s-1} and $T\varphi_{s-1}$.

Definition 2.9. For a $C \in \mathbb{B}(\mathbb{A})$ the Q -groups of C are defined by

$$Q_n(C) = H_n(W_{\%}(C)), \quad Q^n(C) = H_n(W^{\%}(C)), \quad \widehat{Q}^n(C) = H_n(\widehat{W}^{\%}(C)).$$

Proposition 2.10. [16, Prop. 1.2] *For a chain complex $C \in \mathbb{B}(\mathbb{A})$ we have a long exact sequence of Q -groups*

$$\dots \longrightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \longrightarrow \dots$$

The sequence is induced from the short exact sequence of chain complexes

$$0 \longrightarrow W^{\%}(C) \longrightarrow \widehat{W}^{\%}(C) \longrightarrow \Sigma W_{\%}(C) \longrightarrow 0.$$

The connecting map

$$(2.2) \quad 1 + T : W_{\%}(C) \rightarrow W_{\%}(C), \quad ((1 + T)\psi)_s = \begin{cases} (1 + T)\psi_0, & \text{if } s = 0, \\ 0, & \text{if } s \geq 1, \end{cases}$$

is called the *symmetrization map*.

Definition 2.11. An n -dimensional *symmetric algebraic complex* (SAC) in $\mathbb{A}$ is a pair (C, φ) where $C \in \mathbb{B}(\mathbb{A})$ and φ is an n -dimensional symmetric structure on C . It is called *Poincaré* (SAPC) if φ_0 is a chain homotopy equivalence.

An n -dimensional *quadratic algebraic complex* (QAC) in $\mathbb{A}$ is a pair (C, ψ) where $C \in \mathbb{B}(\mathbb{A})$ and ψ is an n -dimensional quadratic structure on C . It is called *Poincaré* (QAPC) if $((1 + T) \cdot \psi)_0$ is a chain homotopy equivalence.

An analogous notion for hyperquadratic complexes is not defined. The following construction helps to understand the exact sequence of Proposition 2.10.

Definition 2.12. Let C be a chain complex $\mathbb{B}(\mathbb{A})$. The *suspension* maps

$$S : W_{\%}(C) \rightarrow \Sigma^{-1}(W_{\%}(\Sigma C)), \quad S : \widehat{W}_{\%}(C) \rightarrow \Sigma^{-1}(\widehat{W}_{\%}(\Sigma C)),$$

are defined by

$$(S(\varphi))_k := \varphi_{k-1}, \quad (S(\theta))_k := \theta_{k-1}.$$

Proposition 2.13. *The hyperquadratic Q -groups are the stabilization of the symmetric Q -groups:*

$$\widehat{Q}^n(C) = \operatorname{colim}_{k \rightarrow \infty} Q^{n+k}(\Sigma^k C).$$

Moreover, the suspension induces an isomorphism on hyperquadratic Q -groups:

$$S : \widehat{Q}^n(C) \xrightarrow{\cong} \widehat{Q}^{n+1}(\Sigma C).$$

The proposition is proved in [16, Sec. 1]. It follows that a symmetric structure has a quadratic refinement if and only if its suspension S^k is zero in $Q^{n+k}(\Sigma^k C)$ for some k . This can be improved in a sense that a preferred quadratic refinement can be chosen if a preferred path of the suspension S^k to 0 is chosen in $\Sigma^{-k}W_{\%}(\Sigma^k C)$.

Remark 2.14. There exists a direct sum operation on the structured chain complexes [16, Sec. 1]. We remark that the quadratic and symmetric Q -groups do not respect this operation, but the hyperquadratic Q -groups do. In fact the assignments $C \mapsto \widehat{Q}^n(C)$ constitute a generalized cohomology theory on the category of chain complexes in $\mathbb{A}$, see [24, Thm. 1.1].

Now we proceed to explain how the above structures arise from geometric examples.

Construction 2.15. [17, Prop. 1.1,1.2] Let X be a topological space with the singular chain complex $C(X)$. The Alexander–Whitney diagonal approximation gives a chain map

$$\varphi : C(X) \rightarrow W_{\%}(C(X)),$$

called the *symmetric construction on X* , such that for every n -dimensional cycle $[X] \in C(X)$, the component $\varphi([X])_0 : C^{n-*}(X) \rightarrow C(X)$ is the cap product with the cycle $[X]$.

There exists an equivariant version as follows. Let $\tilde{X}$ be the universal cover of X . The singular chain complex $C(\tilde{X})$ is a chain complex over $\mathbb{Z}[\pi_1(X)]$. The symmetric construction $\varphi_{\tilde{X}}$ on $\tilde{X}$ produces a chain map of $\mathbb{Z}[\pi_1(X)]$ -modules. Applying $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]}$ we obtain a chain map of chain complexes of abelian groups

$$\varphi : C(X) \rightarrow W^{\%}(C(\tilde{X})) = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{X})),$$

also called the *symmetric construction of X* , such that for every n -dimensional cycle $[X] \in C(X)$, the component $\varphi([X])_0 : C^{n-*}(\tilde{X}) \rightarrow C(\tilde{X})$ is the cap product with the cycle $[X]$, but now we obtain a map of $\mathbb{Z}[\pi_1(X)]$ -module chain complexes. There is also a version of it for pointed spaces where one works with reduced chain complexes $\tilde{C}(\tilde{X})$.

If X is an n -dimensional geometric Poincaré complex with the fundamental class $[X]$, then $\varphi([X])_0$ is the Poincaré duality chain equivalence. In this case we obtain an n -dimensional SAPC over $\mathbb{Z}[\pi_1(X)]$,

$$(C(\tilde{X}), \varphi([X])).$$

The symmetric construction is functorial with respect to maps of topological spaces and natural with respect to the suspension of chain complexes, as shown in [17, Prop. 1.1,1.2]. However, a chain map $C(X) \rightarrow C(Y)$ that is not induced by a map of spaces might not commute with the symmetric constructions of X and Y . This is one motivation for the quadratic construction below.

Construction 2.16. [17, Prop. 1.5] Let X, Y be pointed spaces and let $F : \Sigma^k X \rightarrow \Sigma^k Y$ be a map. Denote

$$f : C(X) \xrightarrow{\Sigma} \Sigma^{-k} C(\Sigma^k X) \xrightarrow{F} \Sigma^{-k} C(\Sigma^k Y) \xrightarrow{\Sigma^{-1}} C(Y),$$

where Σ^{-1} is some homotopy inverse of Σ from Notation 2.7. The following diagram does not necessarily commute, since f does not come from a geometric map

$$\begin{array}{ccc} C(X) & \xrightarrow{\varphi} & W^{\%}(C(X)) \\ \downarrow f_* & & \downarrow f^{\%} \\ C(Y) & \xrightarrow{\varphi} & W^{\%}(C(Y)). \end{array}$$

There is a chain map, called the *quadratic construction on F* ,

$$\psi : C(X) \rightarrow W_{\%}(C(Y)) \text{ such that } (1 + T) \cdot \psi \equiv f^{\%} \varphi - \varphi f_*.$$

To show that such a map exists we look at the difference $f^{\%}\varphi - \varphi f_*$, and use Proposition 2.10 to obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & H_n(X) & & & & \\
 & \swarrow \Psi_F & \downarrow f^{\%}\varphi - \varphi f_* & \searrow \equiv 0 & & & \\
 \dots & \longrightarrow & Q_n(C(Y)) & \longrightarrow & Q^n(C(Y)) & \longrightarrow & \hat{Q}^n(C(Y)) \longrightarrow \dots
 \end{array}$$

The map $H_n(X) \rightarrow \hat{Q}^n(C(Y))$ is the stabilization of the map $f^{\%}\varphi - \varphi f_*$. But when we stabilize f we recover the map $F: C(\Sigma^k X) \rightarrow C(\Sigma^k Y)$, up to a preferred chain homotopy. This map comes from a geometric map, and so, by the naturality of the symmetric construction, the map $H_n(X) \rightarrow \hat{Q}^n(C(Y))$ is zero.

Then exactness tells us there is a lift. However, we are allowed to look on the chain level, and we observe that there is a preferred null-homotopy of the difference $S^k(f^{\%}\varphi - \varphi f_*) \simeq F^{\%}\varphi - \varphi F_*$ in the chain complex $\Sigma^{-k}W^{\%}(C(\Sigma^k Y))$. By the remark following Proposition 2.13 we obtain a preferred lift. This describes the map ψ , the full details can be found in [17].

Similarly as in the symmetric construction there is an equivariant version, also called *quadratic construction on F* ,

$$\psi: C(X) \rightarrow W_{\%}(C(\tilde{Y})) \text{ such that } (1 + T) \cdot \psi \equiv f^{\%}\varphi - \varphi f_*.$$

Construction 2.17. Let M, X be geometric Poincaré complexes with a degree one normal map $(f, b): M \rightarrow X$. Using $\pi_1(X)$ -equivariant S -duality (see Section 3) we obtain a stable equivariant map $F: \Sigma^k \tilde{X}_+ \rightarrow \Sigma^k \tilde{M}_+$ for some $k \in \mathbb{N}$. Consider the Umkehr map

$$f^!: C(\tilde{X}) \rightarrow \Sigma^{-k}C(\Sigma^k \tilde{X}_+) \xrightarrow{F} \Sigma^{-k}C(\Sigma^k \tilde{M}_+) \rightarrow C(\tilde{M})$$

and its mapping cone $\mathcal{C}(f^!)$ with the inclusion map $e: C(\tilde{M}) \rightarrow \mathcal{C}(f^!)$. We obtain an n -dimensional QAPC over $\mathbb{Z}[\pi_1(X)]$,

$$(\mathcal{C}(f^!), e_{\%}\psi([X])).$$

An example of a hyperquadratic structure on a chain complex coming from geometry is relegated to Section 3. Now we present the relative versions of the above concepts.

Definition 2.18. An $(n+1)$ -dimensional *symmetric algebraic pair* over $\mathbb{A}$ is a chain map $f: C \rightarrow D$ in $\mathbb{B}(\mathbb{A})$ together with an $(n+1)$ -dimensional cycle $(\delta\varphi, \varphi) \in \mathcal{C}(f^{\%})$. An $(n+1)$ -dimensional *quadratic algebraic pair* over $\mathbb{A}$ is a chain map $f: C \rightarrow D$ in $\mathbb{B}(\mathbb{A})$ together with an $(n+1)$ -dimensional cycle $(\delta\psi, \psi) \in \mathcal{C}(f_{\%})$.

Notice that an $(n+1)$ -dimensional symmetric pair contains an n -dimensional symmetric complex (C, φ) , and similarly an $(n+1)$ -dimensional quadratic pair contains an n -dimensional quadratic complex (C, ψ) . The condition that $(\delta\varphi, \varphi)$ is a cycle in $\mathcal{C}(f^{\%})_{n+1}$ translates to a relation between $\delta\varphi \in W^{\%}(D)_{n+1}$

and $\varphi \in W^\%(C)_n$ via the equation $d(\delta\varphi) = (-1)^n f^\%(\varphi)$. It is also helpful to define the evaluation map

$$\text{ev} : \mathcal{C}(f^\%) \rightarrow \text{Hom}_{\mathbb{A}}(D^{n+1-*}, \mathcal{C}(f)), \quad \text{ev}(\delta\varphi, \varphi) = \begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n+1-*} \rightarrow \mathcal{C}(f)$$

and likewise in the quadratic case.

Definition 2.19. An $(n+1)$ -dimensional *symmetric algebraic Poincaré pair* (SAPP) in $\mathbb{A}$ is a symmetric pair $(f : C \rightarrow D, (\delta\varphi, \varphi))$ such that

$$\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n+1-*} \rightarrow \mathcal{C}(f)$$

is a chain equivalence.

An $(n+1)$ -dimensional *quadratic algebraic Poincaré pair* (QAPP) in $\mathbb{A}$ is a quadratic pair $(f : C \rightarrow D, (\delta\psi, \psi))$ such that

$$(1+T) \cdot \begin{pmatrix} \delta\psi_0 \\ \psi_0 f^* \end{pmatrix} : D^{n+1-*} \rightarrow \mathcal{C}(f)$$

is a chain equivalence.

Construction 2.20. Let (X, Y) be a pair of topological spaces, and denote the inclusion $i : Y \rightarrow X$. By the naturality of the symmetric construction we obtain a chain map

$$\varphi : C(X, Y) \rightarrow \mathcal{C}(i^\%)$$

which is called the *relative symmetric construction*.

If (X, Y) is an $(n+1)$ -dimensional Poincaré pair with the fundamental class $[X] \in C_{n+1}(X, Y)$ then the evaluation

$$\text{ev} \circ \varphi([X]) : C^{n+1-*}(X) \rightarrow C(X, Y)$$

is a chain homotopy equivalence. There also exists an equivariant version.

Construction 2.21. Let (X, A) and (Y, B) be pairs of pointed topological spaces and let

$$\begin{array}{ccc} \Sigma^k A & \xrightarrow{\partial F} & \Sigma^k B \\ \downarrow i & & \downarrow j \\ \Sigma^k X & \xrightarrow{F} & \Sigma^k Y \end{array}$$

be a commutative diagram. Let ∂f and f be maps defined analogous to the map f in Construction 2.16. There is a chain map, the *relative quadratic construction*,

$$\psi : C(X, A) \rightarrow \mathcal{C}(j^\%)$$

such that $(1+T) \cdot \psi = (f, \partial f)^\% \varphi - \varphi(f, \partial f)_*$. Again, there is also an equivariant version.

Construction 2.22. Let $((f, b), \partial(f, b)) : (M, N) \rightarrow (X, Y)$ be a degree one normal map of manifolds with boundary. Here we do not assume that the restriction of ∂f on the boundary N is a homotopy equivalence. The S -duality yields in this case the commutative diagrams

$$\begin{array}{ccc} T(\nu_M)/T(\nu_N) & \longrightarrow & T(\nu_X)/T(\nu_Y) \\ \downarrow & & \downarrow \\ \Sigma T(\nu_N) & \longrightarrow & \Sigma T(\nu_Y) \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \Sigma^k Y_+ & \xrightarrow{\partial F} & \Sigma^k N_+ \\ \downarrow i & & \downarrow j \\ \Sigma^k X_+ & \xrightarrow{F} & \Sigma^k M_+. \end{array}$$

We have two Umkehr maps $\partial f^!$ and $f^!$ and a commutative square

$$\begin{array}{ccc} C(N) & \xrightarrow{\partial e} & \mathcal{C}(\partial f^!) \\ \downarrow j & & \downarrow k \\ C(M) & \xrightarrow{e} & \mathcal{C}(f^!). \end{array}$$

We obtain an $(n+1)$ -dimensional QAPP,

$$(k : \mathcal{C}(\partial f^!) \rightarrow \mathcal{C}(f^!), (e, \partial e) \% \psi([X])).$$

The notion of a pair allows us to define the notion of a cobordism of structured chain complexes.

Definition 2.23. A *cobordism* of n -dimensional SAPCs $(C, \varphi), (C', \varphi')$ in $\mathbb{A}$ is an $(n+1)$ -dimensional SAPP in $\mathbb{A}$

$$((f f') : C \oplus C' \rightarrow E, (\delta \varphi, \varphi \oplus -\varphi')).$$

Analogously, a *cobordism* of n -dimensional QAPCs $(C, \psi), (C', \psi')$ in $\mathbb{A}$ is an $(n+1)$ -dimensional QAPP in $\mathbb{A}$

$$((f f') : C \oplus C' \rightarrow E, (\delta \psi, \psi \oplus -\psi')).$$

There is a notion of a *union* of two adjoining cobordisms in $\mathbb{A}$ is defined in [16, Sec. 3]. Using it one obtains transitivity for the cobordisms and hence an equivalence relation.

Geometrically, one obtains a symmetric cobordism from a geometric Poincaré triad and a quadratic cobordism from a degree one normal map of geometric Poincaré triads.

Recall the well-known fact that using Morse theory any geometric cobordism can be decomposed into elementary cobordisms which are in turn obtained via surgery. Although it has slightly different properties there exists an analogous notion of algebraic surgery which we now recall. For simplicity we will only discuss it in the symmetric case, although there is an analogous notion for quadratic complexes.

Construction 2.24. [19, Def. 1.12] Let (C, φ) be an n -dimensional SAC. The *data* for an algebraic surgery on (C, φ) is an $(n+1)$ -dimensional symmetric

pair $(f: C \rightarrow D, (\delta\varphi, \varphi))$. The *effect* of the algebraic surgery on (C, φ) using $(f: C \rightarrow D, (\delta\varphi, \varphi))$ is the n -dimensional SAC (C', φ') defined by

$$C' = \Sigma^{-1}\mathcal{C}\left(\begin{smallmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{smallmatrix}\right), \quad \varphi' = \Sigma^{-1}(e')^{\%}(\delta\varphi/\varphi).$$

Here the map e' is defined by the diagram

$$\begin{array}{ccccc} C & \xrightarrow{f} & D & & \\ & & \searrow & & \nearrow \Sigma C' \\ & & \mathcal{C}(f) & \xrightarrow{e'} & \\ C' & \longrightarrow & D^{n+1-*} & \xrightarrow{\left(\begin{smallmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{smallmatrix}\right)} & \end{array}$$

The symmetric structure on the pair $f: C \rightarrow D$ defines a symmetric structure $\delta\varphi/\varphi$ on $\mathcal{C}(f)$ by the formula as in [19, Prop. 1.15]. It is pushed forward by e' to an $(n+1)$ -cycle of $W^{\%}(\Sigma C')$ which turns out to have a preferred desuspension and so we obtain an n -cycle φ' .

A geometric analogue is obtained from a cobordism W between closed manifolds M and M' . Then we have a diagram

$$\begin{array}{ccc} C(M) & \longrightarrow & C(W, M') \\ & & \searrow \\ & & C(W, M \cup M') \\ & \nearrow & \\ C(M') & \longrightarrow & C(W, M) \end{array}$$

where the chain complexes $C(W, M')$ and $C(W, M)$ are Poincaré dual.

Definition 2.25. Let (C, φ) be an n -dimensional SAC. The *boundary* of (C, φ) is the $(n-1)$ -dimensional SAC $\partial(C, \varphi) = (\partial C, \partial\varphi)$ obtained from surgery on the symmetric pair $(0 \rightarrow C, (\varphi, 0))$. Explicitly we have $\partial C = \Sigma^{-1}\mathcal{C}(\varphi_0)$ and $\partial\varphi = S^{-1}e^{\%}(\varphi)$, where $e: C \rightarrow \mathcal{C}(\varphi_0)$.

Here the geometric analogue arises from considering an n -dimensional manifold with boundary, say $(N, \partial N)$. Consider the chain complex $C(N, \partial N)$ and its suspended dual $C^{n-*}(N, \partial N)$. There is a symmetric structure on $C(N, \partial N)$, which is not necessarily Poincaré. However, there is the Poincaré duality $C^{n-*}(N, \partial N) \simeq C(N)$. Thus the mapping cone of the duality map $C^{n-*}(N, \partial N) \rightarrow C(N, \partial N)$ becomes homotopy equivalent to the mapping cone of the map $C(N) \rightarrow C(N, \partial N)$ which is $\Sigma C(\partial N)$.

Remark 2.26. Notice that an n -dimensional SAC is Poincaré if and only if its boundary is contractible.

We also have the following proposition which is proven in [16] by writing out the formulas.

Proposition 2.27. [16, Prop. 4.1] *Algebraic surgery preserves the homotopy type of the boundary of (C, φ) . In particular we have that*

$$(C, \varphi) \text{ is Poincaré} \Leftrightarrow (C', \varphi') \text{ is Poincaré}.$$

An algebraic surgery on (C, φ) using $(f : C \rightarrow D, (\delta\varphi, \varphi))$ gives rise to a symmetric pair $(f, f' : C \oplus C' \rightarrow D', (\delta\varphi', \varphi \oplus \varphi))$ with $D' = \mathcal{C}(\varphi_0 f^*)$. If (C, φ) is Poincaré then, as noted above, (C', φ') is also Poincaré, and in addition the pair is a cobordism. We remark that the data for algebraic surgery might not be a Poincaré pair, in fact this is a typical case, since if it is a Poincaré pair, then it already defines a null-cobordism of (C, φ) and hence C' is contractible.

The relationship between the algebraic cobordism and algebraic surgery turns out to be as follows:

Proposition 2.28. [16, Prop. 4.1] *The equivalence relation generated by surgery and homotopy equivalence is the same as the equivalence relation given by cobordism.*

Definition 2.29. [16, Prop. 3.2]

The *symmetric L -groups* of an additive category with chain duality $\mathbb{A}$ are

$$L^n(\mathbb{A}) := \{\text{cobordism classes of } n\text{-dimensional SAPCs in } \mathbb{A}\}.$$

The *quadratic L -groups* of an additive category with chain duality $\mathbb{A}$ are

$$L_n(\mathbb{A}) := \{\text{cobordism classes of } n\text{-dimensional QAPCs in } \mathbb{A}\}.$$

The group operation is the direct sum of the structured chain complexes in both cases. The inverse of a SAPC (C, φ) is given by $(C, -\varphi)$, and the inverse of a QAPC (C, ψ) is given by $(C, -\psi)$.

Remark 2.30. It is proven in [16, Sec. 5,6,7] for $\mathbb{A} = \mathbb{A}(R)$, where R is a ring with involution, that the groups $L_n(\mathbb{A}(R))$ are isomorphic to the surgery obstruction groups $L_n(R)$ of Wall. Both symmetric and quadratic groups $L^n(\mathbb{A})$ and $L_n(\mathbb{A})$ are 4-periodic for any $\mathbb{A}$ [19, Prop. 1.10].

Definition 2.31. Let X be an n -dimensional Poincaré complex. The cobordism class of the n -dimensional SAPC obtained from any choice of the fundamental class $[X] \in C_n(X)$ in Construction 2.15 does not depend on the choice of $[X]$ and hence defines an element

$$\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet}(X) = [(C(\tilde{X}), \varphi([X]))] \in L^n(\mathbb{Z}[\pi_1(X)]).$$

called the *symmetric signature* of X .¹¹ If X is an oriented n -dimensional topological manifold, then the symmetric signature only depends on the oriented cobordism class of X , and so it provides us with a homomorphism¹²

$$\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet} : \Omega_n^{\text{STOP}}(K(\pi_1(X), 1)) \rightarrow L^n(\mathbb{Z}[\pi_1(X)]).$$

¹¹The notation is somewhat premature, the symbol $\mathbf{L}^\bullet$ denotes the symmetric L -spectrum and will be defined later in Section 7 Likewise in the quadratic case.

¹²If X is not a manifold we can still say that the symmetric signature only depends on the oriented cobordism class of X in the Poincaré cobordism group Ω_n^P , but we will not need this point of view later.

Definition 2.32. Let $(f, b) : M \rightarrow X$ be a degree one normal map of Poincaré complexes. The cobordism class of the n -dimensional QAPC obtained from any choice of the fundamental class $[X] \in C_n(X)$ in Construction 2.17 does not depend on the choice of $[X]$ and hence defines an element

$$\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet}(f, b) = [(C(f^!), e_{\%}\psi([X]))] \in L_n(\mathbb{Z}[\pi_1(X)])$$

called the *quadratic signature* of the degree one normal map (f, b) . If M is an n -dimensional oriented manifold then the quadratic signature only depends on the normal cobordism class of (f, b) in the set of normal invariants $\mathcal{N}(X)$ and provides us with a function¹³

$$\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet} : \mathcal{N}(X) \rightarrow L_n(\mathbb{Z}[\pi_1(X)]).$$

Remark 2.33. [17, Prop. 2.2] The symmetrization map (2.2) carries over to the L -groups as

$$(1 + T) : L_n(\mathbb{A}) \rightarrow L^n(\mathbb{A})$$

and for $\pi = \pi_1(X)$ we have

$$(1 + T) \text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}^\bullet}(f, b) = \text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}^\bullet}(M) - \text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}^\bullet}(X).$$

Remark 2.34. If M is a closed n -dimensional topological manifold then the quadratic signature from Definition 2.32 coincides with the classical surgery obstruction by the result of [17, Prop. 7.1].

Remark 2.35. Notice that we did not define a hyperquadratic version of the L -groups. In fact, hyperquadratic structures are useful when we have a fixed chain complex C and we study the relationship between the symmetric and quadratic structures on C via the sequence in Proposition 2.10. When comparing the symmetric and quadratic L -groups, hence cobordism groups of complexes equipped with a symmetric and quadratic structure a new concept of an algebraic normal complex is needed. It is discussed in the next section.

3. NORMAL COMPLEXES

A geometric normal complex is a notion generalizing a geometric Poincaré complex. It is motivated by the observation that although a Poincaré complex is not necessarily locally Poincaré, it is locally normal. On the other hand a manifold is also locally Poincaré. Hence the question whether a Poincaré complex can be modified within the homotopy type so that the locally normal structure becomes Poincaré is central to our main problem.

In this section we will recall the definition of an algebraic normal complex. In addition we recall that cobordism groups of algebraic normal complexes, the so-called NL -groups, which measure the difference between the symmetric and quadratic L -groups. Another viewpoint on that same fact is that the quadratic L -groups measure the difference between the symmetric L -groups

¹³Recall that for X an n -dimensional GPC the set of normal invariants if it is nonempty is a group with respect to the group structure given by the Whitney sum. The quadratic signature is NOT a homomorphism with respect to this group structure.

and the NL -groups. This will be crucially used in the proof of the Main Technical Theorem.

The material from this section comes from [19, Sec. 2], [24, 25] and [18, Sec. 7.3 and 7.4].

Definition 3.1. An n -dimensional *geometric normal complex* (GNC) is a triple (X, ν, ρ) consisting of a space X with a k -dimensional oriented spherical fibration ν and a map $\rho: S^{n+k} \rightarrow \text{Th}(\nu)$ to the Thom space of ν .

The *fundamental class* of (X, ν, ρ) is the n -dimensional homology class in $H_n(X)$ represented by the cycle $[X] := u(\nu) \cap h(\rho) \in C_n(X)$ where h is the Hurewicz homomorphism, and $u(\nu) \in C^k(\text{Th}(\nu))$ is some choice of the Thom class of ν .

Note that the dimension of a GNC is the dimension of the source sphere of the map ρ minus the dimension of the spherical fibration. It does not necessarily have anything to do with a geometric dimension of X . Also the cap product with the fundamental class does not necessarily induce an isomorphism between cohomology and homology of X .

Example 3.2. Let X be an n -dimensional geometric Poincaré complex (GPC). Let $[X]$ be the fundamental class given by Poincaré duality. Then, for k large enough, the space X has the Spivak normal fibration (SNF) $\nu_X: X \rightarrow \text{BSG}(k)$, which has the property that there is a map $\rho_X: S^{n+k} \rightarrow \text{Th}(\nu_X)$ such that

$$[X] = u(\nu_X) \cap h(\rho_X) \in H_n(X).$$

Thus we get an n -dimensional geometric normal complex (X, ν_X, ρ_X) with the fundamental class equal to the fundamental class in the sense of Poincaré duality.

Some properties of normal complexes can be stated in terms of the S -duality from stable homotopy theory. For pointed spaces X, Y the symbol $[X, Y]$ denotes the abelian group of stable homotopy classes of stable pointed maps from X to Y . Here, for simplicity, we confine ourselves to a nonequivariant S -duality. An equivariant version, which is indeed needed for our purposes is presented in detail in [16, Sec. 3].

Definition 3.3. Let X, Y be pointed spaces. A map $\alpha: S^N \rightarrow X \wedge Y$ is an N -dimensional S -duality map if the slant product maps

$$\alpha_*([S^N]) \backslash_-: \tilde{C}(X)^{N-*} \rightarrow \tilde{C}(Y) \text{ and } \alpha_*([S^N]) \backslash_-: \tilde{C}(Y)^{N-*} \rightarrow \tilde{C}(X)$$

are chain equivalences. We say the spaces X, Y are S -dual.

Example 3.4. Let X be an n -dimensional GPC with the k -dimensional SNF $\nu_X: X \rightarrow \text{BSG}(k)$. Then $\text{Th}(\nu_X)$ is an $(n+k)$ -dimensional S -dual to X_+ .

Proposition 3.5. *The S -duality satisfies:*

- (1) *For every finite CW-complex X there exists an N -dimensional S -dual, which we denote X^* , for some large $N \geq 1$.*

- (2) If X^* is an N -dimensional S -dual of X then its reduced suspension ΣX^* is an $(N+1)$ -dimensional S -dual of X .
- (3) For any space Z we have isomorphisms

$$S : [X, Z] \cong [S^N, Z \wedge Y] \quad \gamma \mapsto S(\gamma) = (\gamma \wedge \text{id}_Y) \circ \alpha,$$

$$S : [Y, Z] \cong [S^N, X \wedge Z] \quad \gamma \mapsto S(\gamma) = (\text{id}_X \wedge \gamma) \circ \alpha.$$

- (4) A map $f : X \rightarrow Y$ induces a map $f^* : Y^* \rightarrow X^*$ for N large enough via the isomorphism

$$[X, Y] \cong [S^N, Y \wedge X^*] \cong [Y^*, X^*].$$

- (5) If $X \rightarrow Y \rightarrow Z$ is a cofibration sequence then $Z^* \rightarrow Y^* \rightarrow X^*$ is a cofibration sequence for N large enough.

The S -dual is also unique in some sense. In fact the assignment $X \mapsto X^*$ can be made into a functor in an appropriate stable homotopy category. As this requires a certain amount of technicalities and we do not really need it, we skip this aspect. The reader can find the details for example in [1].

Now we present a generalization of Example 3.4.

Construction 3.6. Let (X, ν, ρ) be an n -dimensional GNC. Let V be the mapping cylinder of the projection map of ν with ∂V being the total space of the spherical fibration ν . Then we have the generalized diagonal map

$$\tilde{\Delta} : \text{Th}(\nu) \simeq \frac{V}{\partial V} \xrightarrow{\Delta} \frac{V \times V}{V \times \partial V} \simeq \text{Th}(\nu) \wedge X_+$$

where Δ is the actual diagonal map. Consider the composite

$$S^{n+k} \xrightarrow{\rho} \text{Th}(\nu) \xrightarrow{\tilde{\Delta}} \text{Th}(\nu) \wedge X_+.$$

By Proposition 3.5 part (1) we have an S -duality map $S^N \rightarrow \text{Th}(\nu) \wedge \text{Th}(\nu)^*$ for N large enough. Setting $p = N - (n+k)$ we obtain from part (3) the one-to-one correspondence:

$$S^{-1} : [S^{n+k}, \text{Th}(\nu) \wedge X_+] \cong [\text{Th}(\nu)^*, \Sigma^p X_+]$$

$$\tilde{\Delta} \circ \rho \mapsto \Gamma_X := S^{-1}(\tilde{\Delta} \circ \rho).$$

Moreover, we obtain the following homotopy commutative diagram in which γ_X is the chain map induced by Γ_X :

$$(3.1) \quad \begin{array}{ccc} C(X)^{n-*} & \xrightarrow{-\cup u(\nu)} & \tilde{C}(\text{Th}(\nu))^{n+k-*} \xrightarrow{S\text{-dual}} C(\text{Th}(\nu)^*)_{p+*} \\ -\cap [X] \downarrow & & \downarrow \gamma_X \\ C(X) = \tilde{C}(X_+) & \xrightarrow{\Sigma^p} & \tilde{C}(\Sigma^p X_+)_{p+*}. \end{array}$$

If X is Poincaré then p can be chosen to be 0 and the maps Γ_X and γ_X the identity. Hence the Poincaré duality is seen as the composition of the Thom isomorphism for the SNF and the S -duality.

Now we turn to algebraic normal complexes. As a first step we discuss the following notion which is an algebraic analogue of a spherical fibration.

Definition 3.7. Let C be a chain complex over an additive category with chain duality $\mathbb{A}$. A *chain bundle* over C is a 0-dimensional cycle γ in $\widehat{W}^{\%}(TC)$.

