On transverse triangulations

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Abstract. We show that every smooth manifold admits a smooth triangulation transverse to a given smooth map. This removes the properness assumption on the smooth map used in an essential way in Scharlemann’s construction [6].

1. Introduction

For \( l \in \mathbb{Z}^\geq 0 \), let \( \Delta^l \subset \mathbb{R}^l \) denote the standard \( l \)-simplex. If \(|K| \subset \mathbb{R}^N\) is a geometric realization of a simplicial complex \( K \) in the sense of [5, Sec. 3], for each \( l \)-simplex \( \sigma \) of \( K \) there is an injective linear map\(^1\) \( \iota_\sigma : \Delta^l \rightarrow |K| \) taking \( \Delta^l \) to \(|\sigma|\). If \( X \) is a smooth manifold, a topological embedding \( \mu : \Delta^l \rightarrow X \) is a smooth embedding if there exist an open neighborhood \( \Delta^l_\mu \) of \( \Delta^l \) in \( \mathbb{R}^l \) and a smooth embedding \( \tilde{\mu} : \Delta^l_\mu \rightarrow X \) so that \( \tilde{\mu}|_{\Delta^l} = \mu \). A triangulation of a smooth manifold \( X \) is a pair \( T = (K, \eta) \) consisting of a simplicial complex and a homeomorphism \( \eta : |K| \rightarrow X \) such that

\[
\eta \circ \iota_\sigma : \Delta^l \rightarrow X
\]

is a smooth embedding for every \( l \)-simplex \( \sigma \) in \( K \) and \( l \in \mathbb{Z}^\geq 0 \). If \( T = (K, \eta) \) is a triangulation of \( X \) and \( \psi : X \rightarrow X \) is a diffeomorphism, then \( \psi_* T = (K, \psi \circ \eta) \) is also a triangulation of \( X \).

Theorem 1.1. If \( X, Y \) are smooth manifolds and \( h : Y \rightarrow X \) is a smooth map, there exists a triangulation \( (K, \eta) \) of \( X \) such that \( h \) is transverse to \( \eta|_{\text{Int} \sigma} \) for every simplex \( \sigma \in K \).

This theorem is stated in [8] as Lemma 2.3 and described as an obvious fact. As pointed out to the author by Matthias Kreck, Scharlemann [6] proves Theorem 1.1 under the assumption that the smooth map \( h \) is proper, and his argument makes use of this assumption in an essential way. For the purposes of [8], a transverse \( C^1 \)-triangulation would suffice, and the existence of a such

\(^1\)i.e. \( \iota_\sigma \) takes the vertices of \( \Delta^l \) to the vertices of \(|\sigma|\) and is linear between them, as in [8, Footnote 5]
triangulation is fairly evident from the point of view of the Sard-Smale Theorem [7, (1.3)]. On the other hand, according to Matthias Kreck, the existence of smooth transverse triangulations without the properness assumption is related to subtle issues arising from the topology of stratifolds [2]. In this note we give a detailed proof of Theorem 1.1 as stated above, using Sard’s theorem [3, Section 2].

2. Outline of the proof of Theorem 1.1

If \( K \) is a simplicial complex, we denote by \( sd\ K \) the barycentric subdivision of \( K \). For any nonnegative integer \( l \), let \( K_l \) be the \( l \)-th skeleton of \( K \), i.e. the subcomplex of \( K \) consisting of the simplices in \( K \) of dimension at most \( l \). If \( \sigma \) is a simplex in a simplicial complex \( K \) with geometric realization \( |K| \), let

\[
St(\sigma, K) = \bigcup_{\sigma' \subset \sigma} \text{Int} \sigma'
\]

be the star of \( \sigma \) in \( K \), as in [5, Sec. 62], and \( b_\sigma \in sd\ K \) the barycenter of \( \sigma \). The main step in the proof of Theorem 1.1 is the following observation.

