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An enrichment of KK-theory over the category of symmetric spectra

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Dedicated to Joachim Cuntz on the occasion of his 60th birthday

Abstract. In [6] Higson showed that the formal properties of the Kasparov KK-theory groups are best understood if one regards $KK(A,B)$ for separable C^*-algebras A, B as the morphism set of a category KK. In category language the composition and exterior KK-product give KK the structure of a symmetric monoidal category which is enriched over abelian groups. We show that the enrichment of KK can be lifted to an enrichment over the category of symmetric spectra.

1. Introduction

A fundamental tool in Index theory and in the theory of C^*-algebras is Kasparov's bivariant K-theory which associates to C^*-algebras A, B an abelian group $KK(A,B)$ which is contravariant in A and covariant in B. Central to Kasparov's theory is the construction of a product

$$KK(A_1, C_1 \otimes B) \otimes KK(A_2 \otimes B, C_2) \to KK(A_1 \otimes A_2, C_1 \otimes C_2), \tag{1.1}$$

which contains and generalizes a number of constructions from K-theory and index theory. For example, if π is a discrete group with classifying space $B\pi$ and reduced C^*-algebra $C^*\pi$, and we set $A_1 = A_2 = \mathbb{R}$, $B = C_0(B\pi)$, $C_1 = C^*\pi$, $C_2 = C_0(\mathbb{R}^n)$, then the Kasparov product with the so-called Mischenko-Fomenko element $\nu \in KK(\mathbb{R}, C^*\pi \otimes C_0(B\pi))$ gives a homomorphism

$$KO_n(B\pi) = KK(C_0(B\pi), C_0(\mathbb{R}^n)) \to KK(\mathbb{R}, C^*\pi \otimes C_0(\mathbb{R}^n)) = KO_n(C^*\pi), \tag{1.2}$$

which is known as the *assembly map* in (real) K-theory.

As Higson explains very nicely in [6], the plethora of formal properties of the Kasparov product (1.1) is best organized by thinking of an element of

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$KK(A,B)$ as a "generalized" $*$-homomorphism from A to B. Specializing the Kasparov product (1.1), there is a product

$$KK(A,B) \otimes KK(B,C) \longrightarrow KK(A,C), \tag{1.3}$$

to be thought of as "composition" of generalized homomorphisms, and a product

$$KK(A_1,A_2) \otimes KK(C_1,C_2) \longrightarrow KK(A_1 \otimes C_1, A_2 \otimes C_2) \tag{1.4}$$

that is interpreted as "tensor product" of generalized homomorphisms. The general Kasparov product (1.1) of $f \in KK(A_1, C_1 \otimes B)$ and $g \in KK(A_2 \otimes B, C_2)$ is then given by the composition

$$A_1 \otimes A_2 \xrightarrow{f\otimes 1} C_1 \otimes B \otimes A_2 \xrightarrow{1 \otimes g} C_1 \otimes C_2.$$

In particular, the general Kasparov product (1.1) can be expressed in terms of the composition product (1.3) and the external product (1.4). All the formal properties of these products can be nicely expressed by saying that there is a category KK with the following properties

- the objects of KK are the separable C^*-algebras
- the morphisms from A to B form the abelian group $KK(A,B)$, and the composition of morphisms is given by the product (1.3)
- KK is a symmetric monoidal category; the tensor product for objects is given by the (spatial) tensor product of C^*-algebras; the tensor product for morphisms is given by the product (1.4).
- there is a functor $C\colon C^* \to KK$ from the category of (separable) C^*-algebras to KK which is the identity on objects and which is compatible with the symmetric monoidal structure on both categories.

We note that the functor C gives for C^*-algebras A, B a map $C^*(A,B) \to KK(A,B)$, where $C^*(A,B)$ is the space of morphisms from A to B in the category C^* consisting of all $*$-homomorphisms from A to B. This map can be thought of as associating a "generalized" homomorphism to each $*$-homomorphism in a way that is compatible with composition product and external product.