Construction 3.8. Let X be a finite CW-complex and let $\nu : X \rightarrow \text{BSG}(k)$ be a k -dimensional spherical fibration over X . The Thom space $\text{Th}(\nu)$ is also a finite CW-complex and hence has an N -dimensional S-dual $\text{Th}(\nu)^*$ for some N . The *hyperquadratic construction* is the chain map given by the following composition:

$$\begin{aligned} \gamma_{\nu} : \tilde{C}^k(\text{Th}(\nu)) &\xrightarrow{\text{S-duality}} \tilde{C}_{N-k}(\text{Th}(\nu)^*) \\ &\xrightarrow{\varphi_{\text{Th}(\nu)^*}} W^{\%}(\tilde{C}(\text{Th}(\nu)^*))_{N-k} \\ &\xrightarrow{\text{S-duality}} W^{\%}(\tilde{C}(\text{Th}(\nu))^{N-*})_{N-k} \\ &\xrightarrow{\text{Thom}} W^{\%}(C(X)^{N-k-*})_{N-k} \\ &\xrightarrow{J} \widehat{W}^{\%}(C(X)^{N-k-*})_{N-k} \\ &\xrightarrow{S^{-(N-k)}} \widehat{W}^{\%}(C(X)^{-*})_0. \end{aligned}$$

Given a choice of the Thom class $u(\nu) \in \tilde{C}^k(\text{Th}(\nu))$, the cycle $\gamma_{\nu}(u(\nu))$ becomes a chain bundle over $C(X)$. An equivariant version produces a chain bundle over $\mathbb{Z}[\pi_1(X)]$,

$$(C(\tilde{X}), \gamma_{\nu}(u(\nu))).$$

Now we can define an algebraic analogue of a geometric normal complex.

Definition 3.9. An n -dimensional normal algebraic complex (NAC) in $\mathbb{A}$ is a pair (C, θ) where θ is a triple (φ, γ, χ) such that

- (C, φ) is an n -dimensional SAC,
- $\gamma \in (\widehat{W}^{\%}(TC))_0$ is a chain bundle over C ,
- $\chi \in (\widehat{W}^{\%}(C))_{n+1}$ satisfies $d\chi = J(\varphi) - (\widehat{\varphi}_0)^{\%}(S^n\gamma)$.

As we indicate below in the geometric example the third condition is a consequence of the homotopy commutativity of the diagram (3.1) and as such can be seen as a generalization of the equation in Example 3.2. Notice that there is no requirement on φ_0 being a chain equivalence, that means normal complexes are in no sense Poincaré. Now we indicate the *normal construction* which to an n -dimensional GNC functorially associates an n -dimensional NAC. The full details are somewhat complicated, the reader can find them in [24, 25].

Construction 3.10. Let (X, ν, ρ) be an n -dimensional GNC with a choice of the Thom class $u(\nu) \in \tilde{C}^k(\text{Th}(\nu))$ whose associated fundamental class is denoted $[X]$. We would like to associate to it an n -dimensional NAC over $\mathbb{Z}[\pi_1 X]$. We start with

- $C = C(\tilde{X})$,
- $\varphi = \varphi([X])$,

- $\gamma = \gamma_\nu(u(\nu))$.

Now we will only show that an element χ with required properties exists. In other words we show that $J(\varphi) = (\widehat{\varphi}_0)^\% (S^n \gamma)$ in $\widehat{Q}^n(C(X))$. Consider in our case the symmetric construction, the hyperquadratic construction and the diagram (3.1). We obtain the following commutative diagram:

$$\begin{array}{ccccc}
 & & H^k(\mathrm{Th}(\nu)) & & \\
 & \swarrow -\cap h(\rho) & \downarrow & \searrow S\text{-dual} & \\
 H_n(X) & \xrightarrow{\Sigma^p} & H_{n+p}(\Sigma^p X) & \xleftarrow{\gamma_X} & H_{n+p}(\mathrm{Th}(\nu)^*) \\
 \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
 Q^n(C(X)) & \xrightarrow{\Sigma^p} & Q^{n+p}(\tilde{C}(\Sigma_+^p X)) & \xleftarrow{\gamma_X^\%} & Q^{n+p}(\tilde{C}(\mathrm{Th}(\nu)^*)) \\
 \downarrow J & & \downarrow J & & \downarrow J \\
 \widehat{Q}^n(C(X)) & \xrightarrow[\Sigma^p]{\cong} & \widehat{Q}^{n+p}(\tilde{C}(\Sigma_+^p X)) & \xleftarrow[\widehat{\gamma}_X^\%]{\gamma_X^\%} & \widehat{Q}^{n+p}(\tilde{C}(\mathrm{Th}(\nu)^*)) \\
 \uparrow \widehat{\varphi}_0^\% & & \uparrow \widehat{S^p \varphi}_0^\% & & \downarrow S\text{-dual} \\
 \widehat{Q}^n(C^{n-*}(X)) & \xrightarrow[\Sigma^p]{\cong} & \widehat{Q}^{n+p}(C^{n+p-*}(X)) & \xleftarrow{\mathrm{Thom}} & \widehat{Q}^{n+p}(\tilde{C}^{N-*}(\mathrm{Th}(\nu))).
 \end{array}$$

The commutativity of the upper left part follows from the basic properties of the symmetric construction. The commutativity of the lower left part follows from the existence of the map Γ_X and naturality of the symmetric construction. The commutativity of the right part follows from the commutativity of the diagram (3.1).

As mentioned above, the construction can be made sufficiently functorial, that means there is a preferred choice of χ . We obtain an n -dimensional NAC over $\mathbb{Z}[\pi_1(X)]$,

$$(C(\tilde{X}), \theta(u(\nu))).$$

As in the previous section, we also need to discuss the relative versions.

Definition 3.11. An $(n+1)$ -dimensional *geometric normal pair* $((X, Y), \nu, \rho)$ consists of a pair of spaces (X, Y) together with a k -dimensional spherical fibration $\nu : X \rightarrow \mathrm{BSG}(k)$ and a map $\rho : (D^{n+1+k}, S^{n+k}) \rightarrow (\mathrm{Th}(\nu), \mathrm{Th}(\nu|_Y))$. We abbreviate “geometric normal pair” to “GNP”.

The *fundamental class* of a GNP $((X, Y), \nu, \rho)$ is the $(n+1)$ -dimensional homology class represented by the cycle $[X, Y] \in C_n(X, Y)$ given by the formula $[X, Y] := u(\nu) \cap h(\rho)$ where h is the Hurewicz homomorphism, and $u(\nu) \in \tilde{C}^k(\mathrm{Th}(\nu))$ is some choice of the Thom class of ν .

A *geometric normal cobordism* between two n -dimensional GNCs (X, ν, ρ) and (X', ν', ρ') is an $(n+1)$ -dimensional normal pair $((Z, X \sqcup X'), \nu'', \rho'')$ which restricts accordingly over X and X' .

The *normal cobordism group* $\Omega_n^N(K)$ is defined as the abelian group of normal cobordism classes of n -dimensional GNCs with a reference map $r : X \rightarrow K$ and with the group operation given by the disjoint union operation.

Notice that in the above setting, the triple $(Y, \nu|_Y, \rho|_{S^{n+k}})$ is an n -dimensional GNC. The relative algebraic analogues come next.

Definition 3.12. A map of chain bundles $(f, b) : (C, \gamma) \rightarrow (C', \gamma')$ in $\mathbb{A}$ is a map $f : C \rightarrow C'$ of chain complexes in $\mathbb{B}(\mathbb{A})$ together with a chain $b \in \widehat{W}^{\%}(TC)_1$ such that

$$d(b) = \widehat{f}^{\%}(\gamma') - \gamma \in \widehat{W}^{\%}(TC)_0.$$

Definition 3.13. An $(n+1)$ -dimensional *normal pair* $(f : C \rightarrow D, (\delta\theta, \theta))$ in $\mathbb{A}$ is an $(n+1)$ -dimensional symmetric pair $(f : C \rightarrow D, (\delta\varphi, \varphi))$ together with a map of chain bundles $(f, b) : (C, \gamma) \rightarrow (D, \delta\gamma)$ and chains $\chi \in \widehat{W}^{\%}(C)_{n+1}$ and $\delta\chi \in \widehat{W}^{\%}(D)_{n+2}$ such that

$$\begin{aligned} J(\varphi) - (\widehat{\varphi}_0)^{\%}(S^n\gamma) &= d\chi \in \widehat{W}^{\%}(C)_n, \\ J(\delta\varphi) - \widehat{\delta\varphi}_0^{\%}(S^{n+1}\delta\gamma) + \widehat{f}^{\%}(\chi - \widehat{\varphi}_0^{\%}(S^n b)) &= d(\delta\chi) \in \widehat{W}^{\%}(D)_{n+1}, \end{aligned}$$

where we abbreviate $(\delta\theta, \theta)$ for $((\delta\varphi, \delta\gamma, \delta\chi), (\varphi, \gamma, \chi))$.

Again notice that in the above setting (C, θ) is an n -dimensional NAC.

Definition 3.14. A *normal cobordism* between normal complexes (C, θ) and (C', θ') is a normal pair $((f, f') : C \oplus C' \rightarrow D, (\delta\theta, \theta \oplus -\theta'))$.

The direct sum operation is defined analogously to the direct sum for the symmetric and quadratic complexes. Also there is a notion of a union of adjoining normal cobordisms and we obtain an equivalence relation. Again notice that a cobordism of normal complexes is in no sense a Poincaré pair.

There exists a relative normal construction. Given an $(n+1)$ -dimensional geometric normal pair it produces an $(n+1)$ -dimensional algebraic normal pair in a functorial way. An $(n+1)$ -dimensional geometric normal cobordism induces an $(n+1)$ -dimensional algebraic normal cobordism in this way. These constructions are quite complicated and therefore we again refer at this place to [25, Sec. 7].

Now we are ready to define the NL -groups, alias normal L -groups.

Definition 3.15. The *normal L -groups* of an additive category with chain duality $\mathbb{A}$ are

$$NL^n(\mathbb{A}) := \{\text{normal cobordism classes of } n\text{-dimensional NACs in } \mathbb{A}\}.$$

Definition 3.16. Let (X, ν, ρ) be an n -dimensional GNC. The cobordism class of the n -dimensional NAC obtained in Construction 3.10 is independent of the choice of Thom class $u(\nu) \in \tilde{C}^k(\text{Th}(\nu))$ and hence defines an element

$$\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\text{NL}^\bullet}(X) = [(C(\tilde{X}), \theta(u(\nu)))] \in NL^n(\mathbb{Z}[\pi_1(X)])$$

called the *normal signature* of (X, ν, ρ) .

In fact the element $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{NL}^\bullet}(X)$ only depends on the normal cobordism class of (X, ν, ρ) and hence we obtain a homomorphism

$$\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{NL}^\bullet} : \Omega_n^N(K(\pi_1(X), 1) \rightarrow NL^n(\mathbb{Z}[\pi_1(X)]).$$

See also Remark 3.23 for a note on the notation.

Now we discuss the relation between the groups $L_n(\mathbb{A})$, $L^n(\mathbb{A})$ and $NL^n(\mathbb{A})$. The details can be found in [19, Sec. 2] and [25, 25]. Here we confine ourselves to the main ideas. We start with a lemma.

Lemma 3.17. [19, Prop. 2.6 (i)] *Let (C, φ) be an n -dimensional SAC. Then (C, φ) can be extended to a normal complex $(C, \varphi, \gamma, \chi)$ if and only if the boundary $(\partial C, \partial \varphi)$ has a quadratic refinement.*

Proof. Consider the following long exact sequences

$$\begin{array}{ccccccc} \dots & \longrightarrow & Q_{n-1}(\partial C) & \xrightarrow{1+T} & Q^{n-1}(\partial C) & \xrightarrow{J} & \widehat{Q}^{n-1}(\partial C) \longrightarrow \dots \\ & & & & & & \downarrow S \cong \\ \dots & \longrightarrow & \widehat{Q}^n(C^{n-*}) & \xrightarrow{\widehat{\varphi}_0^\%} & \widehat{Q}^n(C) & \xrightarrow{\widehat{e}^\%} & \widehat{Q}^n(\mathcal{C}(\varphi_0)) \longrightarrow \dots \end{array}$$

We have $\partial \varphi = S^{-1}(e^\%(\varphi)) \in Q^{n-1}(\partial C)$. A diagram chase (using a slightly larger diagram than the one above) gives the equation

$$\widehat{e}^\%(J(\varphi)) = S(J(\partial \varphi)) \in \widehat{Q}^n(\mathcal{C}(\varphi_0)).$$

It follows that $\partial \varphi$ has a preimage in $Q_{n-1}(\partial C)$, that means a quadratic refinement, if and only if $J(\varphi)$ has a preimage in $\widehat{Q}^n(C^{n-*}) \cong \widehat{Q}^0(C^{-*})$, that means a chain bundle whose suspension maps to $J(\varphi)$ via $(\widehat{\varphi}_0)^\%$, in other words there is a normal structure refining φ . $\square$

The lemma can be improved so that one obtains a one-to-one correspondence between the normal structures extending (C, φ) and quadratic refinements of $(\partial C, \partial \varphi)$, the details are to be found in [25, Sec. 4,5].

Construction 3.18. [19, Def. 2.9] The map

$$\partial : NL^n(\mathbb{A}) \rightarrow L_{n-1}(\mathbb{A}), \quad \partial(C, \varphi, \gamma, \chi) = (\partial C, \partial \psi)$$

is defined so that $\partial \psi$ is the quadratic refinement of $\partial \varphi$ described in Lemma 3.17.

Lemma 3.19. [19, Prop. 2.6 (ii)] *There is a one-to-one correspondence between the homotopy classes of n -dimensional SAPCs and the homotopy classes n -dimensional NACs such that φ_0 is a chain homotopy equivalence.*

Proof. Let (C, φ) be an n -dimensional SAPC in $\mathbb{A}$ so that $\varphi_0 : \Sigma^n TC \rightarrow C$ is a chain homotopy equivalence. One can associate a normal structure to (C, φ) as follows. The chain bundle $\gamma \in \widehat{W}^\%(TC)_0$ is the image of $\varphi \in W^\%(C)_n$ under

$$W^\%(C)_n \xrightarrow{J} \widehat{W}^\%(C)_n \xrightarrow{(\widehat{\varphi}_0^\%)^{-1}} \widehat{W}^\%(\Sigma^n TC)_n \xrightarrow{S^{-n}} \widehat{W}^\%(TC)_0$$

where $(\widehat{\varphi}_0^\%)^{-1}$ is a chain homotopy inverse to $\widehat{\varphi}_0^\%$. The chain $\chi \in \widehat{W}^\%(C)_{n+1}$ comes from the chain homotopy $(\widehat{\varphi}_0^\%) \circ (\widehat{\varphi}_0^\%)^{-1} \simeq 1$. $\square$

Construction 3.20. [19, Prop. 2.6 (ii)] The map

$$J : L^n(\mathbb{A}) \rightarrow NL^n(\mathbb{A}), \quad J(C, \varphi) = (C, \varphi, \gamma, \chi)$$

is constructed using the above Lemma 3.19.

The maps we just described in fact fit into a long exact sequence.

Proposition 3.21. [19, Def. 2.10, Prop. 2.8] [25, Ex. 6.7] *Let $\mathbb{A}$ be an additive category with chain duality. Then there is a long exact sequence*

$$\dots \longrightarrow L_n(\mathbb{A}) \xrightarrow{1+T} L^n(\mathbb{A}) \xrightarrow{J} NL^n(\mathbb{A}) \xrightarrow{\partial} L_{n-1}(\mathbb{A}) \longrightarrow \dots$$

Sketch of proof. In [18, Chap. 2] Ranicki defines the concept of a triad of structured chain complexes and shows that one can define a cobordism group of pairs of structured chain complexes, where the structure on the boundary is some refinement of the structure inherited from the pair. Such cobordism groups then fit into a corresponding long exact sequence. The whole setup is analogous to the definition of relative cobordism groups for a pair of spaces and the associated long exact sequence.

In our special case we consider the map $J : L^n(\mathbb{A}) \rightarrow NL^n(\mathbb{A})$. So the n -th relative group is the cobordism group of n -dimensional (normal, symmetric Poincaré) pairs, that means we have a normal pair $f : C \rightarrow D$ such that the symmetric structure on C is Poincaré. This together with the following lemma establish the proposition. $\square$

Lemma 3.22. [19, Prop. 2.8 (ii)] *Let $\mathbb{A}$ be an additive category with chain duality. There is a one-to-one correspondence between the cobordism classes of n -dimensional (normal, symmetric Poincaré) pairs in $\mathbb{A}$ and the cobordism classes of $(n-1)$ -dimensional QAPCs in $\mathbb{A}$.*

Sketch of proof. Let $(f : C \rightarrow D, (\delta\theta, \theta))$ be an n -dimensional (normal, symmetric Poincaré) pair in $\mathbb{A}$. In particular we have an n -dimensional symmetric pair $(f : C \rightarrow D, (\delta\varphi, \varphi))$, which we can use as data for an algebraic surgery on the $(n-1)$ -dimensional SAPC (C, φ) . The effect (C', φ') is again an $(n-1)$ -dimensional SAPC. It turns out to have a quadratic refinement, by a generalization of the proof of Lemma 3.17 (the lemma is a special case when $f : 0 \rightarrow C$). The assignment $(f : C \rightarrow D, (\delta\theta, \theta)) \mapsto (C', \partial\psi')$ turns out to induce a one-to-one correspondence on cobordism classes. $\square$

Remark 3.23. Proposition 3.21 provides us with an isomorphism between the groups $NL^n(\mathbb{A})$ and the groups $\widehat{L}^n(\mathbb{A})$ defined in [18] and used in [15].

3.24. The quadratic boundary of a GNC. In this subsection we study in more detail the passage from a GNC to the boundary of its associated NAC. This means that from an n -dimensional GNC we pass to an $(n - 1)$ -dimensional QAPC. The construction was described in [18, Sec. 7.4] even before the invention of NAC in [24, 25]. It will be useful for geometric applications in later sections.

Before we start we need more basic technology. First we describe the spectral quadratic construction:

Construction 3.25. Let $F : X \rightarrow \Sigma^p Y$ be a map between pointed spaces (a map of this shape is called a *semistable map*) inducing the chain map

$$f : \tilde{C}(X)_{p+*} \rightarrow \tilde{C}(\Sigma^p Y)_{p+*} \simeq \tilde{C}(Y).$$

The *spectral quadratic construction* on F is a chain map

$$\Psi : \tilde{C}(X)_{p+*} \rightarrow W_{\%}(\mathcal{C}(f))$$

such that

$$(1 + T) \circ \Psi \equiv e^{\%} \circ \varphi \circ f$$

where $\varphi : C(Y) \rightarrow W^{\%}(C(Y))$ is the symmetric construction on Y and $e^{\%}$ is induced by the inclusion map $e : C(Y) \rightarrow \mathcal{C}(f)$. The existence of Ψ can be read off the following commutative diagram in which the lower horizontal sequence is exact by Remark 2.14 and the right vertical sequence is exact by Proposition 2.10

$$\begin{array}{ccccccc}
 & & \tilde{H}_{n+p}(X) & & & & \\
 & \swarrow \cong & & \searrow f & \dashrightarrow \Psi & & \\
 \tilde{H}_{n+p}(X) & \xrightarrow{F} & \tilde{H}_{n+p}(\Sigma^p Y) & \xleftarrow{\cong} & \tilde{H}_n(Y) & \xrightarrow{\quad} & Q_n(\mathcal{C}(f)) \\
 \downarrow \varphi_X & & \downarrow \varphi_{\Sigma^p Y} & & \downarrow \varphi_Y & & \downarrow 1+T \\
 Q^{n+p}(\tilde{C}(X)) & \xrightarrow{F^{\%}} & Q^{n+p}(\tilde{C}(\Sigma^p Y)) & \xleftarrow{S^p} & Q^n(\tilde{C}(Y)) & \xrightarrow{e^{\%}} & Q^n(\mathcal{C}(f)) \\
 & \searrow J & & \searrow J & \downarrow J & & \downarrow J \\
 & & \hat{Q}^n(\Sigma^{-p}\tilde{C}(X)) & \xrightarrow{\hat{f}^{\%}} & \hat{Q}^n(\tilde{C}(Y)) & \xrightarrow{\hat{e}^{\%}} & \hat{Q}^n(\mathcal{C}(f)).
 \end{array}$$

The spectral quadratic construction Ψ on F has the property that if $X = \Sigma^p X_0$ for some X_0 then it coincides with the quadratic construction on F as presented in Construction 2.16 composed with $e^{\%}$.

Recall that we have already encountered the semistable map Γ_Y coming from an n -GNC (Y, ν_Y, ρ_Y) in Construction 3.6. The spectral quadratic construction on Γ_Y is identified below.

Construction 3.26. See [18, Prop. 7.4.1] and [25, Thm. 7.1].

Let $\Gamma_Y : \mathrm{Th}(\nu_Y)^* \rightarrow \Sigma^p Y_+$ be the semistable map obtained in Construction 3.6

and let $\gamma_Y : C(\mathrm{Th}(\nu_Y)^*)_{*+p} \rightarrow C(Y)$ denote the induced map. Recall diagram (3.1) in Construction 3.6 which identifies

$$\mathcal{C}(\varphi_0) \simeq \mathcal{C}(\gamma_Y)$$

via the Thom isomorphism and S -duality. The spectral quadratic construction on the map Γ_Y produces a quadratic structure

$$\Psi(u(\nu_Y)^*) \in W_{\%}\mathcal{C}(\gamma_Y)_n$$

where $u(\nu_Y)^*$ denotes the S -dual of the Thom class of ν_Y . We also have

$$(1 + T) \circ \Psi(u(\nu_Y)^*) \equiv e^{\%}(\varphi([Y])) \stackrel{\mathrm{def}}{=} S(\partial\varphi([Y])).$$

From the cofibration sequence of chain complexes (with $C' = \partial C(Y)$),

$$\Sigma W_{\%}(C') \xrightarrow{\begin{pmatrix} 1+T \\ S \end{pmatrix}} \Sigma W^{\%}(C') \oplus W_{\%}(\Sigma C') \xrightarrow{S-(1+T)} W^{\%}(\Sigma C'),$$

we see that there exists a $\psi(Y) \in (W_{\%}(\partial C(Y)))_{n-1}$, unique up to equivalence, such that $(1 + T)\psi(Y) \simeq \partial\varphi([Y])$. Hence we obtain an $(n - 1)$ -dimensional QAPC over $\mathbb{Z}$ giving an element

$$[(\partial C(Y), \psi(Y))] \in L_{n-1}(\mathbb{Z}).$$

Recall from Construction 3.10 that for any GNC (Y, ν_Y, ρ_Y) there is defined an n -dimensional NAC $\mathrm{sign}^{\mathrm{NL}\bullet}(Y)$ over $\mathbb{Z}$, which, as such, has a quadratic boundary

$$\partial \mathrm{sign}^{\mathrm{NL}\bullet}(Y) = [(C', \psi')] \in L_{n-1}(\mathbb{Z})$$

defined via Lemma 3.17. Inspecting the definitions we see that $C' \simeq \partial C(Y)$ and further inspection of commutative diagrams defining the respective quadratic structures shows that $\psi(Y)$ and ψ' are equivalent.

Example 3.27. See [19, Rem. 2.16], [18, Prop. 7.4.1] and [25, Thm. 7.1]. Recall from the sketch proof of Lemma 3.22 that there is an equivalence between cobordism classes of n -dimensional algebraic (normal, symmetric Poincaré) pairs and cobordism classes of $(n - 1)$ -dimensional QAPCs, and that, in the special case that the boundary in the pair we start with is 0 the construction giving the equivalence specializes to the construction of the quadratic boundary of a normal complex.

In Construction 3.26 it is shown how the spectral quadratic construction can be used to construct the quadratic boundary when we have a geometric normal complex as input. In this example it is shown how the equivalence of Lemma 3.22 can be realized using the spectral quadratic construction when we have a degree one normal map of Poincaré complexes as input. In that case the mapping cylinder of the map gives a normal pair, with Poincaré boundary. Furthermore, it is shown that the quadratic complex obtained in this way coincides with the surgery obstruction associated to the degree one normal map. This is crucially used in the proof of part (i) of the Main technical theorem (see proof of Theorem 13.16).

Let $(f, b) : M \rightarrow X$ be a degree one normal map of n -GPC. Denote by ν_M, ν_X the respective SNFs. We form the $(n+1)$ -dimensional geometric (normal, Poincaré) pair

$$((W, M \sqcup X), (\nu_W, \nu_{M \sqcup X}), (\rho_W, \rho_{M \sqcup X}))$$

with $W = \text{cyl}(f)$. The symbol ν_W denotes the k -spherical fibration over W induced by b and

$$(\rho_W, \rho_{M \sqcup X}) : (D^{n+1+k}, S^{n+k}) \rightarrow (\text{Th}(\nu_W), \text{Th}(\nu_M \sqcup \nu_X))$$

is the map induced by ρ_M and ρ_X . Denote $j : M \sqcup X \hookrightarrow W$, $j_M : M \hookrightarrow W$, and $j_X : X \hookrightarrow W$ the inclusions, by $\text{pr}_X : W \rightarrow X$ the projection which is also a homotopy inverse to j_X and observe that $f = \text{pr}_X \circ j_M$.

Now we describe the passage

$$(3.2) \quad \text{Lemma 3.22 : } (\text{sign}^{\mathbf{NL}^\bullet}(W), \text{sign}^{\mathbf{L}^\bullet}(M) - \text{sign}^{\mathbf{L}^\bullet}(X)) \mapsto [(C', \psi')].$$

According to the proof of Lemma 3.22 the underlying chain complex C' is obtained by algebraic surgery on the $(n+1)$ -dimensional symmetric pair

$$(j_* : C(M) \oplus C(X) \rightarrow C(W), (\delta\varphi, \varphi)).$$

This is just the desuspension of the mapping cone of the 'want to be' Poincaré duality map

$$C' = S^{-1}\mathcal{C}(C^{n+1-*}(W) \xrightarrow{\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 j_*^* \end{pmatrix}} C(W, M \sqcup X)).$$

If we want to use the spectral quadratic construction we need a semistable map inducing the map in the above display. Consider the map

$$S^N \xrightarrow{\rho_W / \rho_{M \sqcup X}} \text{Th}(\nu_W) / \text{Th}(\nu_{M \sqcup X}) \xrightarrow{\Delta} \Sigma^p(W / (M \sqcup X)) \wedge \text{Th}(\nu_W)$$

which has an S -dual

$$\Gamma_W : \text{Th}(\nu_W)^* \rightarrow \Sigma^p(W / (M \sqcup X))$$

which in turn induces a map of chain complexes

$$\gamma_W : C_{*+p}(\text{Th}(\nu_W)^*) \rightarrow C_*(W / (M \sqcup X)).$$

The map γ_W coincides with the map $\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 j_*^* \end{pmatrix}$ under Thom isomorphism and S -duality (by a relative version of Diagram 3.1, see also Section 14).

The spectral quadratic construction on Γ_W

$$\Psi : C_{n+1+p}(\text{Th}(\nu_W)^*) \rightarrow W_{\%}(\mathcal{C}(\gamma_W))_{n+1}$$

produces from the dual of the Thom class $u(\nu_W)^* \in C_{n+1+p}(\text{Th}(\nu_W)^*)$ an $(n+1)$ -dimensional quadratic structure on $\mathcal{C}(\gamma_W)$ which has a desuspension unique up to equivalence and that is our desired ψ' such that

$$\Psi(u(\nu_W)^*) = S(\psi').$$

The construction just described comes from [18, Prop. 7.4.1]. By [25, Proof of Thm. 7.1] we obtain that (3.2) holds.

Now recall from Definition 2.32 that we have another way of assigning n -dimensional quadratic Poincaré complex to (f, b) , namely the surgery obstruction $\text{sign}^{\mathbf{L}\bullet}(f, b) \in L_n(\mathbb{Z})$.

We claim that

$$[(C', \psi')] = \text{sign}^{\mathbf{L}\bullet}(f, b) \in L_n(\mathbb{Z}).$$

The following commutative diagram identifies $C' \simeq \mathcal{C}(f^!)$:

$$(3.3) \quad \begin{array}{ccc} & C^{n+1-*}(W) & \xrightarrow{\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 j^* \end{pmatrix}} C(W, M \sqcup X) \\ & \downarrow j_M^* & \downarrow \simeq \\ \text{pr}_X^* \simeq & C^{n+1-*}(M) & \xrightarrow[\simeq]{\varphi_0|_M} C(\Sigma M) \\ & \uparrow S(f^*) & \uparrow S(f^!) \\ & C^{n+1-*}(X) & \xrightarrow[\simeq]{\varphi_0|_X} C(\Sigma X). \end{array}$$

To identify the quadratic structures recall first that the spectral quadratic construction Ψ on a semistable map F is the same as the quadratic construction ψ composed with $e_{\%}$ if the semistable map $F : X \rightarrow \Sigma^p Y$ is in fact a stable map $F = \Sigma^p X_0 = X \rightarrow \Sigma^p Y$. Furthermore the homotopy equivalence j_X and the S -duality are used to show that Diagram 3.3 is induced by the commutative diagram of maps of spaces as follows:

$$(3.4) \quad \begin{array}{ccc} & \text{Th}(\nu_W)^* & \xrightarrow{\Gamma_W} \Sigma^p(W/(M \sqcup X)) \\ & \downarrow T(j_M)^* & \downarrow \simeq \\ T(\text{pr}_X)^* \simeq & \text{Th}(\nu_M)^* & \xrightarrow[\simeq]{\gamma_M} \Sigma^{p+1} M_+ \\ & \uparrow T(b)^* & \uparrow F \\ & \text{Th}(\nu_X)^* & \xrightarrow[\simeq]{\gamma_X} \Sigma^{p+1} X_+ \end{array}$$

which identifies F and Γ_W .

The Thom class $u(\nu_W)$ restricts to $u(\nu_X)$ and hence the duals $u(\nu_W)^*$ and $u(\nu_X)^* = \Sigma[X]$ are also identified. The uniqueness of desuspensions gives the identification of the equivalence classes of the quadratic structures

$$e_{\%}\psi([X]) \sim \psi'.$$

4. ALGEBRAIC BORDISM CATEGORIES AND EXACT SEQUENCES

In previous sections we recalled the notions of certain structured chain complexes over an additive category with chain duality $\mathbb{A}$ and the corresponding L -groups. In this section we review a generalization where the category we work with is an algebraic bordism category. This eventually allows us to vary $\mathbb{A}$ and we also obtain certain localization sequences.

Definition 4.1. An algebraic bordism category $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}, (T, e))$ consists of an additive category with chain duality $(\mathbb{A}, (T, e))$, a full subcategory $\mathbb{B} \subseteq \mathbb{B}(\mathbb{A})$ and another full subcategory $\mathbb{C} \subseteq \mathbb{B}$ closed under taking cones.

Definition 4.2. Let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}, (T, e))$ be an algebraic bordism category.

An n -dimensional symmetric algebraic complex in Λ is a pair (C, φ) where $C \in \mathbb{B}$ and $\varphi \in (W^\%C)_n$ is an n -dimensional cycle such that the boundary $\partial C = \Sigma^{-1}\mathcal{C}(\varphi_0 : \Sigma^n TC \rightarrow C)$ is an element of $\mathbb{C}$.

An $(n+1)$ -dimensional symmetric algebraic pair in Λ is a pair $(f : C \rightarrow D, (\delta\varphi, \varphi))$ in Λ where $f : C \rightarrow D$ is a chain map with $C, D \in \mathbb{B}$, the pair $(\delta\varphi, \varphi) \in \mathcal{C}(f^\%)$ is an $(n+1)$ -cycle and $\mathcal{C}(\delta\varphi_0, \varphi_0 f^*) \in \mathbb{C}$.

A cobordism between two n -dimensional symmetric algebraic complexes (C, φ) and (C', φ') in Λ is an $(n+1)$ -dimensional symmetric pair in Λ of the form $(C \oplus C' \rightarrow D, (\delta\psi, \varphi \oplus -\varphi'))$.

So informally these are complexes, pairs and cobordisms of chain complexes which are in $\mathbb{B}$ and which are Poincaré modulo $\mathbb{C}$. There are analogous definitions in quadratic and normal case. The L -groups are generalized to this setting as follows.

Definition 4.3. The symmetric, quadratic, and normal L -groups

$$L^n(\Lambda), L_n(\Lambda) \text{ and } NL^n(\Lambda)$$

are defined as the cobordism groups of n -dimensional symmetric, quadratic, and normal algebraic complexes in Λ respectively.

Example 4.4. Let R be a ring with involution. By

$$\Lambda(R) = (\mathbb{A}(R), \mathbb{B}(R), \mathbb{C}(R), (T, e))$$

is denoted the algebraic bordism category with

$\mathbb{A}(R)$ the category of R -modules from Example 2.2,

$\mathbb{B}(R)$ the bounded chain complexes in $\mathbb{A}(R)$,

$\mathbb{C}(R)$ the contractible chain complexes of $\mathbb{B}(R)$.