**Proposition 2.1.** Let \( h : Y \to X \) be a smooth map between smooth manifolds. If \( (K, \eta) \) is a triangulation of \( X \) and \( \sigma \) is an \( l \)-simplex in \( K \), there exists a diffeomorphism \( \psi_\sigma : X \to X \) restricting to the identity outside of \( \eta(St(b_\sigma, sd\ K)) \) so that \( \psi_\sigma \circ \eta|\text{Int} \sigma \) is transverse to \( h \).

If \( \sigma \) and \( \sigma' \) are two distinct simplices in \( K \) of the same dimension \( l \),

\[
\text{St}(b_\sigma, sd\ K) \cap \text{St}(b_{\sigma'}, sd\ K) = \emptyset.
\]

Since \( \psi_\sigma \) is the identity outside of \( \eta(St(b_\sigma, sd\ K)) \) and the collection \( \{\text{St}(b_\sigma, sd\ K)\} \) is locally finite, the composition \( \psi_l : X \to X \) of all diffeomorphisms \( \psi_\sigma : X \to X \) taken over all \( l \)-simplices \( \sigma \) in \( K \) is a well-defined diffeomorphism\(^2\) of \( X \). Since \( \psi_l \circ \eta|\sigma = \psi_\sigma \circ \eta|\sigma \) for every \( l \)-simplex \( \sigma \) in \( K \), we obtain the following conclusion from Proposition 2.1.

**Corollary 2.2.** Let \( h : Y \to X \) be a smooth map between smooth manifolds. If \( (K, \eta) \) is a triangulation of \( X \), for every \( l = 0,1, \ldots, \dim X \), there exists a diffeomorphism \( \psi_l : X \to X \) restricting to the identity on \( \eta(|K|_{l-1}) \) so that \( \psi_l \circ \eta|\text{Int} \sigma \) is transverse to \( h \) for every \( l \)-simplex \( \sigma \) in \( K \).

This corollary implies Theorem 1.1. By [4, Chap. II], \( X \) admits a triangulation \( (K, \eta_{-1}) \). By induction and Corollary 2.2, for each \( l = 0,1, \ldots, \dim X - 1 \) there exists a triangulation \( (K, \eta_l) = (K, \psi_l \circ \eta_{l-1}) \) of \( X \) which is transverse to \( h \) on every open simplex in \( K \) of dimension at most \( l \).

\(^2\)The locally finite property implies that the composition of these diffeomorphisms in any order is a diffeomorphism; by (1), these diffeomorphisms commute and so the composition is independent of the order.
3. Proof of Proposition 2.1

**Lemma 3.1.** For every \( l \in \mathbb{Z}^+ \), there exists a smooth function \( \rho_l : \mathbb{R}^l \to \mathbb{R}^+ \) such that
\[
\rho_l^{-1}(\mathbb{R}^+) = \text{Int} \, \Delta^l.
\]

**Proof.** Let \( \rho : \mathbb{R} \to \mathbb{R} \) be the smooth function given by
\[
\rho(r) = \begin{cases} 
    e^{-1/r}, & \text{if } r > 0, \\
    0, & \text{if } r \leq 0.
\end{cases}
\]
The smooth function \( \rho_l : \mathbb{R}^l \to \mathbb{R} \) given by
\[
\rho_l(t_1, \ldots, t_l) = \rho \left( 1 - \sum_{i=1}^{l} t_i \right) \cdot \prod_{i=1}^{l} \rho(t_i)
\]
then has the desired property. \( \square \)

**Lemma 3.2.** Let \((K, \eta)\) be a triangulation of a smooth manifold \( X \) and \( \sigma \) an \( l \)-simplex in \( K \). If
\[
\tilde{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \to U_\sigma \subset X
\]
is a diffeomorphism onto an open neighborhood \( U_\sigma \) of \( \eta(|\sigma|) \) in \( X \) such that \( \tilde{\mu}_\sigma(t,0) = \eta(\iota_\sigma(t)) \) for all \( t \in \Delta_\sigma \), there exists \( c_\sigma \in \mathbb{R}^+ \) such that
\[
\{(t,v) \in (\text{Int} \, \Delta^l) \times \mathbb{R}^{m-l} \mid |v| \leq c_\sigma \rho_l(t)\} \subset \tilde{\mu}_\sigma^{-1}(\eta(\text{St}(b_\sigma, \text{sd} K))).
\]