We observe that KK is a category *enriched over the category of abelian groups* (also called an *preadditive category*) in the sense that the morphisms $KK(A,B)$ form an abelian group and that the composition law (1.3) is a homomorphism of abelian groups. The formal properties of the composition and external products can be expressed in the lingo of category theory by saying that KK is a symmetric monoidal preadditive category.

The main result of this paper is that the above statement can be "spectrified" in the following sense. Let $\mathbb{K} = \mathbb{K}(\mathbb{F})$ be the (real resp. complex) K-theory spectrum, which is a *commutative ring spectrum* in the world of symmetric spectra (see Section 7). As in the category of modules over a commutative ring, there is a product $M \wedge_{\mathbb{K}} N$ of $\mathbb{K}$-module spectra M, N, which is again a $\mathbb{K}$-module spectrum. This "smash" product over $\mathbb{K}$ gives the category $\mathbb{K}$-Mod of $\mathbb{K}$-module spectra the structure of a a symmetric monoidal category. Associating to a $\mathbb{K}$-module spectrum X its homotopy group $\pi_0(X)$ gives a functor

$$\pi_0 \colon \mathbb{K}\text{-Mod} \longrightarrow \text{Ab}$$

to the category of abelian groups which is compatible with the symmetric monoidal structure on these categories.

Theorem 1.5. *There is a symmetric monoidal category $\mathbb{KK}$ enriched over the category of $\mathbb{K}$-module spectra such that the symmetric monoidal preadditive category obtained from $\mathbb{KK}$ by applying the functor π_0 is KK.*

This result can be expressed in a less technical, but also less precise form in the following way.

Theorem 1.6. *For separable, $\mathbb{Z}/2$-graded C^*-algebras A and B there is a $\mathbb{K}$-module spectrum $\mathbb{KK}(A,B)$ and there are maps*

$$c\colon \mathbb{KK}(A,B) \wedge_{\mathbb{K}} \mathbb{KK}(B,C) \longrightarrow \mathbb{KK}(A,C) \tag{1.7}$$

$$m\colon \mathbb{KK}(A_1,A_2) \wedge_{\mathbb{K}} \mathbb{KK}(C_1,C_2) \longrightarrow \mathbb{KK}(A_1 \otimes A_2, C_1 \otimes C_2), \tag{1.8}$$

with the following properties:

(1) *there are isomorphisms*

$$\pi_0\mathbb{KK}(A,B) \cong KK(A,B)$$

for C^*-algebras A, B.

(2) *the map* (1.7) *induces on π_0 the composition product* (1.3).

(3) *the map* (1.8) *induces on π_0 the external product* (1.4).

(4) *The products* (1.7) *and* (1.8) *satisfy a collection of (quite natural) associativity and compatibility conditions (which are spelled out in the definition of an enriched symmetric monoidal category in Section 6).*

Here $\pi_0\mathbb{KK}(A,B)$ is the zeroth homotopy group of the symmetric spectrum $\mathbb{KK}(A,B)$ (cp. Definition 7.8). Applying a criterion of Hovey, Shipley and Smith (the second part of their Proposition 5.6.4), we see that $\mathbb{KK}(A,B)$ is a *semistable* symmetric spectrum, which in turn implies that $\pi_0(\mathbb{KK}(A,B))$ can be identified with $[S,\mathbb{KK}(A,B)]$, the group of homotopy classes of spectrum maps from the sphere spectrum to $\mathbb{KK}(A,B)$ (the morphisms from S to $\mathbb{KK}(A,B)$ in the associated "homotopy category"). Since $\mathbb{KK}(A,B)$ is a $\mathbb{K}$-module spectrum, we may identify $[S,\mathbb{KK}(A,B)]$ with the group $[\mathbb{K},X]_{\mathbb{K}}$ of homotopy classes of $\mathbb{K}$-module maps. In particular, the composition product map c of (1.7) induces a homomorphism of abelian groups