We also consider the algebraic bordism category

$$\widehat{\Lambda}(R) = (\mathbb{A}(R), \mathbb{B}(R), \mathbb{B}(R), (T, e)).$$

The L -groups of Section 2 and the L -groups of Definition 4.3 are related by

$$L^n(R) \cong L^n(\Lambda(R)) \text{ and } L_n(R) \cong L_n(\Lambda(R)).$$

For the NL -groups of Section 3 we have:

$$NL^n(R) \cong NL^n(\widehat{\Lambda}(R)) \text{ and } L^n(R) \cong NL^n(\Lambda(R)).$$

The second isomorphism is due to Lemma 3.19.

The notion of a functor of algebraic bordism categories

$$F : \Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}) \rightarrow \Lambda' = (\mathbb{A}', \mathbb{B}', \mathbb{C}')$$

is defined in [19, Def. 3.7]. Any such functor induces a map of the corresponding L -groups.

Proposition 4.5. [19, Prop. 3.8] *For a functor $F : \Lambda \rightarrow \Lambda'$ of algebraic bordism categories there are relative L -groups $L_n(F)$, $L^n(F)$ and $NL^n(F)$ which fit into the long exact sequences*

$$\begin{aligned} \dots &\rightarrow L_n(\Lambda) \rightarrow L_n(\Lambda') \rightarrow L_n(F) \rightarrow L_{n-1}(\Lambda) \rightarrow \dots, \\ \dots &\rightarrow L^n(\Lambda) \rightarrow L^n(\Lambda') \rightarrow L^n(F) \rightarrow L^{n-1}(\Lambda) \rightarrow \dots, \\ \dots &\rightarrow NL^n(\Lambda) \rightarrow NL^n(\Lambda') \rightarrow NL^n(F) \rightarrow NL^{n-1}(\Lambda) \rightarrow \dots. \end{aligned}$$

These exact sequences are produced by the technology of [18, Chap. 2] which was mentioned in the previous section. An element of the group $L_n(F)$ is an $(n-1)$ -dimensional quadratic complex (C, ψ) in Λ with an n -dimensional quadratic pair $(F(C) \rightarrow D, (\delta\psi, F(\psi)))$ in Λ' . There is a notion of a cobordism of such pairs and the group $L_n(F)$ is defined as such a cobordism group. Analogously in the symmetric and normal case.

The following proposition improves the above statement in the sense that the relative terms are given as cobordism groups of complexes rather than pairs.

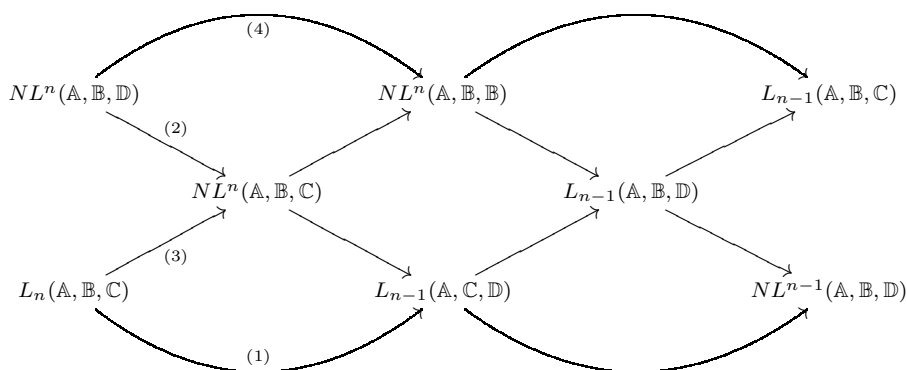
Proposition 4.6. [19, Prop. 3.9] *Let $\mathbb{A}$ be an additive category with chain duality and let $\mathbb{D} \subset \mathbb{C} \subset \mathbb{B} \subset \mathbb{B}(\mathbb{A})$ be subcategories closed under taking cones. The relative symmetric L -groups for the inclusion $F : (\mathbb{A}, \mathbb{B}, \mathbb{D}) \rightarrow (\mathbb{A}, \mathbb{B}, \mathbb{C})$ are given by*

$$(i) \quad L^n(F) \cong L^{n-1}(\mathbb{A}, \mathbb{C}, \mathbb{D})$$

and in the quadratic and normal case by

$$(ii) \quad L_n(F) \cong L_{n-1}(\mathbb{A}, \mathbb{C}, \mathbb{D}) \cong NL^n(F).$$

Part (ii) of the proposition allows us to produce interesting relations between the long exact sequences for various inclusions combining the quadratic L_n -groups and the normal NL^n -groups. In the following commutative braid we have 4 such sequences. Sequence (1) is given by the inclusion $(\mathbb{A}, \mathbb{B}, \mathbb{D}) \rightarrow (\mathbb{A}, \mathbb{B}, \mathbb{C})$ in the quadratic theory, sequence (2) by the inclusion $(\mathbb{A}, \mathbb{B}, \mathbb{D}) \rightarrow (\mathbb{A}, \mathbb{B}, \mathbb{C})$, sequence (3) by the inclusion $(\mathbb{A}, \mathbb{B}, \mathbb{C}) \rightarrow (\mathbb{A}, \mathbb{B}, \mathbb{B})$, and sequence (4) by the inclusion $(\mathbb{A}, \mathbb{B}, \mathbb{D}) \rightarrow (\mathbb{A}, \mathbb{B}, \mathbb{B})$, all last three in the normal theory:



Comments on the proof Proposition 4.6. Recall that an element in $L_n(F)$ is an $(n-1)$ -dimensional quadratic complex (C, ψ) in $(\mathbb{A}, \mathbb{B}, \mathbb{D})$ together with an n -dimensional quadratic pair $(C \rightarrow D, (\delta\psi, \psi))$ in $(\mathbb{A}, \mathbb{B}, \mathbb{C})$. The isomorphism $L_n(F) \cong L_{n-1}(\mathbb{A}, \mathbb{C}, \mathbb{D})$ is given by

$$((C, \psi), C \rightarrow D, (\delta\psi, \psi)) \mapsto (C', \psi')$$

where (C', ψ') is the effect of algebraic surgery on (C, ψ) using as data the pair $(C \rightarrow D, (\delta\psi, \psi))$. We have $C' \in \mathbb{C}$ since $C \rightarrow D$ is Poincaré modulo $\mathbb{C}$. Furthermore, the observation that (C', ψ') is Poincaré modulo $\mathbb{D}$ follows from the assumption that (C, ψ) is Poincaré modulo $\mathbb{D}$ and from Proposition 2.27 which says that the homotopy type of the boundary is preserved by algebraic surgery.

The inverse map is given by

$$(C, \psi) \mapsto ((C, \psi), C \rightarrow 0, (0, \psi)).$$

Similarly for the isomorphism $NL^n(F) \cong L_{n-1}(\mathbb{C}, \mathbb{D})$. Consider an element $((C, \theta), C \rightarrow D, (\delta\theta, \theta))$ of $NL^n(F)$ and perform algebraic surgery on (C, θ) with data $(C \rightarrow D, (\delta\theta, \theta))$. We obtain an $(n-1)$ -dimensional symmetric complex in $\mathbb{C}$ which is Poincaré modulo $\mathbb{D}$. Using [19, 2.8(ii)] we see that the symmetric structure has a quadratic refinement. $\square$

Example 4.7. Let R be a ring with involution and consider the inclusion of the algebraic bordism categories $\Lambda(R) \rightarrow \widehat{\Lambda}(R)$ from Example 4.4. Then the long exact sequence of the associated NL -groups (sequence (3) in the diagram above) becomes the long exact sequence of Proposition 3.21, thanks to Lemma 3.19.

5. CATEGORIES OVER COMPLEXES

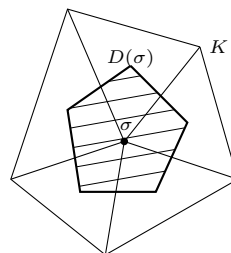
In this section we recall the setup for studying local Poincaré duality over a locally finite simplicial complex K . For a simplex $\sigma \in K$ we will use the notion of a dual cell $D(\sigma, K)$ which is a certain subcomplex of the barycentric subdivision K' , see [19, Rem. 4.10] for the definition if needed.¹⁴

Observe first that there are two types of such a local duality for a triangulated n -manifold K .

- (1) Each simplex σ of K is a $|\sigma|$ -dimensional manifold with boundary and so there is a duality between $C_*(\sigma, \partial\sigma)$ and $C^{|\sigma|-*}(\sigma)$.

¹⁴Note that in general the dual cell $D(\sigma, K)$ is not a “cell” in the sense that it is not homeomorphic to D^l for any l . Nevertheless the terminology is used in [19] and we keep it.

- (2) Each dual cell $D(\sigma, K)$ is an $(n - |\sigma|)$ -dimensional manifold with boundary and so there is a duality between the chain complexes $C_*(D(\sigma, K), \partial D(\sigma, K))$ and $C^{n-|\sigma|-*}(D(\sigma, K))$.



This observation leads to two notions of additive categories with chain duality over K .

Definition 5.1. Let $\mathbb{A}$ be an additive category with chain duality and K as above. The additive categories of K -based objects $\mathbb{A}^*(K)$ and $\mathbb{A}_*(K)$ are defined by

$$\text{Obj}(\mathbb{A}^*(K)) = \text{Obj}(\mathbb{A}_*(K)) = \left\{ \sum_{\sigma \in K} M_\sigma \mid M_\sigma \in \mathbb{A} \right\},$$

$$(1) \text{ Mor}(\mathbb{A}^*(K)) = \left\{ \sum_{\sigma \geq \tau} f_{\tau, \sigma} : \sum_{\sigma \in K} M_\sigma \rightarrow \sum_{\tau \in K} N_\tau \mid (f_{\tau, \sigma} : M_\sigma \rightarrow N_\tau) \in \text{Mor}(\mathbb{A}) \right\},$$

$$(2) \text{ Mor}(\mathbb{A}_*(K)) = \left\{ \sum_{\sigma \leq \tau} f_{\tau, \sigma} : \sum_{\sigma \in K} M_\sigma \rightarrow \sum_{\tau \in K} N_\tau \mid (f_{\tau, \sigma} : M_\sigma \rightarrow N_\tau) \in \text{Mor}(\mathbb{A}) \right\}.$$

A chain complex (C, d) over $\mathbb{A}^*(K)$, respectively $\mathbb{A}_*(K)$, is a collection $(C(\sigma), d(\sigma))_{\sigma \in K}$ of chain complexes together with additional boundary maps $d(\sigma, \tau) : C(\sigma)_* \rightarrow C(\tau)_{*-1}$ for each $\tau \leq \sigma$, respectively $\sigma \leq \tau$.

Example 5.2. The simplicial chain complex $C = \Delta(K)$ is a chain complex in $\mathbb{B}(\mathbb{A}^*(K))$, by defining $C(\sigma) := \Delta(\sigma, \partial\sigma) = S^{|\sigma|}\mathbb{Z}$.

The simplicial chain complex $C = \Delta(K')$ of the barycentric subdivision K' is a chain complex in $\mathbb{B}(\mathbb{A}_*(K))$ by $C(\sigma) = \Delta(D(\sigma), \partial D(\sigma))$.

The picture depicts the simple case of the simplicial chain complex $\Delta_*(\Delta^1)$ as a chain complex in $\mathbb{A}(\mathbb{Z})^*(\Delta^1)$:

$$\begin{array}{c}
 \begin{array}{ccc}
 \sigma_0 & \xrightarrow{\tau} & \sigma_1 \\
 \bullet & & \bullet
 \end{array} \\
 C(\sigma_0) = \Delta_*(\sigma_0, \partial\sigma_0) & C(\tau) = \Delta_*(\tau, \partial\tau) & C(\sigma_1) = \Delta_*(\sigma_1, \partial\sigma_1) \\
 \\
 \begin{array}{ccccc}
 C_2 : & 0 & & 0 & \\
 & \downarrow & \nearrow & \searrow & \downarrow \\
 C_1 : & 0 & & \mathbb{Z} & 0 \\
 & \downarrow & \nearrow \partial_0 & \searrow \partial_1 & \downarrow \\
 C_0 : & \mathbb{Z} & & 0 & \mathbb{Z}
 \end{array}
 \end{array}$$

Now we recall the extension of the chain duality from $\mathbb{A}$ to the two new categories.

Definition 5.3.

$$T^* : \mathbb{A}^*(K) \rightarrow \mathbb{B}(\mathbb{A}^*(K)), \quad T^*\left(\sum_{\sigma \in K} M_\sigma\right)_r(\tau) = (T(\bigoplus_{\tau \geq \tilde{\tau}} M_{\tilde{\tau}}))_{r-|\tau|}.$$

$$T_* : \mathbb{A}_*(K) \rightarrow \mathbb{B}(\mathbb{A}_*(K)), \quad T_*\left(\sum_{\sigma \in K} M_\sigma\right)_r(\tau) = (T(\bigoplus_{\tau \leq \tilde{\tau}} M_{\tilde{\tau}}))_{r+|\tau|}.$$

Example 5.4. The dual $T^*(C)$ of the simplicial chain complex $C = \Delta(K)$ is a chain complex in $\mathbb{B}(\mathbb{A}^*(K))$ given by $(T^*C)(\sigma) = \Delta^{|\sigma|-*}(\sigma)$.

The dual T_*C of the simplicial chain complex $C = \Delta(K')$ of the barycentric subdivision K' is a chain complex in $\mathbb{B}(\mathbb{A}_*(K))$ whose piece over $\sigma \in K$ is given by $(T_*C)(\sigma) = \Delta^{-|\sigma|-*}(D(\sigma))$.

In the example when K is a triangulated manifold recall that the chain duality functor T^* on $\mathbb{A}^*(K)$ is supposed to encode the local Poincaré duality of all simplices of K . But the dimensions of these local Poincaré dualities vary with the dimension of the simplices and we have to deal with the boundaries. So the dimension shift in the above formula comes from the varying dimensions and the direct sum comes from “dealing with the boundary”. In the example $C = \Delta_*(\Delta^1)$ we obtain the following picture

$$\begin{array}{ccccccc}
 \Delta_*(\sigma_0, \partial\sigma_0) & \Delta_*(\tau, \partial\tau) & \Delta_*(\sigma_1, \partial\sigma_1) & \Delta^*(\sigma_0) & \Delta^*(\tau) & \Delta^*(\sigma_1) & \\
 C_1 : \begin{array}{ccc} 0 & & \mathbb{Z} \\ \downarrow & \nearrow \partial_0 & \downarrow \\ \mathbb{Z} & & 0 \end{array} & & \begin{array}{ccc} 0 & & \mathbb{Z} \\ \downarrow & \nearrow \partial_1 & \downarrow \\ \mathbb{Z} & & 0 \end{array} & \xleftarrow{\varphi_0 -} & \begin{array}{ccc} (\mathbb{Z} \oplus \mathbb{Z})^* & & \\ \swarrow i_0^* & \downarrow \begin{pmatrix} \partial_0^* \\ \partial_1^* \end{pmatrix} & \searrow i_1^* \\ \mathbb{Z}^* & & \mathbb{Z}^* \end{array} & : (T^*C)_1 \\
 C_0 : \begin{array}{ccc} \mathbb{Z} & & 0 \\ \downarrow & \nearrow & \downarrow \\ 0 & & \mathbb{Z} \end{array} & & \begin{array}{ccc} \mathbb{Z}^* & & \mathbb{Z}^* \\ \downarrow & \nearrow & \downarrow \\ 0 & & 0 \end{array} & \xleftarrow{\varphi_0 -} & \begin{array}{ccc} \mathbb{Z}^* & & \mathbb{Z}^* \\ \downarrow & \nearrow & \downarrow \\ 0 & & 0 \end{array} & : (T^*C)_0 \\
 & & & & & & : (T^*C)_{-1}
 \end{array}$$

In $\mathbb{A}_*(K)$ the role of simplices is replaced by the dual cells and so the formulas are changed accordingly.

The additive categories with chain duality $\mathbb{A}^*(K)$ and $\mathbb{A}_*(K)$ can be made into algebraic bordism categories in various ways yielding chain complexes with various types of Poincaré duality. Now we introduce the local duality, in the next section we will have the global duality.

Proposition 5.5. *Let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ be an algebraic bordism category and K a locally finite simplicial complex. Then the triples*

$$\Lambda^*(K) = (\mathbb{A}^*(K), \mathbb{B}^*(K), \mathbb{C}^*(K)), \quad \Lambda_*(K) = (\mathbb{A}_*(K), \mathbb{B}_*(K), \mathbb{C}_*(K)),$$

where $\mathbb{B}^*(K)$, $\mathbb{B}_*(K)$, $(\mathbb{C}^*(K), \mathbb{C}_*(K))$ are the full subcategories of $\mathbb{B}(\mathbb{A}^*(K))$, respectively $\mathbb{B}(\mathbb{A}_*(K))$, consisting of the chain complexes C such that $C(\sigma) \in \mathbb{B}(C(\sigma) \in \mathbb{C})$ for all $\sigma \in K$ are algebraic bordism categories.

See [19, Prop. 5.1] for the proof. We remark that other useful algebraic bordism categories associated to Λ and K will be defined in Definitions 6.4 and 8.15.

Proposition 5.6. *Let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ be an algebraic bordism category and let $f : J \rightarrow K$ be a simplicial map. Then f induces contravariantly (covariantly) a covariant functor of algebraic bordism categories*

$$f^* : \Lambda^*(K) \rightarrow \Lambda^*(J), \quad (f_* : \Lambda_*(J) \rightarrow \Lambda_*(K)).$$

See [19, Prop. 5.6]. A consequence is that we obtain induced maps on the L -groups as well, which we do not write down explicitly at this stage.

Now we present constructions over the category $\mathbb{A}(\mathbb{Z})^*(K)$ analogous to the symmetric and quadratic construction in Section 2. Examples of chain complexes over $\mathbb{A}(\mathbb{Z})^*(K)$ were already presented in Examples 5.2 and 5.4. The underlying chain complexes below are generalizations of those. We will write $\mathbb{Z}^*(K)$ as short for $\mathbb{A}(\mathbb{Z})^*(K)$ and $\mathbb{Z}_*(K)$ as short for $\mathbb{A}(\mathbb{Z})_*(K)$.

Construction 5.7. Consider a topological k -ad $(X, (\partial_\sigma X)_{\sigma \in \Delta^k})$ and the chain subcomplex of the singular chain complex $C(X)$ consisting of simplices that respect the k -ad structure in a sense that each singular simplex is contained in $\partial_\sigma X$ for some $\sigma \in \Delta^k$. By a Mayer–Vietoris type argument this chain subcomplex is chain homotopy equivalent to $C(X)$ and in an abuse of notation we still denote it by $C(X)$. It becomes a chain complex over $\mathbb{Z}^*(\Delta^k)$ by $C(X)(\sigma) = C(\partial_\sigma X, \partial(\partial_\sigma X))$. Its dual is a chain complex $T^*C(X)$ given by $(T^*C(X))(\sigma) = C^{|\sigma|-*}(\partial_\sigma X)$ for $\sigma \in \Delta^k$. A generalization of the relative symmetric construction 2.20 gives a chain map

$$\varphi_{\Delta^k} : \Sigma^{-k}C(X, \partial X) \rightarrow W^\%(C(X)) \text{ over } \mathbb{Z}^*(\Delta^k)$$

called the *symmetric construction over Δ^k* which when evaluated on a cycle $[X] \in C_{n+k}(X, \partial X)$ gives an n -dimensional symmetric algebraic complex $(C(X), \varphi_{\Delta^k}[X])$ in $\mathbb{Z}^*(\Delta^k)$ whose component

$$\varphi_{\Delta^k}([X])(\sigma)_0 : C^{n+|\sigma|-*}(\partial_\sigma X) \rightarrow C(\partial_\sigma X, \partial(\partial_\sigma X))$$

is the cap product with the cycle $[\partial_\sigma X] \in C_{n+|\sigma|}(\partial_\sigma X, \partial(\partial_\sigma X))$. Here the map $\partial_\sigma : C(\Delta^k) \rightarrow C(\sigma)$ is the map defined as in [19, Def. 8.2].

Construction 5.8. Consider now the special case of an $(n+k)$ -dimensional manifold k -ad $(M, (\partial_\sigma M)_{\sigma \in \Delta^k})$. Let $\Lambda(\mathbb{Z})$ be the algebraic bordism category from Example 4.4. Construction 5.7 applied to the fundamental class $[M] \in C_{n+k}(M, \partial M)$ produces an n -dimensional symmetric algebraic complex $(C(M), \varphi_{\Delta^k}([M]))$ in the category $\Lambda(\mathbb{Z})^*(\Delta^k)$ since the maps

$$\varphi_{\Delta^k}([M])(\sigma)_0 : C^{(n-k)+|\sigma|-*}(\partial_\sigma M) \rightarrow C(\partial_\sigma M, \partial(\partial_\sigma M))$$

are given by taking the cap products with the appropriate fundamental classes $[\partial_\sigma M] \in C_{n-k+|\sigma|}(\partial_\sigma M, \partial(\partial_\sigma M))$ and hence are chain homotopy equivalences and hence their mapping cones are contractible.

Construction 5.9. Analogously, when we have a degree one normal map of manifold k -ads

$$((f, b), (f_\sigma, b_\sigma)) : (M, \partial_\sigma M) \rightarrow (X, \partial_\sigma X)$$

with $\sigma \in \Delta^k$, the stable Umkehr map $F : \Sigma^p X_+ \rightarrow \Sigma^p M_+$ for some p induces by a generalization of the relative quadratic construction 2.21 a chain map

$$\psi_{\Delta^k} :: \Sigma^{-k} C(X, \partial X) \rightarrow W_{\%}(C(M)) \text{ over } \mathbb{Z}^*(\Delta^k)$$

called the *quadratic construction over Δ^k* . Evaluated on the fundamental class $[X] \in C_{n+k}(X, \partial X)$ it produces an n -dimensional quadratic algebraic complex in the category $\Lambda(\mathbb{Z})^*(\Delta^k)$. The mapping cone $\mathcal{C}(f^!)$ becomes a complex over $\mathbb{Z}^*(\Delta^k)$ by $\mathcal{C}(f^!)(\sigma) = \mathcal{C}(f^!_\sigma, \partial f^!_\sigma)$. The chain map $e : C(M) \rightarrow \mathcal{C}(f^!)$ in $\mathbb{Z}^*(\Delta^k)$ produces an n -dimensional quadratic complex in $\Lambda(\mathbb{Z})^*(\Delta^k)$,

$$(C(f^!), e_{\%} \psi_{\Delta^k}[X]).$$

Now we move to the constructions in the category $\mathbb{Z}_*(K)$.

Construction 5.10. Let $r : X \rightarrow K$ be a map of simplicial complexes. Denoting for $\sigma \in K$

$$X[\sigma] = r^{-1}(D(\sigma, K)) \subset X' \text{ we obtain } X = \bigcup_{\sigma \in K} X[\sigma].$$

This decomposition is called a K -dissection of X . Consider the subcomplex of the singular chain complex of $C(X)$ consisting of the singular chains which respect the dissection in the sense that each singular simplex is contained in some $X[\sigma]$. This chain complex is chain homotopy equivalent to $C(X)$ and by abuse of notation we still denote it $C(X)$. It becomes a chain complex in $\mathbb{B}(\mathbb{Z}_*(K))$ by

$$C(X)(\sigma) = C(X[\sigma], \partial X[\sigma]) \text{ for } \sigma \in K$$

with n -dual

$$\Sigma^n T_* C(X)(\sigma) = C^{n-|\sigma|-*}(X[\sigma]).$$

There is a chain map $\partial_\sigma : C(X) \rightarrow S^{|\sigma|} C(X[\sigma], \partial X[\sigma])$, defined in [19, Def. 8.2], and the image of a chain $[X] \in C(X)_n$ under the map ∂_σ is denoted by $[X[\sigma]] \in C(X[\sigma], \partial X[\sigma])_{n-|\sigma|}$. A generalization of the relative symmetric construction 2.20 gives a chain map

$$\varphi_K : C(X) \rightarrow W_{\%}(C(X)) \text{ over } \mathbb{Z}_*(K)$$

called the *symmetric construction over K* , which when evaluated on a cycle $[X] \in C(X)_n$ produces an n -dimensional symmetric complex $(C(X), \varphi_K[X])$ over $\mathbb{Z}_*(K)$ whose component

$$\varphi_K([X])(\sigma)_0 : C^{n-|\sigma|-*}(X[\sigma]) \rightarrow C(X[\sigma], \partial X[\sigma])$$

is the cap product with the class $[X[\sigma]]$.

Construction 5.11. More generally, let X be an n -dimensional topological manifold and let $r : X \rightarrow K$ be a map, transverse to the dual cells $D(\sigma, K)$ for all $\sigma \in K$. Any map can be so deformed by topological transversality. In this situation we obtain an analogous K -dissection. The resulting complex $(C(X), \varphi_K[X])$ is now an n -dimensional symmetric algebraic complex in $\Lambda(\mathbb{Z})_*(K)$ since the maps

$$\varphi(\sigma)_0 : C^{(n-|\sigma|-*)}(X[\sigma]) \rightarrow C(X[\sigma], \partial X[\sigma])$$

are the cap products with the fundamental classes $[X[\sigma]] \in C_{n-|\sigma|}(X[\sigma], \partial X[\sigma])$ and hence chain homotopy equivalences and hence their mapping cones are contractible. Here we are using the fact that each $X[\sigma]$ is an $n - |\sigma|$ -dimensional manifold with boundary and hence satisfies Poincaré duality.

Construction 5.12. Analogously let $(f, b) : M \rightarrow X$ be a degree one normal map of closed n -dimensional topological manifolds. We can make f transverse to the K -dissection of X in a sense that each preimage

$$(M[\sigma], \partial M[\sigma]) := f^{-1}(X[\sigma], \partial X[\sigma])$$

is an $(n - |\sigma|)$ -dimensional manifold with boundary and each restriction

$$(f[\sigma], f[\partial\sigma]) : (M[\sigma], \partial M[\sigma]) \rightarrow (X[\sigma], \partial X[\sigma])$$

is a degree one normal map. The stable Umkehr map $F : \Sigma^p X_+ \rightarrow \Sigma^p M_+$ for some p induces by a generalization of the relative quadratic construction 2.21 a chain map

$$\psi_K : \Sigma^{-k} C(X) \rightarrow W_{\%}(C(M)) \text{ over } \mathbb{Z}_*(K)$$

called the *quadratic construction over K* . Evaluated on the fundamental class $[X] \in C_n(X, \partial X)$ produces an n -dimensional quadratic algebraic complex in the category $\Lambda(\mathbb{Z})_*(K)$. The mapping cone $\mathcal{C}(f^!)$ becomes a complex over $\mathbb{Z}_*(K)$ by $\mathcal{C}(f^!)(\sigma) = \mathcal{C}(f(\sigma)^!, f(\partial\sigma)^!)$. The chain map $e : C(M) \rightarrow \mathcal{C}(f^!)$ in $\mathbb{Z}_*(K)$ produces an n -dimensional quadratic complex in $\Lambda(\mathbb{Z})_*(K)$,

$$(C(f^!), e_{\%} \psi_K[X]).$$

6. ASSEMBLY

Assembly is a map that allows us to compare the concepts of the local Poincaré duality introduced in Section 5 and the global Poincaré duality in Section 2. It is formulated as a functor of algebraic bordism categories.

Proposition 6.1. *The functor of additive categories $A : \mathbb{Z}_*(K) \rightarrow \mathbb{Z}[\pi_1(K)]$ defined by*

$$M \mapsto \sum_{\tilde{\sigma} \in \tilde{K}} M(p(\tilde{\sigma}))$$

defines a functor of algebraic bordism categories,

$$A : \Lambda(\mathbb{Z})_*(K) \rightarrow \Lambda(\mathbb{Z}[\pi_1(K)])$$

and hence homomorphisms

$$\begin{aligned} A : L^n(\Lambda(\mathbb{Z})_*(K)) &\rightarrow L^n(\Lambda(\mathbb{Z}[\pi_1(K)])), \\ A : L_n(\Lambda(\mathbb{Z})_*(K)) &\rightarrow L_n(\Lambda(\mathbb{Z}[\pi_1(K)])). \end{aligned}$$

Example 6.2. [19, Ex. 9.6] Let X be an n -dimensional topological manifold with a map $r : X \rightarrow K$. In Construction 5.10 there is described how to associate to X an n -dimensional SAC (C, φ) in $\Lambda(\mathbb{Z})_*(K)$. The assembly $A(C, \varphi)$ is then the n -dimensional SAPC $\text{sign}^{\mathbf{L}^\bullet}(X) = (C(\tilde{X}), \varphi([X]))$ in $\Lambda(\mathbb{Z}[\pi_1(K)])$ described in Construction 2.15.

Example 6.3. [19, Ex. 9.6] Let $(f, b) : M \rightarrow X$ be a degree one normal map of closed n -dimensional topological manifolds. In Construction 5.12 we produced from (f, b) an n -dimensional QAC (C, ψ) in $\Lambda(\mathbb{Z})_*(K)$. The assembly $A(C, \varphi)$ is then the n -dimensional QAPC $\text{sign}^{\mathbf{L}^\bullet}(f, b) = (C(f^!), e_\psi \psi[X])$ in $\Lambda(\mathbb{Z}[\pi_1(K)])$ described in Construction 2.16.

It is convenient to factor the assembly map into two maps. The reason is that we have nice localization sequences for a functor of algebraic bordism categories when the underlying category with chain duality is fixed and the functor is an inclusion. Hence we define

Definition 6.4. Let $\Lambda(\mathbb{Z})$ be the algebraic bordism category of Example 4.4 and K a locally finite simplicial complex. Then the triple

$$\Lambda(\mathbb{Z})(K) = (\mathbb{A}_*(K), \mathbb{B}_*(K), \mathbb{C}(K))$$

where the subcategory $\mathbb{C}(K)$ consists of the chain complexes $C \in \mathbb{B}_*(K)$ such that $A(C) \in \mathbb{C}(\mathbb{Z}[\pi_1(K)])$.

Hence, for example, an n -dimensional symmetric complex (C, φ) in $\Lambda(\mathbb{Z})(K)$ will be a complex over $\mathbb{Z}_*(K)$, which will only be globally Poincaré in the sense that $A(C, \varphi)$ will be an n -dimensional SAPC over $\mathbb{Z}[\pi_1 K]$, but the duality maps $\varphi(\sigma) : \Sigma^n TC(\sigma) \rightarrow C(\sigma)$ do not have to be chain homotopy equivalences for a particular simplex $\sigma \in K$.

Proposition 6.5. *The assembly functor factors as*

$$A : \Lambda(\mathbb{Z})_*(K) \rightarrow \Lambda(\mathbb{Z})(K) \rightarrow \Lambda(\mathbb{Z}[\pi_1(K)]).$$

Furthermore Ranicki proves the following algebraic $\pi - \pi$ -theorem:¹⁵

Proposition 6.6. [19, Chap. 10] *The functor $\Lambda(\mathbb{Z})(K) \rightarrow \Lambda(\mathbb{Z}[\pi_1(K)])$ induces an isomorphism on quadratic L -groups*

$$L_n(\Lambda(\mathbb{Z})(K)) \cong L_n(\mathbb{Z}[\pi_1(K)]).$$

It follows that when we want to compare local and global Poincaré duality it is enough to study the map

$$(6.1) \quad A : L_n(\Lambda(\mathbb{Z})_*(K)) \rightarrow L_n(\Lambda(\mathbb{Z})(K)).$$

¹⁵The name is explained at the beginning of [19, Chap. 10].

7. L-SPECTRA

The technology of the previous sections also allows us to construct L -theory spectra whose homotopy groups are the already defined L -groups. Spectra give rise to generalized (co-)homology theories via the standard technology of stable homotopy theory. That is also the main reason for their introduction in L -theory.

These spectra are constructed as spectra of Δ -sets, alias simplicial sets without degeneracies. We refer the reader to [19, Chap. 11] for the detailed definition as well as for the notions of Kan Δ -sets, the geometric product $K \otimes L$, the smash product $K \wedge L$, the function Δ -sets L^K , the fiber and the cofiber of a map of Δ -sets, the loop Δ -set ΩK and the suspension ΣK as well as the notion of an Ω -spectrum of Δ -sets.

