Nuclear dimension and $n$-comparison

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Abstract. It is shown that if a $C^*$-algebra has nuclear dimension $n$ then its Cuntz semigroup has the property of $n$-comparison. It then follows from results by Ortega, Perera and Rørdam that $\sigma$-unital $C^*$-algebras of finite nuclear dimension (and even of nuclear dimension at most $\omega$) are stable if and only if they have no nonzero unital quotients and no nonzero bounded traces.

1. Introduction

In [8], Winter and Zacharias define nuclear dimension for $C^*$-algebras. This is a form of noncommutative dimension which directly generalizes the covering dimension of topological spaces. Finite nuclear dimension is especially relevant to the classification of $C^*$-algebras. The simple $C^*$-algebras of finite nuclear dimension have been proposed as a likely class of $C^*$-algebras for which Elliott’s classification in terms of $K$-theory and traces holds true.

In the main result of this paper it is shown that the Cuntz semigroup of a $C^*$-algebra of finite nuclear dimension $n$ satisfies the $n$-comparison property. For $n = 0$, this property is the same as almost unperforation in the Cuntz semigroup. For arbitrary $n$, it is reminiscent of the comparability between vector bundles whose fibrewise dimensions differ sufficiently relative to the dimension of the base space.

The $n$-comparison property for the Cuntz semigroup was first considered by Toms and Winter (see [7, Lemma 6.1]). They showed that $n$-comparison holds under the more restrictive assumption that the $C^*$-algebra is simple unital of decomposition rank $n$ (the decomposition rank bounds the nuclear dimension, and, unlike nuclear dimension, is infinite for UCT Kirchberg algebras). The $n$-comparison property was subsequently studied, and more precisely defined, by Ortega, Perera and Rørdam in [6]. These authors obtained a simple criterion of stability for $\sigma$-unital $C^*$-algebras with $n$-comparison in their Cuntz semigroup: the $C^*$-algebra is stable if and only if it has no nonzero unital quotients and no nonzero bounded 2-quasitraces. By Theorem 1.3 below, this stability criterion
applies to all C*-algebras of finite nuclear dimension. It then follows that σ-unital C*-algebras of finite nuclear dimension have the corona factorization property.

Let us recall the definition of nuclear dimension given in [8].

**Definition 1.1.** The C*-algebra $A$ has nuclear dimension $n$ if $n$ is the smallest natural number for which there exist nets of completely positive contractive (henceforth abbreviated as c.p.c.) maps

$$
\psi^i_\lambda : A \to F^i_\lambda \text{ and } \phi^i_\lambda : F^i_\lambda \to A,
$$

with $i = 0, 1, \ldots, n$, $\lambda \in \Lambda$, and $F^i_\lambda$ finite dimensional C*-algebras for all $i$ and $\lambda$, such that

1. $\phi^i_\lambda$ is an order 0 map (i.e., preserves orthogonality) for all $i$ and $\lambda$,
2. $\lim_{\lambda} \sum_{i=1}^{n} \phi^i_\lambda \psi^i_\lambda(a) = a$ for all $a \in A$.

If no such $n$ exists then $A$ has infinite nuclear dimension.

Let us recall the definition given in [6] of the $n$-comparison property of an ordered semigroup. For $x, y$ elements of an ordered semigroup $S$, let us write $x \leq_{s} y$ if $(k + 1)x \leq k y$ for some $k \in \mathbb{N}$.

**Definition 1.2.** The ordered semigroup $S$ has the $n$-comparison property if $x \leq_{s} y_i$ for $x, y_i \in S$ and $i = 0, 1, \ldots, n$ implies $x \leq \sum_{i=0}^{n} y_i$.

Let $\text{Cu}(A)$ denote the stabilized Cuntz semigroup of the C*-algebra $A$ (i.e., the semigroup $W(A \otimes \mathcal{K})$; see [2]).

