Multiplier algebras of $C_0(X)$-algebras

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Abstract. If a $C^*$-algebra $A$ is a $C_0(X)$-algebra then the multiplier algebra $M(A)$ is a $C(\beta X)$-algebra in a canonical way. In the case where $A$ is $\sigma$-unital, we give necessary and sufficient conditions on $A$ and $X$ for $M(A)$ to be a continuous $C(\beta X)$-algebra.

INTRODUCTION

Let $A$ be a $C^*$-algebra which is a $C_0(X)$-algebra over a locally compact Hausdorff space $X$. Then the multiplier algebra $M(A)$ may be regarded in a natural way as a $C(\beta X)$-algebra over $\beta X$, the Stone-Čech compactification of $X$. The purpose of this paper is to characterize, for $A$ $\sigma$-unital, when $M(A)$ is a continuous $C(\beta X)$-algebra. An elementary necessary condition is that the $C_0(X)$-algebra $A$ should be continuous. The additional conditions for the characterization involve the interplay between the base map $\phi : \text{Prim}(A) \to X$ (where $\text{Prim}(A)$ is the primitive ideal space of $A$ with the hull-kernel topology), the structure map $\mu : C_0(X) \to ZM(A)$ (where $ZM(A)$ is the center of $M(A)$), and the topology of $X$ (Theorem 3.8). In the special case where $A$ is separable and the base map $\phi$ is surjective it follows that $M(A)$ is a continuous $C(\beta X)$-algebra if and only if $X$ is a disjoint union $X = U \cup D$ where $U := \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$ is clopen and $D$ is a discrete set (see Corollary 3.9). The maps $\phi$ and $\mu$, and the ideals $J_x$ of $A$, are described in the next section.

The structure of the paper is as follows. In the first section we collect some general results about $C_0(X)$-algebras. The second and third sections work gradually towards the main result. The fourth section gives applications to various classes of $C_0(X)$-algebras. For example, it is shown that if $A$ is a stable, $\sigma$-unital $C^*$-algebra with $\text{Prim}(A)$ Hausdorff then $M(A)$ is a continuous $C(\beta X)$-algebra if and only if $X = \text{Prim}(A)$ is basically disconnected.

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1. Preliminaries on $C_0(X)$-algebras

Let $A$ be a $C^*$-algebra and $X$ a locally compact Hausdorff space. Then $A$ is a $C_0(X)$-algebra if there is a $^*$-homomorphism $\mu : C_0(X) \to ZM(A)$ such that $\mu(C_0(X))A$ is norm-dense in $A$.

The map $\mu$ is called the structure map. If $X$ is compact then $\mu$ is necessarily unital, and in this case it is usual to speak of a "$C(X)$-algebra" rather than a "$C_0(X)$-algebra". An equivalent definition is that $A$ is a $C_0(X)$-algebra if and only if the mapping $\phi : \text{Prim}(A) \to X$ \cite[Prop. C.5]{31}, \cite[Prop. 4.1]{3}. The map $\phi$ is called the base map.

The maps $\mu$ and $\phi$ uniquely determine each other as follows. Let $\theta_A : C^b(\text{Prim}(A)) \to ZM(A)$ be the Dauns-Hofmann $^*$-isomorphism. This has the property that

$$\left( \theta_A(f)a + P = f(P)(a + P) \right) \quad (f \in C^b(\text{Prim}(A)), \ a \in A, \ P \in \text{Prim}(A))$$

or, equivalently, $\theta_A(f) - f(P)1 \in \tilde{P}$ (where $\tilde{P}$ is the ideal of $M(A)$ defined prior to Proposition 1.1 below). Then $\mu$ and $\phi$ are related by the equation $\mu(f) = \theta_A(f \circ \phi)$ for all $f \in C_0(X)$ \cite[Prop. C.5]{31}. Strictly, a $C_0(X)$-algebra is a triple $(A, X, \mu)$ (or $(A, X, \phi)$), but we generally find it less cumbersome, and more in accord with common usage, to say that $A$ is a $C_0(X)$-algebra with respect to $\mu$ or $\phi$. Elementary examples show that it is not enough to state the space $X$ and that one must specify $\mu$ or $\phi$ as well; and furthermore we shall see that the answer to our main question depends not only on $X$ but also on $\mu$ and $\phi$.

The definition of a $C_0(X)$-algebra was introduced by Kasparov \cite{24} as the culmination of work by Fell \cite{18}, Tomiyama \cite{30}, Dauns and Hofmann \cite{11}, Lee \cite{25}, and others over the previous three decades. An account of the somewhat tangled history can be found in \cite{31}. Other useful references are \cite{15}, \cite{16}, \cite{10}, \cite{27}, \cite{17}, \cite{22}, and \cite{3}.

For $x \in X$, let $J_x = \mu\{f \in C_0(X) \mid f(x) = 0\}A$, a norm-closed two-sided ideal of $A$ by the Cohen factorization theorem (see \cite[Thm. 16.1]{14}). For $a \in A$ and $x \in X$, we often write $a_x = a + J_x \in A/J_x$. Then

$$\left( \mu(f)a_x = f(x)a_x \right) \quad (a \in A, \ f \in C_0(X), \ x \in X).$$

This observation will be strengthened in Proposition 1.2 below. For $x \in X$ and $P \in \text{Prim}(A)$, $J_x \subseteq P$ if and only if $\phi(P) = x$. Indeed, $J_x \subseteq P$ if and only if $f(\phi(P))(a + P) = (\theta_A(f \circ \phi)a) + P = 0$ for all $a \in A$ and all $f \in C_0(X)$ such that $f(x) = 0$. The latter holds if and only if $f(\phi(P)) = 0$ for all such $f$, that is, if and only if $\phi(P) = x$. It follows that $J_x = A$ if and only if $x \notin \text{Im}(\phi)$. Note, too, that $\bigcap_{x \in X} J_x \subseteq \bigcap_{P \in \text{Prim}(A)} P = \{0\}$.

For each $a \in A$, the norm function $x \to \|a_x\|$ $(x \in X)$ is upper semicontinuous \cite[Prop. C.10]{31}. The $C_0(X)$-algebra $A$ is said to be continuous if, for all $a \in A$, the norm function $x \to \|a_x\|$ $(x \in X)$ is continuous. By Lee’s theorem this happens if and only if the mapping $\phi : \text{Prim}(A) \to X$ is open \cite[Prop. C.10 and Thm. C.26]{31}. In particular, if $A$ is a continuous $C_0(X)$-algebra
then \( \text{Im}(\phi) \) is open in \( X \), and if \( \text{Prim}(A) \) is also compact then \( \text{Im}(\phi) \) is clopen in \( X \).

Note that the question of whether \( A \) is a continuous \( C_0(X) \)-algebra depends crucially on the base map \( \phi \). For example, let \( A \) be a continuous \( C_0(X) \)-algebra where \( X \) has a nonisolated point \( x_0 \). Define a new map \( \psi : \text{Prim}(A) \to X \) by \( \psi(P) = x_0 \) (\( P \in \text{Prim}(A) \)). Then \((A, X, \psi)\) is a noncontinuous \( C_0(X) \)-algebra because \( \psi \) is not open. On the other hand, if \( A \) is a noncontinuous \( C_0(X) \)-algebra where \( X \) has an isolated point \( x_0 \) then, with \( \psi \) as above, \((A, X, \psi)\) is a continuous \( C_0(X) \)-algebra.

Our next step is to show that if \( A \) is a \( C_0(X) \)-algebra with structure map \( \mu \) then \( \mu \) has a unique extension \( \overline{\mu} \) such that \( M(A) \) is a \( C(\beta X) \)-algebra with structure map \( \overline{\mu} \). First, however, it is convenient to collect some elementary facts about the strict closure in \( M(A) \) of an ideal \( J \) in \( A \).

Let \( J \) be a proper, closed, two-sided ideal of a \( C^* \)-algebra \( A \). The quotient map \( q_J : A \to A/J \) has a canonical extension \( \overline{q_J} : M(A) \to M(A/J) \) such that, for all \( b \in M(A) \) and \( a \in A \),

\[
\overline{q_J}(b)(a + J) = ba + J \text{ and } (a + J)\overline{q_J}(b) = ab + J.
\]

We define a proper, closed, two-sided ideal \( \tilde{J} \) of \( M(A) \) by

\[
\tilde{J} = \ker \overline{q_J} = \{ b \in M(A) \mid ba, ab \in J \text{ for all } a \in A \}.
\]

**Proposition 1.1.** Let \( J \) be a proper, closed, two-sided ideal of a \( C^* \)-algebra \( A \). Then

(i) \( \tilde{J} \) is the strict closure of \( J \) in \( M(A) \);

(ii) \( \tilde{J} \cap A = J \);

(iii) if \( P \in \text{Prim}(A) \) then \( \tilde{P} \) is primitive (and hence is the unique ideal in \( \text{Prim}(M(A)) \) whose intersection with \( A \) is \( P \));

(iv) \( \tilde{J} = \bigcap \{ \tilde{P} \mid P \in \text{Prim}(A) \text{ and } P \supseteq J \} \) and for all \( b \in M(A) \)

\[
\|b + \tilde{J}\| = \sup\{|b + \tilde{P}|| P \in \text{Prim}(A) \text{ and } P \supseteq J\};
\]

(v) \((A + \tilde{J})/\tilde{J}\) is an essential ideal in \( M(A)/\tilde{J} \).

**Proof.** Let \((u_\lambda)\) be an approximate identity for \( A \).

(i) Suppose that \((b_\alpha)\) is a net in \( J \) which is strictly convergent to some \( b \in M(A) \). Let \( a \in A \). Then \( ba \) is the norm-limit of \((b_\alpha a)\) and hence belongs to \( J \). Similarly, \( ab \in J \) and so \( b \in \tilde{J} \).

Conversely, suppose that \( b \in \tilde{J} \). Then \( bu_\lambda \in J \) for all \( \lambda \). For \( a \in A \), we have

\[
\|bu_\lambda a - ba\| \leq \|b\|\|u_\lambda a - a\| \to 0
\]

and \( \|(ab)u_\lambda - ab\| \to 0 \), and so \( bu_\lambda \to b \) strictly.

(ii) Let \( b \in \tilde{J} \cap A \). Then \( bu_\lambda \in J \) for all \( \lambda \) and so \( b \in J \). Thus \( \tilde{J} \cap A \subseteq J \) and the reverse inclusion is clear.

(iii) Let \( \pi : A \to B(H) \) be an irreducible representation of \( A \) with kernel \( P \) and let \( \tilde{\pi} : M(A) \to B(H) \) be the canonical extension to an irreducible representation of \( M(A) \). It suffices to show that \( \tilde{P} = \ker \tilde{\pi} \). Let \( b \in \tilde{P} \) and
Theorem 1.3. Let $\mathcal{A}$ be a $C_0(X)$-algebra with structure map $\mu$. Then $\mu$ has a unique extension to a $*-$homomorphism $\overline{\mu} : C(\beta X) \to ZM(\mathcal{A})$ such that

\[ \overline{\mu}(J) = \mu(f) \quad (f \in C_0(X)). \]

Moreover $\overline{\mu}(1) = 1_{M(\mathcal{A})}$ and $\overline{\mu}(f)a_x = f(x)a_x$ for all $f \in C(\beta X)$, $a \in A$, $x \in X$.

Hence $M(\mathcal{A})$ is a $C(\beta X)$-algebra with structure map $\overline{\mu}$, and the corresponding base map $\overline{\phi} : \text{Prim}(M(\mathcal{A})) \to \beta X$ satisfies $\overline{\phi}(\mathcal{P}) = \phi(P)$ for all $P \in \text{Prim}(A)$.

Proof. Let $\phi : \text{Prim}(\mathcal{A}) \to X$ be the base map such that $\mu(f) = \theta_A(f \circ \phi)$ for all $f \in C_0(X)$. Define $\overline{\mu}(f) = \theta_A(f \circ \phi)$ for all $f \in C(\beta X)$. Then (1) holds and also $\overline{\mu}(1) = \theta_A(1) = 1_{M(\mathcal{A})}$. Let $f \in C(\beta X)$, $a \in A$ and $x \in X$, and let $P \in \text{Prim}(\mathcal{A})$ with $P \supseteq J_x$. Recall that $\phi(P) = x$. Then

\[ \overline{\mu}(f)a - f(x)a \in \bigcap_{P \supseteq J_x} P = J_x, \] 

as required.