$$[\mathbb{K},\mathbb{KK}(A,B)]_{\mathbb{K}} \otimes [\mathbb{K},\mathbb{KK}(B,C)]_{\mathbb{K}} \longrightarrow [\mathbb{K},\mathbb{KK}(A,C)]_{\mathbb{K}}$$

given by sending the tensor product of maps $f\colon \mathbb{K} \longrightarrow \mathbb{KK}(A,B)$ and $g\colon \mathbb{K} \longrightarrow \mathbb{KK}(B,C)$ to the composition

$$\mathbb{K} = \mathbb{K} \wedge_{\mathbb{K}} \mathbb{K} \xrightarrow{f\wedge g} \mathbb{KK}(A,B) \wedge_{\mathbb{K}} \mathbb{KK}(B,C) \xrightarrow{c} \mathbb{KK}(A,C).$$

The claim in (2) is that this pairing is equal to the composition pairing (1.3) via the identification $[\mathbb{K},\mathbb{KK}(A,B)]_{\mathbb{K}} = \pi_0(\mathbb{KK}(A,B)) = KK(A,B)$ (the second equality is provided by part (1) of the theorem). Part (3) of the theorem is completely analogous.

We like to think of this result as a "dictionary" that allows to translate between KK-groups—the central objects in index theory and the theory of C^*-algebras—and maps between spectra, the primary objects in stable homotopy theory. We want to illustrate this in two simple examples.

We note that for any (separable) C^*-algebra B we have the $\mathbb{K}$-module spectrum $\mathbb{K}B \stackrel{\text{def}}{=} \mathbb{KK}(\mathbb{F}, B)$, whose homotopy group $\pi_n(\mathbb{K}B)$ is the K-theory group $K_n(B)$. Taking the adjoint of the map c in the special case $A = \mathbb{F}$, we obtain a $\mathbb{K}$-module map

$$\mathbb{KK}(B, C) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}B, \mathbb{K}C).$$

The associativity properties of c imply that this map provides us with a functor

$$\mathbb{KK} \longrightarrow \mathbb{K}\text{-Mod}$$

of categories enriched over the category of $\mathbb{K}$-module spectra; here the objects of $\mathbb{K}$-Mod are the $\mathbb{K}$-module spectra, and the morphisms from X to Y is the $\mathbb{K}$-module spectrum $\mathrm{Hom}_{\mathbb{K}}(X, Y)$. Moreover, this functor is a functor of symmetric monoidal categories, where the monoidal structure in $\mathbb{K}$-Mod is given by the smash product over $\mathbb{K}$. Passing to the homotopy category, the functor gives on morphisms a homomorphism

$$KK(B, C) = \pi_0(KK(B, C)) \longrightarrow \pi_0(\mathrm{Hom}_{\mathbb{K}}(\mathbb{K}B, \mathbb{K}C)) = [\mathbb{K}B, \mathbb{K}C]_{\mathbb{K}},$$

which is compatible with the composition and the external product on both sides (such a map compatible with composition was constructed before by Schlichting [10, 15]). Summarizing, we see that this construction produces from KK-elements (the kind of objects index theory people play with) maps between symmetric module spectra (the toys of stable homotopy theorists); moreover, the composition resp. external product of KK-elements corresponds to the composition resp. smash product of module maps.

This shows that *any* map between K-theory groups that has been produced by constructing certain KK-elements is induced by a map of the corresponding K-theory spectra, simply by replacing all KK-elements by the corresponding maps between spectra, and all Kasparov products by the appropriate compostions/smash products of these maps. To illustrate this, let us construct the map of $\mathbb{K}$-theory spectra inducing the assembly map (1.2) (of course the fact that the assembly map comes from a map of spectra is well-known).