It is shown in Lemma 2.1 below that for $\text{Cu}(A)$ the $n$-comparison property can be reformulated as follows: if $[a], [b_i] \in \text{Cu}(A)$, with $i = 0, 1, \ldots, n$, satisfy for each $i$ there is $\varepsilon_i > 0$ such that $d_{\tau}(a) \leq (1 - \varepsilon_i)d_{\tau}(b_i)$ for all dimension functions $d_{\tau}$ induced by lower semicontinuous 2-quasitraces, then $[a] \leq \sum_{i=0}^{n} [b_i]$. It is this formulation of the $n$-comparison property that is used by Toms and Winter in [7], and that may potentially have the most applications.

**Theorem 1.3.** If $A$ has nuclear dimension $n$ then $\text{Cu}(A)$ has the $n$-comparison property.

The following section is dedicated to the proof of Theorem 1.3. The last section discusses the application of Theorem 1.3, and of a variation on it that relates to $\omega$-comparison, to establish the stability of C*-algebras of finite (or at most $\omega$) nuclear dimension.

## 2. Proof of Theorem 1.3

Let us start by proving that the property of $n$-comparison may be formulated using comparison by lower semicontinuous 2-quasitraces in place of the relation $\leq_{s}$. This result, however, will not be needed in the proof of Theorem 1.3.

For $[a], [b] \in \text{Cu}(A)$, elements of the Cuntz semigroup of $A$, let us write $[a] <_{\tau} [b]$ if there is $\varepsilon > 0$ such that $d_{\tau}(a) \leq (1 - \varepsilon)d_{\tau}(b)$ for all dimension functions $d_{\tau}$ induced by lower semicontinuous 2-quasitraces $\tau : A^{+} \to [0, \infty]$.
(see [3, Sec. 4]). We do not assume that the 2-quasitraces are necessarily finite on a dense subset of $A^+$.  

**Lemma 2.1.** The ordered semigroup $\text{Cu}(A)$ has the n-comparison property if and only if for $[a], [b_i] \in \text{Cu}(A)$, with $i = 0, 1, \ldots, n$, $[a] \prec [b_i]$ for all $i$ implies that $[a] \leq \sum_{i=0}^{n}[b_i]$.  

**Proof.** It is clear that if $[a] \leq_s [b]$ then $[a] \prec [b]$. Thus, $n$-comparison is implied by the property stated in the lemma. Suppose that we have $n$-comparison in $\text{Cu}(A)$. Let $[a], [b_i] \in \text{Cu}(A)$, with $i = 0, 1, \ldots, n$, be such that $[a] \prec [b_i]$ for all $i$. That is, there exist $\varepsilon_i > 0$ such that  

\[(1) \quad d_\tau(a) \leq (1 - \varepsilon_i)d_\tau(b_i)\]

for all $i$. Let us show that $[(a - \varepsilon) \leq_s [b_i]$ for all $\varepsilon > 0$ and all $i$.

Let $I_i \subseteq A \otimes K$ denote the closed two-sided ideal generated by $b_i$ and $\tau_{I_i}$ the trace that is 0 on $I_i^+$ and $\infty$ elsewhere. Setting $\tau = \tau_{I_i}$ in (1) we get that $a \in I_i$. This implies that $(a - \varepsilon) = \sum_{j=1}^{k_i} c_j^*b_ic_j$, which in turn yields

\[[(a - \varepsilon) \leq \sum_{i=0}^{n}[b_i].\]

Let $\lambda: \text{Cu}(A) \to [0, \infty]$ be additive and order preserving. Let us define $\tilde{\lambda}: \text{Cu}(A) \to [0, \infty]$ by

\[\tilde{\lambda}([c]) := \sup_{\delta > 0} \lambda([(c - \delta) +])\]

It is clear from the definition of $\tilde{\lambda}$ that $\lambda([(c - \delta) +]) \leq \tilde{\lambda}([c]) \leq \lambda([c])$ for all $[c] \in \text{Cu}(A)$ and $\delta > 0$. By [3, Prop. 4.2 and Lemma 4.7], there exists a lower semicontinuous 2-quasitrace $\tau$ such that $\tilde{\lambda}([c]) = d_\tau(c)$ for all $c \in (A \otimes K)^+$. Thus,  

\[\lambda([(a - \varepsilon) +]) \leq \tilde{\lambda}([a]) \leq (1 - \varepsilon_i)\tilde{\lambda}([b_i]) \leq (1 - \varepsilon_i)\lambda([b_i])\]

for all $i$. That is, $\lambda([(a - \varepsilon) +]) \leq (1 - \varepsilon_i)\lambda([b_i])$ for any $\lambda$ that is additive and order preserving. We conclude by [6, Prop. 2.1] that $[(a - \varepsilon) +] \leq_s [b_i]$ for all $i$.