Thus $M(\mathcal{A})$ is a $C(\beta X)$-algebra with structure map $\overline{\mu}$. Let $\overline{\phi}$ denote the corresponding base map. Let $P \in \text{Prim}(\mathcal{A})$, $a \in A \setminus P$ and $f \in C_0(X)$. Then $\theta_A(f \circ \phi) + P = f(\phi(P))(a + P)$ and so

\[ f(\phi(P))a + \mathcal{P} = \theta_A(f \circ \phi)a + \mathcal{P} = \theta_M(\mathcal{A})(\overline{\mu}(f \circ \phi)a + \mathcal{P}) = \overline{\phi}(\mathcal{P})(a + \mathcal{P}). \]

So $f(\phi(P)) - \overline{\phi}(\mathcal{P}))a \in \mathcal{P} \cap A = P$. Hence $f(\phi(P)) = \overline{\phi}(\mathcal{P})$. Since $f$ was arbitrary, $\phi(P) = \overline{\phi}(\mathcal{P})$ (by the remark immediately preceding this proposition). Thus $\overline{\phi}$ has the required property.
Finally, suppose that $\rho : C(\beta X) \to ZM(A)$ is a $^*$-homomorphism such that $\rho(\overline{f}) = \mu(f)$ for all $f \in C_0(X)$. Then $\rho(C(\beta X))A$ is norm-dense in $A$ and so the central projection $\rho(1)$ in $ZM(A)$ must be $1_{M(A)}$ because $A$ is an essential ideal of $M(A)$. Hence $M(A)$ is a $C(\beta X)$-algebra with structure map $\rho$. Let $\sigma : \text{Prim}(M(A)) \to \beta X$ be the corresponding base map such that $\rho(g) = \theta_{M(A)}(g \circ \sigma)$ \quad $(g \in C(\beta X))$.

Then the same argument given for $\overline{\sigma}$ applies to $\sigma$ and so $\sigma(P) = \sigma(\overline{P})$ for all $P \in \text{Prim}(A)$. Hence $\overline{\sigma} = \sigma$, since $\{\overline{P} \mid P \in \text{Prim}(A)\}$ is dense in $\text{Prim}(M(A))$. Thus $\rho = \overline{\sigma}$.

Thus in the language of Proposition 1.2, the main question of this paper is as follows. Suppose that $A$ is a $C_0(X)$-algebra with base map $\phi$. Under what circumstances is $\overline{\phi}$ open? Note that since the canonical embedding of $\text{Prim}(A)$ in $\text{Prim}(M(A))$ is an open map, the openness of $\phi$ is certainly a necessary condition for the openness of $\overline{\phi}$.

Proposition 1.2 has a useful corollary.

**Corollary 1.3.** Let $A$ be a $C_0(X)$-algebra with structure map $\mu$ and base map $\phi$. The following are equivalent.

(i) the $^*$-homomorphism $\mu : C_0(X) \to ZM(A)$ is injective;
(ii) the mapping $\phi : \text{Prim}(A) \to X$ has dense range;
(iii) the mapping $\overline{\phi} : \text{Prim}(M(A)) \to \beta X$ is surjective;
(iv) the $^*$-homomorphism $\overline{\mu} : C(\beta X) \to ZM(A)$ is injective.

**Proof.** (i) $\implies$ (ii). If $\text{Im}(\phi)$ is not dense in $X$, there exists a nonzero $f \in C_0(X)$ such that $f \circ \phi = 0$. Then $\mu(f) = \theta_A(f \circ \phi) = 0$.

(ii) $\implies$ (iii). Since $\{\overline{P} \mid P \in \text{Prim}(A)\}$ is dense in the compact space $\text{Prim}(M(A))$, $\text{Im}(\overline{\phi})$ is the closure of $\text{Im}(\phi)$ in $\beta X$. So if $\text{Im}(\phi)$ is dense in $X$ then $\overline{\phi}$ is surjective.

(iii) $\implies$ (iv). Suppose that (iii) holds and that $\overline{\mu}(g) = 0$ for some $g \in C(\beta X)$. Then $\theta_{M(A)}(g \circ \overline{\phi}) = \overline{\mu}(g) = 0$ and so $g \circ \overline{\phi} = 0$ since $\theta_{M(A)}$ is injective. Since $\overline{\phi}$ is surjective, $g = 0$.

(iv) $\implies$ (i). Suppose that (iv) holds and that $\mu(f) = 0$ for some $f \in C_0(X)$. Then $\overline{\mu}(f) = \mu(f) = 0$. Hence $\overline{f} = 0$ and so $f = 0$. $\square$

**Definition.** Let $A$ be a $C_0(X)$-algebra with structure map $\mu$ and let $\overline{\mu} : C(\beta X) \to ZM(A)$ be as in Proposition 1.2. For $x \in \beta X$, we define

$$H_x = \overline{\mu}\{f \in C(\beta X) \mid f(x) = 0\}M(A),$$

a closed two-sided ideal of $M(A)$.

Note that $H_x$ is defined in relation to $(M(A), \beta X, \overline{\mu})$ in the same way that $J_x$ (for $x \in X$) is defined in relation to $(A, X, \mu)$. It follows, in particular, that for $Q \subseteq \text{Prim}(M(A))$: $Q \supseteq H_x$ if and only if $\overline{\phi}(Q) = x$. Also, for each $b \in M(A)$, the function $x \to \|b + H_x\|$ $(x \in \beta X)$ is upper semi-continuous.
Note, too, that if \( x \in \beta X \), \( f \in C(\beta X) \) and \( g := f - f(x)1 \) then \( \overline{\mu}(g)1 \in H_x \) and hence \( \overline{\mu}(f) + H_x = f(x)(1 + H_x) \).

**Proposition 1.4.** Let \( A \) be a \( C_0(X) \)-algebra with structure map \( \mu \).

(i) For all \( x \in X \), \( J_x = \overline{\mu}\{f \in C(\beta X) \mid f(x) = 0\}A \).

(ii) For all \( x \in X \), \( J_x \subseteq H_x \subseteq \tilde{J}_x \) and \( J_x = H_x \cap A \).

(iii) For all \( b \in M(A) \),

\[
\|b\| = \sup\{\|b + \tilde{J}_x\| \mid x \in X\} = \sup\{\|b + H_x\| \mid x \in X\}.
\]

**Proof.** (i) Let \( f \in C(\beta X) \) with \( f(x) = 0 \) and let \( a \in A \). It suffices to show that \( \overline{\mu}(f)a \in J_x \). Let \( \epsilon > 0 \). There is a compact subset \( K \) of \( \text{Prim}(A) \) such that \( \|(a + P)\| < \epsilon/(1 + \|f\|) \) for all \( P \in \text{Prim}(A) \setminus K \). There exists \( g \in C_0(X) \) with \( 0 \leq g \leq 1 \) such that the restriction of \( g \) to the compact set \( \phi(K) \) is 1. Then \( h := (f|X)g \in C_0(X) \) and \( h(x) = 0 \).

For \( P \in \text{Prim}(A) \),

\[
\overline{\mu}(f)\{a + P = \theta_A(f \circ \phi)a + P = f(\phi(P))(a + P)\).
\]

and \( \mu(h)a + P = f(\phi(P))g(\phi(P))(a + P) \). So for \( P \in K \), \( \overline{\mu}(f)a - \mu(h)a \in P \).

For \( P \in \text{Prim}(A) \setminus K \),

\[
\|\overline{\mu}(f)a - \mu(h)a\| + P = |f(\phi(P))|(1 - g(\phi(P)))\|a + P\| < \epsilon.
\]

So \( \|\overline{\mu}(f)a - \mu(h)a\| < \epsilon \). Hence \( \overline{\mu}(f)a \in J_x \).

(ii) It follows from (i) and the definition of \( H_x \) that \( J_x \subseteq H_x \). Let \( b \in H_x \) and \( a \in A \). Then \( ab, ba \in J_x \) by (i). Hence \( b \in \tilde{J}_x \). It now follows from Proposition 1.2 (ii) that \( H_x \cap A = J_x \).

(iii) Suppose that \( c \in \tilde{J}_x \) for all \( x \in X \) and that \( a \in A \). Then \( ac, ca \in J_x \) for all \( x \in X \) and so \( ac = ca = 0 \). Hence \( c = 0 \). Thus the canonical *-homomorphism from \( M(A) \) into \( \tilde{J}_x \) is injective and hence isometric, establishing the first equality. The second follows from the fact that \( \|b + \tilde{J}_x\| \leq \|b + H_x\| \leq \|b\| \) \( (x \in X) \). \( \square \)

The next lemma establishes a crucial link between the ideal \( H_x \) and the ideals \( \tilde{J}_y \) for \( y \) close to \( x \).

**Lemma 1.5.** Let \( A \) be a \( C_0(X) \)-algebra with structure map \( \mu \). Let \( x \in \beta X \) and \( b \in M(A) \).

(i) Let \( W \) be a neighborhood of \( x \) in \( \beta X \). Then

\[
\|b + H_x\| \leq \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\}.
\]

(ii) Taking the infimum over neighborhoods \( W \) of \( x \) in \( \beta X \), we have

\[
\|b + H_x\| = \inf_{W} \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\}.
\]

**Proof.** (i) Choose \( f \in C(\beta X) \) with \( 0 \leq f \leq 1 \) such that \( f(x) = 1 \) and \( f(y) = 0 \) for all \( y \in \beta X \setminus W \). Then \( \overline{\mu}(1 - f)b \in H_x \) and so \( b - \overline{\mu}(f)b \in H_x \). So

\[
\|b + H_x\| = \|\overline{\mu}(f)b + H_x\| \leq \|\overline{\mu}(f)b\| \leq \sup\{\|\overline{\mu}(f)b + \tilde{J}_y\| \mid y \in X\}
\]

by Proposition 1.4 (iii).

Suppose that \( y \in X \setminus W \) and \( a \in A \). Then
\[
(\overline{\pi}(f)ba)_y = f(y)(ba)_y = 0
\]
by Proposition 1.2. So \( \overline{\pi}(f)ba \in J_y \) and similarly \( a\overline{\pi}(f)b = \overline{\pi}(f)ab \in J_y \). Hence \( \overline{\pi}(f)b \in J_y \). It follows that
\[
\|b + H_x\| \leq \sup\{\|\overline{\pi}(f)b + \tilde{J}_y\| \mid y \in W \cap X\}
\]
\[
\leq \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\}.
\]
(ii) By (i), the norm of \( b + H_x \) is majorized by the infimum. So let \( \epsilon > 0 \). By upper semi-continuity, there is a neighborhood \( W \) of \( x \) in \( \beta X \) such that, for \( y \in W \cap X \),
\[
\|b + \tilde{J}_y\| \leq \|b + H_y\| \leq \|b + H_x\| + \epsilon.
\]
Hence
\[
\inf_{W} \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\} \leq \|b + H_x\| + \epsilon.
\]
Since \( \epsilon \) was arbitrary, the result follows. \( \square \)

2. Sufficient conditions for continuity

In this section we establish conditions which are sufficient for the continuity of norm functions of elements of \( M(A) \) on \( \beta X \) (Theorem 2.9). In the next section we shall show that these conditions are also necessary for continuity.

Let \( A \) be a \( C_0(X) \)-algebra with structure map \( \mu \) and define
\[
U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.
\]
Note that if \( x \in U \) then \( J_x \neq A \) and so \( x \in \operatorname{Im}(\phi) \). It will follow from Lemma 2.1 (ii) that \( \operatorname{Im}(\phi) \setminus U \) is closed in \( \operatorname{Im}(\phi) \) and hence \( U \) is an open subset of \( \operatorname{Im}(\phi) \). If \( A \) is a continuous \( C_0(X) \)-algebra then \( \operatorname{Im}(\phi) \) is open, as we noted in Section 1, and so in this case \( U \) is open in \( X \).

There are three subsets of \( X \) which require separate consideration. The first is the set \( U \) itself, which consists of the easiest points to deal with (Proposition 2.7). The next is the set \( X \setminus \operatorname{cl}(U) \), where \( \operatorname{cl}(U) \) is the closure of \( U \) in \( X \). These points, too, are fairly tractable (Proposition 2.4). The third, and the most difficult to deal with, consists of those points which lie in the boundary of \( U \) (Proposition 2.7).

To illustrate two elementary examples, first let \( A \) be the \( C^* \)-algebra of all sequences \( x = (x_n)_{n \geq 1} \) of \( 2 \times 2 \) complex matrices such that \( x_n \to \operatorname{diag}(\lambda(x), 0) \) as \( n \to \infty \). Set \( P_n = \{x \in A \mid x_n = 0\} \) (\( n \geq 1 \)) and \( P_\infty = \ker \lambda \). Then \( \operatorname{Prim}(A) = \{P_n \mid n \geq 1\} \cup \{P_\infty\} \) with the topology induced from the space \( X = \mathbb{N} \cup \{\infty\} \) (the 1-point compactification of \( \mathbb{N} \)) by the map \( \phi: \operatorname{Prim}(A) \to X \) for which \( \phi(P_n) = n \) and \( \phi(P_\infty) = \infty \). Then \( U = \mathbb{N} \) and the point \( \infty \) lies in the boundary of \( U \). Note that \( U \) is not \( C^* \)-embedded in \( X \) (see Lemma 2.6).