We recall that the assembly map is given by the Kasparov product with the Fomenko-Mischenko element $\nu \in KK(\mathbb{R}, C^*\pi \otimes C_0(B\pi)) = [\mathbb{K}, \mathbb{K}(C^*\pi \otimes C_0(B\pi))]_{\mathbb{K}}$ (assume that $B\pi$ is compact for simplicity). Identifying ν with (the homotopy class of) a $\mathbb{K}$-module map we obtain a map of $\mathbb{K}$-module spectra

$$\begin{aligned} &\mathbb{K} \wedge B\pi_+ \xrightarrow{\simeq} \mathbb{K} \wedge_{\mathbb{K}} \mathbb{KK}(C_0(B\pi_+), \mathbb{R}) \longrightarrow \\ &\qquad \mathbb{KK}(\mathbb{R}, C_0(B\pi_+) \otimes C^*\pi) \wedge_{\mathbb{K}} \mathbb{KK}(C_0(B\pi_+) \otimes C^*\pi, C^*\pi) \xrightarrow{c} \mathbb{K}(C^*\pi) \end{aligned}$$

whose induced map on homotopy groups

$$A_*\colon \pi_n(\mathbb{K}\wedge B\pi_+) = KO_n(B\pi) \to \pi_n(\mathbb{K}(C^*\pi)) = KO_n(C^*\pi)$$

is the assembly map. One advantage of this spectrum level version of the assembly map is that it allows us to "introduce coefficients"; this is an important move, since it turns out that for a finite group π the assembly map is essentially trivial, while it is an *isomorphism* with coefficients in $\mathbb{Q}/\mathbb{Z}$.

It should be emphasized that the point of the paper is a translation between the Kasparov product on the one hand and the composition/smash product of maps between $\mathbb{K}$-module spectra on the other hand; in particular, no new operator theoretic statement is obtained—except a generalization of Higson's axiomatic characterization of the Kasparov product to the case of $\mathbb{Z}/2$-graded C^*-algebras.

This paper is organized as follows. In Section 2 we describe the elements of $KK(A,B)$ following Cuntz as "generalized" $*$-homomorphisms from A to B. In Section 3 we review Higson's axiomatic characterization of the KK-groups. This leads in particular to a uniqueness statement concerning the composition and tensor product of KK-elements. In Sections 4 (resp. 5) we describe the composition (resp. tensor product) of KK-elements in the Cuntz picture. The definition of an "enriched" category is reviewed in Section 6 (for objects A, B in an "enriched" category $\mathcal{D}$, the morphisms $\mathcal{D}(A,B)$ from A to B form not just a set, but have more structure: $\mathcal{D}(A,B)$ could be a topological space, an abelian group, or—more generally—an object in a symmetric monoidal category). In Section 7 we define symmetric spaces and spectra. In Sections 8 and 9 we describe a recipe to produce (symmetric monoidal) categories which are enriched over the category of symmetric spaces (or spectra). In Section 10 we apply this recipe to produce the category $\mathbb{KK}$ (which is enriched over the category of symmetric spectra meaning that for objects A, B in this category, the morphisms from A to B constitute a symmetric spectrum). Finally, in Section 11, we extend these results to $\mathbb{Z}/2$-graded C^*-algebras.

For simplicity we shall assume that all C^*-algebras considered in the following are separable.

2. The Cuntz picture of KK-theory

There are basically three descriptions of the abelian groups $KK(A,B)$:

- The original definition of Kasparov [8], where $KK(A,B)$ is defined as the set of equivalence classes of "Kasparov $A-B$-bimodules";
- The "Cuntz picture", where $KK(A,B)$ consists of homotopy classes of "quasi-homomorphisms" from A to B;
- Higson's axiomatic characterization of $KK(A,B)$.

This paper is based on the Cuntz picture of KK-theory; basically the spectrum $\mathbb{KK}(A,B)$ is build from spaces of quasi-homomorphisms. As a fairly direct consequence of the construction we obtain an isomorphism of *sets*

$$\pi_0\mathbb{KK}(A,B) \longleftrightarrow KK(A,B).$$

Then it will be convenient for us to use Higson's axiomatic characterization as a tool to check that our map is compatible with the group structure, the composition product and the external product on both sides.