Below, Δ^n is the standard n -simplex, Λ is an algebraic bordism category and K is a finite Δ -set.

Definition 7.1. Let $\mathbf{L}_n(\Lambda)$, $\mathbf{L}^n(\Lambda)$ and $\mathbf{NL}^n(\Lambda)$ be pointed Δ -sets defined by

$$\begin{aligned}\mathbf{L}^n(\Lambda)^{(k)} &= \{n\text{-dim. symmetric complexes in } \Lambda^*(\Delta^k)\}, \\ \mathbf{L}_n(\Lambda)^{(k)} &= \{n\text{-dim. quadratic complexes in } \Lambda^*(\Delta^k)\}, \\ \mathbf{NL}^n(\Lambda)^{(k)} &= \{n\text{-dim. normal complexes in } \Lambda^*(\Delta^{k+n})\}.\end{aligned}$$

The face maps are induced by the face inclusions $\partial_i : \Delta^{k-1} \rightarrow \Delta^k$ and the base point is the 0-chain complex.

Proposition 7.2. *We have Ω -spectra of pointed Kan Δ -sets*

$$\begin{aligned}\mathbf{L}^\bullet(\Lambda) &:= \{\mathbf{L}^n(\Lambda) \mid n \in \mathbb{Z}\}, \\ \mathbf{L}_\bullet(\Lambda) &:= \{\mathbf{L}_n(\Lambda) \mid n \in \mathbb{Z}\}, \\ \mathbf{NL}^\bullet(\Lambda) &:= \{\mathbf{NL}^n(\Lambda) \mid n \in \mathbb{Z}\},\end{aligned}$$

with homotopy groups

$$\begin{aligned}\pi_n(\mathbf{L}^\bullet(\Lambda)) &\cong L^n(\Lambda), \\ \pi_n(\mathbf{L}_\bullet(\Lambda)) &\cong L_n(\Lambda), \\ \pi_n(\mathbf{NL}^\bullet(\Lambda)) &\cong NL^n(\Lambda).\end{aligned}$$

Remark 7.3. The indexing of the L -spectra above is the opposite of the usual indexing in stable homotopy theory. Namely, if $\mathbf{E}$ is any of the spectra above we have $\mathbf{E}_{n+1} \simeq \Omega \mathbf{E}_n$.

Notation 7.4. To save space we will abbreviate

$$\mathbf{L}^\bullet = \mathbf{L}^n(\Lambda(\mathbb{Z})), \quad \mathbf{L}_\bullet = \mathbf{L}_n(\Lambda(\mathbb{Z})), \quad \mathbf{NL}^\bullet = \mathbf{NL}^n(\widehat{\Lambda}(\mathbb{Z})).$$

We note that the exact sequences from Propositions 3.21, 4.5, and 4.6 can be seen as the long exact sequences of the homotopy groups of fibration sequences of spectra. We are mostly interested in the following special case.

Proposition 7.5. *Let R be a ring with involution. Then we have a fibration sequence of spectra*

$$\mathbf{L}_\bullet(\Lambda(R)) \rightarrow \mathbf{L}^\bullet(\Lambda(R)) \rightarrow \mathbf{NL}^\bullet(\Lambda(R)).$$

Proof. Consider the fiber of the map of spectra $\mathbf{L}^\bullet(\Lambda(R)) \rightarrow \mathbf{NL}^\bullet(\Lambda(R))$. Use algebraic surgery to identify it with $\mathbf{L}_\bullet(\Lambda(R))$ just as in the proof of Proposition 3.21. $\square$

In fact the L -theory spectra are modeled on some geometric spectra. We will use the notion of a $(k+2)$ -ad (of spaces) and manifold $(k+2)$ -ads as defined in [23, § 0].

Definition 7.6. Let $n \in \mathbb{Z}$ and Ω_n^{STOP} and Ω_n^{N} be pointed Δ -sets defined by

$$\begin{aligned} (\Omega_n^{\text{STOP}})^{(k)} &= \{(M, \partial_0 M, \dots, \partial_k M) \mid (n+k)\text{-dimensional manifold} \\ &\quad (k+2)\text{-ad such that } \partial_0 M \cap \dots \cap \partial_k M = \emptyset\}, \\ (\Omega_n^{\text{N}})^{(k)} &= \{(X, \nu, \rho) \mid (n+k)\text{-dimensional normal space } (k+2)\text{-ad} \\ &\quad X = (X, \partial_0 X, \dots, \partial_k X) \text{ such that } \partial_0 X \cap \dots \cap \partial_k X = \emptyset, \\ &\quad \nu : X \rightarrow \text{BSG}(r) \text{ and } \rho : \Delta^{n+k+r} \rightarrow \text{Th}(\nu_X) \text{ such that} \\ &\quad \rho(\partial_i \Delta^{n+k+r}) \subset \text{Th}(\nu_{\partial_i X})\}. \end{aligned}$$

Face maps $\partial_i : (\Omega_n^?)^{(k)} \rightarrow (\Omega_n^?)^{(k-1)}$, $0 \leq i \leq k$ are given in both cases by

$$\partial_i(X) = (\partial_i X, \partial_0 X \cap \partial_i X, \dots, \partial_{i-1} X \cap \partial_i X, \partial_{i+1} X \cap \partial_i X, \dots, \partial_n X \cap \partial_i X,).$$

Here a convention is used that an empty space is a manifold (normal space) of any dimension $n \in \mathbb{Z}$ and it is a base point in all the dimensions.

Proposition 7.7. *We have Ω -spectra of pointed Kan Δ -sets*

$$\Omega_\bullet^{\text{STOP}} := \{\Omega_n^{\text{STOP}} \mid n \in \mathbb{Z}\}, \quad \Omega_\bullet^{\text{N}} := \{\Omega_n^{\text{N}} \mid n \in \mathbb{Z}\},$$

with homotopy groups

$$\pi_n(\Omega^{\text{STOP}}) = \Omega_n^{\text{STOP}}, \quad \pi_n(\Omega^{\text{N}}) = \Omega_n^{\text{N}}.$$

Definition 7.8. For $n \in \mathbb{Z}$ let $\Sigma^{-1}\Omega_n^{\text{N},\text{STOP}}$ be the pointed Δ -set defined as the fiber of the map of Δ -sets

$$\Sigma^{-1}\Omega_n^{\text{N},\text{STOP}} = \text{Fiber}(\Omega_n^{\text{STOP}} \rightarrow \Omega_n^{\text{N}}).$$

The collection $\Sigma^{-1}\Omega_n^{\text{N},\text{STOP}}$ becomes an Ω -spectrum of Δ -sets.

Remark 7.9. Again, the indexing of the above spectra is the opposite of the usual indexing in stable homotopy theory. To see that the spectra are indeed Ω -spectra observe that an $(n+1+k-1)$ -dimensional $(k-1+2)$ -ad is the same as an $(n+k)$ -dimensional $(k+2)$ -ad whose faces ∂_0 and $\partial_1 \dots \partial_k$ are empty. Similar observation is used in the algebraic situation.

Hence we have a homotopy fibration sequence of spectra

$$(7.1) \quad \Sigma^{-1}\Omega_{\bullet}^{\mathbf{N},\text{STOP}} \rightarrow \Omega_{\bullet}^{\text{STOP}} \rightarrow \Omega_{\bullet}^{\mathbf{N}}.$$

The fibration sequences from Proposition 7.5 and of (7.1) are related by the signature maps as follows.

Proposition 7.10. *The relative symmetric construction produces*

$$(1) \text{ sign}^{\mathbf{L}^{\bullet}} : \Omega_{\mathbf{n}}^{\text{STOP}} \rightarrow \mathbf{L}^n(\Lambda(\mathbb{Z})) \rightsquigarrow \text{sign}^{\mathbf{L}^{\bullet}} : \Omega_{\bullet}^{\text{STOP}} \rightarrow \mathbf{L}^{\bullet}.$$

The relative normal construction produces

$$(2) \text{ sign}^{\mathbf{NL}^{\bullet}} : \Omega_{\mathbf{n}}^{\mathbf{N}} \rightarrow \mathbf{NL}^n(\widehat{\Lambda}(\mathbb{Z})) \rightsquigarrow \text{sign}^{\mathbf{NL}^{\bullet}} : \Omega_{\bullet}^{\mathbf{N}} \rightarrow \mathbf{NL}^{\bullet}.$$

The relative normal construction together with the fibration sequence from Proposition 7.5 produces

$$(3) \text{ sign}^{\mathbf{L}^{\bullet}} : \Sigma^{-1}\Omega_{\mathbf{n}}^{\mathbf{N},\text{STOP}} \rightarrow \mathbf{L}_n(\Lambda(\mathbb{Z})) \rightsquigarrow \text{sign}^{\mathbf{L}^{\bullet}} : \Sigma^{-1}\Omega_{\bullet}^{\mathbf{N},\text{STOP}} \rightarrow \mathbf{L}_{\bullet}.$$

Proof. For (1) use Construction 5.7 which is just a generalization of the relative symmetric construction. For (2) the relative normal construction can be used. The full details are complicated, they can be found in [25, Sec. 7]. For (3) observe that the relative normal construction provides us with a map to the fiber of the map $\mathbf{L}^n(\Lambda(\mathbb{Z})) \rightarrow \mathbf{NL}^n(\Lambda(\mathbb{Z}))$. The identification of this fiber from Proposition 7.5 produces the desired map. $\square$

8. GENERALIZED HOMOLOGY THEORIES

Now we come to the use of the spectra just defined to produce (co-)homology. Definition 8.1 below contains the formulas. In addition the S -duality gives an opportunity to express homology as cohomology and vice versa. In our application it turns out that the input we obtain is of cohomological nature, but we would like to think of it in terms of homology. Therefore the strategy is adopted which comes under the slogan: “homology is the cohomology of the S -dual”. Here in fact a simplicial model for the S -duality will be useful when we work with particular cycles. For L -theory spectra a relation to the L -groups of algebraic bordism categories from Section 5 will be established.

The following definitions are standard.

Definition 8.1. Let $\mathbf{E}$ be an Ω -spectrum of Kan Δ -sets and let K be locally finite Δ -set.

(1) The *cohomology with $\mathbf{E}$ -coefficients* is defined by

$$H^n(K; \mathbf{E}) = \pi_{-n}(\mathbf{E}^{K_+}) = [K_+, \mathbf{E}_{-n}]$$

where $\mathbf{E}^{K_+}$ is the mapping Δ -set given by

$$(\mathbf{E}_{-n}^{K_+})^{(p)} = \{K_+ \otimes \Delta^p \rightarrow \mathbf{E}_{-n}\}.$$

(2) The *homology with $\mathbf{E}$ -coefficients* is defined by

$$H_n(K; \mathbf{E}) = \pi_n(K_+ \wedge \mathbf{E}) = \text{colim } \pi_{n+j}(K_+ \wedge \mathbf{E}_{-j})$$

where $K_+ \wedge \mathbf{E}$ is the Ω -spectrum of Δ -sets given by

$$(K_+ \wedge \mathbf{E}) = \{\text{colim } \Omega^j(K_+ \wedge \mathbf{E}_{n-j}) \mid n \in \mathbb{Z}\}.$$

What follows is a combinatorial description of S -duality from [27] and [19].

Definition 8.2. Let $K \subset L$ be an inclusion of a simplicial subcomplex. The *supplement of K in L* is the subcomplex of the barycentric subdivision L' defined by

$$\overline{K} = \{\sigma' \in L' \mid \text{no face of } \sigma' \text{ is in } K'\} = \bigcup_{\sigma \in L, \sigma \notin K} D(\sigma, L) \subset L'.$$

Next we come to the special case when $L = \partial\Delta^{m+1}$. In this case the dual cell decomposition of $\partial\Delta^{m+1}$ can in fact be considered as a simplicial complex, which turns out to be convenient. First a definition.

Definition 8.3. Define the simplicial complex Σ^m by

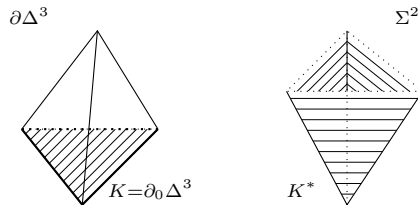
$$(\Sigma^m)^{(k)} = \{\sigma^* \mid \sigma \in (\partial\Delta^{m+1})^{(m-k)}\}$$

$$\partial_i : (\Sigma^m)^{(k)} \rightarrow (\Sigma^m)^{(k-1)} \text{ for } 0 \leq i \leq k \text{ is } \partial_i : \sigma^* \mapsto (\delta_i \sigma)^*$$

with $\delta_i : (\partial\Delta^{m+1})^{(m-k)} \rightarrow (\partial\Delta^{m+1})^{(m-k+1)}$ given by

$$\delta_i : \sigma = \{0, \dots, m+1\} \setminus \{j_0, \dots, j_k\} \mapsto \sigma \cup \{j_i\}, \quad (j_0 < j_1 < \dots < j_k).$$

So Σ^m has one k -simplex σ^* for each $(m-k)$ -simplex σ of $\partial\Delta^{m+1}$ and $\sigma^* \leq \tau^*$ if and only if $\sigma \geq \tau$.



The usefulness of this definition is apparent from the following proposition, namely that each dual cell in $\partial\Delta^{m+1}$ appears as a simplex in Σ^m .

Proposition 8.4. *There is an isomorphism of simplicial complexes*

$$\Phi : (\Sigma^m)' \xrightarrow{\cong} (\partial\Delta^{m+1})'$$

such that for each $\sigma \in K \subset \partial\Delta^{m+1}$ we have

$$\Phi(\sigma^*) = D(\sigma, \partial\Delta^{m+1}) \quad \text{and} \quad \Phi(\sigma^*) \cap K' = D(\sigma, K).$$

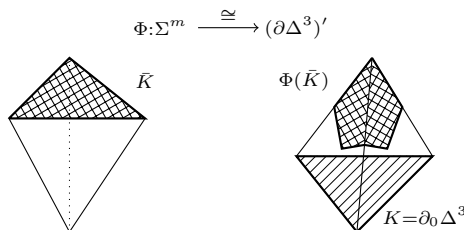
Notice that since $\partial\Delta^{m+1}$ is an m -dimensional manifold, every dual cell $D(\sigma, \partial\Delta^{m+1})$ is a submanifold with boundary of dimension $(m - |\sigma|)$ which coincides with the dimension of σ^* .

Proof. The isomorphism Φ is given by the formula

$$\begin{array}{ccc} (\sigma^*)' = \{\hat{\sigma}_0^* \hat{\sigma}_1^* \dots \hat{\sigma}_p^* \mid \sigma_p^* < \dots < \sigma_1^* < \sigma_0^* \leq \sigma^*\}, & & \hat{\sigma}_0^* \hat{\sigma}_1^* \dots \hat{\sigma}_p^* \\ \downarrow \Phi & & \downarrow \\ D(\sigma, \partial \Delta^{m+1}) = \{\hat{\sigma}_0 \hat{\sigma}_1 \dots \hat{\sigma}_p \mid \sigma \leq \sigma_0 < \sigma_1 < \dots < \sigma_p\}, & & \hat{\sigma}_0 \hat{\sigma}_1 \dots \hat{\sigma}_p \end{array}$$

□

The isomorphism of course induces a homeomorphism of geometric realizations. For $m = 2$ it looks like this:



Proposition 8.5. [19, Prop. 12.4] *Let $\mathbf{E}$ be a Ω -spectrum of Kan Δ -sets and K a finite simplicial complex. Then for $m \in \mathbb{N}$ large enough we have*

$$H_n(K; \mathbf{E}) \cong H^{m-n}(\Sigma^m, \overline{K}; \mathbf{E}).$$

Proof. The above proposition allows us to think of K as being embedded in Σ^m and the complex $\Sigma^m / \overline{K}$ is the quotient of Σ^m by the complement of a neighborhood of K . This is a well known construction of an m -dimensional S -dual of K , which is proved in detail for example in [27, p. 265]. The construction there provides an explicit simplicial construction of a map $\Delta' : \Sigma^m \rightarrow K_+ \wedge (\Sigma^m / \overline{K})$ which turns out to be such an S -duality. □

We remark that if K is an n -dimensional Poincaré complex with the SNF $\nu_K : K \rightarrow \text{BSG}(m-n)$ then $\Sigma^m / \overline{K} \simeq \text{Th}(\nu_K)$.

Now we come to the promised alternative definition of the homology of K .

Definition 8.6. Let $\mathbf{E}$ be an Ω -spectrum of Δ -sets. An n -dimensional $\mathbf{E}$ -cycle in K is a collection

$$x = \{x(\sigma) \in \mathbf{E}_{n-m}^{(m-|\sigma|)} \mid \sigma \in K\}$$

$$\text{such that } \partial_i x(\sigma) = \begin{cases} x(\delta_i \sigma), & \text{if } \delta_i \sigma \in K, \\ \emptyset, & \text{if } \delta_i \sigma \notin K, \end{cases} \quad (0 \leq i \leq m - |\sigma|).$$

A cobordism of n -dimensional $\mathbf{E}$ -cycles x_0, x_1 in K is a Δ -map

$$y : (\Sigma^m, \overline{K}) \otimes \Delta^1 \rightarrow (\mathbf{E}_{n-m}, \emptyset)$$

such that $y(\sigma \otimes i) = x_i(\sigma) \in \mathbf{E}_{n-m}^{m-|\sigma|}$ for $\sigma \in K$ and $i = 0, 1$.

Proposition 8.7. [19, Prop. 2.8] *There is a bijection between the set of cobordism equivalence classes of n -dimensional $\mathbf{E}$ -cycles in K and the n -dimensional $\mathbf{E}$ -homology group $H_n(K, \mathbf{E})$.*

Proof. A n -dimensional $\mathbf{E}$ -cycle x defines a Δ -map

$$(\Sigma^m, \overline{K}) \rightarrow \mathbf{E}_{n-m}, \sigma^* \mapsto \begin{cases} x(\sigma), & \sigma \in K, \\ \emptyset, & \sigma \notin K, \end{cases}$$

and the cobordism relation of cycles corresponds to the homotopy relation of Δ -maps. $\square$

Proposition 8.8. [19, Prop. 13.7], [10, Rem. 14.2] *Let K be a finite simplicial complex and Λ an algebraic bordism category. Then*

- (i) $\mathbf{L}_\bullet(\Lambda)^{K+} \simeq \mathbf{L}_\bullet(\Lambda^*(K))$ and $\mathbf{L}^\bullet(\Lambda)^{K+} \simeq \mathbf{L}^\bullet(\Lambda^*(K))$,
- (ii) $K_+ \wedge \mathbf{L}_\bullet(\Lambda) \simeq \mathbf{L}_\bullet(\Lambda_*(K))$ and $K_+ \wedge \mathbf{L}^\bullet(\Lambda) \simeq \mathbf{L}^\bullet(\Lambda_*(K))$.

Corollary 8.9. *For the algebraic bordism category $\Lambda = \Lambda(\mathbb{Z})$ we have*

$$L_n(\Lambda(\mathbb{Z})_*(K)) \cong H_n(K, \mathbf{L}_\bullet) \text{ and } L^n(\Lambda(\mathbb{Z})_*(K)) \cong H_n(K, \mathbf{L}^\bullet).$$

Proof of Corollary. For any Λ we have

$$L_n(\Lambda_*(K)) \cong \pi_n(\mathbf{L}_\bullet(\Lambda_*(K))) \cong \pi_n(K_+ \wedge \mathbf{L}_\bullet(\Lambda)) \cong H_n(K, \mathbf{L}_\bullet(\Lambda))$$

and similarly in the symmetric case. $\square$

Proof of (i). Since the morphisms in the category $\Lambda^*(K)$ only go from bigger to smaller simplices we can split an n -dimensional QAC $(C, \varphi) \in \Lambda^*(K)$ over K into a collection of n -dimensional QACs $\{(C_\sigma, \varphi_\sigma) \in \Lambda^*(\Delta^{|\sigma|})\}$ over standard simplices such that the $(C_\sigma, \varphi_\sigma)$ are related to each other in the same way the corresponding simplices are related to each other in K , i.e. $C_\sigma(\partial_i \sigma) = C_{\partial_i \sigma}(\partial_i \sigma)$ for all $\sigma \in K$. The complex $(C_\sigma, \varphi_\sigma)$ is a $|\sigma|$ -simplex in $\mathbf{L}_n(\Lambda)$ and the compatibility conditions are contained in the notion of Δ -maps.

Hence we get

$$\begin{aligned} (C, \varphi) &= \{n\text{-dim. QAC } (C_\sigma, \varphi_\sigma) \in \Lambda^*(\Delta^{|\sigma|}) \mid \\ &\quad \sigma \in K \text{ and } C_\sigma(\partial_i \sigma) = C_{\partial_i \sigma}(\partial_i \sigma)\} \\ &= \Delta\text{-map } f_C : K_+ \rightarrow \mathbf{L}_n(\Lambda) \text{ with } f(\sigma) = (C_\sigma, \varphi_\sigma) \text{ for } \sigma \in K_+. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{L}_n(\Lambda^*(K))^{(k)} &= \{n\text{-dim. QAC } (C, \varphi) \in \Lambda^*(K)^*(\Delta^k) \simeq \Lambda^*(K \otimes \Delta^k)\} \\ &= \{f : (K \otimes \Delta^k)_+ \rightarrow \mathbf{L}_n(\Lambda) \mid f \text{ is a pointed } \Delta\text{-map}\} \\ &= (\mathbf{L}_n(\Lambda)^{K+})^{(k)}. \end{aligned}$$

$\square$

Proof of (ii). For $m \in \mathbb{N}$ large enough consider an embedding $i : K \rightarrow \partial\Delta^{m+1}$, the complex Σ^m and the supplement $\bar{K}$ in Σ^m as in Definition 8.3. The first observation is that there is an isomorphism of algebraic bordism categories

$$\Lambda_*(K) \cong \Lambda^*(\Sigma^m, \bar{K}).$$

This follows from the existence of the one-to-one correspondence $\sigma \leftrightarrow \sigma^*$ between k -simplices of K and $(m-k)$ -simplices of $\Sigma^m \setminus \bar{K}$ which have the property $\sigma \leq \tau$ if and only if $\sigma^* \geq \tau^*$ and the symmetry in the definition of the dualities T_* and T^* .

The observation leads to

$$\mathbf{L}_\bullet(\Lambda_*(K)) \cong \mathbf{L}_\bullet(\Lambda^*(\Sigma^m, \bar{K})) \simeq \mathbf{L}_\bullet(\Lambda)^{(\Sigma^m, \bar{K})} \simeq K_+ \wedge \mathbf{L}_\bullet(\Lambda),$$

where the last homotopy equivalence is a spectrum version of the isomorphism in Proposition 8.5. $\square$

Remark 8.10. Recall that in Section 5 we have defined various structured algebraic complexes over X . By theorems of this section some of them represent homology classes with coefficients in the L -theory spectra. Alternatively to the explicit construction above a different approach in [26] proves that these homology groups $H_n(K, \mathbf{E})$ are induced by homotopy invariant and excisive functors $K \rightarrow \mathbf{E}(\Lambda_*(K))$ and hence this construction is natural in K .

Definition 8.11. Let X be an n -dimensional closed topological manifold with a map $r : X \rightarrow K$ to a simplicial complex. The cobordism class of the n -dimensional SAC in $\Lambda(\mathbb{Z})_*(K)$ obtained from any choice of the fundamental class $[X] \in C_n(X)$ in Construction 5.11 does not depend on the choice of $[X]$ and hence defines an element

$$\text{sign}_K^{\mathbf{L}^\bullet}(X) = (C(X), \varphi_K([X])) \in H_n(K; \mathbf{L}^\bullet)$$

called the *symmetric signature* of X over K .

Definition 8.12. Let X be an n -dimensional closed topological manifold with a map $r : X \rightarrow K$ to a simplicial complex. Recall the spectrum $\Omega_\bullet^{\text{STOP}}$ from Section 7. Note that the K -dissection of X obtained by making r transverse to the dual cells gives a compatible collection of manifolds with boundary so that the assignment $\sigma \rightarrow X[\sigma]$ is precisely an n -dimensional $\Omega_\bullet^{\text{STOP}}$ -cycle. We call it the *STOP-signature* of X over K and denote

$$\text{sign}_K^{\Omega_\bullet^{\text{STOP}}}(X) \in H_n(K; \Omega_\bullet^{\text{STOP}}).$$

Remark 8.13. The symmetric signature $\text{sign}_K^{\mathbf{L}^\bullet}(X)$ can be seen as obtained from the STOP-signature $\text{sign}_K^{\Omega_\bullet^{\text{STOP}}}(X)$ by applying the symmetric signature map on the level of spectra, that means the map $\text{sign}^{\mathbf{L}^\bullet}$ from Proposition 7.10. In fact the STOP-signature and hence the symmetric signature only depend on the oriented cobordism class of X , and so we obtain a homomorphism

$$\text{sign}_K^{\mathbf{L}^\bullet} : \Omega_n^{\text{STOP}}(K) \rightarrow H_n(K; \mathbf{L}^\bullet).$$

Definition 8.14. Let $(f, b) : M \rightarrow X$ be a degree one normal map of n -dimensional closed topological manifolds and let $r : X \rightarrow K$ be a map to a simplicial complex. The cobordism class of the n -dimensional QAC in $\Lambda(\mathbb{Z})_*(K)$ obtained from any choice of the fundamental class $[X] \in C_n(X)$ in Construction 5.12 does not depend on the choice of $[X]$ and hence defines an element

$$\text{sign}_K^{\mathbf{L}^\bullet}(f, b) = (\mathcal{C}(f^!), e\% \psi_K([X])) \in H_n(K; \mathbf{L}_\bullet)$$

called the *quadratic signature* of the degree one normal map (f, b) over K . In fact the quadratic signature only depends on the normal cobordism class of (f, b) in the set of normal invariants $\mathcal{N}(X)$ and provides us with a function

$$\text{sign}_K^{\mathbf{L}^\bullet} : \mathcal{N}(X) \rightarrow H_n(K; \mathbf{L}_\bullet).$$

In order to obtain an analogue of Proposition 8.8 for $\mathbf{NL}^\bullet$ spectra we need to introduce yet another algebraic bordism category associated to Λ and K .

Definition 8.15. Let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ be an algebraic bordism category and K a locally finite simplicial complex. Define the algebraic bordism category

$$\widehat{\Lambda}(K) = (\mathbb{A}_*(K), \mathbb{B}_*(K), \mathbb{B}_*(K))$$

where $\mathbb{A}_*(K)$ and $\mathbb{B}_*(K)$ are as in Section 5.

Proposition 8.16. [19, Prop. 14.5] *Let K be a finite simplicial complex and Λ an algebraic bordism category. Then*

$$K_+ \wedge \mathbf{NL}^\bullet(\Lambda) \simeq \mathbf{NL}^\bullet(\widehat{\Lambda}(K)).$$

To complete the picture we present the following proposition which follows from Lemma 3.19 and Proposition 8.8

Proposition 8.17. *We have*

$$\mathbf{NL}^\bullet(\Lambda_*(K)) \simeq \mathbf{L}^\bullet(\Lambda_*(K)) \simeq K_+ \wedge \mathbf{L}^\bullet(\Lambda).$$

Remark 8.18. Recall the idea of the assembly map from Section 6. Via Proposition 8.8 it induces a map

$$A : H_n(K; \mathbf{L}_\bullet) = \pi_n(K_+ \wedge \mathbf{L}_\bullet) \rightarrow L_n(\mathbb{Z}[\pi_1(K)]) = \pi_n(\mathbf{L}_\bullet(\Lambda(\mathbb{Z}[\pi_1(K)]))).$$

If $\pi_1(K) = 0$ then this map can be thought of as an induced map on homology by the collapse map $K \mapsto *$. Similarly for spectra $\mathbf{L}^\bullet$ and $\mathbf{NL}^\bullet$. However, this is not a phenomenon special to these spectra. In fact in [19, Chap. 12] an assembly map

$$A : H_n(K; \mathbf{E}) \rightarrow \pi_n(\mathbf{E})$$

is discussed for any spectrum $\mathbf{E}$, hence any homology theory. On the level of chains this map can be described via certain “gluing” procedure. For the spectra $\Omega_\bullet^N$ and $\Omega_\bullet^{\text{STOP}}$ this procedure coincides with the geometric gluing.

9. CONNECTIVE VERSIONS

An important technical aspect of the theory is the use of connective versions of the L -theory spectra. This is related to the difference between topological manifolds and ANR-homology manifolds. In principle there are two ways how to impose connectivity restrictions. One is to fix the algebraic bordism category and modify the definition of the L -groups and L -spectra. The other is to modify the algebraic bordism category and keep the definition of the L -groups and L -spectra. Both ways are convenient at some stages.

Proposition 9.1. *Let Λ be an algebraic bordism category and let $q \in \mathbb{Z}$. Then there are Ω -spectra of Kan Δ -sets $\mathbf{L}_\bullet\langle q\rangle(\Lambda)$, $\mathbf{L}^\bullet\langle q\rangle(\Lambda)$, $\mathbf{NL}^\bullet\langle q\rangle(\Lambda)$ with homotopy groups*

$$\begin{aligned}\pi_n \mathbf{L}_\bullet\langle q\rangle(\Lambda) &= \mathbf{L}_n(\Lambda) \text{ for } n \geq q, \text{ } 0 \text{ for } n < q, \\ \pi_n \mathbf{L}^\bullet\langle q\rangle(\Lambda) &= \mathbf{L}^n(\Lambda) \text{ for } n \geq q, \text{ } 0 \text{ for } n < q, \\ \pi_n \mathbf{NL}^\bullet\langle q\rangle(\Lambda) &= \mathbf{NL}^n(\Lambda) \text{ for } n \geq q, \text{ } 0 \text{ for } n < q.\end{aligned}$$

Definition 9.2. Let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ be an algebraic bordism category, and let $q \in \mathbb{Z}$. Denote $\mathbb{B}\langle q\rangle \subset \mathbb{B}$ the category of chain complexes in $\mathbb{B}$ which are homotopy equivalent to q -connected chain complexes and $\mathbb{C}\langle q\rangle = \mathbb{B}\langle q\rangle \cap \mathbb{C}$. The algebraic bordism categories $\Lambda\langle q\rangle$ and $\Lambda\langle 1/2\rangle$ are defined by

$$\Lambda\langle q\rangle = (\mathbb{A}, \mathbb{B}\langle q\rangle, \mathbb{C}\langle q\rangle) \quad \text{and} \quad \Lambda\langle 1/2\rangle = (\mathbb{A}, \mathbb{B}\langle 0\rangle, \mathbb{C}\langle 1\rangle).$$

Notation 9.3. In accordance with [19] we will now use the notation $\Lambda\langle q\rangle(R)$ for $\Lambda(R)\langle q\rangle$ where R is a ring with involution, $q \in \mathbb{Z}$ or $q = 1/2$ and $\Lambda(R)$ is the algebraic bordism category from Example 4.4. Similarly $\widehat{\Lambda}\langle q\rangle(R)$ stands for $\widehat{\Lambda}(R)\langle q\rangle$.

In general the groups $L^n(\Lambda\langle q\rangle(R)) = \pi_n \mathbf{L}^\bullet(\Lambda\langle q\rangle(R))$ need not be isomorphic to $\pi_n \mathbf{L}^\bullet\langle q\rangle(\Lambda(R))$, likewise for quadratic and normal L -groups. However, in certain special cases this holds, for example we have ([19, Ex. 15.8]):

$$\begin{aligned}\pi_n \mathbf{L}_\bullet\langle 0\rangle(\Lambda(\mathbb{Z})) &\cong L_n(\Lambda\langle 0\rangle(\mathbb{Z})), \\ \pi_n \mathbf{L}^\bullet\langle 0\rangle(\Lambda(\mathbb{Z})) &\cong L^n(\Lambda\langle 0\rangle(\mathbb{Z})).\end{aligned}$$

Notation 9.4. Again, to save space we abbreviate for $q \in \mathbb{Z}$ or $q = 1/2$:

$$\mathbf{L}^\bullet\langle q\rangle = \mathbf{L}^n(\Lambda\langle q\rangle(\mathbb{Z})), \quad \mathbf{L}_\bullet\langle q\rangle = \mathbf{L}_n(\Lambda\langle q\rangle(\mathbb{Z})), \quad \mathbf{NL}^\bullet\langle q\rangle = \mathbf{NL}^n(\widehat{\Lambda}\langle q\rangle(\mathbb{Z})).$$

We also obtain a connective version of Proposition 7.5.