**Proof.** It is sufficient to show\(^3\) that there exists \( c_\sigma > 0 \) such that
\[
\{(t,v) \in (\text{Int} \, \Delta^l) \times \mathbb{R}^{m-l} \mid |v| \leq c_\sigma \rho_l(t)\} \subset \tilde{\mu}^{-1}_\sigma(\eta(\text{St}(\sigma, K))).
\]
We assume that \( 0 < l < m \). Suppose \((t_p, v_p) \in (\text{Int} \, \Delta^l) \times (\mathbb{R}^{m-l} - 0)\) is a sequence such that
\[
(t_p, v_p) \notin \tilde{\mu}_\sigma^{-1}(\eta(\text{St}(\sigma, K))), \quad |v_p| \leq \frac{1}{p} \rho_l(t_p).
\]
Since \( \eta(\text{St}(\sigma, K)) \) is an open neighborhood of \( \eta(\text{Int} \, \sigma) \) in \( X \), by shrinking \( v_p \) and passing to a subsequence we can assume that
\[
(t_p, v_p) \in \tilde{\mu}_\sigma^{-1}(\eta(|\tau'|)) \subset \tilde{\mu}_\sigma^{-1}(\eta(|\tau|))
\]
for an \( m \)-simplex \( \tau \) in \( K \) and a face \( \tau' \) of \( \tau \) so that \( \sigma \not\supset \tau', \tau' \not\subset \sigma, \text{ and } \sigma \subset \tau \).

Let \( \iota_\tau : \Delta^m \to |K| \) be an injective linear map taking \( \Delta^m \) to \( |\tau| \) so that
\[
\iota_\tau^{-1}(|\sigma|) = \Delta^m \cap \mathbb{R}^l \times 0 \subset \mathbb{R}^l \times \mathbb{R}^{m-l},
\]
\[
\iota_\tau^{-1}(|\tau'|) = \Delta^m \cap 0 \times \mathbb{R}^{m-1} \subset \mathbb{R}^l \times \mathbb{R}^{m-1}.
\]
Choose a smooth embedding \( \mu_\tau : \Delta^m_\tau \to X \) from an open neighborhood of \( \Delta^m \) in \( \mathbb{R}^m \) such that \( \mu_\tau|\Delta^m = \eta \circ \iota_\tau \). Let \( \phi \) be the first component of the diffeomorphism
\[
\mu_\tau^{-1} \circ \tilde{\mu}_\sigma : \tilde{\mu}_\sigma^{-1}(\mu_\tau(\Delta^m_\tau)) \to \mu_\tau^{-1}(\tilde{\mu}_\sigma(\Delta^l_\sigma \times \mathbb{R}^{m-l})) \subset \mathbb{R}^l \times \mathbb{R}^{m-1}.
\]

\(^3\)If \( K' \) is the subdivision of \( K \) obtained by adding the vertices \( b_\sigma', \) with \( \sigma' \supseteq \sigma \), then \( \text{St}(b_\sigma, \text{sd} K) = \text{St}(\sigma, K') \).

By (3), the second assumption in (4), the continuity of $d\phi$, and the compactness of $\Delta^l$,
\begin{equation}
|\phi(t_p,0)| = |\phi(t_p,0) - \phi(t_p,v_p)| \leq C|v_p| \quad \forall \ p,
\end{equation}
for some $C > 0$. On the other hand, by the first assumption in (4), the vanishing of $\rho_i$ on $\text{Bd} \Delta^l$, the continuity of $d\rho_i$, and the compactness of $\Delta^l$,
\begin{equation}
|\rho_i(t_p)| \leq C|\phi(t_p,0)| \quad \forall \ p,
\end{equation}
for some $C > 0$. The second assumption in (2), (5), and (6) give a contradiction for $p > C^2$. □