Since $\text{Cu}(A)$ has $n$-comparison, $[(a - \varepsilon) +] \leq \sum_{i=0}^{n}[b_i]$. Passing on the left to the supremum with respect to $\varepsilon > 0$ we get that $[a] \leq \sum_{i=0}^{n}[b_i]$. \hfill $\square$

Given an upward directed set $\Lambda$ and a family of $C^*$-algebras $(A_\lambda)_{\lambda \in \Lambda}$, let $\bigoplus_\Lambda A_\lambda$ denote the $C^*$-algebra of nets $(x_\lambda)$ such that $\|x_\lambda\| \to 0$ and $\prod_\Lambda A_\lambda$ the $C^*$-algebra of nets of uniformly bounded norm. For $A$ a $C^*$-algebra, let $A_\Lambda$ denote the $C^*$-algebra $\prod_\Lambda A / \bigoplus_\Lambda A$ and $\iota: A \to A_\Lambda$ the diagonal embedding of $A$ into $A_\Lambda$.

**Notation convention.** Given a homomorphism or c.p.c. order 0 map $\phi: A \to B$, we shall also denote by $\phi$ the map $\phi \otimes \text{id}$ from $A \otimes K$ to $B \otimes K$.

In [8, Prop. 3.2] Winter and Zacharias show that if $A$ has nuclear dimension $n$ then the maps $\psi^i_\Lambda$ in the definition of nuclear dimension may be chosen to be asymptotically of order 0, i.e., such that the induced maps $\psi^i: A \to \prod_\Lambda F^i_\Lambda / \bigoplus_\Lambda F^i_\Lambda$ have order 0 for $i = 0, 1, \ldots, n$. We get the following proposition as a result of this.
Proposition 2.2. If $A$ has nuclear dimension $n$ then for $i = 0, 1, \ldots, n$ there are c.p.c. order 0 maps $\psi_i^0 : A \to \prod_\lambda F_i^0 / \bigoplus_\lambda F_i^0$ and $\phi^i : \prod_\lambda F_i^0 / \bigoplus_\lambda F_i^0 \to A_\Lambda$ such that

$$\nu = \sum_{i=0}^n \phi^i \psi^0_i,$$

where $\nu : A \to A_\Lambda$ is the diagonal embedding of $A$ into $A_\Lambda$.

Proof. As pointed out in the previous paragraph, by [8, Prop. 3.2] the maps $\psi^0_\lambda$ in the definition of nuclear dimension may be chosen so that the induced maps $\phi^i$ are of order 0.

The equation (2) is a consequence of Definition 1.1 (ii).

Let us show that the maps $\phi^i : \prod_\lambda F_i^0 / \bigoplus_\lambda F_i^0 \to A_\Lambda$, induced by the order 0 maps $\phi^i$, also have order 0. It is clear that $(\phi^0_\lambda) : \prod_\lambda F_i^0 \to \prod_\lambda A$ has order 0. Hence, $\alpha \circ (\phi^i_\lambda) : \prod_\lambda F_i^0 \to A_\Lambda$, where $\alpha$ is the quotient onto $A_\Lambda$, has order 0. We will be done once we show that if $\phi : C \to D$ is a c.p.c. map of order 0 and $\phi|_I = 0$ for some closed two-sided ideal $I$, then the induced map $\tilde{\phi} : C/I \to D$ has order 0. By [9, Cor. 4.1], there is a *-homomorphism $\pi : C \otimes C_0(0, 1) \to D$ such that $\phi(c) = \pi(c \otimes t)$ for all $c \in C$. From $\pi(I \otimes t) = 0$ we get that $\pi(I \otimes C_0(0, 1)) = 0$. Thus, $\pi$ induces a *-homomorphism $\tilde{\pi} : C/I \otimes C_0(0, 1) \to D$. Since $\tilde{\phi}(c) = \tilde{\pi}(c \otimes t)$ for all $c \in C/I$, $\tilde{\phi}$ has order 0.