Proposition 9.5. [19, Prop. 15.16] *We have a homotopy fibration sequence*

$$(9.1) \quad \mathbf{L}_\bullet\langle 1\rangle \rightarrow \mathbf{L}^\bullet\langle 0\rangle \rightarrow \mathbf{NL}^\bullet\langle 1/2\rangle.$$

As a consequence we have

$$\pi_0 \mathbf{L}^\bullet\langle 0\rangle \cong \pi_0 \mathbf{NL}^\bullet\langle 1/2\rangle \cong \mathbb{Z}.$$

Let K be a simplicial complex. Now we consider the connective versions of the L -theory groups/spectra of algebraic bordism categories associated to K in Sections 5 and 8. Specifically we are interested in the algebraic bordism category $\Lambda\langle q\rangle(\mathbb{Z})_*(K)$ from Definition 5.5 and $\widehat{\Lambda}\langle q\rangle(\mathbb{Z})(K)$ from Definition 8.15. The following proposition is an improvement on Propositions 8.8, 8.17 and 6.6.

Proposition 9.6. [19, Prop. 15.9,15.11] *There are isomorphisms*

$$\begin{aligned}\pi_n \mathbf{L}^\bullet(\Lambda\langle q\rangle(\mathbb{Z})_*(K)) &\cong H_n(K; \mathbf{L}^\bullet\langle q\rangle(\mathbb{Z})), \\ \pi_n \mathbf{L}_\bullet(\Lambda\langle q\rangle(\mathbb{Z})_*(K)) &\cong H_n(K; \mathbf{L}_\bullet\langle q\rangle(\mathbb{Z})), \\ \pi_n \mathbf{NL}^\bullet(\widehat{\Lambda}\langle q\rangle(\mathbb{Z})(K)) &\cong H_n(K; \mathbf{NL}^\bullet\langle q\rangle(\mathbb{Z})), \\ \pi_n \mathbf{L}_\bullet(\Lambda\langle q\rangle(K)) &\cong L_n(\mathbb{Z}[\pi_1[K]]) \quad \text{for } n \geq 2q.\end{aligned}$$

In addition there is an improved version of Proposition 7.10 as follows:

Proposition 9.7. *The relative symmetric construction produces*

$$(1) \operatorname{sign}^{\mathbf{L}^\bullet} : \Omega_n^{\mathbf{S}^{\text{TOP}}} \rightarrow \mathbf{L}^n(\Lambda\langle 0\rangle(\mathbb{Z})) \rightsquigarrow \operatorname{sign}^{\mathbf{L}^\bullet} : \Omega_\bullet^{\mathbf{S}^{\text{TOP}}} \rightarrow \mathbf{L}^\bullet\langle 0\rangle.$$

The relative normal construction produces

$$(2) \operatorname{sign}^{\mathbf{NL}^\bullet} : \Omega_n^{\mathbf{N}} \rightarrow \mathbf{NL}^n(\widehat{\Lambda}\langle 1/2\rangle(\mathbb{Z})) \rightsquigarrow \operatorname{sign}^{\mathbf{NL}^\bullet} : \Omega_\bullet^{\mathbf{N}} \rightarrow \mathbf{NL}^\bullet\langle 1/2\rangle.$$

The relative normal construction produces

$$(3) \operatorname{sign}^{\mathbf{L}^\bullet} : \Sigma^{-1}\Omega_n^{\mathbf{N},\text{TOP}} \rightarrow \mathbf{L}_n(\Lambda\langle 1\rangle(\mathbb{Z})) \rightsquigarrow \operatorname{sign}^{\mathbf{L}^\bullet} : \Sigma^{-1}\Omega_\bullet^{\mathbf{N},\text{TOP}} \rightarrow \mathbf{L}_\bullet\langle 1\rangle.$$

Part (1) is obvious, since a geometric situation provides only chain complexes concentrated in nonnegative dimensions. Part (2) is shown in [19, p. 178]. Part (3) follows from part (2) and the fibration sequence from Proposition 9.5.

10. SURGERY SEQUENCES AND THE STRUCTURE GROUPS $\mathbb{S}_n(X)$

Now we have assembled all the tools needed to define, for a finite simplicial complex X , the group $\mathbb{S}_n(X)$, which is the home of the total surgery obstruction if X is an n -dimensional Poincaré complex. It is important and useful to define not only the group $\mathbb{S}_n(X)$ itself, but also to relate it to other groups which we might understand better. So the group $\mathbb{S}_n(X)$ is placed into a commutative braid, which is obtained from the braid in Section 4 by plugging in suitable algebraic bordism categories. We recall these now.

In fact the categories are, as indicated below, various connective versions of the following categories. The underlying additive category with chain duality is

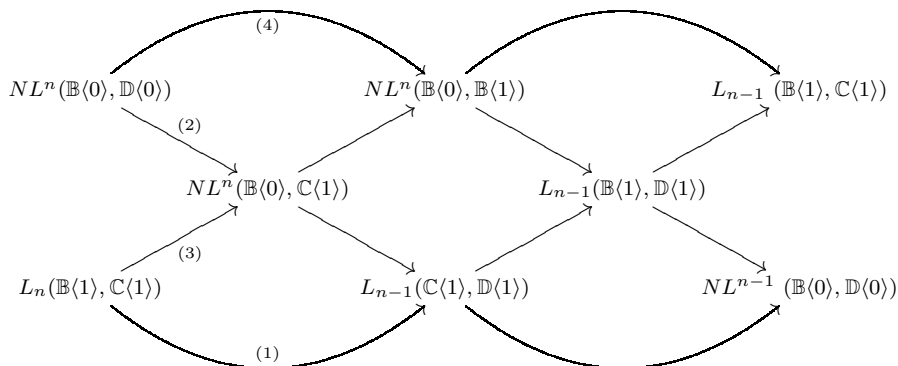
- $\mathbb{A} = \mathbb{Z}_*(X)$ is the additive category of finitely generated free $\mathbb{Z}$ -modules over X .

Now we specify the subcategories of $\mathbb{B}(\mathbb{A})$ needed to construct the braid.

- $\mathbb{B} = \mathbb{B}(\mathbb{Z}_*(X))$ are the bounded chain complexes in $\mathbb{A}$,
- $\mathbb{C} = \{C \in \mathbb{B} \mid A(C) \simeq *\}$ are the *globally contractible* chain complexes in $\mathbb{B}(\mathbb{A})$,

- $\mathbb{D} = \{C \in \mathbb{B} \mid C(\sigma) \simeq * \ \forall \sigma \in K\}$ are the *locally contractible* chain complexes in $\mathbb{B}(\mathbb{A})$.

The precise connective versions used are indicated in the braid diagram below, which is taken from [19, Prop. 15.18]. Due to the lack of space we have omitted the underlying category $\mathbb{A}$ from the notation as it is the same everywhere. We also note that obviously $\mathbb{D}\langle 0 \rangle = \mathbb{D}\langle 1 \rangle$:



Notice that the exact sequence labeled (1) is induced by the assembly functor $A : \Lambda(\mathbb{Z})\langle 1 \rangle_*(X) \rightarrow \Lambda(\mathbb{Z})\langle 1 \rangle(X)$ from Section 6. The other sequences are induced by analogous functors, the precise statements are left to the reader. It is more interesting at this stage that we have already identified various groups in the braid. We recapitulate using Propositions 9.6 and 8.17:

$$\begin{aligned} L_n(\mathbb{B}\langle 1 \rangle, \mathbb{C}\langle 1 \rangle) &= L_n(\mathbb{Z}[\pi_1(X)]), \\ L_n(\mathbb{B}\langle 1 \rangle, \mathbb{D}\langle 1 \rangle) &= H_n(X; \mathbf{L}_\bullet\langle 1 \rangle), \\ NL^n(\mathbb{B}\langle 0 \rangle, \mathbb{D}\langle 0 \rangle) &= H_n(X; \mathbf{L}^\bullet\langle 0 \rangle), \\ NL^n(\mathbb{B}\langle 0 \rangle, \mathbb{B}\langle 1 \rangle) &= H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle). \end{aligned}$$

The sequence containing only homology theories can be thought of as induced by the cofibration sequence from Proposition 9.5. In addition we have new terms as follows. We keep the notation from the beginning of this section.

Definition 10.1. [19, Chap. 17] Let X be a finite simplicial complex. Define the n -dimensional *structure group* of X to be

$$\mathbb{S}_n(X) = L_{n-1}(\mathbb{A}, \mathbb{C}\langle 1 \rangle, \mathbb{D}\langle 1 \rangle).$$

So an element in $\mathbb{S}_n(X)$ is represented by an $(n-1)$ -dimensional 1-connective QAC in $\mathbb{Z}_*(X)$ which is globally contractible and locally Poincaré. We will see in the following section how to obtain such a chain complex from a geometric situation.

Definition 10.2. [19, Chap. 17] Let X be a finite simplicial complex. Define the n -dimensional *visible symmetric L -group* of X to be

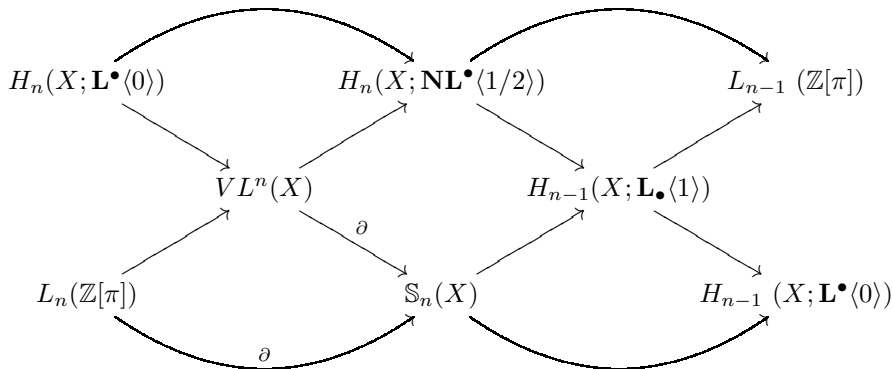
$$VL^n(X) = NL^n(\mathbb{A}, \mathbb{B}\langle 0 \rangle, \mathbb{C}\langle 1 \rangle).$$

The visible symmetric L -groups were defined by Weiss in [26] to clarify certain relations between the symmetric and quadratic L -groups. We will not need this aspect, what is important is that an element in $VL^n(X)$ is represented by an n -dimensional 0-connective NAC in $\mathbb{Z}_*(X)$ whose underlying symmetric structure is locally 0-connective and globally Poincaré. We will see in the following section a geometric situation which yields such a chain complex.

The sequence labeled (1) in the braid is known as the *algebraic surgery exact sequence*:

$$(10.1) \quad \rightarrow H_n(X, \mathbf{L}_\bullet\langle 1 \rangle) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{\partial} \mathbb{S}_n(X) \rightarrow H_{n-1}(X; \mathbf{L}_\bullet\langle 1 \rangle) \rightarrow .$$

Summarizing the above identification we obtain the commutative braid which will be our playground in the rest of the paper:



11. NORMAL SIGNATURES OVER X

As indicated in the introduction in order to define the total surgery obstruction of X we need to discuss the normal and the visible signature of X . In this section we define the visible signature $\text{sign}_X^{\mathbf{VL}_\bullet}(X) \in VL^n(X)$ as a refinement of the normal signature $\text{sign}_X^{\mathbf{NL}_\bullet}(X) \in H_n(X; \mathbf{NL}_\bullet\langle 1/2 \rangle)$. In fact, it should be expected that any n -dimensional GNC (X, ν, ρ) has an associated normal signature over X which assembles to the normal signature over $\mathbb{Z}[\pi_1 X]$ defined in Section 3. However, we are not able to show such a general statement. We need to assume that the normal complex comes from a Poincaré complex with its SNF.¹⁶ Before we start we still need some technical preliminaries.

¹⁶In [22, Errata for p. 103] a construction for the normal signature over X for a normal complex X is actually given. However, since in the proof of the subsequent sections we

Construction 11.1. Recall some more ideas from [27] surrounding the concept of supplement described in Section 8. Let $K \subseteq L$ be a simplicial subcomplex. The supplement is a subcomplex $\bar{K} \subseteq L'$. As explained in [27] there is an embedding $|L| \subset |K'| * |\bar{K}|$ into the join of the two realizations. A point in $|K'| * |\bar{K}|$ can be described as $t \cdot x + (1-t) \cdot y$ for $x \in |K'|$, $y \in |\bar{K}|$, and $t \in [0, 1]$. The space $|L|$ can be decomposed as the union of two subspaces

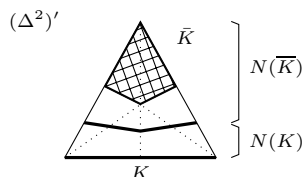
$$N = N(K') = \{t \cdot x + (1-t) \cdot y \mid t \geq 1/2\} \cap L,$$

$$\bar{N} = N(\bar{K}) = \{t \cdot x + (1-t) \cdot y \mid t \leq 1/2\} \cap L.$$

These come with obvious deformation retractions $r : N \rightarrow |K|$ and $\bar{r} : \bar{N} \rightarrow |\bar{K}|$.

Next denote $N(\sigma) = N \cap (|D(\sigma, K)| * |\bar{K}|)$ for $\sigma \in K$. Then we have the dissection $N = \cup_{\sigma \in K} N(\sigma)$ and the retraction r respects the dissections of N and $|K|$,

$$r|_{N(\sigma)} = r(\sigma) : N(\sigma) \rightarrow |D(\sigma, K)|.$$



Construction 11.2. Consider now the case when X is a finite simplicial Poincaré complex of dimension n which we embed into $\partial\Delta^{m+1}$, that means $K = X$ and $L = \partial\Delta^{m+1}$ in the above notation. For m large enough the homotopy fiber of the projection map $\partial r : \partial N = N \cap \bar{N} \rightarrow X$ is homotopy equivalent to S^{m-n-1} and the associated spherical fibration is the SNF ν_X . In more detail, there is a (D^{m-n}, S^{m-n-1}) -fibration $p : (D(\nu_X), S(\nu_X)) \rightarrow X$ and a homotopy equivalence of pairs $i : (N, \partial N) \rightarrow (D(\nu_X), S(\nu_X))$ such that the following diagram commutes

$$\begin{array}{ccc} (N, \partial N) & \xrightarrow{i} & (D(\nu_X), S(\nu_X)) \\ & \searrow r & \swarrow p \\ & X & \end{array}$$

The map p is now an honest fibration. Recall from Definition 8.3 the complex Σ^m and that we have an embedding $\bar{X} \subset \Sigma^m$. It follows that

$$|\Sigma^m / \bar{X}| \simeq N / \partial N \simeq \text{Th}(\nu_X).$$

directly use the specific properties of the construction presented in this section in the case when X is Poincaré, we only discuss this special case.

Construction 11.3. Now we would like to present an analogue of Construction 5.10 for normal complexes. What we are aiming for is an assignment

$$\text{sign}_X^{\Omega_\bullet^N}(X) : \sigma \mapsto (X(\sigma), \nu(\sigma), \rho(\sigma)) \in (\Omega_{n-m}^N)^{(m-|\sigma|)} \text{ for each } \sigma \in X.$$

The first two entries are defined as follows:

$$X(\sigma) = |D(\sigma, X)|, \quad \nu(\sigma) = \nu_X \circ \text{incl} : X(\sigma) \subset X \rightarrow \text{BSG}(m - n - 1).$$

To define $\rho(\sigma)$ consider the following commutative diagram

$$\begin{array}{ccc} N(\sigma) & \xrightarrow{\quad} & N \\ & \searrow \rho(N(\sigma)) & \searrow i \\ & D(\nu(\sigma)) & \xrightarrow{\quad} D(\nu_X) \\ & \downarrow & \downarrow \\ |D(\sigma, X)| & \xrightarrow{\quad} & |X|. \end{array}$$

In general the homotopy fibers of the projections $r(\sigma) : \partial N(\sigma) \rightarrow |D(\sigma, X)|$ are not S^{k-1} . On the other hand the pullback of ν_X along the inclusion $D(\sigma, X) \subset X$ yields an S^{m-n-1} -fibration $\nu(\sigma)$. The associated disc fibration is a pullback as indicated by the diagram, and we obtain the dashed map since the two compositions $N(\sigma) \rightarrow |X|$ commute.

Recall the $(m - |\sigma|)$ -dimensional simplex $\sigma^* \in \Sigma^m$. Observe that we have

$$\Delta^{m-|\sigma|} \cong |\sigma^*| = N(\sigma) \cup \overline{N}(\sigma)$$

where $\overline{N}(\sigma) = \overline{N} \cap (|D(\sigma, X)| * |\overline{X}|)$. Define the map

$$\begin{aligned} \rho(\sigma) &= \rho(N(\sigma)) \cup \rho(\overline{N}(\sigma)) : \\ \Delta^{m-|\sigma|} &\cong N(\sigma) \cup \overline{N}(\sigma) \rightarrow D(\nu(\sigma)) \cup \{*\} \cong \text{Th}(\nu(\sigma)) \end{aligned}$$

where the map $\rho(\overline{N}(\sigma))$ is the collapse map.

Definition 11.4. Let X be an n -dimensional finite Poincaré simplicial complex with the associated n -dimensional GNC (X, ν_X, ρ_X) . Then the assignment from Construction 11.3 defines an element

$$\text{sign}_X^{\Omega_\bullet^N}(X) \in H_n(X; \Omega_\bullet^N)$$

and is called the *geometric normal signature of X over X* .

Remark 11.5. The assembly of Remark 8.18 satisfies

$$A(\text{sign}_X^{\Omega_\bullet^N}(X)) = (X, \nu_X, \rho_X) \in \Omega_n^N.$$

Definition 11.6. Let X be an n -dimensional finite Poincaré simplicial complex with the associated n -dimensional GNC (X, ν_X, ρ_X) . The composition

of the geometric normal signature over X from Definition 11.4 with the normal signature map on the level of spectra from Proposition 9.7 produces a well-defined element

$$\text{sign}_X^{\mathbf{NL}^\bullet}(X) = \text{sign}^{\mathbf{NL}^\bullet} \circ \text{sign}_X^{\Omega^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle)$$

called the *normal signature of X over X* .

Remark 11.7. We have

$$A(\text{sign}_X^{\mathbf{NL}^\bullet}(X)) = \text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{NL}^\bullet}(X) \in NL^n(\mathbb{Z}[\pi_1(X)]).$$

Recall that an n -dimensional NAC has an underlying symmetric complex. In the case of $\text{sign}_X^{\mathbf{NL}^\bullet}(X)$ this is the complex obtained in Construction 5.10. It is not locally Poincaré, that means, it does not give a symmetric complex in $\Lambda(\mathbb{Z})_*(X)$. Its assembly is $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}^\bullet}(X) \in L^n(\mathbb{Z}[\pi_1(X)])$.

Definition 11.8. Let X be an n -dimensional finite Poincaré simplicial complex with the associated n -dimensional GNC (X, ν_X, ρ_X) . The assembly of the normal complex over X which defines the normal signature over X is Poincaré and hence produces an n -dimensional NAC in the algebraic bordism category $\Lambda(\mathbb{Z})(X)$ from Definition 6.4 and as such a well-defined element

$$\text{sign}_X^{\mathbf{VL}^\bullet}(X) \in VL^n(X)$$

called the *visible symmetric signature of X over X* .

Now we present another related construction. It will not be needed for the definition of the total surgery obstruction, but it will be used in the proof of the main theorem. (See Theorem 13.16.)

Let $(f, b) : M \rightarrow X$ be a degree one normal map of n -dimensional topological manifolds such that X is triangulated. As discussed in Example 3.27 the pair $(W, M \sqcup X)$, where W is the mapping cylinder of f , possesses a structure of an $(n+1)$ -dimensional geometric (normal, topological manifold) pair: the spherical fibration denoted by $\nu(b)$ is obtained as the mapping cylinder of $b : \nu_M \rightarrow \nu_X$ and the required map $\rho(b) : D^{n+k+1} \rightarrow \text{Th}(\nu(b))$ as the composition $D^{n+k+1} \rightarrow S^{n+k} \times [0, 1] \rightarrow \text{Th}(\nu(b))$. We can also view the pair $(W, M \sqcup X)$ as an n -simplex in the space $\Sigma^{-1}\Omega_0^{N, \text{STOP}}$. Hence via the relative normal construction we can associate to (f, b) an $(n+1)$ -dimensional (normal, symmetric Poincaré) algebraic pair

$$\text{sign}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(f, b) = (\text{sign}^{\mathbf{NL}^\bullet}(W), \text{sign}^{\mathbf{L}^\bullet}(M) - \text{sign}^{\mathbf{L}^\bullet}(X)) \in \pi_n(\mathbf{F})$$

where $\mathbf{F} := \text{Fiber } \mathbf{L}^\bullet\langle 0 \rangle \rightarrow \mathbf{NL}^\bullet\langle 1/2 \rangle$.

We would like to associate to (f, b) , respectively the pair $(W, M \sqcup X)$, an $(n+1)$ -dimensional (normal, symmetric Poincaré) algebraic pair over $\mathbb{Z}_*(X)$. This is not exactly a relative version of the previous definitions since the pair is not Poincaré. Nevertheless, in this special case we are able to obtain what we want.

Construction 11.9. Let $(f, b) : M \rightarrow X$ be a degree one normal map between n -dimensional topological manifolds such that X is triangulated. We can assume that f is transverse to the dual cell decomposition of X . Consider the dissection

$$X = \bigcup_{\sigma \in X} X(\sigma), \quad (f, b) = \bigcup_{\sigma \in X} (f(\sigma), b(\sigma)) : M(\sigma) \rightarrow X(\sigma)$$

where each $(f(\sigma), b(\sigma))$ is a degree one normal map of $(n - |\sigma|)$ -dimensional manifolds $(m - |\sigma|)$ -ads. We obtain an assignment which to each $\sigma \in X$ associates an $(n + 1 - |\sigma|)$ -dimensional pair of normal $(m - |\sigma|)$ -ads

$$\sigma \mapsto ((W(\sigma), \nu(b(\sigma)), \rho(b(\sigma))), M(\sigma) \sqcup X(\sigma)).$$

These fit together to produce an $\Omega_{\bullet}^N$ -cobordism of $\Omega_{\bullet}^{\text{STOP}}$ -cycles in the sense of Definition 8.6, or equivalently a $\Sigma^{-1}\Omega_{\bullet}^{N, \text{STOP}}$ -cycle, providing us with an element

$$\text{sign}_X^{\text{G/TOP}}(f, b) \in H_n(X; \Sigma^{-1}\Omega_{\bullet}^{N, \text{STOP}}).$$

Composing with the normal signature map $\text{sign}^{\text{NL}^{\bullet}} : \Omega_{\bullet}^N \rightarrow \text{NL}^{\bullet}\langle 1/2 \rangle$ then produces a $\text{NL}^{\bullet}\langle 1/2 \rangle$ -cobordism, which can be seen as an $(n + 1)$ -dimensional (normal, symmetric Poincaré) pair over $\mathbb{Z}_*(X)$

$$\text{sign}_X^{\text{NL}^{\bullet}, \text{L}^{\bullet}}(f, b) = \text{sign}^{\text{NL}^{\bullet}, \text{L}^{\bullet}}(\text{sign}_X^{\text{G/TOP}}(f, b)) \in H_n(X; \mathbf{F}).$$

Then the homological assembly of Remark 8.18 gives the $(n + 1)$ -dimensional (normal, symmetric Poincaré) pair

$$\text{sign}^{\text{NL}^{\bullet}, \text{L}^{\bullet}}(f, b) \in \pi_n(\mathbf{F}).$$

Remark 11.10. Recall from Example 3.27 the correspondence

$$(\text{sign}^{\text{NL}^{\bullet}}(W), \text{sign}^{\text{L}^{\bullet}}(M) - \text{sign}^{\text{L}^{\bullet}}(X)) \longleftrightarrow \text{sign}^{\text{L}^{\bullet}}(f, b)$$

where $\text{sign}^{\text{L}^{\bullet}}(f, b) \in L_n(\mathbb{Z})$ is the quadratic signature (=surgery obstruction) of the degree one normal map (f, b) . Using the relative version of Example 3.27 we obtain in this situation an identification of $\text{sign}_X^{\text{NL}^{\bullet}, \text{L}^{\bullet}}(f, b)$ with the a quadratic signature of Construction 5.12

$$\text{sign}_X^{\text{NL}^{\bullet}, \text{L}^{\bullet}}(f, b) = \text{sign}_X^{\text{L}^{\bullet}}(f, b) \in H_n(X; \mathbf{L}_{\bullet}\langle 1 \rangle).$$

12. DEFINITION OF $s(X)$

Definition 12.1. [19, 17.1] Define

$$(12.1) \quad s(X) := \partial(\text{sign}_X^{\text{VL}^{\bullet}}(X)) \in \mathbb{S}_n(X).$$

Let us have a close look at the $(n - 1)$ -dimensional QAC (C, ψ) in the category $\Lambda(\mathbb{Z})\langle 1 \rangle_*(X)$ representing $s(X)$. By definition of $\text{sign}_X^{\text{VL}^{\bullet}}(X)$ and of the map $\partial : VL^n(X) \rightarrow \mathbb{S}_n(X)$ the subcomplex $C(\sigma)$ is the mapping cone of the duality map

$$\varphi[X(\sigma)] : \Sigma^n TC(\sigma) = C^{n-|\sigma|}(D(\sigma)) \rightarrow C(\sigma) = C(D(\sigma), \partial D(\sigma)).$$

The quadratic structure $\psi(\sigma)$ is more subtle to describe. It corresponds to the normal structure on $X(\sigma)$ via Lemma 3.17.

We clearly see that if X is a manifold then the mapping cones of the maps $\varphi[X(\sigma)]$ above are contractible and the total surgery obstruction equals 0. If X is homotopy equivalent to a manifold then the mapping cylinder of the homotopy equivalence provides via the constructions in Construction 11.9 a cobordism from (C, ψ) to 0.

13. PROOF OF THE MAIN TECHNICAL THEOREM (I)

Recall the statement. For an n -dimensional finite Poincaré complex X with $n \geq 5$ let $t(X)$ be the image of $s(X)$ under the map $\mathbb{S}_n(X) \rightarrow H_{n-1}(X; \mathbf{L}_\bullet \langle 1 \rangle)$. Then $t(x) = 0$ if and only if there exists a topological block bundle reduction of the SNF ν_X . The main idea of the proof is to translate the statement about the reduction of ν_X into a statement about orientations with respect to L -theory spectra. The principal references for this section are [15, pp. 280–292] and [19, Sec. 16].

13.1. Topological surgery theory.

Before we start we offer some comments about the topological surgery and about the bundle theories used. The topological surgery is a modification of the surgery in the smooth and PL-category, due to Browder-Novikov-Sullivan-Wall as presented in [4] and [23], by the work of Kirby and Siebenmann as presented in [8]. This book also discusses various bundle theories and transversality theorems for topological manifolds. From our point of view the notion of a “stable normal bundle” for topological manifolds is of prominent importance. As explained in Essay III, §1, the notion of a stable microbundle is appropriate and there exists a corresponding transversality theorem, whose dimension and codimension restrictions are removed by [6, Chap. 9]. It is also explained that when enough triangulations are in sight, one can use block bundles and the stable microbundle transversality can be replaced by block transversality. This is thanks to the fact that for the classifying spaces we have $\text{BSTOP} \simeq \widetilde{\text{BSTOP}}$. Since for our problem we can suppose that the Poincaré complex X is in fact a simplicial complex we can ask about the reduction of the SNF to a stable topological block bundle. When we talk about the degree one normal maps $(f, b) : M \rightarrow X$ we mean the stable microbundle normal data, since we need to work in full generality.

13.2. Orientations.

Let $\mathbf{E}$ be a ring spectrum. An $\mathbf{E}$ -orientation of a $\mathbb{Z}$ -oriented spherical fibration $\nu : X \rightarrow \text{BSG}(k)$ is an element of $u_{\mathbf{E}}(\nu) \in H^k(\text{Th}(\nu); \mathbf{E})$ that means a homotopy class of maps $u_{\mathbf{E}}(\nu) : \text{Th}(\nu) \rightarrow \mathbf{E}$, where $\text{Th}(\nu)$ denotes the Thom spectrum of ν , such that for each $x \in X$, the restriction $u_{\mathbf{E}}(\nu)_x : \text{Th}(\nu_x) \rightarrow \mathbf{E}$ to the fiber ν_x of ν over x represents a generator of $\mathbf{E}^*(\text{Th}(\nu_x)) \cong \mathbf{E}^*(S^k)$ which under the Hurewicz homomorphism $\mathbf{E}^*(\text{Th}(\nu_x)) \rightarrow H^*(\text{Th}(\nu_x); \mathbb{Z})$ maps to the chosen $\mathbb{Z}$ -orientation.

13.3. Canonical orientations.

Denote by **MSG** the Thom spectrum of the universal stable $\mathbb{Z}$ -oriented spherical fibrations over the classifying space BSG . Its k -th space is the Thom space $\mathbf{MSG}(k) = \text{Th}(\gamma_{\text{SG}}(k))$ of the canonical k -dimensional spherical fibration $\gamma_{\text{SG}}(k)$ over $\text{BSG}(k)$. Similarly denote by **MSTOP** the Thom spectrum of the universal stable $\mathbb{Z}$ -oriented topological block bundles over the classifying space $\text{BSTOP} \simeq \widetilde{\text{BSTOP}}$. Its k -th space is the Thom space $\mathbf{MSTOP}(k) = \text{Th}(\gamma_{\widetilde{\text{STOP}}}(k))$ of the canonical k -dimensional block bundle $\gamma_{\widetilde{\text{STOP}}}(k)$ over $\text{BSTOP}(k)$. There is a map $J : \mathbf{MSTOP} \rightarrow \mathbf{MSG}$ defined by viewing the canonical block bundle $\gamma_{\widetilde{\text{STOP}}}(k)$ as a spherical fibration.

Both **MSG** and **MSTOP** are ring spectra. The multiplication on **MSTOP** is given by the Cartesian product of block bundles. The multiplication on **MSG** is given by the sequence of the operations: take the associated disk fibrations, form the product disk fibration and take the associated spherical fibration. Upon precomposition with the diagonal map the multiplication on **MSTOP** becomes the Whitney sum and the multiplication on **MSG** becomes fiberwise join. The map $J : \mathbf{MSTOP} \rightarrow \mathbf{MSG}$ is a map of ring spectra.

Proposition 13.4. [15, pp. 280–283]

- (1) A k -dimensional $\mathbb{Z}$ -oriented spherical fibration $\alpha : X \rightarrow \text{BSG}(k)$ has a canonical orientation $u_{\mathbf{MSG}}(\alpha) \in H^k(\text{Th}(\alpha); \mathbf{MSG})$.
- (2) A k -dimensional $\mathbb{Z}$ -oriented topological block bundle $\beta : X \rightarrow \widetilde{\text{BSTOP}}(k)$ has a canonical orientation $u_{\mathbf{MSTOP}}(\beta) \in H^k(\text{Th}(\beta); \mathbf{MSTOP})$.

Moreover $J(u_{\mathbf{MSTOP}}(\beta)) = u_{\mathbf{MSG}}(J(\beta))$.

This follows since any spherical fibration (or a topological block bundle) is a pullback of the universal via the classifying map.

13.5. Transversality.

By transversality one often describes statements which assert that a map from a manifold to some space with a closed subspace can be deformed by a small homotopy to a map such that the inverse image of the closed subspace is a submanifold. Such notion of transversality can then be used to prove various versions of the Pontrjagin–Thom isomorphism. For example topological transversality of Kirby–Siebenmann [8, Essay III] and Freedman–Quinn [6, Chap. 9] implies that the classifying map induces

$$(13.1) \quad c : \Omega_{\bullet}^{\text{STOP}} \simeq \mathbf{MSTOP}.$$

On the other hand normal transversality used here has a different meaning, no statement invoking preimages is required.¹⁷ It just means that there is the homotopy equivalence (13.2) below inducing a Pontrjagin–Thom isomorphism. To arrive at it one can use the ideas described in [19, Errata]. Recall the spectrum $\Omega_{\bullet}^N$ from Section 7. Further recall for a space X with a k -dimensional spherical fibration $\nu : X \rightarrow \text{BSG}(k)$ the space $\Omega_n^N(X, \nu)$ of normal spaces

¹⁷Although there are some such statements [7], we will not need them.

with a degree one normal map to (X, ν) . The normal transversality described in [19, Errata] says that the classifying map induces

$$c : \Omega_0^N(X, \nu) \simeq \text{Th}(\nu).$$

We have the classifying space $\text{BSG}(k)$ with the canonical k -dimensional spherical fibration $\gamma_{\text{SG}}(k)$. The spectrum $\Omega_\bullet^N$ can be seen as the colimit of spectra $\Omega_\bullet^N(\text{BSG}(k), \gamma_{\text{SG}}(k))$. The normal transversality from [19, Errata] translates into homotopy equivalence

$$(13.2) \quad \Omega_\bullet^N \simeq \mathbf{MSG}.$$

There are multiplication operations on the spectra $\Omega_\bullet^{\text{STOP}}$ and $\Omega_\bullet^N$, which make the above Pontrjagin–Thom maps to ring spectra homotopy equivalences. These operations are given by Cartesian products. However, we will not use this point of view later.