Next, let \( A = C[0, 1] \otimes K(H) \), where \( K(H) \) is the algebra of compact linear operators on an infinite-dimensional Hilbert space \( H \). For \( x \in X = [0, 1] \), set \( P_x = \{f \in A \mid f(x) = 0\} \). Then \( \operatorname{Prim}(A) = \{P_x \mid x \in X\} \) with the topology
induced from $X$ by the map $\phi : \text{Prim}(A) \to X$ for which $\phi(P_x) = x$. In this case the set $U$ is empty because $ZM(A) \cap A$ (the center of $A$) is $\{0\}$.

We begin with a simple lemma.

**Lemma 2.1.** Let $A$ be a $C_0(X)$-algebra with structure map $\mu$ and base map $\phi$ and let $x \in \text{Im}(\phi)$. The following are equivalent.

(i) $\mu(C_0(X)) \cap A \subseteq J_x$;
(ii) $\{f \in C_0(X) \mid \mu(f) \in A\} \subseteq \{f \in C_0(X) \mid f(x) = 0\}$;
(iii) there exists $R \in \text{Prim}(M(A))$ such that $R \supseteq A$ and $\overline{\phi}(R) = x$.

**Proof.** (i) $\implies$ (ii). Assume (i) and let $P \in \text{Prim}(A/J_x)$. Suppose that $f \in C_0(X)$ and that $\mu(f) \in A$. Choose $a \in A \setminus P$. Then

$$0 = \mu(f)a + P = \theta_A(f \circ \phi)a + P = f(\phi(P))a + P = f(x)a + P.$$ 

Hence $f(x) = 0$ as required.

(ii) $\implies$ (i). Assuming (ii), we have

$$\mu(C_0(X)) \cap A = \mu(\{f \in C_0(X) \mid \mu(f) \in A\}) \subseteq \text{cl}(\mu(\{f \in C_0(X) \mid \mu(f) \in A\})A) \subseteq J_x.$$ 

(iii) $\implies$ (ii). Assume (iii) and let $f \in C_0(X)$ with $\mu(f) \in A$. Then

$$0 = \mu(f) + R = \overline{\mu(f)} + R = \theta_{M(A)}(\overline{f} \circ \overline{\phi}) + R = \overline{f}(\overline{\phi}(R))1 + R = f(x)1 + R$$

and so $f(x) = 0$ as required.

(ii) $\implies$ (iii). Suppose that (iii) fails, so that $x$ is not contained in the compact subset $\overline{\phi}(\{R \in \text{Prim}(M(A)) \mid R \supseteq A\})$ of $\beta X$. Then there exists $g \in C(\beta X)$ such that $g(x) = 1$ and $g(\overline{\phi}(R)) = 0$ for all $R \in \text{Prim}(M(A))$ such that $R \supseteq A$. Then $\overline{\mu(g)} + R = g(\overline{\phi}(R))1 + R = 0$ for all such $R$ and so $\overline{\mu(g)} \in A$.

Choose $f \in C_0(X)$ such that $f(x) = 1$. Then $f|_X \in C_0(X)$ and takes the value $1$ at $x$. On the other hand, $\mu(f|_X) = \overline{\mu(f|_X)} = \mu(f)\overline{\mu(g)} \in A$. Thus (ii) fails to hold. \qed

**Proposition 2.2.** Let $A$ be a $C_0(X)$-algebra with structure map $\mu$ and let

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$ 

Let $x \in U$. Then

(i) $H_x = \tilde{J}_x$ and $H_x$ is strictly closed in $M(A)$;
(ii) $A/J_x$ is unital and $\mu(f) + J_x = f(x)1_{A/J_x}$ for all $f \in C_0(X)$ such that $\mu(f) \in A$;
(iii) $A/J_x$ is canonically isomorphic to $M(A)/H_x$ via the map $a + J_x \to a + H_x$ ($a \in A$).

**Proof.** (i) Since $J_x \neq A$, it follows that $\tilde{J}_x$ is a proper ideal of $M(A)$ and hence so is $H_x$. Let $R \in \text{Prim}(M(A))$ and suppose that $R \supseteq H_x$ (equivalently, $\overline{\phi}(R) = x$). By Lemma 2.1, $R$ does not contain $A$ and so $R = \overline{P}$ for some $P \in \text{Prim}(A)$. By Proposition 1.2, $\phi(P) = \overline{\phi(P)} = x$ and so $P \supseteq J_x$. Hence

$R = \tilde{P} \supseteq J_x$. It follows that $H_x \supseteq J_x$ and the reverse inclusion always holds (Proposition 1.4 (ii)). So $H_x$ is strictly closed in $M(A)$ by Proposition 1.1 (i).

(ii) Suppose that $f \in C_0(X)$ satisfies $\mu(f) \in A$. Note that, by hypothesis, $J_x \neq A$. Let $P \in \text{Prim}(A)$ with $P \supseteq J_x$. Then, for all $a \in A$,

$$\mu(f)a + P = a\mu(f) + P = f(x)a + P.$$ 

Hence $\mu(f)a - f(x)a, a\mu(f) - f(x)a \in J_x$. All that remains is to show that $A/J_x$ is unital. By Lemma 2.1, we may choose $f$ such that $f(x) = 1$ and then $\mu(f) + J_x$ is an identity element for $A/J_x$.

(iii) Since $J_x \cap A = J_x$, the map $a + J_x \rightarrow a + \tilde{J}_x$ ($a \in A$) gives a *-isomorphism of $A/J_x$ onto $(A + \tilde{J}_x)/\tilde{J}_x$. By Proposition 1.1 (v), $(A + \tilde{J}_x)/\tilde{J}_x$ is a unital, essential ideal of $M(A)/\tilde{J}_x$ and hence must equal $M(A)/\tilde{J}_x$. Since $\tilde{J}_x = H_x$, the result follows.

Proposition 2.3. Let $A$ be a continuous $C_0(X)$-algebra with structure map $\mu$ and let $U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$. Let $x \in U$. Then for all $b \in M(A)$, the norm function $y \rightarrow \|b + H_y\|$ ($y \in X$) is continuous at $x$.

Proof. Since $J_x \neq A$, $x \in \text{Im}(\phi)$ and it follows from Lemma 2.1 that there exists $f \in C_0(X)$ such that $\mu(f) \in A$ and $f(x) = 1$. Replacing $f$ by $|f|^2$, we may assume that $f \geq 0$. There is an open neighborhood $V$ of $x$ in $X$, contained in the open subset $\text{Im}(\phi)$ of $X$, such that $f(y) \geq \frac{1}{2}$ for all $y \in V$. Let $g : [0, \infty) \rightarrow [0, 1]$ be the continuous function defined by $g(t) = 2t$ ($0 \leq t \leq \frac{1}{2}$) and $g(t) = 1$ ($t > \frac{1}{2}$). Applying functional calculus, we may form $h := g(f) = g \circ f \in C_0(X)$. Then $\mu(h) = g(\mu(f)) \in A$ and $h(y) = 1$ for all $y \in V \subseteq \text{Im}(\phi)$. For all $y \in V$, $\mu(C_0(X)) \cap A \not\subseteq J_y$ by Lemma 2.1 and so $\mu(h) + H_y$ is the identity of $M(A)/H_y$ by Proposition 2.2.

Now let $b \in M(A)$ and set $a = \mu(h)b \in A$. By hypothesis, the function $y \rightarrow \|a + J_y\|$ is continuous on $X$. For $y \in V$, we have

$$\|a + J_y\| = \|\mu(h)b + H_y\| = \|b + H_y\|.$$ 

So the function $y \rightarrow \|b + H_y\|$ ($y \in X$) is continuous on $V$ and in particular at $x$. 

For the next class of points we need some definitions. Recall that a subset $U$ of a topological space $X$ is a cozero set if there is a continuous real-valued function $f$ on $X$ which vanishes precisely on the complement of $U$ in $X$. Now let $X$ be a completely regular topological space. A point $x \in X$ is a BD-point (standing for basically disconnected) if whenever $U$ is a cozero set in $X$ and $V$ an open set in $X$ such that $x \in \text{cl}(U) \cap \text{cl}(V)$ then $x \in \text{cl}(U \cap V)$. If each point in $X$ is a BD-point then $X$ is basically disconnected.

Before establishing a connection between BD-points and continuity of norm functions, we make an observation on open sets and cozero sets. Let $A$ be a continuous $C_0(X)$-algebra and let $b \in M(A)$. Then the set $Y = \{x \in X \mid \|b + \tilde{J}_x\| > 0\}$ is open in $X$, being the image under the open mapping $\phi$ of the open set $\{P \in \text{Prim}(A) \mid \|b + \tilde{P}\| > 0\}$ by Proposition 1.1 (iv). Now suppose
furthermore that $A$ is $\sigma$-unital with a strictly positive element $u$. Then $bu \in A$
and, for $P \in \text{Prim}(A)$, $b \in \bar{P}$ if and only if $bu \in \bar{P}$ (to see this, use the notation
of the proof of Proposition 1.1 (iii) and note that if $0 = \tilde{\pi}(bu) = \tilde{\pi}(b)\pi(u)$ then
$b \in \ker \tilde{\pi} = \bar{P}$ because the operator $\pi(u)$ has dense range). Hence for $x \in X,
 b \in \bar{J}_x$ if and only if $bu \in \bar{J}_x$. Thus, in this case, the set $Y$ is the cozero set of a
continuous function on $X$, namely the function $x \rightarrow \|bu + \bar{J}_x\|$ ($x \in X$).

Up till now we have worked with general $C^*$-algebras $A$ but for many of the
subsequent results we have to assume that $A$ is $\sigma$-unital. When $A$ is a $\sigma$-unital
$C^*$-algebra which is also a $C_0(X)$-algebra we shall say that $A$ is a $\sigma$-unital
$C_0(X)$-algebra.

In the next proposition, we have chiefly in mind points $x \in X \setminus \text{cl}(U)$ but
we do not require this restriction.

**Proposition 2.4.** Let $A$ be a $\sigma$-unital continuous $C_0(X)$-algebra and let $x$ be
a BD-point in $X$. Then for all $b \in M(A)$, the norm function $y \rightarrow \|b + H_y\|
(y \in X)$ is continuous at $x$.

**Proof.** By the $C^*$-condition, it suffices to consider $b \in M(A)^+$. Suppose that
there exists $b \in M(A)^+$ such that the norm function of $b$ is discontinuous at $x$.
Since the function $y \rightarrow \|b + H_y\|$ ($y \in \beta X$) is upper semi-continuous on
$\beta X$, its restriction to $X$ must fail to be lower semi-continuous at $x$. Hence, by
scaling $b$, we may suppose that $\|b + H_x\| = 1$ and that there exists $\delta \in (0, 1)$
such that $x$ lies in the closure of the set $V = \{y \in X | \|b + H_y\| < \delta\}$, a set
which is open in $X$ by upper semi-continuity.

On the other hand, by Lemma 1.5 (i), $x$ lies in the closure of the set $W = \{y \in X | \|b + \bar{J}_y\| > \frac{1+\delta}{2}\}$. Let $g : [0, \infty) \rightarrow [0, \infty)$ be the continuous function
defined by $g(t) = 0$ for $0 \leq t \leq \frac{1+\delta}{2}$ and $g(t) = t - \frac{1+\delta}{2}$ for $t > \frac{1+\delta}{2}$. Then
$W = \{y \in X | \|g(b) + \bar{J}_y\| > 0\}$ and so $W$ is a cozero set in $X$ by the discussion
above. Evidently $V$ and $W$ are disjoint and $x$ is in the closure of each of them,
contradicting the fact that $x$ is a BD-point.

The final set of points consists of those in the boundary of $U$. For these, we
need the following two lemmas.