An ordered semigroup $S$ is called unperforated if $kx \leq ky$ for $x, y \in S$ and $k \in \mathbb{N}$ implies $x \leq y$.

In the following two lemmas the index $\lambda$ ranges through an upward directed set $\Lambda$.

Lemma 2.3. (i) If $Cu(A)$ is unperforated then so is $Cu(A/I)$ for any closed two-sided ideal $I$.

(ii) If $A_\lambda$ are $C^*$-algebras such that $Cu(A_\lambda)$ is unperforated for all $\lambda$ then so are $Cu(\prod_\lambda A_\lambda)$ and $Cu(\prod_\lambda A_\lambda / \bigoplus_\lambda A_\lambda)$.

Proof. (i) Let $[\tilde{a}], [\tilde{b}] \in Cu(A/I)$ be such that $k[\tilde{a}] \leq [\tilde{b}]$ for some $k \in \mathbb{N}$. Then for $[a], [b] \in Cu(A)$ lifts of $[\tilde{a}]$ and $[\tilde{b}]$ we have

$$k[a] \leq k[b] + [i] \leq k([b] + [i])$$

for some $i \in (I \otimes \mathcal{K})^+$ (by [1, Prop. 1]). Since $Cu(A)$ is unperforated, we have $[a] \leq [b] + [i]$, and passing to $Cu(A/I)$ we get that $[\tilde{a}] \leq [\tilde{b}]$.

(ii) Let $(a_\lambda)_{\lambda}, (b_\lambda)_{\lambda} \in (\prod_\lambda A_\lambda) \otimes \mathcal{K} \subseteq \prod_\lambda (A_\lambda \otimes \mathcal{K})$ be positive elements of norm at most 1 such that $k[(a_\lambda)\lambda] \leq k[(b_\lambda)\lambda]$ for some $k$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$k[((a_\lambda - \varepsilon)_+)_\lambda] \leq k[((b_\lambda - \delta)_+)_\lambda].$$

Hence, $k[(a_\lambda - \varepsilon)_+] \leq k[(b_\lambda - \delta)_+]$ for all $\lambda$. Since $Cu(A_\lambda)$ is unperforated, $[(a_\lambda - \varepsilon)_+] \leq [(b_\lambda - \delta)_+]$. Let $x_\lambda \in A_\lambda \otimes \mathcal{K}$ be such that

$$(a_\lambda - 2\varepsilon)_+ = x_\lambda^* x_\lambda \text{ and } x_\lambda x_\lambda^* \in \text{her}(b_\lambda - \delta)_+).$$
Since \(\|x_\lambda\|^2 \leq \|a_\lambda\| \leq 1\), we have \((x_\lambda)\lambda \in \prod_\lambda (A_\lambda \otimes K)\). We also have \(x_\lambda x_\lambda^* \leq \frac{1}{2} b_\lambda\), whence
\[
((a_\lambda - 2\varepsilon)_+) = (x_\lambda)\lambda (x_\lambda)\lambda \text{ and } (x_\lambda)\lambda (x_\lambda)\lambda^* \in \text{her}(b_\lambda)\lambda).
\]
But \((\prod_\lambda A_\lambda) \otimes K\) sits as a closed hereditary subalgebra of \(\prod_\lambda (A_\lambda \otimes K)\). Therefore, \((x_\lambda)\lambda \in (\prod_\lambda A_\lambda) \otimes K\) and so \([(a_\lambda - 2\varepsilon)_+) \leq [(b_\lambda)\lambda]\). Since \(\varepsilon\) may be arbitrarily small, we have \([(a_\lambda)\lambda] \leq [(b_\lambda)\lambda]\) as desired. \(\square\)

**Lemma 2.4.** Let \(a, b \in (A \otimes K)^+\). If \(\text{Cu}(\iota)([a]) \leq \text{Cu}(\iota)([b])\) in \(\text{Cu}(A_\Lambda)\), then \([a] \leq [b]\).