To complete the picture we denote

$$(13.3) \quad \mathbf{MS}(\mathbf{G}/\mathbf{TOP}) := \text{Fiber}(\mathbf{MSTOP} \rightarrow \mathbf{MSG})$$

and observe that the above classifying maps induce yet another Pontrjagin–Thom isomorphism

$$(13.4) \quad \Sigma^{-1}\Omega_\bullet^{N, \text{STOP}} \simeq \mathbf{MS}(\mathbf{G}/\mathbf{TOP}).$$

Furthermore we have that $\mathbf{MS}(\mathbf{G}/\mathbf{TOP})$ is a module spectrum over $\mathbf{MSTOP}$ and similarly $\Sigma^{-1}\Omega_\bullet^{N, \text{STOP}}$ is a module spectrum over $\Omega_\bullet^{\text{STOP}}$.

13.6. L -theory orientations. Here we use the signature maps between the spectra from Section 9 to construct orientations with respect to the L -theory spectra. We recall that $\mathbf{NL}^\bullet\langle 1/2 \rangle$ and $\mathbf{L}^\bullet\langle 0 \rangle$ are ring spectra with the multiplication given by the products of [16, Sec. 8] and [19, Appendix B]. The spectrum $\mathbf{L}_\bullet\langle 1 \rangle$ is a module over $\mathbf{L}^\bullet\langle 0 \rangle$ again by the products of [16, Sec. 8] and [19, Appendix B].

Proposition 13.7. [15, pp. 284–289]

- (1) A k -dimensional $\mathbb{Z}$ -oriented spherical fibration $\alpha : X \rightarrow \text{BSG}(k)$ has a canonical orientation $u_{\mathbf{NL}^\bullet}(\alpha) \in H^k(\text{Th}(\alpha); \mathbf{NL}^\bullet\langle 1/2 \rangle)$.
- (2) A k -dimensional $\mathbb{Z}$ -oriented topological block bundle $\beta : X \rightarrow \widetilde{\text{BSTOP}}(k)$ has a canonical orientation $u_{\mathbf{L}^\bullet}(\beta) \in H^k(\text{Th}(\beta); \mathbf{L}^\bullet\langle 0 \rangle)$.

Moreover $J(u_{\mathbf{L}^\bullet}(\beta)) = u_{\mathbf{NL}^\bullet}(J(\beta))$.

Proof. These orientations are obtained from maps between spectra using the following up to homotopy commutative diagram of spectra:

$$\begin{array}{ccccc} \mathbf{MSTOP} & \longrightarrow & \Omega_\bullet^{\text{STOP}} & \xrightarrow{\text{sign}^{\mathbf{L}^\bullet}} & \mathbf{L}^\bullet\langle 0 \rangle \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{MSG} & \longrightarrow & \Omega_\bullet^N & \xrightarrow[\text{sign}^{\mathbf{NL}^\bullet}]{} & \mathbf{NL}^\bullet\langle 1/2 \rangle, \end{array}$$

where the maps in the left hand part of the diagram are the homotopy inverses of the transversality homotopy equivalences. $\square$

13.8. S -duality.

If X is a Poincaré complex with the SNF $\nu_X : X \rightarrow \text{BSG}(k)$ then we have the S -duality $\text{Th}(\nu_X)^* \simeq X_+$ producing isomorphisms

$$\begin{aligned} S : H^k(\text{Th}(\nu_X); \mathbf{NL}^\bullet\langle 1/2 \rangle) &\cong H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle), \\ S : H^k(\text{Th}(\nu_X); \mathbf{L}^\bullet\langle 0 \rangle) &\cong H_n(X; \mathbf{L}^\bullet\langle 0 \rangle). \end{aligned}$$

The following proposition describes a relation between the signatures over X in homology, obtained in Sections 8 and 11 and orientations in cohomology from this section.

Proposition 13.9. [19, Prop. 16.1.] *If X is an n -dimensional geometric Poincaré complex with the Spivak normal fibration $\nu_X : X \rightarrow \text{BSG}(k)$ then we have*

$$S(u_{\mathbf{NL}^\bullet}(\nu_X)) = \text{sign}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet\langle 1/2 \rangle).$$

If $\bar{\nu}_X$ is a topological block bundle reduction of the SNF of X and $(f, b) : M \rightarrow X$ is the associated degree one normal map, then we have

$$S(u_{\mathbf{L}^\bullet}(\bar{\nu}_X)) = \text{sign}_X^{\mathbf{L}^\bullet}(M) \in H_n(X; \mathbf{L}^\bullet\langle 0 \rangle).$$

Proof. The identification of the normal signature $\text{sign}_X^{\mathbf{NL}^\bullet}(X)$ as the canonical orientation follows from the commutative diagram of simplicial sets

$$\begin{array}{ccc} \Sigma^m / \overline{X} & \xrightarrow{\text{sign}_X^{\Omega^N}(X)} & \Omega_{-k}^N \\ \downarrow i & & \downarrow c \\ \text{Sing Th}(\nu_X) & \xrightarrow{u_{\mathbf{MSG}}(\nu_X)} & \text{Sing } \mathbf{MSG}(k). \end{array}$$

This in turn is seen by inspecting the definitions of the maps in the diagram. The upper horizontal map comes from 11.3, the map i from 11.2 and the other two maps were defined in this section. Note that the classifying map c is characterized by the property that the classified spherical fibration is obtained as the pullback of the canonical γ_{SG} along c . But this is also the characterization of the canonical orientation $u_{\mathbf{MSG}}(\nu_X)$. The desired statement is obtained by composing with $\text{sign}^{\mathbf{NL}^\bullet} : \Omega_{\bullet}^N \rightarrow \mathbf{NL}^\bullet\langle 1/2 \rangle$.

For the second part recall how the degree one normal map (f, b) associated to $\bar{\nu}_X$ is constructed. Consider the composition $\Sigma^m \rightarrow \Sigma^m / \overline{X} \rightarrow \text{Th}(\bar{\nu}_X)$. Since $\bar{\nu}_X$ is a stable topological block bundle this map can be made transverse to X and M is the preimage, f is the restriction of the map to M and it is covered by a map of stable microbundles $\nu_M \rightarrow \bar{\nu}_X$, where ν_M is the stable normal microbundle of M . In addition this can be made in such a way that f is transverse to the dual cells of X . Hence we obtain a dissection of M which gives

rise to the symmetric signature $\text{sign}_X^{\mathbf{L}^\bullet}(M)$ over X as in Construction 5.11. It fits into the following diagram

$$\begin{array}{ccc} \Sigma^m/\overline{X} & \xrightarrow{\text{sign}_X^{\Omega_{-k}^{\text{STOP}}}(M)} & \Omega_{-k}^{\text{STOP}} \\ \downarrow i & & \downarrow c \\ \text{Sing Th}(\nu_X) & \xrightarrow{u_{\mathbf{MSTOP}}(\nu_X)} & \text{Sing } \mathbf{MSTOP}(k). \end{array}$$

The desired statement is obtained by composing with $\text{sign}^{\mathbf{L}^\bullet} : \Omega_{-k}^{\text{STOP}} \rightarrow \mathbf{L}^\bullet\langle 0 \rangle$. $\square$

Suppose now that we are given a degree one normal map $(f, b) : M \rightarrow X$ between n -dimensional topological manifolds with X triangulated. In Construction 11.9 we defined the (normal, symmetric Poincaré) signature $\text{sign}_X^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(f, b)$ over X associated to (f, b) . In analogy with the previous proposition we would like to interpret this signature as an orientation via the S -duality. For this recall first that specifying the degree one normal map (f, b) is equivalent to specifying a pair (ν, h) , with $\nu : X \rightarrow \mathbf{BSTOP}$ and $h : J(\nu) \simeq \nu_X$, where in our situation the SNF ν_X has a preferred topological block bundle lift, also denoted ν_X , coming from the stable normal bundle of X (see Subsection 14.23 if needed). The homotopy h gives us a spherical fibration over $X \times I$ with the canonical orientation $u^{\mathbf{MSG}}(h)$ which we view as a homotopy between the orientations $J(u^{\mathbf{MSTOP}}(\nu))$ and $J(u^{\mathbf{MSTOP}}(\nu_X))$. In this way we obtain an element

$$u^{\mathbf{G/TOP}}(\nu, h) \in H^k(\text{Th}(\nu_X); \mathbf{MS}(\mathbf{G/TOP}))$$

given by

$$u^{\mathbf{G/TOP}}(\nu, h) = (u^{\mathbf{MSG}}(h), u^{\mathbf{MSTOP}}(\nu) - u^{\mathbf{MSTOP}}(\nu_X)).$$

The Pontrjagin-Thom isomorphism (13.4) together with the normal and symmetric signature $\text{sign}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet} : \Sigma^{-1}\Omega_{-k}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet} \rightarrow \mathbf{F}$ provide us with the pair

$$u^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(\nu, h) = (u^{\mathbf{NL}^\bullet}(h), u^{\mathbf{L}^\bullet}(\nu) - u^{\mathbf{L}^\bullet}(\nu_X)) \in H^k(\text{Th}(\nu_X); \mathbf{F}).$$

Proposition 13.10. *Let $(f, b) : M \rightarrow X$ be a degree one normal map between n -dimensional simply connected topological manifolds with X triangulated, corresponding to the pair (ν, h) , where $\nu : X \rightarrow \mathbf{BSTOP}$ and $h : J(\nu) \simeq \nu_X$. Then we have*

$$S(u^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(\nu, h)) = \text{sign}_X^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(f, b) \in H_n(X; \mathbf{F}).$$

Proof. The proof is analogous to the proof of Proposition 13.9. Recall that the signature $\text{sign}_X^{\mathbf{G/TOP}}(f, b)$ is constructed using a dissection of the degree one normal map (f, b) . Using this dissection we inspect that we have a commutative

diagram

$$\begin{array}{ccc}
 \Sigma^m / \overline{X} & \xrightarrow{\text{sign}_X^{G/\text{TOP}}(f,b)} & \Sigma^{-1} \Omega_{-k}^{N,\text{STOP}} \\
 \downarrow i & & \downarrow c \\
 \text{Sing } F(\nu, \nu_X) & \xrightarrow{u^{G/\text{TOP}}(\nu,h)} & \text{Sing } \mathbf{MS}(\mathbf{G}/\mathbf{TOP})(k)
 \end{array}$$

where we set $\mathbf{MS}(\mathbf{G}/\mathbf{TOP})(k) := \text{Fiber } (\mathbf{MSTOP}(k) \rightarrow \mathbf{MSG}(k))$ and $F(\nu, \nu_X) := \text{Pullback } (\text{Th}(\nu) \rightarrow \text{Th}(\nu_X) \leftarrow \text{Th}(\nu_X))$. Composing with the signature map $\text{sign}^{\mathbf{NL}\bullet, \mathbf{L}\bullet} : \Sigma^{-1} \Omega_{-k}^{N,\text{STOP}} \rightarrow \mathbf{F}$ proves the claim. $\square$

13.11. Assembly.

Keep X a Poincaré complex with the SNF ν_X and suppose there exists a topological block bundle reduction $\bar{\nu}_X$. Recall that orientations with respect to ring spectra induce Thom isomorphisms in corresponding cohomology theories. Hence we have Thom isomorphisms induced by $u_{\mathbf{NL}\bullet}(\nu_X)$ and $u_{\mathbf{L}\bullet}(\bar{\nu}_X)$ and these are compatible. Also recall that although the spectrum $\mathbf{L}\bullet\langle 1 \rangle$ is not a ring spectrum, it is a module spectrum over $\mathbf{L}\bullet\langle 0 \rangle$, see [19, Appendix B]. Therefore $u_{\mathbf{L}\bullet}(\bar{\nu}_X)$ also induces a compatible Thom isomorphism in $\mathbf{L}\bullet\langle 1 \rangle$ -cohomology. In fact we have a commutative diagram relating these Thom isomorphisms, the S -duality and the assembly maps:

$$\begin{array}{ccccccc}
 H^0(X; \mathbf{L}\bullet\langle 1 \rangle) & \xrightarrow{\cong} & H^k(\text{Th}(\nu_X); \mathbf{L}\bullet\langle 1 \rangle) & \xrightarrow{\cong} & H_n(X; \mathbf{L}\bullet\langle 1 \rangle) & \xrightarrow{A} & L_n(\mathbb{Z}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^0(X; \mathbf{L}\bullet\langle 0 \rangle) & \xrightarrow{\cong} & H^k(\text{Th}(\nu_X); \mathbf{L}\bullet\langle 0 \rangle) & \xrightarrow{\cong} & H_n(X; \mathbf{L}\bullet\langle 0 \rangle) & \xrightarrow{A} & L^n(\mathbb{Z}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^0(X; \mathbf{NL}\bullet\langle 1/2 \rangle) & \xrightarrow{\cong} & H^k(\text{Th}(\nu_X); \mathbf{NL}\bullet\langle 1/2 \rangle) & \xrightarrow{\cong} & H_n(X; \mathbf{NL}\bullet\langle 1/2 \rangle) & \xrightarrow{A} & NL^n(\mathbb{Z})
 \end{array}$$

If $X = S^n$ then the map $A : H_n(S^n; \mathbf{L}\bullet\langle 1 \rangle) \rightarrow L_n(\mathbb{Z})$ is an isomorphism. This follows from the identification of the assembly map with the surgery obstruction map, which is presented in Proposition 14.18 and the fact that the surgery obstruction map for S^n is an isomorphism due to Kirby and Siebenmann [8, Essay V, Thm. C.1]. We note that Proposition 14.18 is presented in a greater generality than needed here. We just need the case when $X = S^n$ and so it is a manifold and hence the degree one normal map (f_0, b_0) in the statement of Proposition 14.18 can be taken to be the identity on S^n , which is the version we need at this place.

Further observe that all the homomorphisms in the diagram are induced homomorphisms on homotopy groups by maps of spaces, see definitions in Sections 7, 8 for the underlying spaces.

13.12. Classifying spaces for spherical fibrations with an orientation.

Let $\mathbf{E}$ be an Omega ring spectrum with $\pi_0(\mathbf{E}) = \mathbb{Z}$ and recall the notion of an $\mathbf{E}$ -orientation of a $\mathbb{Z}$ -oriented spherical fibration $\alpha : X \rightarrow \text{BSG}(k)$. In [12] a construction of a classifying space BEG for spherical fibrations with such a structure was given. The construction itself is not so important for us. Of more significance for our purposes is a description of what it means to have a map from a space to one of these classifying spaces. If X is a finite complex then there is a one-to-one correspondence between homotopy classes of maps $\alpha_{\mathbf{E}} : X \rightarrow \text{BEG}$ and homotopy classes of pairs $(\alpha, u_{\mathbf{E}}(\alpha))$ where $\alpha : X \rightarrow \text{BSG}(k)$ and $u_{\mathbf{E}}(\alpha) : \text{Th}(\alpha) \rightarrow \mathbf{E}_k$ is an $\mathbf{E}$ -orientation of α .

Proposition 13.13. *There is a commutative diagram*

$$\begin{array}{ccc} \text{BSTOP} & \xrightarrow{\text{sign}^{\mathbf{L}^\bullet}} & \text{BL}^\bullet\langle 0 \rangle \text{G} \\ J \downarrow & & \downarrow J \\ \text{BSG} & \xrightarrow{\text{sign}^{\mathbf{NL}^\bullet}} & \text{BNL}^\bullet\langle 1/2 \rangle \text{G}. \end{array}$$

Proof. This follows from Proposition 13.7. $\square$

We need to study what orientations do there exist for a fixed spherical fibration $\alpha : X \rightarrow \text{BSG}(k)$. Denote by $\mathbf{E}_\otimes$ the component of $1 \in \mathbb{Z}$ in any Omega ring spectrum $\mathbf{E}$ with $\pi_0(\mathbf{E}) = \mathbb{Z}$. Then there is a homotopy fibration sequence [12, Sec. III.2]

$$(13.5) \quad \mathbf{E}_\otimes \xrightarrow{i} \text{BEG} \rightarrow \text{BSG}.$$

The map i can be interpreted via the Thom isomorphism. Let $c : X \rightarrow \mathbf{E}_\otimes$ be a map. Then $i(c) : X \rightarrow \text{BEG}$ is the map given by the trivial fibration $\varepsilon : X \rightarrow \text{BSG}(k)$ with an $\mathbf{E}$ -orientation given by the composition

$$u_{\mathbf{E}}(i(c)) : \text{Th}(\varepsilon) \xrightarrow{\tilde{\Delta}} X_+ \wedge \text{Th}(\varepsilon) \xrightarrow{c \wedge \Sigma^k(1)} \mathbf{E}_\otimes \wedge \mathbf{E}_k \rightarrow \mathbf{E}_k.$$

We will use the spectra $\mathbf{L}^\bullet\langle 0 \rangle$ and $\mathbf{NL}^\bullet\langle 1/2 \rangle$, which are both ring spectra with $\pi_0 \cong \mathbb{Z}$. We will need the following proposition.

Proposition 13.14. *There is the following homotopy fibration sequence of spaces*

$$\mathbf{L}_0\langle 1 \rangle \rightarrow \text{BL}^\bullet\langle 0 \rangle \text{G} \rightarrow \text{BNL}^\bullet\langle 1/2 \rangle \text{G}.$$

Proof. Consider the sequences (13.5) for the spectra $\mathbf{L}^\bullet\langle 0 \rangle$ and $\mathbf{NL}^\bullet\langle 1/2 \rangle$ and the map between them. The induced map between the homotopy fibers fits into the fibration sequence

$$(13.6) \quad \mathbf{L}_0\langle 1 \rangle \rightarrow \mathbf{L}^\otimes\langle 0 \rangle \rightarrow \mathbf{NL}^\otimes\langle 1/2 \rangle$$

which is obtained from the fibration sequence of Proposition 9.5 (more precisely from the space-level version of it on the 0-th spaces) by replacing the

symmetrization map $(1 + T) : \mathbf{L}_0\langle 1 \rangle \rightarrow \mathbf{L}^0\langle 0 \rangle$ by the map given on the l -simplices as

$$(1 + T)^\otimes : \mathbf{L}_0\langle 1 \rangle \rightarrow \mathbf{L}^\otimes\langle 0 \rangle,$$

$$(C, \psi) \mapsto (1 + T)(C, \psi) + (C(\Delta^l), \varphi([\Delta^l])).$$

Its effect is to map the component of 0 (which is the only component of $\mathbf{L}_0\langle 1 \rangle$) to the component of 1 in $\mathbf{L}^0\langle 0 \rangle$ instead of the component of 0. The proposition follows. $\square$

13.15. $\mathbf{L}$ -theory orientations versus reductions.

The following theorem is a crucial result.

Theorem 13.16. [15, pp. 290–292] *There is a one-to-one correspondence between the isomorphism classes of*

- (1) *stable oriented topological block bundles over X , and*
- (2) *stable oriented spherical fibrations over X with an $\mathbf{L}^\bullet\langle 0 \rangle$ -lift of the canonical $\mathbf{NL}^\bullet\langle 1/2 \rangle$ -orientation.*

Proof. In Proposition 13.13 a map from (1) to (2) was described. To prove that it gives a one-to-one correspondence is equivalent to showing that the square in Proposition 13.13 is a homotopy pullback square. This is done by showing that the induced map between the homotopy fibers of the vertical maps in the square, which is indicated by the dashed arrow in the diagram below, is a homotopy equivalence.

$$\begin{array}{ccc} \mathbf{G}/\mathbf{TOP} & \xrightarrow{\text{sign}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}} & \mathbf{L}_0\langle 1 \rangle \\ \downarrow & & \downarrow \\ \mathbf{BSTOP} & \xrightarrow{\text{sign}^{\mathbf{L}^\bullet}} & \mathbf{BL}^\bullet\langle 0 \rangle \mathbf{G} \\ \downarrow J & & \downarrow J \\ \mathbf{BSG} & \xrightarrow{\text{sign}^{\mathbf{NL}^\bullet}} & \mathbf{BNL}^\bullet\langle 1/2 \rangle \mathbf{G}. \end{array}$$

For this it is enough to show that it induces an isomorphism on the homotopy groups, that means to show

$$\text{sign}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet} : [S^n; \mathbf{G}/\mathbf{TOP}] \xrightarrow{\cong} [S^n; \mathbf{L}_0\langle 1 \rangle].$$

Recall that since S^n is a topological manifold with the trivial SNF we have a canonical identification of the normal invariants

$$[S^n; \mathbf{G}/\mathbf{TOP}] \cong \mathcal{N}(S^n), \quad ((\alpha, H) : S^n \rightarrow \mathbf{G}/\mathbf{TOP}) \mapsto ((f, b) : M \rightarrow S^n)$$

where $\alpha : S^n \rightarrow \mathbf{BSTOP}$ and $H = J(\alpha) \simeq \varepsilon : S^n \times [0, 1] \rightarrow \mathbf{BSG}$ is a homotopy to a constant map and (f, b) is the associated degree one normal map.

On the other side we have

$$A(S(i(-))) : [S^n; \mathbf{L}_0\langle 1 \rangle] \cong H^0(S^n; \mathbf{L}_\bullet\langle 1 \rangle) \cong H_n(S^n; \mathbf{L}_\bullet\langle 1 \rangle) \cong L_n(\mathbb{Z}).$$

It is well known that the surgery obstruction map $\text{sign}_{\mathbb{Z}}^{\mathbf{L}^\bullet} : \mathcal{N}(S^n) \rightarrow L_n(\mathbb{Z})$ from Definition 2.32 is an isomorphism for $n \geq 1$ [8, Essay V, Thm. C.1]. Therefore it is enough to show that

$$A(S(i(\text{sign}_{\mathbb{Z}}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(\alpha, H)))) = \text{sign}_{\mathbb{Z}}^{\mathbf{L}^\bullet}(f, b).$$

Denote

$$(J(\alpha), u_{\mathbf{L}^\bullet}(\alpha)) = \text{sign}^{\mathbf{L}^\bullet}(\alpha), \quad (H, u_{\mathbf{NL}^\bullet}(H)) = \text{sign}^{\mathbf{NL}^\bullet}(H).$$

Now we need to describe in more detail what the identification of the homotopy fiber of the right hand column map means. That means to produce a map

$$\bar{u}_{\mathbf{L}^\bullet}(\alpha, H) : S^n \rightarrow \mathbf{L}_0\langle 1 \rangle.$$

from $(J(\alpha), u_{\mathbf{L}^\bullet}(\alpha))$ and $(H, u_{\mathbf{NL}^\bullet}(H))$. The spherical fibration $J(\alpha)$ is trivial because of the null-homotopy H and thus we obtain a map $\bar{u}_{\mathbf{L}^\bullet}(\alpha) : S^n \rightarrow \mathbf{L}^\otimes\langle 0 \rangle$ such that $i(\bar{u}_{\mathbf{L}^\bullet}(\alpha)) = (J(\alpha), u_{\mathbf{L}^\bullet}(\alpha))$.

Similarly the homotopy $(H, u_{\mathbf{NL}^\bullet}(H)) : S^n \times [0, 1] \rightarrow \mathbf{BNL}^\bullet\langle 1/2 \rangle \mathbf{G}$ yields a homotopy $\bar{u}_{\mathbf{NL}^\bullet}(H) : S^n \times [0, 1] \rightarrow \mathbf{NL}^\otimes\langle 1/2 \rangle$ between $J(\bar{u}_{\mathbf{L}^\bullet}(\alpha))$ and the constant map.

The pair $(\bar{u}_{\mathbf{NL}^\bullet}(H), \bar{u}_{\mathbf{L}^\bullet}(\alpha))$ produces via the homotopy fibration sequence (13.6) a lift, which is the desired $\bar{u}_{\mathbf{L}^\bullet}(\alpha, H)$. So we have

$$[\text{sign}^{\mathbf{NL}^\bullet, \mathbf{L}^\bullet}(\alpha, H)] = [\bar{u}_{\mathbf{L}^\bullet}(\alpha, H)] \in [S^n; \mathbf{L}_0\langle 1 \rangle]$$

and we want to investigate $A(S(i(\bar{u}_{\mathbf{L}^\bullet}(\alpha, H))))$. Recall now the commutative diagram from Subsection 13.11. It shows that $A(S(i(\bar{u}_{\mathbf{L}^\bullet}(\alpha, H))))$ can be chased via the lower right part of the diagram. Here we consider maps from S^n and $S^n \times [0, 1]$ to the underlying spaces in this diagram rather than just elements in the homotopy groups.

Observe first, using Definition 8.11, Example 6.2 and Proposition 13.9, that the assembly of the S -dual of the class $u_{\mathbf{L}^\bullet}(\alpha)$ is an n -dimensional SAPC $\text{sign}^{\mathbf{L}^\bullet}(M)$ over $\mathbb{Z}$.

Secondly, by Construction 11.9 and Proposition 13.10, the assembly of the S -dual of the class $u_{\mathbf{NL}^\bullet}(H)$ is an $(n+1)$ -dimensional (normal, symmetric Poincaré) pair

$$(13.7) \quad (\text{sign}^{\mathbf{NL}^\bullet}(W), \text{sign}^{\mathbf{L}^\bullet}(M) - \text{sign}^{\mathbf{L}^\bullet}(S^n))$$

over $\mathbb{Z}$, with $W = \text{cyl}(f)$. We consider this as an element in the n -th homotopy group of the relative term in the long exact sequence of the homotopy groups associated to the map $\mathbf{L}^\otimes\langle 0 \rangle \rightarrow \mathbf{NL}^\otimes\langle 1/2 \rangle$. This group is isomorphic to $L_n(\mathbb{Z})$ by the isomorphism of Proposition 3.21. The effect of this isomorphism on an element as in (13.7) is then described in detail in Example 3.27. It tells us that the n -dimensional QAPC corresponding to (13.7) is the surgery obstruction $\text{sign}^{\mathbf{L}^\bullet}(f, b)$. Finally we obtain

$$A(S(i(\bar{u}_{\mathbf{L}^\bullet}(\alpha, H)))) = \text{sign}^{\mathbf{L}^\bullet}(f, b) \in L_n(\mathbb{Z})$$

which is what we wanted to show. $\square$

Recall the exact sequence:

$$\cdots \longrightarrow H_n(X; \mathbf{L}^\bullet \langle 0 \rangle) \longrightarrow H_n(X; \mathbf{NL}^\bullet \langle 1/2 \rangle) \longrightarrow H_{n-1}(X; \mathbf{L}_\bullet \langle 1 \rangle) \longrightarrow \cdots$$

Putting all together we obtain

Corollary 13.17. *Let X be an n -dimensional geometric Poincaré complex with the Spivak normal fibration $\nu_X : X \rightarrow \text{BSG}$. Then the following are equivalent*

- (1) *There is a lift $\bar{\nu}_X : X \rightarrow \text{BSTOP}$ of ν_X*
- (2) *There is a lift of the normal signature $\text{sign}_X^{\mathbf{NL}^\bullet}(X) \in H_n(X; \mathbf{NL}^\bullet \langle 1/2 \rangle)$ in the group $H_n(X; \mathbf{L}^\bullet \langle 0 \rangle)$.*
- (3) *$0 = t(X) \in H_{n-1}(X; \mathbf{L}_\bullet \langle 1 \rangle)$.*

14. PROOF OF THE MAIN TECHNICAL THEOREM (II)

Let X be a finite n -dimensional GPC and suppose that $t(X) = 0$ so that the SNF ν_X has a topological block bundle reduction and hence there exists a degree one normal map $(f, b) : M \rightarrow X$ from some n -dimensional topological manifold M . We want to show that the subset of $L_n(\mathbb{Z}[\pi_1(X)])$ consisting of the inverses of the quadratic signatures of all such degree one normal maps is equal to the preimage of the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ under the boundary map $\partial : L_n(\mathbb{Z}[\pi_1(X)]) \rightarrow \mathbb{S}_n(X)$.

Let us first look at this map. Inspecting the first of the two commutative braids in Section 10 we see that it is in fact obtained from the boundary map $\partial : L_n(\Lambda(\mathbb{Z})(X)) \rightarrow \mathbb{S}_n(X)$ using the algebraic π - π -theorem of Proposition 6.6. This map is more suitable for investigation since both the source and the target are the L -groups of algebraic bordism categories over the same underlying additive category with chain duality, which is $\mathbb{Z}_*(X)$.

On the other hand there is a price to pay for this point of view. Namely, in the present situation we only have the quadratic signatures $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}_\bullet}(f, b)$ as n -dimensional QAPCs over the category $\mathbb{Z}[\pi_1(X)]$, but we need a quadratic signature $\text{sign}_X^{\mathbf{L}_\bullet}(f, b)$ over the category $\mathbb{Z}_*(X)$.¹⁸ A large part of this section will be devoted to constructing such a quadratic signature, it will finally be achieved in Definition 14.15. More precisely, we define the quadratic signature

$$\text{sign}_X^{\mathbf{L}_\bullet}(f, b) \in L_n(\Lambda(\mathbb{Z})(X))$$

which is represented by an n -dimensional QAC in the algebraic bordism category $\Lambda(\mathbb{Z})(X)$ from Definition 6.4, i.e. an n -dimensional quadratic complex over $\mathbb{Z}_*(X)$ which is globally Poincaré, such that it maps to $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}_\bullet}(f, b)$ under the isomorphism of the algebraic π - π -theorem. We emphasize that in general the quadratic signature $\text{sign}_X^{\mathbf{L}_\bullet}(f, b)$ does not produce an element in $H_n(X, \mathbf{L}_\bullet \langle 1 \rangle)$ since it is not locally Poincaré.

¹⁸As shown in Construction 5.12 in case X is a triangulated manifold we have such a signature but here X is only a Poincaré complex with $t(X) = 0$.

Granting the definition of $\text{sign}_X^{\mathbf{L}\bullet}(f, b)$ the proof of the desired statement starts with the obvious observation that the preimage $\partial^{-1}s(X)$ is a coset of $\ker(\partial) = \text{im}(A)$, where $A : H_n(X; \mathbf{L}\bullet\langle 1 \rangle) \rightarrow L_n(\mathbb{Z}[\pi_1(X)])$ is the assembly map. Then the proof proceeds in two steps as follows.

- (1) Show that the set of the inverses of the quadratic signatures $\text{sign}_X^{\mathbf{L}\bullet}(f, b)$ of degree one normal maps with target X is a subset of $\partial^{-1}s(X)$ and hence the two sets have nonempty intersection.
- (2) Show that the set of the inverses of the quadratic signatures $\text{sign}_X^{\mathbf{L}\bullet}(f, b)$ of degree one normal maps with target X is a coset of $\ker(\partial) = \text{im}(A)$. Hence we have two cosets of the same subgroup with a nonempty intersection and so they are equal.

The definition of $\text{sign}_X^{\mathbf{L}\bullet}(f, b)$ and Step (1) of the proof are concentrated in Subsections 14.1 to 14.14. The main technical proposition is Proposition 14.5 which says that the boundary of the quadratic signature of any degree one normal map from a manifold to X is $s(X)$.

Step (2) of the proof is concentrated in Subsections 14.16 to 14.28. It starts with an easy corollary of Proposition 14.5 which says that although, as noted above, $\text{sign}_X^{\mathbf{L}\bullet}(f, b)$ does not produce an element in $H_n(X, \mathbf{L}\bullet\langle 1 \rangle)$, the difference $\text{sign}_X^{\mathbf{L}\bullet}(f, b) - \text{sign}_X^{\mathbf{L}\bullet}(f_0, b_0)$ for two degree one normal maps does produce such an element. Therefore, fixing some (f_0, b_0) and letting (f, b) vary provides us with a map from the normal invariants $\mathcal{N}(X)$ to $H_n(X, \mathbf{L}\bullet\langle 1 \rangle)$. The main technical proposition is then Proposition 14.18. Via the just mentioned difference map it identifies the set of the quadratic signatures of degree one normal maps with the coset of the image of the assembly map containing $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}\bullet}(f_0, b_0)$.