**Lemma 3.3.** Let $h : Y \rightarrow X$ be a smooth map between smooth manifolds, $(K, \eta)$ a triangulation of $X$, $\sigma$ an $l$-simplex in $K$, and
\[ \tilde{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \rightarrow U_\sigma \subset X \]
a diffeomorphism onto an open neighborhood $U_\sigma$ of $\eta(|\sigma|)$ in $X$ such that $\tilde{\mu}_\sigma(t,0) = \eta(t_\sigma(t))$ for all $t \in \Delta^l_\sigma$. For every $\epsilon > 0$, there exists $s_\sigma \in C^\infty(\text{Int} \, \Delta^l_\sigma; \mathbb{R}^{m-l})$ so that the map
\begin{equation}
\tilde{\mu}_\sigma \circ (\text{id}, s_\sigma) : \text{Int} \, \Delta^l \rightarrow X
\end{equation}
is transverse to $h$,
\begin{equation}
|s_\sigma(t)| < \epsilon^2 \rho(t) \quad \forall \ t \in \text{Int} \, \Delta^l,
\end{equation}
\begin{equation}
\lim_{t \rightarrow \text{Bd} \, \Delta^l} \rho(t)^{-i} |\nabla^j s_\sigma(t)| = 0 \quad \forall \ i, j \in \mathbb{Z}^{\geq 0},
\end{equation}
where $\nabla^j s_\sigma$ is the multilinear functional determined by the $j$-th derivatives of $s_\sigma$.

**Proof.** The smooth map
\[ \phi : \text{Int} \, \Delta^l \times \mathbb{R}^{m-l} \rightarrow X, \quad \phi(t,v) = \tilde{\mu}_\sigma(t, e^{-1/\rho(t)}v), \]
is a diffeomorphism onto an open neighborhood $U'_\sigma$ of $\eta(\text{Int} \, \sigma)$ in $X$. The smooth map (7) with $s_\sigma = e^{-1/\rho(t)}v$ is transverse to $h$ if and only if $v \in \mathbb{R}^{m-l}$ is a regular value of the smooth map
\[ \pi_2 \circ \phi^{-1} \circ h : h^{-1}(U'_\sigma) \rightarrow \mathbb{R}^{m-l}, \]
where $\pi_2 : \text{Int} \, \Delta^l \times \mathbb{R}^{m-l} \rightarrow \mathbb{R}^{m-l}$ is the projection onto the second component. By Sard’s Theorem, the set of such regular values is dense in $\mathbb{R}^{m-l}$. Thus, the map (7) with $s_\sigma = e^{-1/\rho(t)}v$ is transverse to $h$ for some $v \in \mathbb{R}^{m-l}$ with $|v| < \epsilon^2$. The second statement in (8) follows from $\rho|_{\text{Bd} \, \Delta^l} = 0$. □

**Corollary 3.4.** Let $h : Y \rightarrow X$ be a smooth map between smooth manifolds, $(K, \eta)$ a triangulation of $X$, $\sigma$ an $l$-simplex in $K$, and
\[ \tilde{\mu}_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \rightarrow U_\sigma \subset X \]
a diffeomorphism onto an open neighborhood $U_\sigma$ of $\eta(|\sigma|)$ in $X$ such that $\tilde{\mu}_\sigma(t,0) = \eta(t_\sigma(t))$ for all $t \in \Delta_\sigma$. For every $\epsilon > 0$, there exists a diffeomorphism $\psi'_\sigma$ of $\Delta^l_\sigma \times \mathbb{R}^{m-l}$ restricting to the identity outside of

$$\{(t,v) \in (\text{Int}\Delta^l) \times \mathbb{R}^{m-l} : |v| \leq \epsilon \rho_l(t)\}$$

so that the map $\tilde{\mu}_\sigma \circ \psi'_\sigma|_{\text{Int}\Delta^l \times 0}$ is transverse to $h$.