**Lemma 2.5.** Let $A$ be a $\sigma$-unital continuous $C_0(X)$-algebra with structure
map $\mu$ and let
$$U = \{x \in X | \mu(C_0(X)) \cap A \not\subseteq \bar{J}_x\}.$$  

Suppose that $\text{cl}(U)$, the closure of $U$ in $X$, is clopen in $X$ and that $x \in \text{cl}(U) \setminus U$.
Let $b \in M(A)$ and let $V$ be any neighborhood of $x$ in $X$. Then
$$\|b + H_x\| \leq \sup\{\|b + \bar{J}_y\| | y \in U \cap V\}.$$  

**Proof.** Replacing $V$ by its interior, we may assume that $V$ is open. Since
$\text{cl}(U) \cap V$ is a neighborhood of $x$ in $X$, there exists $f \in C_0(X)$ such that
$0 \leq f \leq 1$, $f(x) = 1$ and $f(y) = 0$ for all $y \in X \setminus (\text{cl}(U) \cap V)$. Then
$$\mu(f) + H_x = \overline{\mu(f)} + H_x = \overline{f(x)}1 + H_x = 1 + H_x.$$  

On the other hand, if \( y \in X \setminus (\text{cl}(U) \cap V) \) and \( a \in A \) then \( (\mu(f)a)_y = f(y)a_y = 0 \) so that \( \mu(f)a \in J_y \) and hence \( \mu(f) \in J_y \). Since \( V \) is open, \( U \cap V \) is dense in \( \text{cl}(U) \cap V \), and since \( A \) is continuous it follows that \( \bigcap_{y \in U \cap V} J_y = \bigcap_{y \in \text{cl}(U) \cap V} J_y \) and hence \( \bigcap_{y \in U \cap V} J_y = \bigcap_{y \in \text{cl}(U) \cap V} J_y \). Using Proposition 1.4 (iii), we now have

\[
\|b + H_x\| = \|\mu(f)b + H_x\| = \|\mu(f)b + J_y\| = \sup\{\|\mu(f)b + J_y\| \mid y \in \text{cl}(U) \cap V\} = \sup\{\|\mu(f)b + J_y\| \mid y \in U \cap V\} \leq \sup\{\|b + J_y\| \mid y \in U \cap V\}.
\]

\[\square\]

Lemma 2.6. Let \( V \) be a completely regular space and let \( W \) be a dense subset of \( V \). Then the following are equivalent:

(i) disjoint zero sets in \( W \) have disjoint closures in \( V \);
(ii) \( W \) is \( C^* \)-embedded in \( V \);
(iii) \( V \) is canonically homeomorphic (i.e. homeomorphic under a map which extends the identity map on \( W \)) to a subset of \( \beta W \).

Proof. The equivalence of (i) and (ii) is established in [20, Thm. 6.4, (2)\(\Leftrightarrow\)(3)].

(ii)\(\Rightarrow\) (iii) By [20, Thm. 6.4 (1)] the identity map on \( W \) extends to a continuous map \( \Theta \) from \( V \) into \( \beta W \). Suppose that \( g \) is a continuous bounded function on \( V \). Then, by continuity and by agreement on \( W \), we have

\[
(*) \quad g = (\overline{g|W}) \circ \Theta.
\]

Since \( V \) is completely regular, any two points of \( V \) can be separated by a continuous bounded function \( g \), so \( \Theta \) is injective. Now let \( (v_i) \) be a net in \( V \) and suppose that \( \Theta(v_i) \to \Theta(v) \) for some \( v \in V \). Then (*) gives \( g(v_i) \to g(v) \) for all continuous bounded functions \( g \) on \( V \), and hence \( v_i \to v \) since \( V \) is completely regular. Thus \( \Theta \) is a homeomorphism.

(iii)\(\Rightarrow\)(i) Disjoint zero sets in \( W \) have disjoint closures in \( \beta W \) [20, Thm. 6.5], and hence have disjoint closures in \( \Theta(V) \).

\[\square\]

Proposition 2.7. Let \( A \) be a \( \sigma \)-unital continuous \( C_0(X) \)-algebra and let

\[
U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.
\]

Suppose that \( \text{cl}(U) \), the closure of \( U \) in \( X \), is clopen in \( X \) and that \( \text{cl}(U) \) is canonically homeomorphic to a subset of \( \beta U \). Then for each \( b \in M(A) \), the norm function \( x \to \|b + H_x\| (x \in X) \) is continuous at each point of \( \text{cl}(U) \).

Proof. Let \( b \in M(A) \) and suppose that there exists \( y \in \text{cl}(U) \) such that the function \( x \to \|b + H_x\| (x \in X) \) is not continuous at \( y \). Since the function is continuous at all points of \( U \) (Proposition 2.3), it follows that \( y \in \text{cl}(U) \setminus U \). Furthermore, since the function is upper semi-continuous on \( X \) and \( \text{cl}(U) \) is open in \( X \), we may suppose by scaling \( b \) that \( \|b + H_y\| = 1 \) and that \( y \) lies in the closure of the open set \( V = \{x \in \text{cl}(U) \mid \|b + H_x\| < \delta\} \) for some \( \delta \in (0, 1) \). Since \( V \) is open in \( X \), \( V \cap U \) is dense in \( V \) and so \( y \) lies in the closure of \( X \) of

the set \( Y = \{ x \in U \mid \| b + H_x \| \leq \delta \} \). Since the norm function of \( b \) is continuous on \( U \), \( Y \) is a zero set of \( U \) (for the function \( x \to \max\{\| b + H_x \|, \delta \} - \delta \)).

On the other hand, it follows from Lemma 2.6 and Lemma 2.2 (i) that \( y \) lies in the closure in \( X \) of the set \( Z = \{ x \in U \mid \| b + H_x \| \geq \frac{1 + \delta}{2} \} \), which is also a zero set of \( U \). This contradicts the fact that the disjoint zero sets \( Y \) and \( Z \) of \( U \) have disjoint closures in \( \text{cl}(U) \) (Lemma 2.6).

Next we need to know that continuous norm-functions on \( X \) extend continuously to \( \beta X \).

**Proposition 2.8.** Let \( A \) be a \( C_0(X) \)-algebra with structure map \( \mu \) such that, for each \( b \in M(A) \), the norm function \( x \to \| b + H_x \| \ (x \in X) \) is continuous. Then \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \overline{\mu} \).

**Proof.** For each \( b \in M(A) \), let \( f_b : X \to [0, \infty) \) be the bounded function defined by \( f_b(x) = \| b + H_x \| \ (x \in X) \). By hypothesis, \( f_b \) is continuous and so it suffices to show that \( \overline{f_b}(y) = \| b + H_y \| \) for all \( y \in \beta X \setminus X \) and all \( b \in M(A) \).

Let \( y \in \beta X \setminus X \) and let \( F \) be a \( z \)-ultrafilter on \( X \) with limit \( y \). If \( b \in H_y \) then \( \overline{f_b}(y) = 0 \) by the upper semi-continuity of the norm function \( x \to \| b + H_x \| \) (\( x \in \beta X \)). So we may now restrict to the case where \( H_y \neq M(A) \). Since \( b \to \overline{f_b}(y) = \lim_{x \in F} \| b + H_x \| \) defines a \( C^* \)-seminorm on \( M(A) \), it follows from the uniqueness of the \( C^* \)-norm on \( M(A)/H_y \) that it suffices to show that if \( \overline{f_b}(y) = 0 \) then \( b \in H_y \).

Let \( b \in M(A) \) and suppose that \( \overline{f_b}(y) = 0 \). Let \( \epsilon > 0 \). Then there exists \( Z \in \mathcal{F} \) such that \( \| b + H_x \| < \epsilon/2 \) for all \( x \in Z \). So \( Z \) is disjoint from the set \( W = \{ x \in X \mid \| b + H_x \| \geq \epsilon \} \). Since \( W \) is the zero set for the continuous function \( \min\{f_b, \epsilon\} - \epsilon \), it follows from [20, 1.15] that there exists \( f \in C_b(X) \) such that \( 0 \leq f \leq 1 \), \( f(Z) = \{0\} \) and \( f(W) = \{1\} \). Since \( Z \in \mathcal{F} \) and \( f(Z) = \{0\}, f(y) = 0 \). Thus \( \overline{\mu(f)}b \in H_y \) by definition of \( H_y \). On the other hand, it follows from Proposition 1.4 (iii) that

\[
\| b - \overline{\mu(f)}b \| = \sup_{x \in X} \| (b - \overline{\mu(f)}b) + H_x \| = \sup_{x \in X} \|1 - f(x)\| b + H_x \| \leq \epsilon.
\]

Since \( \epsilon \) was arbitrary, \( b \in H_y \). \( \square \)

Finally, we summarize the work of this section in the following theorem.

**Theorem 2.9.** Let \( A \) be a \( \sigma \)-unital continuous \( C_0(X) \)-algebra with structure map \( \mu \) and base map \( \phi \), let

\[
U = \{ x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x \}
\]

and let \( \text{cl}(U) \) and \( \text{cl}(\text{Im}(\phi)) \) be the closures of \( U \) and \( \text{Im}(\phi) \) in \( X \). Then \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \overline{\mu} \) if

(i) \( \text{cl}(U) \) is clopen in \( X \);

(ii) \( \text{cl}(U) \) is canonically homeomorphic to a subset of \( \beta U \);

(iii) every point of \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \) is a BD-point of \( X \).

Proof. By Proposition 2.8, it suffices to show that, for each \( b \in M(A) \), the norm function \( x \to \|b + H_x\| \ (x \in X) \) is continuous at each point \( x \in X \). This continuity was established in Proposition 2.7 for \( x \in \text{cl}(U) \) and in Proposition 2.4 for \( x \in \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \).

Finally, since \( \{P \mid P \in \text{Prim}(A)\} \) is dense in the compact space \( \text{Prim}(M(A)) \), \( \text{Im}(\phi) \) is the closure of \( \text{Im}(\phi) \) in \( \beta X \). Hence \( \text{Im}(\phi) \cap X = \text{cl}(\text{Im}(\phi)) \). If \( x \) belongs to the open set \( X \setminus \text{cl}(\text{Im}(\phi)) \) then \( x \notin \text{Im}(\phi) \) and so \( \|b + H_x\| = 0 \). Thus \( x \) is a point of continuity for the norm function.

\[ \square \]

3. Necessary conditions of continuity and the main theorem

In this section we prove the converse of Theorem 2.9, thus establishing our main result, Theorem 3.8, which characterizes, for \( A \) a \( \sigma \)-unital continuous \( C_0(X) \)-algebra, when \( M(A) \) is a continuous \( C(\beta X) \)-algebra. The main technical result along the way is Theorem 3.2 which constructs a useful multiplier in \( M(A) \).

**Proposition 3.1.** Let \( A \) be a \( C_0(X) \)-algebra and set \( B = \Pi_{x \in X} M(A)/\tilde{J}_x \). Define \( \Phi : M(A) \to B \) by \( \Phi(b) = (b + \tilde{J}_x)_x \) and set \( \iota = \Phi|_A : A \to B \). Then \( \Phi \) is a *-isomorphism from \( M(A) \) onto \( B_{\text{id}} \), the idealizer of \( \iota(A) \) in \( B \).

**Proof.** It is evident that \( \Phi(M(A)) \subseteq B_{\text{id}} \). Moreover, \( \Phi \) is injective by Proposition 1.4 (iii). It follows from Proposition 1.1 (v) that \( \iota(A) \) is an essential ideal of \( B_{\text{id}} \) and so there exists an injective *-homomorphism \( \theta : B_{\text{id}} \to M(A) \) such that \( \theta(\iota(a)) = a \) for all \( a \in A \) [28, 3.12.8].

Let \( b = (b_x)_x \in B_{\text{id}} \). We claim that \( \Phi(\theta(b)) = b \). To see this, first note that for each \( a \in A \), \( (b + \tilde{J}_x)_x(a) = \iota(c) \) for some \( c \in A \). Hence for each \( x \in X \),

\[
\theta(b)a + \tilde{J}_x = \theta(b)\theta(\iota(a)) + \tilde{J}_x = \theta(ba) + \tilde{J}_x = \theta(c) + \tilde{J}_x
\]

the final equality holding because \( b_x(a + \tilde{J}_x) \) is the \( x \)-component of \( \iota(c) \). Similarly, \( a\theta(b) + \tilde{J}_x = (a + \tilde{J}_x)b_x \). Since \( a \) was arbitrary and \( (A + \tilde{J}_x)/\tilde{J}_x \) is essential in \( M(A)/\tilde{J}_x \) (Proposition 1.1 (v)), it follows that \( \theta(b) + \tilde{J}_x = b_x \). Hence \( \Phi(\theta(b)) = b \), as required.

We now define a function \( g \) from the unit interval \([0, 1]\) to the space \( C[0, 1] \) as follows (where for \( r \in [0, 1] \), \( g_r \) is the continuous function on \([0, 1]\) corresponding to \( r \)):

\[
g_0(x) = 1 \text{ for all } x \in [0, 1];
\]

\[
g_r(x) = \begin{cases} 
0 & (0 \leq x \leq r/2) \\
(2x/r) - 1 & (r/2 \leq x \leq r) \\
1 & (r \leq x \leq 1);
\end{cases}
\]

\[
g_r = g_{1/2} \text{ for } r \geq 1/2.
\]
For an element $a$ in a $C^*$-algebra $A$, let $\text{sp}(a)$ denote the spectrum of $a$. For $a \geq 0$ let $\min \text{sp}(a)$ be the smallest number in $\text{sp}(a)$. Note that the arbitrary cozero set $U$ in the following theorem is not to be confused with the set $U$ defined at the start of Section 2. We will see in Lemma 3.3 (i) and (ii), however, that the set $U$ defined at the start of Section 2 is indeed a cozero set in certain circumstances.