The principal references are: for Step (1) [18, Sec. 7.3, 7.4] and for Step (2) [15, pp. 293–298] and [19, Sec. 17].

14.1. A general discussion of quadratic signatures over X .

As noted above, in case X is a triangulated manifold, we have Construction 5.12 which produces from a degree one normal map $(f, b) : M \rightarrow X$ a quadratic signature $\text{sign}_X^{\mathbf{L}\bullet}(f, b) \in H_n(X, \mathbf{L}\bullet\langle 1 \rangle)$. Let us first look at why it is not obvious how to generalize this to our setting. The idea in 5.12 was to make f transverse to the dual cells of X and to consider the restrictions

$$(14.1) \quad (f(\sigma), b(\sigma)) : (M(\sigma), \partial M(\sigma)) \rightarrow (D(\sigma), \partial D(\sigma)).$$

These are degree one normal maps, but the target $(D(\sigma), \partial D(\sigma))$ is only a normal pair which can be non-Poincaré. Consequently we cannot define the Umkehr maps $f(\sigma)!$ as in 5.12.

We need an alternative way to define the Umkehr maps. Such a construction is a relative version of an absolute construction whose starting point is a degree one normal map $(g, c) : N \rightarrow Y$ from an n -dimensional manifold N to an n -dimensional normal space Y . In this case there is the normal signature $\text{sign}^{\mathbf{NL}\bullet}(Y) \in NL^n(\mathbb{Z})$ with boundary $\partial \text{sign}^{\mathbf{NL}\bullet}(Y) \in L_{n-1}(\mathbb{Z})$. In Definition 14.7 below, we recall the definition of a quadratic signature $\text{sign}^{\mathbf{L}\bullet}(g, c)$ in

this setting, which is an n -dimensional QAC over $\mathbb{Z}$, not necessarily Poincaré.¹⁹ As such it has a boundary, which is an $(n-1)$ -dimensional QAPC over $\mathbb{Z}$, and hence defines an element $\partial \text{sign}^{\mathbf{L}^\bullet}(g, c) \in L_{n-1}(\mathbb{Z})$. The following proposition describes the relationship between these signatures.

Proposition 14.2. [18, Prop. 7.3.4] *Let $(g, c) : N \rightarrow Y$ be a degree one normal map from an n -dimensional manifold to an n -dimensional normal space. Then there are homotopy equivalences of symmetric complexes*

$$h : \partial \text{sign}^{\mathbf{L}^\bullet}(g, c) \xrightarrow{\sim} -(1+T)\partial \text{sign}^{\mathbf{NL}^\bullet}(Y)$$

and a homotopy equivalence of quadratic refinements

$$h : \partial \text{sign}^{\mathbf{L}^\bullet}(g, c) \xrightarrow{\sim} -\partial \text{sign}^{\mathbf{NL}^\bullet}(Y).$$

Remark 14.3. Recall the situation in the case Y is Poincaré. Then there is defined the algebraic Umkehr map $g^! : C_*(\tilde{Y}) \rightarrow C_*(\tilde{N})$ and one obtains the symmetric signature $\text{sign}^{\mathbf{L}^\bullet}(g, c)$ with the underlying chain complex the algebraic mapping cone $\mathcal{C}(g^!)$. This can be further refined to a quadratic structure $\text{sign}^{\mathbf{L}^\bullet}(g, c)$. In addition one has (see Remark 2.33)

$$\mathcal{C}(g^!) \oplus C_*(\tilde{Y}) \simeq C_*(\tilde{N}) \quad \text{and} \quad \text{sign}^{\mathbf{L}^\bullet}(g, c) \oplus \text{sign}^{\mathbf{L}^\bullet}(Y) = \text{sign}^{\mathbf{L}^\bullet}(N).$$

In the situation of Proposition 14.2 one obtains instead the formula

$$\text{sign}^{\mathbf{L}^\bullet}(N) \simeq \text{sign}^{\mathbf{L}^\bullet}(g, c) \cup_h \text{sign}^{\mathbf{L}^\bullet}(Y)$$

where $\cup_h$ denotes the algebraic gluing of symmetric pairs from [16, Sec. 3].

Before going into the proof of Proposition 14.2 remember that we still need its relative version. For that again some preparation is needed. In particular we need the concept of a boundary of a symmetric pair. Let $(f : C \rightarrow D, (\delta\varphi, \varphi))$ be an $(n+1)$ -dimensional symmetric pair which is not necessarily Poincaré. Its boundary is the n -dimensional symmetric pair $(\partial f : \partial C \rightarrow \partial_+ D, \partial_+ \delta\varphi, \partial\varphi)$ with the chain complex

$$\partial_+ D = \mathcal{C}\left(\begin{pmatrix} \delta\varphi_0 \\ \varphi_0 f^* \end{pmatrix} : D^{n+1-*} \rightarrow \mathcal{C}(f)\right)$$

defined in [13]. It is Poincaré. Similarly one can define the boundary of a quadratic pair, which is again a quadratic pair and also Poincaré. Finally the boundary of a normal pair is a quadratic Poincaré pair.

Given a degree one normal map $((g, c), (f, b)) : (N, A) \rightarrow (Y, B)$ from an n -dimensional manifold with boundary to an n -dimensional normal pair, there are relative versions of the signatures appearing in Proposition 14.2 that are defined in Definition 14.12 and Construction 14.11 below. Their relationship is described by the promised relative version of the previous proposition:

¹⁹The terminology “signature” is perhaps not the most suitable, since we do not obtain an element in an L -group. It is used because this “signature” is defined analogously to the signatures of Section 2.

Proposition 14.4. *Let $((g, c), (f, b)) : (N, A) \rightarrow (Y, B)$ be a degree one normal map from an n -dimensional manifold with boundary to an n -dimensional normal pair. Then there are homotopy equivalences of symmetric pairs*

$$h : \partial \text{sign}^{\mathbf{L}^\bullet}((g, c), (f, b)) \xrightarrow{\cong} -(1 + T) \partial \text{sign}^{\mathbf{NL}^\bullet}(Y, B)$$

and a homotopy equivalence of quadratic refinements

$$h : \partial \text{sign}^{\mathbf{L}^\bullet}((g, c), (f, b)) \xrightarrow{\cong} -\partial \text{sign}^{\mathbf{NL}^\bullet}(Y, B).$$

Finally recall that our aim is to prove a certain statement about quadratic chain complexes in the category $\Lambda(\mathbb{Z})\langle 1 \rangle(X)$. Generalizing the definitions above one obtains for a degree one normal map $(f, b) : M \rightarrow X$ from an n -dimensional manifold to an n -dimensional Poincaré complex the desired quadratic signature $\text{sign}_X^{\mathbf{L}^\bullet}(f, b)$ in Definition 14.15. Its relationship to the normal signature of X over X , which was already discussed in Section 11, is described in the following proposition, which can be seen as a global version of Proposition 14.4:

Proposition 14.5. [19, p. 192] *Let $(f, b) : M \rightarrow X$ be a degree one normal map from an n -dimensional manifold to an n -dimensional Poincaré complex. Then there is a homotopy equivalence of symmetric complexes over $\mathbb{Z}_*(X)$*

$$h : \partial \text{sign}_X^{\mathbf{L}^\bullet}(f, b) \xrightarrow{\cong} -(1 + T) \partial \text{sign}_X^{\mathbf{NL}^\bullet}(X),$$

a homotopy equivalence of quadratic refinements over $\mathbb{Z}_(X)$*

$$h : \partial \text{sign}_X^{\mathbf{L}^\bullet}(f, b) \xrightarrow{\cong} -\partial \text{sign}_X^{\mathbf{NL}^\bullet}(X)$$

and consequently a homotopy equivalence

$$h : \partial \text{sign}_X^{\mathbf{L}^\bullet}(f, b) \xrightarrow{\cong} -\partial \text{sign}_X^{\mathbf{VL}^\bullet}(X) = -s(X).$$

In the following subsections we will define the concepts used above and provide the proofs.

14.6. Quadratic signature of a degree one normal map to a normal space.

Recall that the quadratic construction 2.16 was needed in order to obtain the quadratic signature $\text{sign}^{\mathbf{L}^\bullet}(f, b)$ of a degree one normal map $(f, b) : M \rightarrow X$ from an n -dimensional manifold to an n -dimensional Poincaré space. To define $\text{sign}^{\mathbf{L}^\bullet}(g, c)$ for a degree one normal map $(g, c) : N \rightarrow Y$ from an n -dimensional manifold to an n -dimensional normal space we need the spectral quadratic construction 3.25.

Definition 14.7. [18, Prop. 7.3.4] Let $(g, c) : N \rightarrow Y$ be a degree one normal map from a Poincaré complex N to a normal complex Y . The *quadratic signature* of (g, c) is the n -dimensional QAC

$$\text{sign}^{\mathbf{L}^\bullet}(g, c) = (C, \psi)$$

which is not necessarily Poincaré, obtained from a choice of the Thom class $u(\nu_Y) \in \tilde{C}^k(\text{Th}(\nu_Y))$ as follows.

Consider the commutative diagrams

$$\begin{array}{ccc}
 C^{n-*}(N) & \xleftarrow{g^*} & C^{n-*}(Y) \\
 (\varphi(N))_0 \downarrow \simeq & & \downarrow (\varphi(Y))_0 \\
 C(N) & \xrightarrow{g_*} & C(Y),
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathrm{Th}(\nu_N)^* & \xleftarrow{\mathrm{Th}(c)^*} & \mathrm{Th}(\nu_Y)^* \\
 \Gamma_N \downarrow \simeq & & \downarrow \Gamma_Y \\
 \Sigma^p N_+ & \xrightarrow{\Sigma^p g_+} & \Sigma^p Y_+.
 \end{array}$$

The maps Γ_- in the right diagram are obtained using the S -duality as in Construction 3.6. In fact using the properties of the S -duality explained in Construction 3.6 we see that the left diagram can be considered as induced from the right diagram by applying the chain complex functor $C_{*+p}(-)$.

Set $g^! = (\varphi(N))_0 \circ g^*$ and define $C = \mathcal{C}(g^!)$ and $\psi = \Psi(u(\nu_Y)^*)$, where Ψ denotes the spectral quadratic construction on the map $\Gamma_N \circ \mathrm{Th}(c)^*$.

Also note that by the properties of the spectral quadratic construction we have

$$(1 + T)\psi \equiv e_{g^!}^{\%}(\varphi(N))$$

where $e_{g^!} : C(N) \rightarrow \mathcal{C}(g^!)$ is the inclusion.

For the proof of Proposition 14.2 we also need an additional property of the spectral quadratic construction.

Proposition 14.8. [18, Prop. 7.3.1.(v)] *Let $F : X \rightarrow \Sigma^p Y$ and $F' : X' \rightarrow \Sigma^p Y'$ be semistable maps fitting into the following commutative diagram*

$$\begin{array}{ccc}
 X & \xrightarrow{F} & \Sigma^p Y \\
 G_X \downarrow & & \downarrow G_Y \\
 X' & \xrightarrow{F'} & \Sigma^p Y'
 \end{array}$$

inducing the commutative diagram of chain complexes

$$\begin{array}{ccccc}
 \Sigma^{-p}\tilde{C}(X) & \xrightarrow{f} & \tilde{C}(Y) & \xrightarrow{e} & \mathcal{C}(f) \\
 \downarrow g_X & & \downarrow g_Y & & \downarrow \begin{pmatrix} g_Y & 0 \\ 0 & g_X \end{pmatrix} \\
 \Sigma^{-p}\tilde{C}(X') & \xrightarrow{f'} & \tilde{C}(Y') & \xrightarrow{e'} & \mathcal{C}(f').
 \end{array}$$

Then the spectral quadratic constructions of F and F' are related by

$$\Psi(F') \circ g_X \equiv \begin{pmatrix} g_Y & 0 \\ 0 & g_X \end{pmatrix}_{\%} \circ \Psi(F) + (e')_{\%} \circ \psi(G_Y) \circ f$$

where $\Psi(-)$ and $\psi(-)$ denote the (spectral) quadratic constructions on the respective maps.

Proof of Proposition 14.2. For ease of notation, let

- $\mathrm{sign}^{\mathbf{L}^\bullet}(g, c) = (\mathcal{C}(g^!), \varphi(g^!))$,
- $\mathrm{sign}^{\mathbf{NL}^\bullet}(Y) = (C(Y), \psi(Y))$,

and set

- $\partial \text{sign}^{\mathbf{L}\bullet}(g, c) = (\partial \mathcal{C}(g^!), \partial \varphi(g^!)),$
- $\partial \text{sign}^{\mathbf{NL}\bullet}(Y) = (\partial C(Y), \partial \psi(Y)).$

Consider the following commutative diagram where all rows and columns are cofibration sequences (the diagram also sets the notation μ , q_g and $e_{g^!}$)

$$(14.2) \quad \begin{array}{ccccc} 0 & \longrightarrow & C(Y)^{n-*} & \xrightarrow{\text{id}} & C(Y)^{n-*} \\ \downarrow & & \downarrow g^! & & \downarrow \varphi(Y)_0 \\ \Sigma^{-1}\mathcal{C}(g) & \xrightarrow{q_g} & C(N) & \xrightarrow{g} & C(Y) \\ \downarrow \text{id} & & \downarrow e_{g^!} & & \\ \Sigma^{-1}\mathcal{C}(g) & \xrightarrow{\mu} & \mathcal{C}(g^!). & & \end{array}$$

We obtain a homotopy equivalence, say $h' : \mathcal{C}(\mu) \xrightarrow{\simeq} \mathcal{C}(\varphi(Y)_0) \simeq \Sigma \partial C(Y)$. Inspection shows that $\mathcal{C}(g^!)^{n-*} \simeq \Sigma^{-1}\mathcal{C}(g)$ and that under this homotopy equivalence the duality map $\varphi(g^!)_0$ is identified with the map μ . Hence we have

$$(14.3) \quad e_Y \circ g = h' \circ e_{\varphi(g^!)_0} \circ e_{g^!}$$

where $e_{\varphi(g^!)_0} : \mathcal{C}(g^!) \rightarrow \Sigma \partial \mathcal{C}(g^!)$ and $e_Y : C(Y) \rightarrow \Sigma \partial C(Y)$ are the inclusions. We also obtain a commutative braid of chain complexes, which we leave for the reader to draw, with chain homotopy equivalences

$$h : \partial \mathcal{C}(g^!) \xrightarrow{\simeq} \partial C(Y) \quad \text{and} \quad h' : \mathcal{C}(\varphi(g^!)_0) \simeq \mathcal{C}(\mu) \xrightarrow{\simeq} \mathcal{C}(\varphi(Y)_0)$$

which are related by $h' = -\Sigma(h)$, thanks to the sign conventions used for definitions of mapping cones and suspensions.

Next we consider the symmetric structures. Recall that, by definition we have $S(\partial \varphi(g^!)) = e_{\varphi(g^!)_0}^{\%} \circ e_{g^!}^{\%}(\varphi(N))$, and $S(\partial \varphi(Y)) = e_Y^{\%}(\varphi(Y))$. Further we have $g^{\%}(\varphi(N)) = \varphi(Y)$ and hence

$$(h')^{\%} S(\partial \varphi(g^!)) = S(\partial \varphi(Y)).$$

By the injectivity of the suspension we also have $h^{\%}(\partial \varphi(g)) = -\partial \varphi(Y)$.

Finally, the quadratic structures: Set $\partial \text{sign}^{\mathbf{NL}\bullet}(Y) = (\partial C(Y), \partial \psi(Y))$, and $\text{sign}^{\mathbf{L}\bullet}(g, c) = (\partial \mathcal{C}(g^!), \partial \psi(g^!))$. Recall that the spectral quadratic constructions are employed to define the quadratic structures $\partial \psi(Y)$ and $\partial \psi(g^!)$. By properties of the S -duality the semistable maps used in these constructions fit into the commutative diagram

$$(14.4) \quad \begin{array}{ccc} \text{Th}(\nu_Y)^* & \xrightarrow{\text{id}} & \text{Th}(\nu_Y)^* \\ \Gamma_N \circ \text{Th}(c)^* \downarrow & & \downarrow \Gamma_Y \\ \Sigma^p N_+ & \xrightarrow{\Sigma^p g} & \Sigma^p Y_+ \end{array}$$

which induces the upper right part of the Diagram (14.2). By Proposition 14.8 the spectral quadratic construction applied to the maps Γ_Y , $\Gamma_N \circ \text{Th}(c)^*$ and

$\Sigma^p g_+$ satisfy the following relation

$$\Psi(\Gamma_Y) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}_{\%} \circ \Psi(\Gamma_N \circ \text{Th}(c)^*) + (e_{g_G})_{\%} \circ \psi(\Sigma^p g_+) \circ g_F$$

where the symbols in the brackets specify the map to which the (spectral) quadratic constructions are applied. However, the map $\Sigma^p g_+$ comes from the map $g : N \rightarrow Y$ and so $\psi(\Sigma^p g_+) = 0$. This leads to the commutative diagram

$$(14.5) \quad \begin{array}{ccc} \tilde{C}_{n+p}(\text{Th}(\nu_Y)^*) & \xrightarrow{\Psi(\Gamma_N \circ \text{Th}(c)^*)} & (W_{\%}(\mathcal{C}(g^!)))_n \\ & \searrow \Psi(\Gamma_Y) & \downarrow \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}_{\%} = (h' \circ e_{\varphi(g^!)_0})_{\%} \\ & & (W_{\%}(\mathcal{C}(\varphi(Y)_0)))_n. \end{array}$$

The identification of the vertical map comes from Diagram (14.2) and equation (14.3). Hence we obtain that

$$\begin{aligned} h'_{\%}(S\partial\psi(g^!)) &= h'_{\%} \circ (e_{\varphi(g^!)_0})_{\%}(S\psi(g^!)) = \\ &= (h' \circ e_{\varphi(g^!)_0})_{\%} \Psi(\Gamma_N \circ \text{Th}(c)^*)(u(\nu_Y)^*) = \\ &= \Psi(\Gamma_Y)(u(\nu_Y)^*) = S\partial\psi(Y). \end{aligned}$$

The uniqueness of desuspension as presented in Construction 3.26 yields the desired

$$h_{\%}(\partial\psi(g^!)) = -\partial\psi(Y)$$

thanks again to $h' = -\Sigma(h)$. $\square$

14.9. Quadratic signature of a degree one normal map to a normal pair.

Now we aim at proving Proposition 14.4, which is a relative version of the proposition just proved. First we need a relative version of Definition 14.7. For that we need to know how to apply the spectral quadratic construction in the relative setting and for that, in turn, how to apply S -duality in the relative setting.

Construction 14.10. [18, Prop. 7.3.1] Given a semistable map between point- ed pairs $(G, F) : (X, A) \rightarrow \Sigma^p(Y, B)$, consider the following diagram of induced chain maps

$$\begin{array}{ccccc} \tilde{C}(A)_{p+*} & \xrightarrow{f} & \tilde{C}(B) & \longrightarrow & \mathcal{C}(f) \\ \downarrow i & & \downarrow j & & \downarrow (j, i) \\ \tilde{C}(X)_{p+*} & \xrightarrow{g} & \tilde{C}(Y) & \longrightarrow & \mathcal{C}(g). \end{array}$$

The *relative spectral quadratic construction* on (G, F) is a chain map

$$\Psi : \Sigma^{-p} \tilde{C}(X, A) \rightarrow \mathcal{C}((j, i)_{\%})$$

such that

$$(1 + T) \circ \Psi \equiv e_{\%} \circ \varphi \circ (g, f)$$

where $\varphi : \tilde{C}(Y, B) \rightarrow \mathcal{C}(j^{\%})$ is the relative symmetric construction on (Y, B) (Construction 2.20) and $e^{\%} : \mathcal{C}(j^{\%}) \rightarrow \mathcal{C}((j, i)^{\%})$ is the map induced by the right hand square in the diagram above. The existence of Ψ follows from the naturality of the diagram in Construction 3.25.

The S -duality is applied to normal pairs as follows. Recall that an $(n+1)$ -dimensional geometric normal pair (Y, B) comes with the map of pairs

$$(\rho_Y, \rho_B) : (D^{n+k+1}, S^{n+k}) \rightarrow (\mathrm{Th}(\nu_Y), \mathrm{Th}(\nu_B)).$$

In the absolute case the map ρ composed with the diagonal map gave rise to a map Γ which was the input for the spectral quadratic construction. Now we have three diagonal maps producing three compositions:

$$\begin{aligned} S^{n+k} &\xrightarrow{\rho_B} \mathrm{Th}(\nu_B) \xrightarrow{\Delta} B_+ \wedge \mathrm{Th}(\nu_B), \\ S^{n+k+1} &\xrightarrow{\rho_Y/\rho_B} \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B) \xrightarrow{\Delta} Y_+ \wedge \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B), \\ S^{n+k+1} &\xrightarrow{\rho_Y/\rho_B} \mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B) \xrightarrow{\Delta} Y/B \wedge \mathrm{Th}(\nu_Y). \end{aligned}$$

These induce three duality maps which fit into a commutative diagram as follows:

$$(14.6) \quad \begin{array}{ccccccc} \Sigma^{-1}\mathrm{Th}(\nu_B)^* & \xrightarrow{i} & (\mathrm{Th}(\nu_Y)/\mathrm{Th}(\nu_B))^* & \longrightarrow & \mathrm{Th}(\nu_Y)^* & \longrightarrow & \mathrm{Th}(\nu_B)^* \\ \downarrow \Sigma^{-1}\Gamma_B & & \downarrow \Gamma_Y & & \downarrow \Gamma_{(Y,B)} & & \downarrow \Gamma_B \\ \Sigma^p B & \xrightarrow{j} & \Sigma^p Y & \longrightarrow & \Sigma^p(Y/B) & \longrightarrow & \Sigma^{p+1} B. \end{array}$$

Construction 14.11. The quadratic boundary of the normal pair (Y, B) is an n -dimensional QAPP

$$\partial \mathrm{sign}^{\mathrm{NL}^\bullet}(Y, B) = (\partial C(B) \rightarrow \partial_+ C(Y), (\psi(Y), \psi(B)))$$

obtained by applying the relative spectral quadratic construction on the pair of maps $(\Gamma_Y, \Sigma^{-1}\Gamma_B)$ with $(\psi(Y), \psi(B))$ the image of $u(\nu(Y))^* \in \tilde{C}(\mathrm{Th}(\nu_Y)^*)_{n+p}$ under

$$\Psi : \Sigma^{-p}\tilde{C}(\mathrm{Th}(\nu_Y)^*) \rightarrow \mathcal{C}((j, i)^{\%})$$

where i and j are as in Diagram (14.6).

Definition 14.12. Let $((g, c), (f, b)) : (N, A) \rightarrow (Y, B)$ be a degree one normal map from a Poincaré pair (N, A) to a normal pair (Y, B) of dimension $(n+1)$. The *quadratic signature* of $((g, c), (f, b))$ is the n -dimensional quadratic pair

$$\mathrm{sign}^{\mathrm{L}^\bullet}((g, c), (f, b)) = (j : C \rightarrow D, (\delta\psi, \psi))$$

which is not necessarily Poincaré, obtained from a choice of the Thom class $u(\nu_Y)$ as follows.

The S -duality produces a commutative diagram

$$\begin{array}{ccc} \Sigma^p A_+ & \xleftarrow{\Gamma_A \circ \text{Th}(b)^*} & \Sigma^{-1} \text{Th}(\nu_B)^* \\ \downarrow & & \downarrow \\ \Sigma^p N_{\Gamma_N \circ (\text{Th}(c)/\text{Th}(b))^*} & \xleftarrow{(\text{Th}(\nu_Y)/\text{Th}(\nu_B))^*} & \end{array}$$

inducing the diagram of chain complexes

$$\begin{array}{ccc} C(A) & \xleftarrow{f^!} & C^{n-*}(B) \\ \downarrow & & \downarrow \\ C(N) & \xleftarrow{g^!} & C^{n+1-*}(Y, B). \end{array}$$

Set $C = \mathcal{C}(f^!)$, $D = \mathcal{C}(g^!)$ and $(\delta\psi, \psi) = \Psi(u(\nu_Y)^*)$, where the spectral quadratic construction is on the pair of maps $(\Gamma_N \circ (\text{Th}(c)/\text{Th}(b))^*, \Gamma_A \circ \text{Th}(b)^*)$.

Proof of Proposition 14.4. The proof follows the same pattern as the proof of Proposition 14.2. With the notation of Construction 14.11 and Definition 14.12 one first observes that we have a homotopy equivalence of pairs

$$(\partial h \rightarrow h) : (\partial C(B) \rightarrow \partial_+ C(Y)) \xrightarrow{\sim} (j : C \rightarrow D)$$

by studying a map of diagrams of the same shape as Diagram (14.2). For the symmetric structures observe that the homotopy equivalence $(\partial h \rightarrow h)$ satisfies an equation analogous to (14.3) and again use the naturality. Finally to obtain the desired equivalence of quadratic structures there is again a map of commutative squares of the form as in Diagram (14.4). A diagram chase shows that the relative spectral construction satisfies a formula analogous to the one appearing in Proposition 14.8. This leads to a diagram analogous to Diagram (14.5). As in the absolute case unraveling what it means and using the desuspension produces the desired equation. $\square$

Remark 14.13. Just as explained in Section 5 the relative version just proved has a generalization for k -ads.

14.14. Quadratic signature over X of a degree one normal map to X .

Now we want to prove Proposition 14.5. The preparation for this starts with discussing $\text{sign}_X^{\text{NL}^\bullet}(X)$ for an n -dimensional GPC. This was defined in Section 11 by first passing to $\text{sign}_X^{\Omega_X^N}(X)$ and then applying the spectrum map $\text{sign}^{\text{NL}^\bullet} : \Omega_X^N \rightarrow \text{NL}^\bullet$ from [25]. By the proof of [25, Thm. 7.1] the spectrum map $\text{sign}^{\text{NL}^\bullet}$ composed with the boundary fits with the quadratic boundary construction as described in Subsection 3.24, so we can think of $\partial \text{sign}_X^{\text{NL}^\bullet}(X)$ as a collection of $(n - |\sigma|)$ -dimensional quadratic $(m - |\sigma|)$ -ads indexed by

simplices of X which fit together and are obtained by the relative spectral construction described in Construction 14.11.

Definition 14.15. Let $(f, b) : M \rightarrow X$ be a degree one normal map from a closed n -dimensional topological manifold to an n -dimensional GPC. Make f transverse to the dual cell $D(\sigma, X)$ for each $\sigma \in X$ so that we have a degree one normal map

$$(f[\sigma], f[\partial\sigma]) : (M[\sigma], \partial M[\sigma]) \rightarrow (X[\sigma], \partial X[\sigma])$$

from an $(n - |\sigma|)$ -dimensional manifold with boundary to an $(n - |\sigma|)$ -dimensional normal pair. Define the *quadratic signature over X* of (f, b) to be the element

$$\text{sign}_X^{\mathbf{L}^\bullet}(f, b) \in L_n(\Lambda(\mathbb{Z})(X))$$

represented by the n -dimensional QAC (C, ψ) in $\Lambda(\mathbb{Z})(X)$ whose component over $\sigma \in X$ is the relative quadratic signature

$$\text{sign}^{\mathbf{L}^\bullet}((f(\sigma), b(\sigma)), \partial(f(\sigma), b(\sigma)))$$

obtained as in Definition 14.12. The resulting element is independent of all the choices.

Proof of Proposition 14.5. In order to prove the proposition it is necessary to prove that one has homotopy equivalences as in the statement but for each simplex and so that they fit together. However, for each simplex this is exactly the statement of Proposition 14.4. Since one can proceed inductively from simplices of the top dimension to smaller simplices the homotopy equivalences can be made to fit together.

To obtain the last homotopy equivalence recall that X is an n -dimensional Poincaré complex and hence the same complex that defines the normal signature defines the visible signature over X , see Section 11 if needed. $\square$

14.16. Identification of the quadratic signature with the assembly.

Now we proceed to Step (2) of the proof of the Main Technical Theorem part (II). We first state the following preparatory proposition which describes what happens when we consider the difference of the quadratic signatures of two degree one normal maps.

Proposition 14.17. *Let $(f_i, b_i) : M_i \rightarrow X$ with $i = 0, 1$ be two degree one normal maps from n -dimensional topological manifolds to an n -dimensional GPC. Then the difference of their quadratic signatures over $\mathbb{Z}_*(X)$ is an n -dimensional QAC in the algebraic bordism category $\Lambda\langle 1\rangle(\mathbb{Z})_*(X)$ and hence represents an element*

$$\text{sign}_X^{\mathbf{L}^\bullet}(f_1, b_1) - \text{sign}_X^{\mathbf{L}^\bullet}(f_0, b_0) \in H_n(X; \mathbf{L}_\bullet\langle 1\rangle).$$

Proof. A quadratic chain complex in $\mathbb{Z}_*(X)$ is an n -dimensional QAC in the algebraic bordism category $\Lambda\langle 1\rangle(\mathbb{Z})_*(X)$ if and only if it is locally Poincaré which is equivalent to saying that its boundary is contractible. So it is enough

to prove that the two quadratic complexes representing $\text{sign}_X^{\mathbf{L}\bullet}(f_i, b_i)$ have homotopy equivalent boundaries. This follows from Proposition 14.5 since they are both homotopy equivalent to $-\partial \text{sign}_X^{\mathbf{NL}\bullet}(X)$. $\square$

The degree one normal maps with the target X are organized in the normal invariants $\mathcal{N}(X)$. The above proposition tells us that the quadratic signature over X relative to (f_0, b_0) defines a map

$$(14.7) \quad \text{sign}_X^{\mathbf{L}\bullet}(-, -) - \text{sign}_X^{\mathbf{L}\bullet}(f_0, b_0) : \mathcal{N}(X) \rightarrow H_n(X; \mathbf{L}\bullet\langle 1 \rangle).$$

The following proposition is the main result in the proof of Step (2). It says that for X an n -dimensional Poincaré complex such that $t(X) = 0$ the surgery obstruction map $\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}\bullet} : \mathcal{N}(X) \rightarrow L_n(\mathbb{Z}[\pi_1(X)])$ can be identified with the assembly map. When X is already a manifold the map (f_0, b_0) can be taken to be the identity.

Proposition 14.18. [15, pp. 293–297], [19, proof of Thm. 17.4] *Let X be an n -dimensional GPC with $\pi = \pi_1(X)$ such that $t(X) = 0$ and let $(f_0, b_0) : M_0 \rightarrow X$ be any choice of a degree one normal map. Then the diagram*

$$\begin{array}{ccc} \mathcal{N}(X) & \xrightarrow{\text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}\bullet}(-, -) - \text{sign}_{\mathbb{Z}[\pi]}^{\mathbf{L}\bullet}(f_0, b_0)} & L_n(\mathbb{Z}[\pi_1(X)]) \\ \text{sign}_X^{\mathbf{L}\bullet}(-, -) - \text{sign}_X^{\mathbf{L}\bullet}(f_0, b_0) \downarrow \cong & & \downarrow = \\ H_n(X; \mathbf{L}\bullet\langle 1 \rangle) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi_1(X)]) \end{array}$$

is commutative and the left vertical map is a bijection.

Overview of the proof.

To see the commutativity consider $(f, b) : M \rightarrow X$ a degree one normal map from an n -dimensional manifold to an n -dimensional Poincaré complex X and $A : \Lambda(X) \rightarrow \Lambda(\mathbb{Z}[\pi_1(X)])$ the assembly functor. Then we have

$$A(\text{sign}_X^{\mathbf{L}\bullet}(f, b)) = \text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}\bullet}(f, b) \in L_n(\Lambda(\mathbb{Z}[\pi_1(X)]))$$

since the assembly corresponds to geometric gluing, see Remark 8.18.