2.1. Quasi-homomorphisms. Let A, B be C^*-algebras. A *quasi-homomorphism* from A to B is a $*$-homomorphism

$$f\colon qA \longrightarrow \mathcal{K} \otimes B.$$

Here

- qA is the *Cuntz algebra*, a C^*-algebra functorially associated to A defined below, and
- $\mathcal{K}$ is the C^*-algebra of compact operators on a fixed (separable) Hilbert space H, and $\mathcal{K} \otimes B$ is the (spatial) tensor product of C^*-algebras [11, Chapter T].

We note that there is a "stabilization map"

$$C^*(A, B) \longrightarrow C^*(qA, \mathcal{K} \otimes B) \qquad f \mapsto (e \otimes 1_B) \circ f \circ \pi_0 \tag{2.2}$$

from $*$-homomorphisms to quasi-homomorphisms. Here $\pi_0\colon qA \to A$ is a $*$-homomorphism defined in 2.5 below, $e\colon \mathbb{F} \to \mathcal{K}$ is the $*$-homomorphism which sends the unit to a fixed rank one projection operator, 1_B is the identity on B, and $e \otimes 1_B\colon \mathcal{K} = \mathbb{F} \otimes B \to \mathcal{K} \otimes B$ is their tensor product.

In the "Cuntz picture" $KK(A, B)$ is defined as

$$KK(A, B) \overset{\text{def}}{=} [qA, \mathcal{K} \otimes B], \tag{2.3}$$

where $[qA, \mathcal{K} \otimes B]$ denotes the homotopy classes of $*$-homomorphisms from qA to $\mathcal{K} \otimes B$; in other words, $KK(A, B)$ is defined as the set of homotopy classes of quasi-homomorphisms from A to B.

The Cuntz algebra is an ideal in the free product $A * A$ of two copies of A. Before defining the Cuntz algebra, we will recall the construction of the free product of C^*-algebras.

2.4. Free product of C^*-algebras. Let A_i, $i \in I$ be a family of C^*-algebras. Then the free product $\mathop{*}_{i \in I} A_i$ is a C^*-algebra which is the coproduct of the A_i's in the category of C^*-algebras; i.e., there are $*$-homomorphisms $\iota_i\colon A_i \to \mathop{*}_{i \in I} A_i$, such that for any C^*-algebra B the map

$$C^*(\mathop{*}_{i \in I} A_i, B) \to \prod_{i \in I} C^*(A_i, B) \qquad f \mapsto f \circ \iota_i$$

is a bijection. The construction of the free product $\mathop{*}_{i \in I} A_i$ is reminiscent of the construction of the free product of groups and goes as follows. Consider "words"

$$a_1 a_2 \dots a_m \qquad a_i \in \bigcup_{i \in I} A^i,$$

whose "letters" $a_1, \dots, a_m$ are elements of the A_is. Here we identify a word

$$a_1 \dots a_{k-1} a_k a_{k+1} a_{k+2} \dots a_m \qquad \text{with} \qquad a_1 \dots a_{k-1} a a_{k+2} \dots a_m$$

if a_k and a_{k+1} belong to the same algebra A^i, and $a = a_k \cdot a_{k+1} \in A^i$. We define the *algebraic free product* $\ast_{i\in I}^{\mathrm{alg}} A_i$ to be the vector space of finite linear combinations of such words. This is a $*$-algebra with multiplication given by concatenation of words and anti-involution $*$ given by $(a_1 \dots a_n)^* = a_n^* \dots a_1^*$. The *free product* $\ast_{i\in I} A^i$ is the completion of $\ast_{i\in I}^{\mathrm{alg}} A_i$ with respect to the maximal C^*-norm

$$||z||_{\max} = \sup_{\pi} \{||\pi(z)||\}$$

Here the supremum is taken over all $*$-homomorphisms $\pi\colon \ast_{i\in I}^{\mathrm{alg}} A_i \to \mathcal{B}(H)$ to the C^*-algebra of bounded operators on some Hilbert space H (this is finite since $||a_1 \dots a_n||_{\max} \leq ||a_1|| \cdot \dots \cdot ||a_n||$).