**Theorem 3.2.** Let $A$ be a $\sigma$-unital $C_0(X)$-algebra with base map $\phi$ and let $u$ be a strictly positive element in $A$ with $\|u\| = 1$. Let $f \in C^b(X)$ with $0 \leq f \leq 1$, let $U$ be the cozero set of $f$ and let $V = \{x \in U \cap \text{Im}(\phi) \mid 2 \min \text{sp}(u_x) \leq f(x)\}$. Let $\overline{U}$ and $\overline{V}$ be the closures of $U$ and $V$ in $\beta X$, respectively. Then there exists $b \in M(A)$ with $0 \leq b \leq 1$ such that

(i) $b + \tilde{J}_x = g_{f(x)}(u + \tilde{J}_x)$ ($x \in X$);

(ii) $b \in A + H_x \subseteq A + \tilde{J}_x$ for all $x \in U$;

(iii) $1 - b \in \tilde{J}_x$ for all $x \in X \setminus U$ and $1 - b \in H_x$ for all $x \in \beta X \setminus \overline{U}$;

(iv) $\|(1 - b) + \tilde{J}_x\| = 1$ for all $x \in V$ and $\|(1 - b) + H_x\| = 1$ for all $x \in \overline{V}$.

Furthermore,

(v) $H_x$ is not strictly closed in $M(A)$ for all $x \in (\overline{V} \cap X) \setminus U$.

**Proof.** (i) Let $B = \Pi_{x \in X} M(A)/\tilde{J}_x$ and define $d \in B$ by

$$d_x = g_{f(x)}(u + \tilde{J}_x) \quad (x \in X).$$

We wish to show that $d \in B_{\text{id}}$. Let $a \in A$ with $\|a\| = 1$, and let $\epsilon > 0$. We first seek $c \in A$ such that $\|d(c(a) - \text{id}(c))\| \leq \epsilon$.

Let $Y = \{P \in \text{Prim}(A) \mid \|a + P\| \geq \epsilon\}$, a compact subset of $\text{Prim}(A)$. Then $Z := \phi(Y)$ is a compact subset of $X$ and, for $x \in X \setminus Z$,

$$\|d_x(a + \tilde{J}_x)\| \leq \|a + \tilde{J}_x\| = \|a_x\| < \epsilon$$

(for, if $P \in \text{Prim}(A)$ and $P \supseteq J_x$ then $\phi(P) = x$ and so $P \notin Y$). For $x \in X$, set $c^x = a$ if $f(x) = 0$ and set $c^x = g_{f(x)}(u)a \in A$ otherwise.

**Case 1:** $x \in X$ with $f(x) = 0$. We claim that there exists $\delta > 0$ such that $\|a - g_{r_k}(u)a\| < \epsilon$ for all $0 < r < \delta$. For, if not, there exists a sequence $(r_k)$ tending to zero such that $\|a - g_{r_k}(u)a\| \geq \epsilon$ for all $k$, contradicting the fact that $g_{r_k}(u)$ is an approximate identity for $A$ (see the proof of [28, 3.10.5]). Hence the claim holds.

Set $N_x = f^{-1}([0, \delta])$, an open neighborhood of $x$ in $X$. Then for all $y \in N_x$,

$$\|d_y(a + \tilde{J}_y) - (c^x + \tilde{J}_y)\| = \|g_{f(y)}(u + \tilde{J}_y)(a + \tilde{J}_y) - (a + \tilde{J}_y)\|
\leq \|g_{f(y)}(u)a - a\| < \epsilon$$

(note that if $f(y) = 0$ then $g_{f(y)}(u) = 1$).

**Case 2:** $x \in X$ with $f(x) \neq 0$. Set $r = f(x)$ and let

$$N_x = \left\{y \in X \mid r/2 < f(y) < 2r \text{ and } \frac{2|f(y) - r|}{r} < \epsilon \right\},$$

an open neighborhood of $x$ in $X$. Then for all $y \in N_x$

$$\|d_y(a + \tilde{J}_y) - (c^x + \tilde{J}_y)\| = \|g_{f(y)}(u + \tilde{J}_y)(a + \tilde{J}_y) - g_{f(x)}(u + \tilde{J}_y)(a + \tilde{J}_y)\|$$

$$\leq \|g_{f(y)} - g_r\| \leq \frac{2|f(y) - r|}{r} < \epsilon.$$  

Since $Z$ is compact, there exist $x_1, \ldots, x_n \in Z$ such that the open sets $N_{x_i} (1 \leq i \leq n)$ cover $Z$. Since $X$ is a locally compact Hausdorff space, there exist $h_i \in C_0(X)^+ (1 \leq i \leq n)$, with each $h_i$ vanishing off $N_{x_i}$, such that $\sum_i h_i(x) = 1$ for all $x \in Z$ and $\sum_i h_i(x) \leq 1$ for all $x \in X \setminus Z$. Let $c = \sum_{i=1}^n \mu(h_i) c^{x_i} \in A$.

For all $x \in X$, $c + J_x = \sum_i h_i(x)(c^{x_i} + J_x)$ and so, since $(A + \tilde{J}_x)/\tilde{J}_x$ is canonically isomorphic to $A/J_x$, $c + \tilde{J}_x = \sum_i h_i(x)(c^{x_i} + \tilde{J}_x)$. For $x \in Z$,

$$\|d_x(a + \tilde{J}_x) - (c + \tilde{J}_x)\| = \|\sum_{i=1}^n h_i(x)(d_x(a + \tilde{J}_x) - (c^{x_i} + \tilde{J}_x))\|$$

$$\leq \sum_{i=1}^n h_i(x)\|d_x(a + \tilde{J}_x) - (c^{x_i} + \tilde{J}_x)\| \leq \epsilon,$$

and for $x \in X \setminus Z$,

$$\|d_x(a + \tilde{J}_x) - (c + \tilde{J}_x)\| = \|d_x(a + \tilde{J}_x) - \sum_{i=1}^n h_i(x)(c^{x_i} + \tilde{J}_x)\|$$

$$\leq (1 - \sum_{i=1}^n h_i(x))\|d_x(a + \tilde{J}_x)\| + \sum_{i=1}^n h_i(x)\|d_x(a + \tilde{J}_x) - (c^{x_i} + \tilde{J}_x)\|$$

$$\leq (1 - \sum_{i=1}^n h_i(x))\epsilon + \sum_{i=1}^n h_i(x)\epsilon = \epsilon.$$

Hence

$$\|d\iota(a) - \iota(c)\| = \sup_{x \in X} \|d_x(a + \tilde{J}_x) - (c + \tilde{J}_x)\| \leq \epsilon.$$  

Since $\epsilon$ was arbitrary and $\iota(A)$ is norm-closed in $B$, it follows that $d\iota(a) \in \iota(A)$. Similarly $\iota(a)d \in \iota(A)$ and so $d \in B_{id}$. Let $b = \Phi^{-1}(d) \in M(A)$ (where $\Phi : M(A) \to B_{id}$ is the *-isomorphism of Proposition 3.1). Then, for all $x \in X$,

$$b + \tilde{J}_x = d_x = g_{f(x)}(u + \tilde{J}_x).$$

(ii) Let $x \in U$ and set $r = f(x) > 0$ and $a = g_r(u) \in A$. Let $\epsilon > 0$. As in Case 2 above, let

$$N_x = \left\{ y \in X \left| \frac{r}{2} < f(y) < 2r \text{ and } \frac{2|f(y) - r|}{r} < \epsilon \right. \right\},$$

an open neighborhood of $x$ in $X$. Then for all $y \in N_x$,

$$\|\tilde{h} - \tilde{h}\| = \|(g_{f(y)}(u) - g_{f(x)}(u)) + \tilde{J}_y\|$$

$$\leq \|g_{f(y)} - g_r\| \leq \frac{2|f(y) - r|}{r} < \epsilon.$$
Hence \( \|(b - a) + H_x\| \leq \epsilon \) by Lemma 1.5 (i). Since \( \epsilon \) was arbitrary, \( b - a \in H_x \subseteq \tilde{J}_x \).

(iii) Let \( x \in X \setminus U \). Then \( f(x) = 0 \) and so \( b + \tilde{J}_x = 1 + \tilde{J}_x \).

Let \( x \in W := \beta X \setminus U \). By Lemma 1.5 (i),
\[
\|(1 - b) + H_x\| = \sup \{|(1 - b) + \tilde{J}_y| : y \in W \cap X\}.
\]
Since \( W \cap X \subseteq X \setminus U \), it follows that \( \|(1 - b) + H_x\| = 0 \).

(iv) Let \( x \in V \). Then \( \min \{\text{sp}(u + \tilde{J}_x) : u \in A \} \leq f(x)/2 \) and so
\[
0 = g_f(x)(\min \{\text{sp}(u + \tilde{J}_x) : u \in A \}) \in \text{sp}(b + \tilde{J}_x)
\]
by the spectral mapping theorem. Hence
\[
1 = \|(1 - b) + \tilde{J}_x\| \leq \|(1 - b) + H_x\| \leq 1.
\]
By upper semi-continuity, \( \|(1 - b) + H_x\| = 1 \) for all \( x \in \overline{V} \).

(v) Let \( x \in (\overline{V} \cap X) \setminus U \). Then \( 1 - b \in \tilde{J}_x \setminus H_x \) by (iii) and (iv). Since \( J_x \subseteq H_x \subseteq \tilde{J}_x \) and \( \tilde{J}_x \) is the strict closure of \( J_x \) in \( M(A) \), \( H_x \) cannot be strictly closed in \( M(A) \).

The next three results go towards establishing conditions (i) and (ii) of Theorem 3.8 when \( M(A) \) is a continuous \( C(\beta X) \)-algebra.

**Lemma 3.3.** Let \( A \) be a \( \sigma \)-unital \( C_0(X) \)-algebra with structure map \( \mu \) and base map \( \phi \) and let \( u \) be a strictly positive element of \( A \) with \( \|u\| = 1 \). Suppose that \( M(A) \) is a continuous \( C(\beta X) \)-algebra with base map \( \overline{\phi} \). Define \( f : X \to [0, 1] \) by \( f(x) = (1 - \|(1 - u) + H_x\|)^{\frac{1}{2}} \) for \( x \in \text{Im}(\overline{\phi}) \cap X \) and \( f(x) = 0 \) otherwise. Then

(i) \( \text{Im}(\overline{\phi}) \cap X \) is clopen in \( X \) and \( f \) is continuous;
(ii) for \( x \in X \), \( f(x) > 0 \) if and only if \( \text{Prim}(C_0(X)) \cap A \subseteq J_x \);
(iii) if \( x \in X \) and \( 0 < f(x) \leq \frac{1}{2} \) then \( 2 \min \{\text{sp}(u_x) : u \in A \} \leq f(x) \).

**Proof.** (i) Since \( \text{Prim}(M(A)) \) is compact, \( \text{Im}(\overline{\phi}) \) is compact and hence is the closure of \( \text{Im}(\phi) \) in \( \beta X \). On the other hand,
\[
\text{Im}(\overline{\phi}) = \{ x \in \beta X \mid H_x \neq M(A) \},
\]
which is the union (over \( b \in M(A) \)) of the cozero sets of the continuous functions \( x \to \|(b + H_x)\| (x \in \beta X) \). Thus \( \text{Im}(\overline{\phi}) \) is clopen in \( \beta X \) and hence \( \text{Im}(\overline{\phi}) \cap X \) (which is the closure of \( \text{Im}(\phi) \) in \( X \)) is clopen in \( X \). Since \( x \to \|(1 - u) + H_x\| \) is continuous on \( \beta X \) and hence on \( \text{Im}(\overline{\phi}) \cap X \), it follows that \( f \) is continuous.

(ii) Let \( x \in X \). Suppose that \( f(x) > 0 \). Then \( x \in \text{Im}(\overline{\phi}) \) and so \( H_x \neq M(A) \) and \( u \notin H_x \). Hence \( J_x \neq A \) and \( x \in \text{Im}(\phi) \). Since \( \|(1 - u) + H_x\| < 1 \), \( u + H_x \) is invertible in \( M(A)/H_x \). So no primitive ideal of \( M(A) \) containing \( H_x \) can contain \( A \). It follows from Lemma 2.1 that \( \mu(C_0(X)) \cap A \subseteq J_x \).

Conversely, suppose that \( \mu(C_0(X)) \cap A \subseteq J_x \). Since \( u + J_x \) is strictly positive in the unital algebra \( A/J_x \), it is invertible. By Proposition 2.2 (iii), \( u + H_x \) is invertible in \( M(A)/H_x \) and hence \( \|(1 - u) + H_x\| < 1 \). Thus \( f(x) > 0 \).
(iii) Let \( x \in X \) and suppose that \( 0 < f(x) \leq \frac{1}{2} \). Then
\[
f(x) \geq 2(f(x))^2 = 2 \min \text{sp}(u + H_x).
\]
But, since \( f(x) > 0 \), \( \mu(C_0(X)) \cap A \not\subset J_x \) and so \( A/J_x \) is canonically isomorphic to \( M(A)/H_x \) (Proposition 2.2 (iii)). Hence \( \text{sp}(u + H_x) = \text{sp}(u_x) \).