Proof. Choose $\beta \in C^\infty(\mathbb{R}; [0,1])$ so that

$$\beta(r) = \begin{cases} 1, & \text{if } r \leq \frac{1}{2}; \\ 0, & \text{if } r \geq 1. \end{cases}$$

Let $C_\beta = \sup_{r \in \mathbb{R}} |\beta'(r)|$. With $s_\sigma$ as provided by Lemma 3.3, define

$$\psi'_\sigma : \Delta^l_\sigma \times \mathbb{R}^{m-l} \rightarrow \Delta^l_\sigma \times \mathbb{R}^{m-l} \quad \text{by}$$

$$\psi'_\sigma(t,v) = \begin{cases} (t, v + \beta \left(\frac{|v|}{\epsilon \rho_l(t)}\right) s_\sigma(t)), & \text{if } t \in \text{Int} \Delta^l, \\ (t, v), & \text{if } t \notin \text{Int} \Delta^l. \end{cases}$$

The restriction of this map to $(\text{Int}\Delta^l) \times \mathbb{R}^{m-l}$ is smooth and its Jacobian is

$$(9) \quad \mathcal{J}\psi'_\sigma|_{(t,v)} = \begin{pmatrix} \mathbb{I}_t & 0 \\ (\mathcal{J}\psi'_\sigma|_{(t,v)})_{2,1} \mathbb{I}_{m-l} + \beta' \left(\frac{|v|}{\epsilon \rho_l(t)}\right) s_\sigma(t) v' \rho_l(t) \end{pmatrix},$$

where

$$(\mathcal{J}\psi'_\sigma|_{(t,v)})_{2,1} = \beta \left(\frac{|v|}{\epsilon \rho_l(t)}\right) \nabla s_\sigma(t) - \beta' \left(\frac{|v|}{\epsilon \rho_l(t)}\right) \frac{|v|}{\epsilon \rho_l(t)} \nabla \rho_l.$$

By the first property in (8), this matrix is non-singular if $\epsilon < 1/C_\beta$. If $W$ is any linear subspace of $\mathbb{R}^{m-l}$ containing $s_\sigma(t)$,

$$\psi'_\sigma(t \times W) \subset t \times W, \quad \psi'_\sigma(t,v) = (t,v) \quad \forall \ v \in W \text{ such that } |v| \geq \epsilon \rho_l(t).$$

Thus, $\psi'_\sigma$ is a bijection on $t \times W$, a diffeomorphism on $(\text{Int}\Delta^l) \times \mathbb{R}^{m-l}$, and a bijection on $\Delta^l_\sigma \times \mathbb{R}^{m-l}$.

Since $\beta(r) = 0$ for $r \geq 1$, $\psi'_\sigma(t,v) = (t,v)$ unless $t \in \text{Int} \Delta^l$ and $|v| < \epsilon \rho_l(t)$. It remains to show that $\psi'_\sigma$ is smooth along

$$\{(t,v) \in (\text{Int}\Delta^l) \times \mathbb{R}^{m-l} : |v| \leq \epsilon \rho_l(t)\} - (\text{Int}\Delta^l) \times \mathbb{R}^{m-l} = (\text{Bd} \Delta^l) \times 0.$$

Since $|s_\sigma(t)| \rightarrow 0$ as $t \rightarrow \text{Bd} \Delta^l$ by the first property in (8), $\psi'_\sigma$ is continuous at all $(t,0) \in (\text{Bd} \Delta^l) \times 0$. By the first property in (8), $\psi'_\sigma$ is also differentiable at all $(t,0) \in (\text{Bd} \Delta^l) \times 0$, with the Jacobian equal to $\mathbb{I}_m$. By (9) and the compactness of $\Delta^l$,

$$|\mathcal{J}\psi'_\sigma|_{(t,v)} - \mathbb{I}_m| \leq C \left(\nabla |s_\sigma(t)| + \rho(t)^{-1} |s_\sigma(t)|\right) \quad \forall \ (t,v) \in (\text{Int}\Delta^l) \times \mathbb{R}^{m-l}$$