2.5. The Cuntz algebra. The Cuntz algebra qA associated to a C^*-algebra A is an ideal in the free product $QA \overset{\mathrm{def}}{=} A * A$ of two copies of A. To describe elements in QA it is convenient to write $QA = A^1 * A^2$, where the superscripts are used to distinguish the two copies of A in QA. In particular for each $a \in A$, there are two 1-letter words, namely a^1 and a^2, where the superscript indicates which copy of A the letter a comes from. The Cuntz algebra is the closed two-sided ideal in QA generated by the elements

$$q(a) \overset{\mathrm{def}}{=} a^1 - a^2 \qquad a \in A.$$

Let π_0 be the C^*-homomorphism

$$\pi_0\colon QA \to A \qquad a^1 \mapsto a, \quad a^2 \mapsto 0.$$

Abusing notation, we will also write $\pi_0\colon qA \to A$ for the restriction of π_0 to $qA \subset QA$.

2.6. The group structure on $KK(A, B)$. A choice of an isomorphism $H \oplus H \cong H$ determines a C^*-homomorphism $\mathcal{K} \oplus \mathcal{K} \to \mathcal{K}$, which is then used to define an "addition".

$$+\colon KK(A, B) \times KK(A, B) \longrightarrow KK(A, B),$$

where $f_1 + f_2$ is defined to be the composition

$$qA \xrightarrow{f_1 \oplus f_2} (\mathcal{K} \otimes B) \oplus (\mathcal{K} \otimes B) = (\mathcal{K} \oplus \mathcal{K}) \otimes B \longrightarrow \mathcal{K} \otimes B.$$

This gives $KK(A, B)$ the structure of an abelian group. Inverses are obtained by precomposition with the natural transposition on qA which is induced by interchanging the two copies of A in QA (cp. [3, p. 37]).

3. Higson's axiomatic characterization of KK-theory

3.1. Properties of $KK(A, B)$. The abelian groups $KK(A, B)$ have a number of functorial properties; in particular, when considered in conjunction with the composition product (1.3) and the external product (1.4). Fortunately, it turns out that all the other properties can be recovered from the following three "basic" properties.

Let us fix a C^*-algebra A. Then we can consider $B \mapsto KK(A, B)$ as a covariant functor

$$F \overset{\text{def}}{=} KK(A, -): C^* \longrightarrow \text{Ab}$$

from the category of C^*-algebras to the category of abelian groups. This functor has the following three properties:

(i) **(Homotopy Invariance)** If the $*$-homomorphisms $f, f': B \to B'$ are homotopic, then $F(f) = F(f')$.

(ii) **(Stability)** Let $\mathcal{K}$ be the C^*-algebra of compact operators, and let $e: \mathbb{F} \to \mathcal{K}$ be the $*$-homomorphism induced by the choice of a rank one projection. Then for any C^*-algebra B the induced homomorphism $F(e \otimes 1_B): F(B) \to F(\mathcal{K} \otimes B)$ is an isomorphism.

(iii) **(Split Exactness)** The functor F applied to a split exact sequence of C^*-algebras gives a split exact sequence (of abelian groups).

We note that the first two properties of $F(B) = [qA, \mathcal{K} \otimes B]$ follow quite directly from the definition.

The following result of Higson gives an axiomatic characterization of the abelian groups $KK(A, B)$.

Theorem 3.2 ([6]). *Given a functor F from C^* to the category of abelian groups with the above properties, and an element $x \in F(A)$, then there exists a unique natural transformation $\alpha: KK(A, -) \to F$ such that $\alpha_A(1_A) = x$.*

We recall that a natural transformation $\alpha: KK(A, -) \to F$ consists of a collection of homomorphisms $\alpha_B: KK(A, B) \to F(B)$, one for each C^*-algebra B which are compatible with induced maps in the sense that for every $*$-homomorphism $f: B \to C$ the following diagram commutes:

$$\begin{array}{ccc} KK(A,B) & \xrightarrow{\alpha_B} & F(B) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ KK(A,C) & \xrightarrow{\alpha_C} & F(C) \end{array}$$

3.3. Addendum. The uniqueness statement in the above theorem can be strengthened: any natural transformation $\alpha: KK(A, \quad) \to F$ between these functors considered as functors with values in the category of *sets* (i.e., the α_Bs are not required to be group homomorphisms) is *automatically* a natural transformation of groups. This is a byproduct of the proof of Theorem 3.2.