**Proposition 3.4.** Let \( A \) be a \( \sigma \)-unital \( C_0(X) \)-algebra with structure map \( \mu \) and suppose that \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \overline{\mu} \). Let
\[
U = \{ x \in X \mid \mu(C_0(X)) \cap A \not\subset J_x \}
\]
and let \( \overline{U} \) be the closure of \( U \) in \( \beta X \). Then \( \overline{U} \cap X \), the closure \( \text{cl}(U) \) of \( U \) in \( X \), is open in \( X \).

**Proof.** With \( u \) and \( f \) as in Lemma 3.3, \( U \) is the cozero set of \( f \) by Lemma 3.3 (ii). If \( U \) is closed in \( X \) then there is nothing to prove. So we may assume that \( (\overline{U} \cap X) \setminus U \) is nonempty. Let \( b \in M(A) \) be constructed as in Theorem 3.2. By Theorem 3.2 (iii), \( \|(1 - b) + H_x\| = 0 \) for all \( x \in \beta X \setminus \overline{U} \) and hence for all \( x \in X \setminus (\overline{U} \cap X) \).

Recalling that \( U \subseteq \text{Im}(\phi) \), let \( V = \{ x \in U \mid 2 \min \text{sp}(u_x) \leq f(x) \} \) and let \( \overline{V} \) be the closure of \( V \) in \( \beta X \). Then \( \{ x \in U \mid 0 < f(x) \leq \frac{1}{2} \} \subseteq V \) by Lemma 3.3 (iii). Let \( x \in (\overline{U} \cap X) \setminus U \) and let \( (x_\alpha) \) be a net in \( U \) that is convergent to \( x \). Then \( f(x_\alpha) \to f(x) = 0 \) and so \( x_\alpha \in V \) eventually, from which it follows that \( x \in \overline{V} \). Thus \( \|(1 - b) + H_x\| = 1 \) for all \( x \in (\overline{U} \cap X) \setminus U \) by Theorem 3.2 (iv). The function \( x \to \|(1 - b) + H_x\| \) is continuous on \( \beta X \), and hence on \( X \), and takes the value 1 on the nonempty set \( (\overline{U} \cap X) \setminus U \) and the value 0 on \( X \setminus (\overline{U} \cap X) \). It follows that \( X \setminus (\overline{U} \cap X) \) is closed in \( X \) and hence \( \overline{U} \cap X \) is open in \( X \).

**Proposition 3.5.** Let \( A \) be a \( \sigma \)-unital \( C_0(X) \)-algebra with structure map \( \mu \) and suppose that \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \overline{\mu} \). Let
\[
U = \{ x \in X \mid \mu(C_0(X)) \cap A \not\subset J_x \}
\]
and let \( \overline{U} \) be the closure of \( U \) in \( \beta X \). Then \( \overline{U} \cap X \), the closure \( \text{cl}(U) \) of \( U \) in \( X \), is canonically homeomorphic to a subset of \( \beta U \).

**Proof.** Suppose that \( \text{cl}(U) \) is not canonically homeomorphic to a subset of \( \beta U \). Then by Lemma 2.5 there is a point \( y \in \text{cl}(U) \setminus U \) and disjoint zero sets \( Y \) and \( Z \) of \( U \) such that \( y \) lies in the closures of both \( Y \) and \( Z \). With \( u \) and \( f \) as in Lemma 3.3, let \( b \in M(A) \) be an element with the properties of Theorem 3.2. Recalling that \( U \subseteq \text{Im}(\phi) \), set \( V = \{ x \in U \mid 2 \min \text{sp}(u_x) \leq f(x) \} \). Then \( \{ x \in U \mid 0 < f(x) \leq \frac{1}{2} \} \subseteq V \) by Lemma 3.3 (iii) and \( \|(1 - b) + H_x\| = 1 \) for all \( x \in V \) by Theorem 3.2 (iv).

By [20, 1.15], there is a continuous function \( g \) on \( U \) with \( 0 \leq g \leq 1 \) such that \( g(Y) = \{ 0 \} \) and \( g(Z) = \{ 1 \} \). Let \( I_b = \text{norm-cl}(A(1 - b)A) \), a closed two-sided ideal of \( A \). If \( P \in \text{Prim}(A) \) and \( I_b \not\subset P \) then \( \phi(P) \in U \) by Theorem 3.2 (iii). It follows that \( g \circ \phi \) defines a continuous bounded function on \( \text{Prim}(I_b) \).

and hence induces a unique central multiplier $z_g$ of $I_b$ via the Dauns-Hofmann isomorphism for $ZM(I_b)$. Extending $z_g$ to be the zero multiplier on $I_b^\perp$, we may regard $z_g$ as a central element of $M(I_b + I_b^\perp)$. Since $I_b + I_b^\perp$ is an essential ideal of $A$, $M(A) \subseteq M(I_b + I_b^\perp)$. Hence $z_g(1 - b) = (1 - b)z_g \in M(I_b + I_b^\perp)$. Using an approximate identity for $A$, we see that $(1 - b)a, (1 - b) \in I_b$ for all $a \in A$, and hence that $z_g(1 - b)a \in I_b \subseteq A$ and $az_g(1 - b) = a(1 - b)z_g \in I_b \subseteq A$. So $z_g(1 - b)$ is in the idealizer $A_{zd}$ of $A$ in $M(I_b + I_b^\perp)$. Since $I_b + I_b^\perp$ is essential in its multiplier algebra, $A$ is essential in $A_{zd}$ and so there is an *-isomorphism $\Phi$ of $A_{zd}$ into $M(A)$ such that $\Phi(a) = a$ for all $a \in A$. It follows that if $a \in I_b + I_b^\perp$ then

$$(z_g(1 - b) - \Phi(z_g(1 - b)))a = 0 = a(z_g(1 - b) - \Phi(z_g(1 - b)))$$

and so $z_g(1 - b) = \Phi(z_g(1 - b)) \in M(A)$.

Let $x \in U$, $a \in A$ and $P \in \text{Prim}(A)$ with $P \supseteq J_x$. If $I_b \not\subseteq P$ then

$$z_g(1 - b)a + (P \cap I_b) = g(x)(1 - b)a + (P \cap I_b)$$

and so $z_g(1 - b)a - g(x)(1 - b)a \in P \cap I_b \subseteq P$. On the other hand, if $I_b \subseteq P$ then $z_g(1 - b)a - g(x)(1 - b)a \in P \cap I_b \subseteq P$. Thus in either case $z_g(1 - b)a - g(x)(1 - b)a \in P$. Since this is true for all such $P$, $(z_g(1 - b) - g(x)(1 - b)a)J_x$ and similarly $a(z_g(1 - b) - g(x)(1 - b))J_x$. Hence, using Proposition 2.2 (i), we obtain that

$$z_g(1 - b) - g(x)(1 - b) \in \tilde{J}_x = H_x \quad (x \in U).$$

It now follows that $\|z_g(1 - b) + H_x\| = \|(1 - b) + H_x\|$ for all $x \in Z$ and $\|z_g(1 - b) + H_x\| = 0$ for all $x \in Y$. Since $M(A)$ is a continuous $C(\beta X)$-algebra and $y$ is in the closure of $Y$, we obtain that $\|z_g(1 - b) + H_y\| = 0$. On the other hand, let $(x_\alpha)$ be a net in $Z$ converging to $y$. Then $f(x_\alpha) \rightarrow f(y) = 0$ and so we may assume that $f(x_\alpha) \leq \frac{1}{\varepsilon}$ for all $\alpha$. Then $x_\alpha \in V$ and

$$\|z_g(1 - b) + H_{x_\alpha}\| = \|(1 - b) + H_{x_\alpha}\| = 1$$

for all $\alpha$, by the first paragraph of the proof. Since $x_\alpha \rightarrow y$, $\|z_g(1 - b) + H_y\| = 1$ by the (upper semi-)continuity of the norm function. This contradiction establishes the result.

The next two lemmas are needed in order to establish condition (iii) of Theorem 3.8 when $M(A)$ is a continuous $C(\beta X)$-algebra.

**Lemma 3.6.** Let $A$ be a $\sigma$-unital continuous $C_0(X)$-algebra with structure map $\mu$ and base map $\phi$ and let $V$ be a nonempty open subset of $X$ such that $A/J_x$ is unital for each $x \in V$. Then there exists $x \in V$ such that $\mu(C_0(X)) \cap A \not\supset J_x$.

**Proof.** For all $x \in V$, $(A + J_x)/J_x$ is canonically isomorphic to $A/J_x$ and so, being a unital essential ideal, must equal $M(A)/J_x$. Let $u$ be a strictly positive element of $A$ with $\|u\| = 1$. Then, for all $x \in V$, $u_x$ is invertible and so $\|(1 - u) + J_x\| < 1$. For every $\epsilon \geq 0$, the set $\{x \in X \mid \|(1 - u) + J_x\| > \epsilon\}$ is open, being the image of the open covering of $\phi$ of the open set $\{P \in \text{Prim}(A) \mid \|(1 - u) + P\| > \epsilon\}$ by Proposition 1.1 (iv) (note that if $\|(1 - u) + J_x\| > 0$ then $J_x \neq A$). Hence the function $x \rightarrow \|(1 - u) + J_x\| (x \in X)$ is lower
semi-continuous on $X$. Since $V$ is open in $X$, $V$ is a locally compact Hausdorff space, hence a Baire space, and so any lower semi-continuous function on $V$ has a point of continuity [13, B18]. Thus there exists $x \in V$ with an open neighborhood $W \subseteq V$ and $\epsilon > 0$ such that $\|(1 - u) + \tilde{J}_y\| < 1 - \epsilon$ for all $y \in W$. Hence $\min \text{sp}(u + \tilde{J}_y) > \epsilon$ for all $y \in W$.

Let $g : [0, \infty) \to [0, 1]$ be a continuous function such that $g(0) = 0$ and $g(t) = 1$ for all $t \geq \epsilon$. Let $w = g(u) \in A$ and observe that $w + \tilde{J}_y$ is the identity of $A/J_y$ for all $y \in W$. Choose $f \in C_0(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X \setminus W$. For all $a \in A$ and all $y \in X$,

$$(\mu(f)a - \mu(f)wa)_y = f(y)(a_y - w_ya_y) = 0.$$  

Thus $\mu(f)a = \mu(f)wa$ and similarly $\mu(f) = \mu(f)w \in A$. Since $A/J_x$ is unital, $J_x \neq A$ and so Lemma 2.1 ((i) implies (ii)) yields the result. \qed

In the next lemma we do not require the $C_0(X)$-algebra $A$ to be continuous.

**Lemma 3.7.** Let $A$ be a $\sigma$-unital $C_0(X)$-algebra and let $Y = \{x \in \text{Im}(\phi) \mid A/J_x \text{ is nonunital}\}$. Suppose that $z$ is a non-BD-point in $X$ and that $z$ has a neighborhood $N$ in $X$ such that $Y \cap N$ is dense in $N$. Then there exists $c \in M(A)$ such that the norm function $x \to \|c + H_x\|$ ($x \in X$) is discontinuous at $z$.

**Proof.** Since $z$ is not a BD-point in $X$, there exists a cozero set $S$ and an open set $T$ such that $z \in \text{cl}(S) \cap \text{cl}(T)$ but $z \notin \text{cl}(S \cap T)$. Replacing $T$ by $T \setminus \text{cl}(S)$, we may assume that $S \cap T = \emptyset$. Let $u$ be a strictly positive element of $A$ with $\|u\| = 1$. Then for all $x \in Y$, $\min \text{sp}(u_x) = 0$ since $A/J_x$ is nonunital. Applying Theorem 3.2 to the cozero set $S$, we obtain $b \in M(A)$ with $0 \leq b \leq 1$ such that $1 - b \in \tilde{J}_x$ for all $x \in X \setminus S$ and $\|(1 - b) + H_x\| = 1$ for all $x \in S \cap Y$.

Let $V$ be an arbitrary open neighborhood of $z$ contained in $N$. Since $z \in \text{cl}(S)$, $V \cap S$ is a nonempty open set contained in $N$ and hence contains some point $y \in Y$. Then $y \in S \cap Y$ and so $\|(1 - b) + H_y\| = 1$. Since $y \in V$ and $V$ was arbitrary, it follows from the upper semi-continuity of the norm function that $\|(1 - b) + H_x\| = 1$. On the other hand, applying Lemma 1.5 (i) to any open subset of $\beta X$ whose intersection with $X$ is $T$, we obtain that $\|(1 - b) + H_x\| = 0$ for all $x \in T$ since $T \subseteq X \setminus S$. Since $z \in \text{cl}(T)$, it follows that the norm function of $1 - b$ is discontinuous at $z$. \qed

We are now in a position to obtain the main result of the paper which is the converse of Theorem 2.9.