**Proof.** Let \(\varepsilon > 0\). Since \([\iota(a)] \leq [\iota(b)]\), there is \(d \in A_\Lambda \otimes K\) such that \(d^* \iota(b)d = \iota(a - \varepsilon)_+\). That is, there are \(d_\lambda \in A_\lambda \otimes K\) such that \(d_\lambda^* bd_\lambda \to (a - \varepsilon)_+\). Thus, \([(a - \varepsilon)_+] \leq [b]\). Since \(\varepsilon > 0\) may be arbitrarily small, we have \([a] \leq [b]\). \(\square\)

**Remark 2.5.** In proving Theorem 1.3, a stronger property than \(n\)-comparison will be shown to hold for \(C^*\)-algebras of nuclear dimension \(n\): if \(x, y_i \in \text{Cu}(A)\) with \(i = 0, 1, \ldots, n\) satisfy that \(k_i x \leq k_i y_i\) for some \(k_i \in \mathbb{N}\) and all \(i\), then \(x \leq \sum_{i=0}^n y_i\). This property, unlike \(n\)-comparison, does not seem to have a formulation in terms of comparison by lower semicontinuous 2-quasitraces.

**Proof of Theorem 1.3.** Suppose that there are \(k_i \in \mathbb{N}\) such that \(k_i [a] \leq k_i [b_i]\) for \(i = 0, 1, \ldots, n\). Since c.p.c. order 0 maps preserve Cuntz comparison (by \([9, \text{Cor. 4.5}]\)), we have that \(k_i [\psi^i(a)] \leq k_i [\psi^i(b_i)]\) for all \(i\). Since the Cuntz semigroup of finite dimensional algebras is unperforated, we have by Lemma 2.3 that the Cuntz semigroup of \(\prod_\lambda F^i_\lambda / \bigoplus_\lambda F^i_\lambda\) is unperforated. Thus, \([\psi^i(a)] \leq [\psi^i(b_i)]\). The maps \(\phi^i\) preserve Cuntz equivalence (since they are c.p.c. of order 0), whence
\[
[\phi^i \psi^i(a)] \leq [\phi^i \psi^i(b_i)] \leq \left[\sum_{j=0}^n \phi^i \psi^j(b_i)\right] = [\iota(b_i)].
\]
So,
\[
[\iota(a)] = \left[\sum_{i=0}^n \phi^i \psi^i(a)\right] \leq \sum_{i=0}^n [\phi^i \psi^i(a)] \leq \sum_{i=0}^n [\iota(b_i)].
\]
By Lemma 2.4, this implies that \([a] \leq \sum_{i=0}^n [b_i]\). \(\square\)

**3. Stability of \(C^*\)-algebras**

A stable \(C^*\)-algebra has no nonzero unital quotients and no nonzero bounded 2-quasitraces (see (i)\(\Rightarrow\)(ii) of \([6, \text{Prop. 4.7}]\)). In \([6, \text{Prop. 4.7}]\) Ortega, Perera and Rørdam show that the converse is true provided that the \(C^*\)-algebra is \(\sigma\)-unital and its Cuntz semigroup has the \(n\)-comparison property. This, combined with Theorem 1.3 and the fact that for exact \(C^*\)-algebras bounded 2-quasitraces are traces, implies that a \(\sigma\)-unital \(C^*\)-algebra of finite nuclear dimension is stable if and only if it has no nonzero unital quotients and no nonzero bounded traces. Ortega, Perera, and Rørdam also show that \(\omega\)-comparison, a weakening of \(n\)-comparison, suffices to obtain the same stability criterion.