for some $C > 0$. So $\mathcal{J}\psi'_\sigma$ is continuous at $(t,0)$ by the second statement in (8), as well as differentiable, with the differential of $\mathcal{J}\psi'_\sigma$ at $(t,0)$ equal
to 0. For \( i \geq 2 \), the \( i \)-th derivatives of the second component of \( \psi'_{\sigma} \) at \((t,v) \in (\text{Int} \, \Delta^l) \times \mathbb{R}^{m-l}\) are linear combinations of the terms

\[
\beta^{(i_1)} \left( \frac{|v|}{\epsilon \rho_l(t)} \right) \cdot \left( \frac{|v|}{\epsilon \rho_l(t)} \right)^{i_1} \cdot \prod_{k=1}^{k=j_l} \left( \frac{\nabla^p \rho_l}{\rho_l(t)} \right) \cdot \frac{v_j}{|v|^{2j_2}} \cdot \nabla^{i_2} s_{\sigma}(t),
\]

where \( i_1, i_2, j_1, j_2 \in \mathbb{Z}^+ \) and \( p_1, \ldots, p_{j_1} \in \mathbb{Z}^+ \) are such that

\[
i_1 + (p_1 + p_2 + \ldots + p_{j_1} - j_1) + i_2 = i, \quad j_1 + j_2 \leq i,
\]

and \( v_j \) is a \( j_2 \)-fold product of components of \( v \). Such a term is nonzero only if \( \epsilon \rho_l(t)/2 < |v| < \epsilon \rho_l(t) \) or \( i_1 = 0 \) and \( |v| < \epsilon \rho_l(t) \). Thus, the \( i \)-th derivatives of \( \psi'_{\sigma} \) at \((t,v) \in (\text{Int} \, \Delta^l) \times \mathbb{R}^{m-l}\) are bounded by

\[
C_i \sum_{i_1 + i_2 \leq i} \rho_l(t)^{-i_1} |\nabla^{i_2} s_{\sigma}(t)|
\]

for some constant \( C_i > 0 \). By the second statement in (8), the last expression approaches 0 as \( t \to \text{Bd} \, \Delta^l \) and does so faster than \( \rho_l \). It follows that \( \psi'_{\sigma} \) is smooth at all \((t,0) \in (\text{Bd} \, \Delta^l) \times 0\). \( \square \)

**Proof of Proposition 2.1.** Let \( \Delta^l_{\sigma} \) be a contractible open neighborhood of \( \Delta^l \) in \( \mathbb{R}^l \) and \( \mu_{\sigma} : \Delta^l_{\sigma} \to X \) a smooth embedding so that \( \mu_{\sigma}|_{\Delta^l} = \eta \circ t_{\sigma} \). By the Tubular Neighborhood Theorem [1, (12.11)], there exist an open neighborhood \( U_{\sigma} \) of \( \mu_{\sigma}(\Delta^l_{\sigma}) \) in \( X \) and a diffeomorphism

\[
\tilde{\mu}_{\sigma} : \Delta^l_{\sigma} \times \mathbb{R}^{m-l} \to U_{\sigma} \quad \text{such that} \quad \tilde{\mu}_{\sigma}(t,0) = \mu_{\sigma}(t) \quad \forall \, t \in \Delta^l_{\sigma}.
\]

Let \( c_{\sigma} > 0 \) be as in Lemma 3.2 and \( \psi'_{\sigma} \) as in Corollary 3.4 with \( \epsilon = c_{\sigma} \). The diffeomorphism

\[
\psi_{\sigma} = \tilde{\mu}_{\sigma} \circ \psi'_{\sigma} \circ \tilde{\mu}_{\sigma}^{-1} : U_{\sigma} \to U_{\sigma}
\]

is then the identity on \( U_{\sigma} - \text{St}(b_{\sigma}, \text{sd} \, K) \). Since \( \psi_{\sigma} \) is also the identity outside of a compact subset of \( U_{\sigma} \), it extends by identity to a diffeomorphism on all of \( X \). \( \square \)

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**References**


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4Since \( \Delta^l_{\sigma} \) is contractible, the normal bundle to the embedding \( \mu_{\sigma} \) is trivial.


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