**Theorem 3.8.** Let $A$ be a $\sigma$-unital continuous $C_0(X)$-algebra with structure map $\mu$ and base map $\phi$, let $U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subset J_x\}$, and let $\text{cl}(U)$ be the closure of $U$ in $X$. Then $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$ if and only if

\[ M(A) = \text{cl}(U). \]
(i) \( \text{cl}(U) \) is clopen in \( X \);
(ii) \( \text{cl}(U) \) is canonically homeomorphic to a subset of \( \beta U \);
(iii) every point of \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \) is a BD-point of \( X \).

Moreover, when these conditions hold, \( \text{cl}(\text{Im}(\phi)) \) is clopen in \( X \) and \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \) is basically disconnected.

\textbf{Proof.} The “if” part of the result is Theorem 2.9. Conversely, suppose that \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \mu \). Then \( \text{cl}(\text{Im}(\phi)) \) is clopen in \( X \) (see the proof of Lemma 3.3 (i)). Also, \( \text{Im}(\phi) \) is open in \( X \) since \( A \) is a continuous \( C_0(X) \)-algebra.

Conditions (i) and (ii) follow from Propositions 3.4 and 3.5 respectively. For condition (iii), we may suppose that \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \) is nonempty (for otherwise there is nothing to prove). Let \( Y = \{ x \in \text{Im}(\phi) \mid A/J_x \text{ is nonunital} \} \). If \( V \) is any nonempty open subset of the clopen set \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \) then \( V \cap \text{Im}(\phi) \) is also a nonempty open subset. Since it is disjoint from \( U \), it must contain an element of \( Y \) by Lemma 3.6. Thus \( Y \cap (\text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U)) \) is dense in \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \). Since \( M(A) \) is a continuous \( C(\beta X) \)-algebra, it follows from Lemma 3.7 that every \( x \in \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \) is a BD-point in \( X \) and hence is a BD-point of the clopen set \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \). \( \square \)

If \( A \) is separable then we can extract some further information from Theorem 3.8.

\textbf{Corollary 3.9.} Let \( A \) be a separable continuous \( C_0(X) \)-algebra with structure map \( \mu \) and base map \( \phi \) and let
\[
U = \{ x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x \}.
\]
If \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \mu \) then
(i) \( U \) is clopen in \( \text{Im}(\phi) \);
(ii) every point of \( \text{Im}(\phi) \setminus U \) is an isolated point of \( X \).

Conversely, if (i) and (ii) hold and \( X = \text{Im}(\phi) \) then \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \mu \).

\textbf{Proof.} Suppose first that \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \mu \). Then \( A \) satisfies conditions (i), (ii), and (iii) of Theorem 3.8. Furthermore, \( \text{Im}(\phi) \) is second countable, being the image of the second countable space \( \text{Prim}(A) \) [13, 3.3.4] under the continuous open map \( \phi \).

In particular, each point \( x \in \text{cl}(U) \cap \text{Im}(\phi) \) has a countable neighborhood base in \( \text{Im}(\phi) \), and hence in \( X \) since \( \text{Im}(\phi) \) is open in \( X \). Thus each \( x \in \text{cl}(U) \cap \text{Im}(\phi) \) has a countable neighborhood base in the subspace \( \text{cl}(U) \) of \( X \). But by condition (ii) of Theorem 3.8 we have the inclusions \( U \subseteq \text{cl}(U) \cap \text{Im}(\phi) \subseteq \text{cl}(U) \subseteq \beta U \). It follows, since \( \text{cl}(U) \) is dense in the compact space \( \beta U \), that each \( x \in \text{cl}(U) \cap \text{Im}(\phi) \) has a countable neighborhood base in \( \beta U \), cp. [20, 9.7]. But no point of \( \beta U \setminus U \) has a countable neighborhood base in \( \beta U \) [20, Cor. 9.6]. Hence \( \text{cl}(U) \cap \text{Im}(\phi) = U \), so \( U \) is clopen in \( \text{Im}(\phi) \), establishing (i).

Similarly, each point \( x \in \text{Im}(\phi) \setminus \text{cl}(U) = \text{Im}(\phi) \setminus U \) has a countable neighborhood base in \( X \), and hence in the basically disconnected space \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \).
But in a basically disconnected space, a point with a countable neighborhood base is isolated [20, 14N]. Thus each \( x \in \text{Im}(\phi) \setminus U \) is isolated in \( \text{cl}(\text{Im}(\phi)) \setminus \text{cl}(U) \), which is a clopen subset of \( X \), and is therefore isolated in \( X \) itself. This establishes (ii).

For the converse, suppose that (i) and (ii) hold and that \( X = \text{Im}(\phi) \). Then it is trivial that conditions (i), (ii), and (iii) of Theorem 3.8 hold. Hence \( M(A) \) is a continuous \( C(\beta X) \)-algebra with structure map \( \bar{\mu} \).

4. Applications

In this section we give some applications of Theorem 3.8. Our first application is to \( C^* \)-algebras \( A \) for which the locally compact space \( \text{Prim}(A) \) is Hausdorff. It is well-known that \( A \) is then a continuous \( C_0(\text{Prim}(A)) \)-algebra [13, 3.9.11]. We may take \( X = \text{Prim}(A) \) and \( \phi = \text{id} \), so that \( \bar{\mu} \) is the restriction to \( C_0(\text{Prim}(A)) \) of the Dauns-Hofmann isomorphism \( \theta_A \). In this case, \( J_P = P \) for all \( P \in \text{Prim}(A) \) and the mapping \( \bar{\phi} : \text{Prim}(M(A)) \to \beta X \) satisfies \( \bar{\phi}(\tilde{P}) = \phi(P) = P \) (\( P \in \text{Prim}(A) \)). Since \( \theta_A^{-1}(Z(A)) \subseteq C_0(\text{Prim}(A)) \), the set

\[
U := \{ P \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_P \}
\]

takes the form \( U = \{ P \in \text{Prim}(A) \mid Z(A) \not\subseteq P \} \), where \( Z(A) \) is the center of \( A \). With this notation, we immediately obtain the following corollary from Theorem 3.8.

**Theorem 4.1.** Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra with Hausdorff primitive ideal space \( X = \text{Prim}(A) \) and let \( U = \{ P \in \text{Prim}(A) \mid Z(A) \not\subseteq P \} \). Then \( M(A) \) is a continuous \( C(\beta X) \)-algebra with base map \( \text{id} : \text{Prim}(M(A)) \to \beta X \) if and only if

(i) \( \text{cl}(U) \) is clopen in \( \text{Prim}(A) \);
(ii) \( \text{cl}(U) \) is canonically homeomorphic to a subset of \( \beta U \);
(iii) every point of \( \text{Prim}(A) \setminus \text{cl}(U) \) is a BD-point of \( \text{Prim}(A) \).

The conditions in Theorem 4.1 simplify substantially in the case \( U = \emptyset \), which holds if and only if \( Z(A) = \{ 0 \} \). In particular this applies when \( A \) is a stable \( C^* \)-algebra.

**Corollary 4.2.** Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra with Hausdorff primitive ideal space \( X = \text{Prim}(A) \) and suppose that \( Z(A) = \{ 0 \} \) (e.g. if \( A \) is stable). Then \( M(A) \) is a continuous \( C(\beta X) \)-algebra with base map \( \text{id} : \text{Prim}(M(A)) \to \beta X \) if and only if \( \text{Prim}(A) \) is basically disconnected.

A second countable, basically disconnected space is discrete [20, 14.N], so Corollary 4.2 implies the following.

**Corollary 4.3.** Let \( A \) be a stable separable \( C^* \)-algebra with Hausdorff primitive ideal space \( X = \text{Prim}(A) \). Then \( M(A) \) is a continuous \( C(\beta X) \)-algebra with base map \( \text{id} : \text{Prim}(M(A)) \to \beta X \) if and only if \( \text{Prim}(A) \) is discrete.
Theorem 4.1 raises the question of characterizing the space $\beta X$ and investigating the nature of the ideals $H_x$ of $M(A)$. In fact, it is a consequence of the Dauns-Hofmann theorem that $\beta X$ in Theorem 4.1 is homeomorphic to the maximal ideal space $\Delta$ of $ZM(A)$ which is in turn homeomorphic to the complete regularization $\text{Glimm}(M(A))$ of $\text{Prim}(M(A))$. The ideals $H_x$ of $M(A)$ ($x \in \beta X$) are the Glimm ideals of $M(A)$ (generated by the ideals in $\Delta$ [21]).

It is easy to see that the ideals $H_x$ need not be maximal ideals in $M(A)$ even when $\text{Prim}(A)$ is Hausdorff. For example, if $A = LC(H)$, the algebra of compact linear operators on an infinite-dimensional Hilbert space, then $X (= \beta X)$ is a singleton set containing the zero ideal $\{0\}$, and $H_{\{0\}} = \{0\}$ which is a primitive but nonmaximal ideal of $M(A) = B(H)$. This phenomenon occurs whenever $A$ has a nonunital primitive quotient.

Even when all the primitive quotients of $A$ are unital and $\text{Prim}(A)$ is Hausdorff, it is still possible for $\text{Prim}(M(A))$ to be non-Hausdorff.

**Example 4.4.** Let $B = C^*_r(F_2)$, where $F_2$ is the free group on two generators, and let $A = c_0 \otimes B$. Then $\text{Prim}(A)$ is homeomorphic to $\mathbb{N}$ and hence is Hausdorff, and the set $U$ of Theorem 4.1 is equal to $\text{Prim}(A)$ so $M(A)$ is a continuous $\beta\mathbb{N}$-algebra. For $x \in \beta\mathbb{N} \setminus \mathbb{N}$, the quotient $M(A)/H_x$ is an ultrapower of $B$, and hence is a primitive but nonsimple $C^*$-algebra [19, Thm. 5.4, Cor. 5.5]. Thus $\text{Prim}(M(A))$ is non-Hausdorff.

Our next application is to quasi-standard $C^*$-algebras. These can be defined in various equivalent ways but perhaps the easiest one for our present purposes is that $A$ is **quasi-standard** if $A$ is a continuous $C_0(X)$-algebra with $X = \text{Im}(\phi)$ such that $J_x$ is a primal ideal of $A$ for each $x \in \text{Im}(\phi)$ [6, Thm. 3.4]. Recall that a closed two-sided ideal $J$ of a $C^*$-algebra $A$ is **primal** if, whenever $n \geq 2$ and $I_1, \ldots, I_n$ are closed two-sided ideals of $A$ with product zero, then there exists $j \in \{1, \ldots, n\}$ such that $I_j \subseteq J$ [5]. Every primitive ideal is prime and hence primal, so the algebra $M(A)$ in Example 4.4 is quasi-standard. Similarly, if $\text{Prim}(A)$ is Hausdorff then $A$ is quasi-standard. Von Neumann algebras are quasi-standard and so too are many group $C^*$-algebras, for example those of the discrete and continuous Heisenberg groups [21], [26], [1], [23].

The main reason for considering primal ideals in this context is that they are the limits of nets of primitive ideals in an appropriate topology [4]. Thus, if $A$ is a continuous $C_0(X)$-algebra and $J_x$ is primitive for $x$ in a dense subset of $X$ then $J_x$ will be primal for all $x \in X$, and a converse statement holds if $A$ is separable [6, 3.4, 3.5]. Primal ideals have found a number of other applications in the theory of $C^*$-algebras. It was shown in [5] that a state of a $C^*$-algebra is a weak*-limit of factorial states if and only if the kernel of the GNS representation is a primal ideal. Primal ideals play a crucial role in the solution of the isometry problem for the central Haagerup tensor product [9] and in the study of norms of inner derivations [29], [7], [8].

**Lemma 4.5.** Let $J$ be a proper, closed, two-sided ideal of a $C^*$-algebra $A$ and suppose that $J$ is a primal ideal of $A$. Then $\tilde{J}$ is a primal ideal of $M(A)$. 

Proof. Suppose that $n \geq 2$ and $I_1, \ldots, I_n$ are closed two-sided ideals of $M(A)$ such that $I_1I_2\ldots I_n = \{0\}$. Then

$$(I_1 \cap A)(I_2 \cap A)\ldots (I_n \cap A) = \{0\}$$

and so there exists $j$ such that $I_j \cap A \subseteq J$. For $b \in I_j$ and $a \in A$, we have $ab, ba \in I_j \cap A \subseteq J$ and so $b \in \tilde{J}$. Thus $I_j \subseteq \tilde{J}$. □

**Proposition 4.6.** Let $A$ be a continuous $C_0(X)$-algebra with base map $\phi$ such that $J_x$ is a primal ideal of $A$ for all $x \in X$. Let $y \in \beta X$ and suppose that for all $b \in M(A)$ the function $x \to \|b + H_x\|$ ($x \in \beta X$) is continuous at $y$. Then $H_y$ is a primal ideal of $M(A)$.