Definition 3.1. (c.f. [6, Def. 2.11]) Let $S$ be an ordered semigroup closed under passage to suprema of increasing sequences. Then $S$ has the $\omega$-comparison property if $x \leq y_i$ for $x, y_i \in S$ and $i = 0, 1, \ldots$ implies $x \leq \sum_{i=0}^{\infty} y_i$.

Remark 3.2. The definition of $\omega$-comparison given above differs slightly from the definition given in [6]. Nevertheless, both definitions agree for ordered semigroups in the category $\text{Cu}$ introduced in [2], and therefore, also for ordered semigroups arising as Cuntz semigroups of C*-algebras.

A notion of nuclear dimension at most $\omega$ may be modelled after the statement of Proposition 2.2.

Definition 3.3. Let us say that a C*-algebra $A$ has nuclear dimension at most $\omega$ if for $i = 0, 1, 2, \ldots$ there are nets of c.p.c. maps $\psi_i^\lambda : A \to F_i^\lambda$ and $\phi_i^\lambda : F_i^\lambda \to A$, with $F_i^\lambda$ finite dimensional C*-algebras and $\lambda \in \Lambda$, such that

(ii) the induced maps $\psi^\lambda : A \to \prod_\Lambda F_i^\lambda / \bigoplus_\Lambda F_i^\lambda$ and $\phi^\lambda : \prod_\Lambda F_i^\lambda / \bigoplus_\Lambda F_i^\lambda \to A_\lambda$ are c.p.c. order 0,

(iii) $\iota(a) = \sum_{i=0}^{\infty} \phi^i \psi^i(a)$ for all $a \in A$ (where the series on the right side is understood to be convergent in the norm topology).

For example, if the C*-algebras $(A_i)_{i=0}^{\infty}$ all have finite nuclear dimension, then $\bigoplus_{i=0}^{\infty} A_i$ has nuclear dimension at most $\omega$. It is not clear whether the assumption that the maps $\psi_i^\lambda$ be asymptotically order 0 may be dropped in Definition 3.3 (and then proved), or if the other results on finite nuclear dimension proved in [8] also hold for nuclear dimension at most $\omega$. In particular, is it true that the property of nuclear dimension at most $\omega$ passes to closed hereditary subalgebras?

The proof of Theorem 1.3 goes through, mutatis mutandis, for nuclear dimension at most $\omega$. We thus have

Theorem 3.4. If $A$ has nuclear dimension at most $\omega$ then $\text{Cu}(A)$ has the $\omega$-comparison property.

Combined with the results of [6], Theorem 3.4 yields the following corollary, which improves on [5, Thm. 0.1], [6, Cor. 4.9] and [6, Cor. 5.12].

Corollary 3.5. Let $A$ be a C*-algebra of nuclear dimension at most $\omega$ and let $B \subseteq A \otimes \mathcal{K}$ be hereditary and $\sigma$-unital. Then $B$ is stable if and only if it has no nonzero unital quotients and no nonzero bounded traces. $B$ has the corona factorization property.

Proof. By Theorem 3.4, $\text{Cu}(A)$ has the $\omega$-comparison property. Since $\text{Cu}(B)$ is an ordered subsemigroup of $\text{Cu}(A)$ (i.e., the order and addition operation on $\text{Cu}(B)$ agree with the ones induced by its inclusion in $\text{Cu}(A)$) the $\omega$-comparison property holds in $\text{Cu}(B)$, too. Hence, by [6, Prop. 4.7], $B$ is stable if and only if it has no nonzero unital quotients and no nonzero bounded traces. (In the hypotheses of [6, Prop. 4.7] the C*-algebra $A$ is assumed to be separable. A closer look into the proof of this result reveals that it suffices to assume that the hereditary subalgebra $B \subseteq A \otimes \mathcal{K}$ be $\sigma$-unital. This justifies the application of [6, Prop.4.7] made here.)
Let $C \subseteq B \otimes K$ be full and hereditary, and suppose that $M_n(C)$ is stable. Then $M_n(C)$, and consequently $C$, cannot have nonzero unital quotients or bounded traces. Thus $C$ is stable. This shows that $B$ has the corona factorization property (see [4, Thm. 4.2]).

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REFERENCES


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