We are left with showing that the left hand vertical map is a bijection which will be done by identifying this map with a composition of four maps which are all bijections. In order to save space we will abbreviate (using x_0 for (f_0, b_0))

$$\text{sign}_X^{\mathbf{L}\bullet}(-; x_0) := \text{sign}_X^{\mathbf{L}\bullet}(-, -) - \text{sign}_X^{\mathbf{L}\bullet}(f_0, b_0)$$

The strategy of the proof can be summarized in the following diagram:

$$\begin{array}{ccc}
 \mathcal{N}(X) & \xrightarrow{=} & \mathcal{N}(X) \\
 \downarrow \cong \scriptstyle t(-, x_0) & & \downarrow \cong \scriptstyle t(-, x_0) \\
 [X; \mathbf{G}/\mathbf{TOP}] & \xrightarrow{=} & [X; \mathbf{G}/\mathbf{TOP}] \\
 \downarrow \scriptstyle \text{sign}_X^{\mathbf{L}\bullet}(-; x_0) \quad \cong \quad \downarrow \scriptstyle \text{sign}^{\mathbf{L}\bullet} & & \downarrow \tilde{\Gamma} \quad \text{sign}_X^{\mathbf{G}/\mathbf{TOP}}(-; x_0) \\
 H^0(X; \mathbf{L}\bullet\langle 1 \rangle) & \xleftarrow{\text{sign}^{\mathbf{L}\bullet}} & H^0(X; \Sigma^{-1}\mathbf{\Omega}_{\bullet}^{N, \text{STOP}}) \\
 \downarrow \scriptstyle -\cup u^{\mathbf{L}\bullet}(\nu_0) \quad \cong & & \downarrow \scriptstyle -\cup u^{\text{STOP}}(\nu_0) \quad \cong \\
 H^k(\text{Th}(\nu_X); \mathbf{L}\bullet\langle 1 \rangle) & \xleftarrow{\text{sign}^{\mathbf{L}\bullet}} & H^k(\text{Th}(\nu_X); \Sigma^{-1}\mathbf{\Omega}_{\bullet}^{N, \text{STOP}}) \\
 \downarrow \scriptstyle S\text{-dual} \quad \cong & & \downarrow \scriptstyle S\text{-dual} \quad \cong \\
 H_n(X; \mathbf{L}\bullet\langle 1 \rangle) & \xleftarrow{\text{sign}^{\mathbf{L}\bullet}} & H_n(X; \Sigma^{-1}\mathbf{\Omega}_{\bullet}^{N, \text{STOP}})
 \end{array}$$

Some of the maps in the diagram have been defined already, the remaining ones will be defined shortly. We will show that those marked with $\cong$ are bijections or isomorphisms. Once this is done it is enough to show that the left hand part of the diagram is commutative, because then we have indeed identified $\text{sign}_X^{\mathbf{L}\bullet}(-; x_0)$ with a composition of four bijections. The commutativity of the left hand part will be shown by proving that:

- (a) the outer square commutes in Subsection 14.19,
- (b) the middle part commutes in Subsection 14.27,
- (c) the right hand part commutes in Subsection 14.28.

The intermediate subsections contain the necessary definitions. $\square$

14.19. Proof of (a) - Quadratic signatures versus normal signatures.

Recall that for a degree one normal map $(f, b) : M \rightarrow X$ from an n -dimensional manifold to an n -dimensional GPC we have two ways how to obtain its quadratic signature over $\mathbb{Z}[\pi_1(X)]$. Namely via the Umkehr map $f^!$ as in Construction 2.16 or via the normal structure on the mapping cylinder W as in Example 3.27. It was further shown in Example 3.27 that these two constructions yield the same result. In Subsection 14.14 we have defined the quadratic signature of (f, b) over $\mathbb{Z}_*(X)$ using Umkehr maps providing an analogue to Construction 2.16. Here we provide an analogue to Example 3.27 over $\mathbb{Z}_*(X)$.

Example 14.20. This is a generalization of the results of Subsection 3.24. Let us consider a degree one normal map $(g, c) : N \rightarrow Y$ from an n -dimensional manifold to an n -dimensional GNC. We obtain an $(n+1)$ -dimensional normal pair $(W, N \sqcup Y)$ where W is the mapping cylinder of f . However, in contrast to the situation in Example 3.27, the disjoint union $N \sqcup Y$ is no longer a Poincaré

complex. Therefore the associated algebraic normal pair

$$(\text{sign}^{\mathbf{NL}^\bullet}(W), \text{sign}^{\mathbf{L}^\bullet}(N) - \text{sign}^{\mathbf{NL}^\bullet}(Y))$$

does not have a Poincaré boundary. Nevertheless we can still perform algebraic surgery on this pair, just as in Lemma 3.22 and thanks to the spectral quadratic construction the result of the surgery is an n -dimensional quadratic complex, which however, will not be Poincaré. The proof from Example 3.27 translates almost word-for-word to an identification of this quadratic complex with $\text{sign}^{\mathbf{L}^\bullet}(g, c)$ from Definition 14.7 (the only difference being the fact that the map $\varphi_0|_Y$ is no longer an equivalence). In symbols we have

$$(14.8) \quad \text{Lemma 3.22} : (\text{sign}^{\mathbf{NL}^\bullet}(W), \text{sign}^{\mathbf{L}^\bullet}(N) - \text{sign}^{\mathbf{NL}^\bullet}(Y)) \mapsto \text{sign}^{\mathbf{L}^\bullet}(g, c).$$

Using the relative version of S -duality and of the spectral quadratic construction from earlier in this section one also obtains a relative version of this identification.

Example 14.21. Given a degree one normal map $(f, b) : M \rightarrow X$ from an n -dimensional manifold to an n -dimensional GPC consider the dissection

$$X = \bigcup_{\sigma \in X} X(\sigma), \quad (f, b) = \bigcup_{\sigma \in X} (f(\sigma), b(\sigma)) : M(\sigma) \rightarrow X(\sigma)$$

where each $(f(\sigma), b(\sigma))$ is a degree one normal map from an $(n - |\sigma|)$ -dimensional manifold $(m - |\sigma|)$ -ad to an $(n - |\sigma|)$ -dimensional normal $(m - |\sigma|)$ -ad. As such it gives rise to an $(n + 1 - |\sigma|)$ -dimensional pair of normal $(m - |\sigma|)$ -ads

$$(W(\sigma), \nu(b(\sigma)), \rho(b(\sigma))).$$

Applying Example 14.20 shows that the quadratic chain complex over $\mathbb{Z}_*(X)$ obtained this way coincides with the quadratic signature $\text{sign}_X^{\mathbf{L}^\bullet}(f, b)$ from Definition 14.15.

The following Lemma is a generalization of ideas from Construction 11.9. In its statement we use the quadratic signature map $\text{sign}^{\mathbf{L}^\bullet} : \Sigma^{-1}\mathbf{\Omega}_\bullet^{N, \text{STOP}} \rightarrow \mathbf{L}_\bullet\langle 1 \rangle$ from Proposition 7.10.

Lemma 14.22. *Let $x_i = (f_i, b_i) : M_i \rightarrow X$ with $i = 0, 1$ be two degree one normal maps from n -dimensional topological manifolds to an n -dimensional GPC. Then there exists a G/TOP-signature*

$$\text{sign}_X^{\text{G/TOP}}(x_1, x_0) \in H_n(X; \Sigma^{-1}\mathbf{\Omega}_\bullet^{N, \text{STOP}})$$

such that

$$\text{sign}_X^{\mathbf{L}^\bullet}(f_1, b_1) - \text{sign}_X^{\mathbf{L}^\bullet}(f_0, b_0) = \text{sign}^{\mathbf{L}^\bullet}(\text{sign}_X^{\text{G/TOP}}(x_1, x_0)) \in H_n(X; \mathbf{L}_\bullet\langle 1 \rangle).$$

Proof. Consider the dissections of $(f_i, b_i) : M_i \rightarrow X$ as in Example 14.21. The assignments

$$\sigma \mapsto (W_i(\sigma), \nu(b_i(\sigma)), \rho(b_i(\sigma)))$$

fit together to produce cobordisms of $\mathbf{\Omega}_\bullet^N$ -cycles in the sense of Definition 8.6. However, they do not produce a $\Sigma^{-1}\mathbf{\Omega}_\bullet^{N, \text{STOP}}$ -cycle since the ends of the cobordisms given by X are not topological manifolds. But the two ends for $i = 0, 1$

are equal and so we can glue the two cobordisms along these ends and we obtain for each $\sigma \in X$ the $(n + 1 - |\sigma|)$ -dimensional pairs of normal $(m - |\sigma|)$ -ads

$$(W_1(\sigma) \cup_{X(\sigma)} W_0(\sigma)), \nu(b_1(\sigma)) \cup_{\nu_{X(\sigma)}} \nu(b_0(\sigma)), \rho(b_1(\sigma)) \cup_{\rho(\sigma)} \rho(b_0(\sigma)))$$

which now fit together to produce a $\Sigma^{-1}\Omega_{\bullet}^{N, \text{STOP}}$ -cycle in the sense of Definition 8.6. This produces the desired G/TOP-signature

$$\text{sign}_X^{\text{G/TOP}}(x_1, x_0) \in H_n(X; \Sigma^{-1}\Omega_{\bullet}^{N, \text{STOP}}).$$

To prove the equation recall from Proposition 9.7 the quadratic signature map $\text{sign}^{\mathbf{L}_{\bullet}} : \Sigma^{-1}\Omega_{\bullet}^{N, \text{STOP}} \rightarrow \mathbf{L}_{\bullet}\langle 1 \rangle$. We have to investigate the value of the induced map on the G/TOP-signature. By definition this value is given on each simplex $\sigma \in X$ as the $(n - |\sigma|)$ -dimensional quadratic Poincaré $(m - |\sigma|)$ -ad obtained by the algebraic surgery on the algebraic pair extracted from the (normal, topological manifold) pair $(W_1(\sigma) \cup_{X(\sigma)} W_0(\sigma), M_1(\sigma) \sqcup M_0(\sigma))$.

Consider now the left hand side of the desired equation. By Example 14.21 the value of each summand on a simplex σ is obtained via algebraic surgery on the algebraic pair extracted from the normal pair $(W_i(\sigma), M_i(\sigma) \sqcup X(\sigma))$ (whose boundaries are not Poincaré and so the resulting complexes are also not Poincaré). Subtracting these corresponds to taking the disjoint union of the normal pairs above and reversing the orientation on the one labeled with $i = 0$. On the other hand there is a geometric normal cobordism between geometric normal pairs

$$(W_1(\sigma) \cup_{X(\sigma)} W_0(\sigma), M_1(\sigma) \sqcup -M_0(\sigma))$$

and

$$(W_1(\sigma) \sqcup -W_0(\sigma), M_1(\sigma) \sqcup X(\sigma) \sqcup -M_1(\sigma) \sqcup -X(\sigma))$$

which induces an algebraic cobordism and hence the extracted algebraic data are also cobordant. $\square$

14.23. Normal invariants revisited.

Let X be an n -dimensional GPC which admits a topological block bundle reduction of its SNF. For such an X we will now discuss in more detail the bijection $\mathcal{N}(X) \cong [X; \text{G/TOP}]$ which was already used in the proof of Theorem 13.16.

We first set up some notation. An element $x \in \mathcal{N}(X)$ is represented either by a degree one normal map $(f, b) : M \rightarrow X$ from an n -dimensional topological manifold M to X or by a pair (ν, h) where $\nu : X \rightarrow \text{BSTOP}$ is a stable topological block bundle on X and $h : J(\nu) \simeq \nu_X$ is a homotopy from the underlying spherical fibration to the SNF. The two descriptions of normal invariants are identified via the usual Pontrjagin-Thom construction, see [23, Chap. 10] if needed.

An element in the set $[X; \text{G/TOP}]$ of homotopy classes of maps from X to G/TOP can be thought of as being represented by a pair $(\bar{\nu}, \bar{h})$, where $\bar{\nu} : X \rightarrow \text{BSTOP}$ is a stable topological block bundle on X and $h : J(\bar{\nu}) \simeq *$ is a homotopy from the underlying spherical fibration to the constant map

(which represents the trivial spherical fibration). The set $[X; \mathbf{G}/\mathbf{TOP}]$ is a group under the Whitney sum operation and it has an action on $\mathcal{N}(X)$ by

$$[X; \mathbf{G}/\mathbf{TOP}] \times \mathcal{N}(X) \rightarrow \mathcal{N}(X) \\ ((\bar{\nu}, \bar{h}), (\nu, h)) \mapsto (\bar{\nu} \oplus \nu, \bar{h} \oplus h).$$

The action is free and transitive [23, Chap. 10] and hence any choice of a point $x_0 = (\nu_0, h_0) = (f_0, b_0) \in \mathcal{N}(X)$ gives a bijection $[X; \mathbf{G}/\mathbf{TOP}] \cong \mathcal{N}(X)$ whose inverse is denoted by

$$(14.9) \quad t(-, x_0) : \mathcal{N}(X) \rightarrow [X; \mathbf{G}/\mathbf{TOP}].$$

So for $x = (\nu, h)$ and $t(x, x_0) = (\bar{\nu}, \bar{h})$ we have

$$(14.10) \quad (\nu, h) = (\bar{\nu} \oplus \nu_0, \bar{h} \oplus h_0).$$

14.24. From normal invariants to cohomology.

Construction 14.25. Now we construct the map

$$\tilde{\Gamma} : \mathbf{G}/\mathbf{TOP} \rightarrow \Sigma^{-1} \Omega_0^{N, \text{STOP}}.$$

To an l -simplex in $\mathbf{G}/\mathbf{TOP}$ alias a degree one normal map $(f, b) : M \rightarrow \Delta^l$ the map $\tilde{\Gamma}$ associates an l -simplex of $\Sigma^{-1} \Omega_0^{N, \text{STOP}}$ alias an $(l+1)$ -dimensional l -ad of (normal, topological manifold) pairs $(W, M \sqcup -\Delta^l)$ where W is the mapping cylinder of f and the normal structure comes from the bundle map b .

The proof of Theorem 13.16 shows that the surgery obstruction alias the quadratic signature map $\text{sign}^{\mathbf{L}\bullet} : \mathbf{G}/\mathbf{TOP} \rightarrow \mathbf{L}_0\langle 1 \rangle$ can be thought of as a composition of two maps:

$$\tilde{\Gamma} : \mathbf{G}/\mathbf{TOP} \rightarrow \Sigma^{-1} \Omega_0^{N, \text{STOP}} \quad \text{and} \quad \text{sign}^{\mathbf{L}\bullet} : \Sigma^{-1} \Omega_0^{N, \text{STOP}} \rightarrow \mathbf{L}_0\langle 1 \rangle,$$

where the second map comes from Proposition 9.7.

14.26. Products and Thom isomorphism.

Now we briefly review the cup products in ring and module spectra. We only concentrate on the cases which are used in this paper. In fact we only need the cup products realizing the Thom isomorphism. Let X be a CW-complex and let ξ be a k -dimensional spherical fibration over X . Further suppose that $\mathbf{F}$ is a module spectrum over the ring spectrum $\mathbf{E}$. Then there are the cup products

$$- \cup - : H^p(X; \mathbf{F}) \otimes H^q(\text{Th}(\xi); \mathbf{E}) \rightarrow H^{p+q}(\text{Th}(\xi); \mathbf{F}) \\ x \otimes y \mapsto x \cup y$$

given by the composition

$$x \cup y : \text{Th}(\xi) \xrightarrow{\Delta} X_+ \wedge \text{Th}(\xi) \xrightarrow{x \wedge y} \mathbf{F}_p \wedge \mathbf{E}_q \rightarrow \mathbf{F}_{p+q}$$

with Δ the diagonal map already mentioned for example in Section 3. If we have an $\mathbf{E}$ -orientation $u \in H^k(\text{Th}(\xi); \mathbf{E})$, then the resulting homomorphism

$$(14.11) \quad - \cup u : H^p(X; \mathbf{F}) \rightarrow H^{p+k}(\text{Th}(\xi); \mathbf{F})$$

is the Thom isomorphism.

Remember now that we are working with simplicial complexes and Δ -sets rather than topological spaces. The above constructions work in this setting as long as we choose a simplicial approximation of the diagonal map Δ . An explicit description of such an approximation in our situation is given in [19, Rem. 12.5]. However, it does not help us much since we do not understand its behavior with respect to the signatures we have defined earlier in this section. On the other hand, as we will see, we understand the behavior of the diagonal map Δ of spaces with respect to the orientations discussed in Section 13. Since we also know that the orientations correspond to the signatures via the S -duality (Proposition 13.9) we can work with them and therefore we can work with the version of the cup product for spaces.

14.27. Proof of (b) - Naturality of orientations.

The commutativity of the second square follows from the second paragraph of Subsection 14.24. The commutativity of the third square follows from the naturality of the cup product with respect to the coefficient spectra and from the fact that the canonical $\mathbf{L}^\bullet$ -orientation of a stable topological block bundle is the image of the canonical **MSTOP**-orientation (Proposition 13.7). The commutativity of the fourth square follows from the naturality of the S -duality with respect to the coefficient spectra.

14.28. Proof of (c) - Signatures versus orientations revisited.

If $x = (f, b) : M \rightarrow X$ represents an element from $\mathcal{N}(X)$ then (c) can be expressed by the formula

$$(14.12) \quad \widetilde{\Gamma}(t(x, x_0)) \cup u^{\text{STOP}}(\nu_0) = S^{-1}(\text{sign}_X^{\text{G/TOP}}(x, x_0))$$

in the group $H^k(\text{Th}(\nu_X); \Sigma^{-1}\Omega_\bullet^{N, \text{STOP}})$. Here $x_i \in \mathcal{N}(X)$ are represented either by degree one normal maps $(f_i, b_i) : M_i \rightarrow X$ or pairs (ν_i, h_i) as in Subsection 14.23 and we keep this notation for the rest of this section. For the proof an even better understanding of the relationship between various signatures and orientations is needed. To put the orientations into the game we use the Pontrjagin-Thom map

$$\Sigma^{-1}\Omega_\bullet^{N, \text{STOP}} \simeq \mathbf{MS}(\mathbf{G}/\mathbf{TOP}) := \text{Fiber}(\mathbf{MSTOP} \rightarrow \mathbf{MSG}).$$

We will show (14.12) in two steps, namely we show that both sides are equal to a certain element $u^{\text{G/TOP}}(\nu, \nu_0) \in H^k(\text{Th}(\nu_X); \mathbf{MS}(\mathbf{G}/\mathbf{TOP}))$.

Construction 14.29. Recall the canonical STOP-orientations (Subsection 13.6)

$$u^{\text{STOP}}(\nu) - u^{\text{STOP}}(\nu_0) \in H^k(\text{Th}(\nu_X); \mathbf{MSTOP})$$

and also the fact that we have the homotopy $h_0 \cup h : \text{Th}(\nu_X) \times [-1, 1] \rightarrow \mathbf{MSG}$ between $J(\nu)$ and $J(\nu_0)$. This homotopy can also be viewed as a null-homotopy of the map $J(u^{\text{STOP}}(\nu) - u^{\text{STOP}}(\nu_0))$. Hence we obtain a preferred lift which we denote

$$u^{\text{G/TOP}}(\nu, \nu_0) \in H^k(\text{Th}(\nu_X); \mathbf{MS}(\mathbf{G}/\mathbf{TOP})).$$

Proposition 14.30. *Let X be an n -dimensional GPC and let x, x_0 be two topological block bundle reductions of the SNF. Then we have*

$$S(u^{G/TOP}(\nu, \nu_0)) = \text{sign}_X^{G/TOP}(x, x_0) \in H_n(X; \mathbf{MS}(\mathbf{G}/\mathbf{TOP})).$$

Proof. The proof is analogous to the proof of Proposition 13.10. Recall that the signature $\text{sign}_X^{G/TOP}(x, x_0)$ is constructed using the dissections of the degree one normal maps (f_i, b_i) . From these dissections we inspect that we have a commutative diagram

$$\begin{array}{ccc} \Sigma^m / \overline{X} & \xrightarrow{\text{sign}_X^{G/TOP}(x_1, x_0)} & \Sigma^{-1} \Omega_{-k}^{N, \text{STOP}} \\ \downarrow i & & \downarrow c \\ \text{Sing } F(\nu_1; \nu_0) & \xrightarrow{u^{G/TOP}(\nu, \nu_0)} & \text{Sing } \mathbf{MS}(\mathbf{G}/\mathbf{TOP})(k) \end{array}$$

where we set $\mathbf{MS}(\mathbf{G}/\mathbf{TOP})(k) := \text{Fiber } (\mathbf{MSTOP}(k) \rightarrow \mathbf{MSG}(k))$ and $F(\nu_1; \nu_0) := \text{Pullback } (\text{Th}(\nu_1) \rightarrow \text{Th}(\nu_X) \leftarrow \text{Th}(\nu_0))$. This proves the claim. $\square$

Now we turn to the left hand side of the formula (14.12). We first need to understand the composition (abusing the notation slightly):

$$\tilde{\Gamma} : [X; \mathbf{G}/\mathbf{TOP}] \xrightarrow{\tilde{\Gamma}} H^0(X; \Sigma^{-1} \Omega_{\bullet}^{N, \text{STOP}}) \rightarrow H^0(X; \mathbf{MS}(\mathbf{G}/\mathbf{TOP})).$$

Let $(\bar{\nu}, \bar{h})$ represent an element on $[X; \mathbf{G}/\mathbf{TOP}]$. Recall that $\bar{h} : J(\bar{\nu}) \simeq \varepsilon$ and that we have the canonical orientations $u^{\mathbf{MSTOP}}(\nu)$ and $u^{\mathbf{MSTOP}}(\varepsilon)$ and the homotopy $u^{\mathbf{MSG}}(\bar{h}) : u^{\mathbf{MSG}}(J(\nu)) \simeq u^{\mathbf{MSG}}(\varepsilon)$. We obtain

$$\tilde{\Gamma}(\bar{\nu}, \bar{h}) = (u^{\mathbf{MSTOP}}(\nu) - u^{\mathbf{MSTOP}}(\varepsilon), u^{\mathbf{MSG}}(\bar{h}) : u^{\mathbf{MSG}}(J(\nu)) - u^{\mathbf{MSG}}(\varepsilon) \simeq *)$$

Hence the element $\tilde{\Gamma}(\bar{\nu}, \bar{h})$ is the unique lift of $u^{\mathbf{MSTOP}}(\nu) - u^{\mathbf{MSTOP}}(\varepsilon)$ obtained from the homotopy $\bar{h}$.

Now consider our $x, x_0 \in \mathcal{N}(X)$ and denote $t := t(x, x_0) = (\bar{\nu}, \bar{h})$. As a warm up before proving the equation (14.12) we consider its push-forward in the group $H^k(\text{Th}(\nu_X); \mathbf{MSTOP})$. Denote the composition

$$\Gamma : [X; \mathbf{G}/\mathbf{TOP}] \xrightarrow{\tilde{\Gamma}} H^0(X; \mathbf{MS}(\mathbf{G}/\mathbf{TOP})) \xrightarrow{\text{incl}} H^0(X; \mathbf{MSTOP})$$

This simply forgets the homotopy $u^{\mathbf{MSG}}(\bar{h})$. So we have

$$\Gamma(\bar{\nu}, \bar{h}) = u^{\mathbf{MSTOP}}(\bar{\nu}) - u^{\mathbf{MSTOP}}(\varepsilon) : \text{Th}(\bar{\nu}) \simeq \Sigma^k \Delta_+^l \simeq \text{Th}(\varepsilon) \rightarrow \mathbf{MSTOP}.$$

Define the following two maps

$$\Phi : [X; \mathbf{G}/\mathbf{TOP}] \rightarrow H^0(X; \mathbf{MSTOP})$$

and

$$1 : [X; \mathbf{G}/\mathbf{TOP}] \rightarrow H^0(X; \mathbf{MSTOP})$$

by

$$\Phi(\bar{\nu}, \bar{h}) = u^{\mathbf{MSTOP}}(\nu) \quad \text{and} \quad 1(\bar{\nu}, \bar{h}) = u^{\mathbf{MSTOP}}(\varepsilon)$$

so that we have $\Gamma = \Phi - 1$ and consider $\Gamma(t) = (\Phi - 1)(t)$. The Thom isomorphism

$$(14.13) \quad - \cup u^{\text{STOP}}(\nu_0) : H^0(X; \mathbf{MSTOP}) \rightarrow H^k(\text{Th}(\nu_X); \mathbf{MSTOP})$$

applied to an element $\Phi(t) \in H^0(X; \mathbf{MSTOP})$ is given by the composition

$$\text{Th}(\nu) \xrightarrow{\Delta} \Sigma^l X_+ \wedge \text{Th}(\nu_0) \xrightarrow{\Phi(t) \wedge u^{\text{STOP}}(\nu_0)} \mathbf{MSTOP} \wedge \mathbf{MSTOP} \xrightarrow{\oplus} \mathbf{MSTOP}.$$

From the relationship between the Whitney sum and the cross product and the diagonal map we obtain that

$$u^{\text{STOP}}(\nu) = \Phi(t(x, x_0)) \cup u^{\text{STOP}}(\nu_0).$$

Analogously we obtain

$$u^{\text{STOP}}(\nu_0) = 1(t(x, x_0)) \cup u^{\text{STOP}}(\nu_0).$$

Lemma 14.31. *Let X be an n -dimensional GPC and let $\nu, \nu_0 : X \rightarrow \text{BSTOP}$ be two topological block bundles such that $J(\nu) \simeq \nu_X \simeq J(\nu_0)$. Then the canonical STOP-orientations satisfy*

$$u^{\text{STOP}}(\nu) - u^{\text{STOP}}(\nu_0) = \Gamma(\tilde{t}(x, x_0)) \cup u^{\text{STOP}}(\nu_0).$$

Proof. The desired equation follows from the definition $\Gamma = \Phi - 1$. □

The final step is the following lemma which is a refinement of Lemma 14.31.

Lemma 14.32. *Let X be an n -dimensional GPC and let $\nu, \nu_0 : X \rightarrow \text{BSTOP}$ be two topological block bundles such that $J(\nu) \simeq \nu_X \simeq J(\nu_0)$. Then the canonical STOP-orientations satisfy*

$$u^{\text{G/STOP}}(\nu, \nu_0) = \tilde{\Gamma}(t(x, x_0)) \cup u^{\text{STOP}}(\nu_0).$$

Proof. The left hand side is obtained from the left hand side of Lemma 14.31 using the null-homotopy of $\Gamma(t(x, x_0))$ coming from $h \cup h_0 : J(\nu) \simeq J(\nu_0)$. The right hand side is obtained from the right hand side of Lemma 14.31 using the null-homotopy of $\Gamma(t(x, x_0))$ coming from $\tilde{h} : J(t(x, x_0)) \simeq J(t(x_0, x_0)) = \nu_X$. Applying the cup product with $u^{\text{STOP}}(\nu_0)$ to this null-homotopy corresponds to taking the Whitney sum with ν_0 and gives a homotopy $\tilde{h} \oplus \text{id}_{\nu_0} : J(\nu) \simeq J(\nu_0)$. The claim now follows from the property of the SNF that any two fiberwise homotopy equivalences between the stable topological block bundle reductions of the SNF are stably fiberwise homotopic. □

14.33. Proof of the Main Technical Theorem (II).

Proof of the Main Technical Theorem (II) assuming Propositions 14.5, 14.18.

Consider the set

$$Q := \{ -\text{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}_\bullet}(f, b) \in L_n(\mathbb{Z}[\pi_1(X)]) \mid (f, b) : M \rightarrow X \text{ degree one normal map, } M \text{ manifold} \}.$$

Fix a degree one normal map $(f_0, b_0): M_0 \rightarrow X$ from a manifold M_0 to our Poincaré complex X . Proposition 14.18 tells us that

$$\begin{aligned} \operatorname{sign}_{\mathbb{Z}[\pi_1(X)]}^{\mathbf{L}_\bullet}(f_0, b_0) + Q &= \operatorname{im} (A : H_n(X; \mathbf{L}_\bullet\langle 1 \rangle \rightarrow L_n(\mathbb{Z}[\pi_1(X)])) \\ &= \ker (\partial : L_n(\mathbb{Z}[\pi_1(X)]) \rightarrow \mathbb{S}_n(X)) \end{aligned}$$

and from this it follows that Q is a coset of $\ker(\partial)$. Furthermore the preimage $\partial^{-1}s(X) \subseteq L_n(\mathbb{Z}[\pi_1(X)])$ is also a coset of $\ker(\partial)$. Moreover, from Proposition 14.5 we have $Q \subseteq \partial^{-1}s(X)$. Hence Q and $\partial^{-1}s(X)$ are the same coset of $\ker(\partial)$ and thus $Q = \partial^{-1}s(X)$. $\square$

15. CONCLUDING REMARKS

In Part II of the book [19] interesting generalizations and applications of the theory can be found.

One important such generalization is the theory when one works with the spectrum $\mathbf{L}_\bullet\langle 0 \rangle$ rather than with $\mathbf{L}_\bullet\langle 1 \rangle$. This yields an analogous theory for the ANR-homology manifolds rather than for topological manifolds. The Quinn resolution obstruction also fits nicely into this theory. For details see [19, Chap. 24,25] and [5].

We note that, as already mentioned in the introduction, this generalization is especially interesting in view of the recent progress in studying the assembly maps associated to the spectrum $\mathbf{L}_\bullet$. For example, thanks to the generalization, the results about the assembly maps in [2] can be used to obtain an application in [3], which discusses when does a torsion-free word-hyperbolic group G have a topological manifold model for its classifying space BG .

Another important application is that the total surgery obstruction can be used to identify the geometric structure set of an n -dimensional manifold M with $\mathbb{S}_{n+1}(M)$. This is closely related to Subsection 14.16 and in fact the geometric surgery exact sequence can be identified with the algebraic surgery exact sequence, with more details to be found in [19, Chap. 18].

Interesting examples of geometric Poincaré complexes with nontrivial total surgery obstruction can be found in [19, Chap. 19].

Alternative approach. We want to conclude with some remarks on how the alternative formulation and proof of the Main Technical Theorem suggested by M. Weiss would go. One important difference is that it would get rid of the orientations which we discussed in Section 13 and only use signatures. This means also that it would only prove a weaker version of Theorem 13.16 which would only be about the reductions of the SNF as opposed to the current formulation which covers any spherical fibration.

The approach starts by rearranging the proof we have presented. Note that the middle part of our proof (Subsections 14.1 to 14.14) is independent of the first part of the proof in Section 13. So one can in fact start from Proposition 14.5. This proposition then becomes the central part of the alternative approach.

Recall that the formula from Proposition 14.5 allows us in Subsection 14.16 to construct a map from $\mathcal{N}(X)$ to $H_n(X; \mathbf{L}_\bullet \langle 1 \rangle)$ for an n -GPC X which has a stable block bundle reduction of the SNF (this ensures in particular that the source of the map is nonempty). But one can in fact think of such a map also for any n -GPC X . The source of the map is the same, but it might be empty. The target has to be reconsidered. It should be π_0 of the space of paths from $t(X)$ to 0, where the latter are points in the L -space whose π_0 is $H_{n-1}(X; \mathbf{L}_\bullet \langle 1 \rangle)$. If X has a reduction, then there exists a path from $t(X)$ to 0 and choosing one gives us a map to the loop space whose π_0 is $H_n(X; \mathbf{L}_\bullet \langle 1 \rangle)$, so we recover the map from 14.16.

A convenient setup would be to consider this map as a map of Δ -sets (spectra) in the sense that our map is the induced map on homotopy groups. The Main Technical Theorem then can be reformulated as saying that this map is a homotopy equivalence for any X an n -GPC. In particular this contains the statement that the source is empty if and only if the target is empty. This is equivalent to the statement of our Corollary 13.17 (the special case of Theorem 13.16 for an SNF). The statement that the map induces an isomorphism on π_0 is equivalent to the bijectivity statement of Proposition 14.18. So this formulation would allow to deal with (the weaker form) of our Section 13 (only for SNF rather than any spherical fibration in Theorem 13.16) and the second part of Section 14 simultaneously.

In the proof of our Main Technical Theorem the orientations played the crucial role. M. Weiss also suggested a different proof of the reformulation. The orientations would be replaced by a Mayer–Vietoris type of argument. For this a trick would be employed which can be found for example in [9, Proof of Lemma 3.1]. It tells us that when we are willing to consider instead of X the Poincaré pair $(X \times D^j, X \times S^{j-1})$ for j large (which we are in our situation since both the source and the target spaces are “homological”), then we can replace it by a pair (Z, Y) where Z is a total space of a spherical fibration over a PL-manifold with boundary. Now PL-manifolds have handlebody decompositions and hence (Z, Y) has a decomposition into Poincaré subspaces. Such a decomposition would allow for Mayer–Vietoris type arguments.

As noted in the introduction this is a broad outline, which has not been worked out in detail, but we include it for the curious reader.

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