**Proof.** Suppose that $n \geq 2$ and $b_1, \ldots, b_n \in M(A) \setminus H_y$. There exists an open neighborhood $V$ of $y$ in $\beta X$ such that $\|b_j + H_x\| > 0$ for $1 \leq j \leq n$ and all $x \in V$. For $1 \leq j \leq n$, let

$$U_j := \{x \in X \mid \|b_j + \tilde{J}_x\| > 0\} = \phi(\{P \in \text{Prim}(A) \mid \|b_j + P\| > 0\}).$$

Since $A$ is a continuous $C_0(X)$-algebra, $\phi$ is open and so $U_j$ is an open subset of $X$. But since $X$ is locally compact, it is open in $\beta X$ [20, 3.15(d)] and so $U_j$ is open in $\beta X$.

Let $W$ be a nonempty open subset of $V$ and let $x \in W$. By Lemma 1.5 (i),

$$0 < \|b_j + H_x\| \leq \sup\{\|b_j + \tilde{J}_i\| \mid t \in W \cap X\}.$$

So $U_j \cap W$ is nonempty and hence $U_j \cap V$ is a dense open subset of $V$. It follows that $\bigcap_{j=1}^n U_j$ is a nonempty subset of $X$. So there exists $x \in X$ such that, for $1 \leq j \leq n$, $b_j \notin \tilde{J}_x$.

By Lemma 4.5, $\tilde{J}_x$ is a primal ideal of $M(A)$ and so

$$b_1M(A)b_2M(A)\ldots b_{n-1}M(A)b_n \neq \{0\}.$$ 

It follows that $H_y$ is a primal ideal of $M(A)$. □

One important fact about a quasi-standard $C^*$-algebra $A$ is that the space $X$ such that $A$ is a continuous $C_0(X)$-algebra with $X = \text{Im}(\phi)$ and $J_x$ primal for all $x \in X$ is unique. Indeed $X$ is the complete regularization of $\text{Prim}(A)$ [20, 3.9], [6, 3.3 and 3.4]. For a general $C^*$-algebra $A$, let $\phi_A : \text{Prim}(A) \to X$ denote the complete regularization map. If $A$ is not quasi-standard then $X$ need not be locally compact. However, it is always possible to form $J_x := \cap\{P \in \text{Prim}(A) \mid \phi_A(P) = x\}$ for each $x \in X$. The ideals $J_x$ ($x \in X$) are called the Glimm ideals of $A$ and we set $\text{Glimm}(A) = \{J_x \mid x \in X\}$, with the complete regularization topology.

If $A$ is quasi-standard then $\text{Glimm}(A)$ coincides with the space of minimal primal ideals of $A$. For convenience, we take $X = \text{Glimm}(A)$ in this case. The corresponding structure map $\mu : C_0(X) \to ZM(A)$ is given by $\mu(f) = \theta_A(f \circ \phi_A)$ ($f \in C_0(X)$). For each $G \in X = \text{Glimm}(A)$,

$$J_G = \cap\{P \in \text{Prim}(A) \mid \phi_A(P) = G\} = \cap\{P \in \text{Prim}(A) \mid P \supset G\} = G.$$
Thus the set $U$ of Section 2 is defined by

$$U = \{ G \in \text{Glimm}(A) \mid \mu(C_0(\text{Glimm}(A))) \cap A \not\subseteq G \}.$$ 

Clearly, $\mu(C_0(\text{Glimm}(A))) \cap A \subseteq \text{ZM}(A) \cap A = \text{Z}(A)$. Conversely, suppose that $z \in \text{Z}(A)$ and let $h = \theta_A^{-1}(z) \in C^b(\text{Prim}(A))$. Then $h(P)1 + \hat{P} = z + \hat{P}$ for all $P \in \text{Prim}(A)$. The function $h$ induces $f \in C^b(\text{Glimm}(A))$ such that $h = f \circ \phi_A$. Let $\epsilon > 0$. Since $z \in A$, there exists a compact subset $K$ of $\text{Prim}(A)$ such that

$$|h(P)| = \|z + \hat{P}\| = \|z + P\| < \epsilon \quad (P \in \text{Prim}(A)).$$

Then $\phi_A(K)$ is a compact subset of $\text{Glimm}(A)$ such that $|f(G)| < \epsilon$ for all $G \in \text{Glimm}(A) \setminus \phi_A(K)$. Thus $f \in C_0(\text{Glimm}(A))$ and

$$\mu(f) = \theta_A(f \circ \phi_A) = \theta(h) = z.$$

It follows that

$$U = \{ G \in \text{Glimm}(A) \mid \text{Z}(A) \not\subseteq G \}.$$ 

The existence of a homeomorphism between $\beta \text{Glimm}(A)$ and $\text{Glimm}(M(A))$ is well-known (see, for example, [2, p. 88]) but we provide some details in order to establish equation (1) below.

**Proposition 4.7.** Let $A$ be a $C^*$-algebra. Then there is a homeomorphism

$$\iota : \beta \text{Glimm}(A) \rightarrow \text{Glimm}(M(A))$$

such that

$$\iota(\phi_A(P)) = \phi_M(A)(\hat{P}) \quad (P \in \text{Prim}(A)).$$

**Proof.** Applying the Dauns-Hofmann theorem both to $A$ and to $M(A)$, we obtain $^*$-isomorphisms

$$\Phi : C(\beta \text{Glimm}(A)) \rightarrow \text{ZM}(A) \quad \text{and} \quad \Psi : C(\text{Glimm}(M(A))) \rightarrow \text{ZM}(A)$$

such that $\Phi(f) = \theta_A(f \circ \phi_A)$ and $\Psi(g) = \theta_M(A)(g \circ \phi_M(A))$. By the Banach-Stone theorem, there exist homeomorphisms

$$j : \beta \text{Glimm}(A) \rightarrow \Delta := \text{Max}(\text{ZM}(A)) \quad \text{and} \quad k : \Delta \rightarrow \text{Glimm}(M(A))$$

such that

$$\Phi(f) + m = f(j^{-1}(m))1 + m \quad \text{and} \quad \Psi(g) + m = g(k(m))1 + m$$

for all $m \in \Delta$, $f \in C(\beta \text{Glimm}(A))$ and $g \in C(\text{Glimm}(M(A)))$. We define $\iota = k \circ j$.

Let $P \in \text{Prim}(A)$ and set $m = \hat{P} \cap \text{ZM}(A) \in \Delta$. Let $f \in C(\beta \text{Glimm}(A))$ and write $z = \Phi(f)$. Since

$$z - f(\phi_A(P))1 \in \hat{P} \cap \text{ZM}(A) = m$$

and $z - f(j^{-1}(m))1 \in m$, we obtain $f(\phi_A(P)) = f(j^{-1}(m))$. Since $f$ was arbitrary, $\phi_A(P) = j^{-1}(m)$.

Now let $g \in C(\text{Glimm}(M(A)))$ and write $z = \Psi(g)$. Since

$$z - g(\phi_M(A)(\hat{P}))1 \in \hat{P} \cap \text{ZM}(A) = m$$

and \( z - g(k(m))1 \in m \), we obtain \( g(\phi_{M(A)}(\tilde{P})) = g(k(m)) \). Since \( g \) was arbitrary, \( \phi_{M(A)}(\tilde{P}) = k(m) \). Hence
\[
\iota(\phi_A(P)) = (k \circ j)(\phi_A(P)) = k(m) = \phi_{M(A)}(\tilde{P}).
\]

\[\square\]

**Theorem 4.8.** Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra and let
\[
U = \{ G \in \text{Glimm}(A) \mid Z(A) \not\subseteq G \}.
\]
Then \( M(A) \) is quasi-standard if and only if

(i) \( A \) is quasi-standard;
(ii) \( \text{cl}(U) \) is clopen in \( \text{Glimm}(A) \);
(iii) \( \text{cl}(U) \) is canonically homeomorphic to a subset of \( \beta U \);
(iv) \( \text{Glimm}(A) \setminus \text{cl}(U) \) is basically disconnected.

**Proof.** Suppose that conditions (i)-(iv) hold. By Theorem 3.8, \( M(A) \) is a continuous \( C(\beta X) \)-algebra (with \( X = \text{Glimm}(A) \)) with respect to the surjective mapping \( \phi_A : \text{Prim}(M(A)) \to \beta X \). Since \( A \) is quasi-standard, every \( G \in \text{Glimm}(A) \) is a primal ideal of \( A \). By Proposition 4.6, \( H_x \) is a primal ideal of \( M(A) \) for all \( x \in \beta X \). Since \( \phi_A \) is surjective, \( H_x \neq M(A) \) for all \( x \in \beta X \) and so it follows from [6, Thm. 3.4] that \( M(A) \) is quasi-standard.

Conversely, suppose that \( M(A) \) is quasi-standard, and set \( Y = \text{Glimm}(M(A)) \). Then \( M(A) \) is a continuous \( C(Y) \)-algebra with respect to the complete regularization map \( \phi_{M(A)} : \text{Prim}(M(A)) \to Y \). By Proposition 4.7, there exists a homeomorphism \( \iota : \beta \text{Glimm}(A) \to \text{Glimm}(M(A)) \) such that
\[
\iota(\phi_A(P)) = \phi_{M(A)}(\tilde{P}) \quad \text{for all } P \in \text{Prim}(A).
\]
Since \( \{ \tilde{P} \mid P \in \text{Prim}(A) \} \) is a dense subset of \( \text{Prim}(M(A)) \), it follows by continuity that \( \iota \circ \phi_A = \phi_{M(A)} \). Hence \( M(A) \) is a continuous \( C(\beta \text{Glimm}(A)) \)-algebra with respect to \( \phi_A \). It follows from Theorem 3.8 that conditions (ii)-(iv) hold. Finally, \( A \) is quasi-standard because it is an ideal of \( M(A) \) (see [6, p. 356]). \[\square\]

As with Theorem 4.1, the conditions in Theorem 4.8 simplify substantially in the case \( U = \emptyset \), which holds if and only if \( Z(A) = \{0\} \).

**Corollary 4.9.** Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra with \( Z(A) = \{0\} \). Then \( M(A) \) is quasi-standard if and only if

(i) \( A \) is quasi-standard;
(ii) \( \text{Glimm}(A) \) is basically disconnected.

At the other extreme, recall that a \( C^* \)-algebra is quasi-central if \( Z(A) \not\subseteq P \) for all \( P \in \text{Prim}(A) \) [12]. It is easily seen that \( A \) is quasi-central if and only if \( U = \text{Glimm}(A) \). In the following result, we do not need to assume that \( A \) is \( \sigma \)-unital.

**Corollary 4.10.** Let \( A \) be a quasi-central \( C^* \)-algebra. Then \( M(A) \) is quasi-standard if and only if \( A \) is quasi-standard.
Proof. Suppose that $A$ is quasi-standard. Since $U = \text{Glimm}(A)$, it follows from Propositions 2.3 and 2.8 that $M(A)$ is a continuous $C(\beta X)$-algebra (with $X = \text{Glimm}(A)$) with respect to the surjective mapping $\phi_A : \text{Prim}(M(A)) \to \beta X$. Hence, as in the proof of Theorem 4.8, $M(A)$ is quasi-standard. \[\Box\]

In particular, if $A$ is an $n$-homogeneous C*-algebra then $\text{Prim}(A)$ is Hausdorff and $A$ is quasi-central so it follows from Corollary 4.10 that $M(A)$ is quasi-standard.

Now let $A$ be a $\sigma$-unital subhomogeneous C*-algebra. Since every nonzero ideal in $A$ is subhomogeneous, and therefore contains a nonzero homogeneous ideal, it follows that the set $\{ P \in \text{Prim}(A) \mid P \not\supseteq Z(A) \}$ is dense in $\text{Prim}(A)$, and hence that the set $U$ of Theorem 4.8 is automatically dense in $\text{Glimm}(A)$. Thus conditions (ii) and (iv) are automatically trivially satisfied and we have the following.

**Corollary 4.11.** Let $A$ be a $\sigma$-unital subhomogeneous C*-algebra and let $U = \{ G \in \text{Glimm}(A) \mid Z(A) \not\subseteq G \}$. Then $M(A)$ is quasi-standard if and only if

(i) $A$ is quasi-standard;

(ii) $\text{Glimm}(A)$ is canonically homeomorphic to a subset of $\beta U$.

If $A$ in Corollary 4.11 is also separable and $M(A)$ is quasi-standard then it follows from condition (i) of Corollary 3.9 that the dense set $U$ equals $\text{Glimm}(A)$, and hence that $A$ is quasi-central. Thus we have that the multiplier algebra of a separable, subhomogeneous, C*-algebra $A$ is quasi-standard if and only if $A$ is quasi-standard and quasi-central.

**References**


Multiplier algebras of $C_0(X)$-algebras


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