Higson proved Theorem 3.2 using the Kasparov definition of $KK(A, B)$ [6, Theorem 4.8]. Using the fact that the Cuntz groups $[qA, \mathcal{K} \otimes B]$ are isomorphic to the Kasparov groups (via a natural transformation), this of course implies the result above. However, since a direct proof is fairly straightforward and might shed a light on the construction of the Cuntz algebra qA, we will prove theorem 3.2.

Proof of Theorem 3.2. The main ingredient of the proof is the result due to Cuntz [3, Prop. 3.1(b)] that for any homotopy invariant, stable, split-exact functor

$$F\colon C^* \to \mathrm{Ab}$$

the induced map

$$(\pi_0)_*\colon F(qA) \to F(A)$$

is an isomorphism. Suppose that $\alpha\colon KK(A,-) \to F$ is a natural transformation (of *set-valued* functors) with $\alpha_A(1_A) = x \in F(A)$. Let $f\colon qA \to \mathcal{K}\otimes B$ be a $*$-homomorphism and let $[f] \in [qA, \mathcal{K}\otimes B] = KK(A,B)$ be its homotopy class. To show that $\alpha([f]) \in F(B)$ is determined by x consider the following commutative diagram

$$\begin{array}{ccc}
KK(A,A) & \xrightarrow{\alpha_A} & F(A) \\
{\scriptstyle (\pi_0)_*}\uparrow{\scriptstyle \cong} & & {\scriptstyle (\pi_0)_*}\uparrow{\scriptstyle \cong} \\
KK(A,qA) & \xrightarrow{\alpha_{qA}} & F(qA) \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\
KK(A,\mathcal{K}\otimes B) & \xrightarrow{\alpha_{\mathcal{K}\otimes B}} & F(\mathcal{K}\otimes B) \\
{\scriptstyle (e\otimes 1)_*}\uparrow{\scriptstyle \cong} & & {\scriptstyle (e\otimes 1)_*}\uparrow{\scriptstyle \cong} \\
KK(A,B) & \xrightarrow{\alpha_B} & F(B)
\end{array}$$

It is easy to check that $[f] \in KK(A,B)$ is the image of $[1_A] \in KK(A,A)$ under the composition of the vertical homomorphisms on the left. Hence $\alpha_B([f])$ equals the image of $x \in F(A)$ under the composition of the vertical homomorphisms on the right. In particular, α_B is determined by $x \in F(B)$ via the formula

$$\alpha_B([f]) = (e\otimes 1)_*^{-1} \circ f_* \circ (\pi_0)_*^{-1}(x).$$

□

4. KK as a category

Following Cuntz [3] we will describe in this section how to "compose" (homotopy classes of) quasi-homomorphisms to obtain an associative bilinear product

$$\circ\colon KK(A,B) \times KK(B,C) \to KK(A,C). \tag{4.1}$$

This extends the usual composition of $*$-homomorphisms in the sense that the natural map

$$C\colon C^*(A,B) \to KK(A,B)$$

from $*$-homomorphisms to quasi-homomorphisms sends the composition $f \circ g$ of two $*$-homomorphisms to the composition $C(f) \circ C(g)$ of the corresponding quasi-homomorphisms. In other words, C is a functor

$$C\colon C^* \to KK,$$

where

- the objects of the category C^* are the C^*-algebras, and the morphisms from A to B are the $*$-homomorphisms;
- the objects of the category KK are the C^*-algebras, and the morphisms from A to B are (homotopy classes of) quasi-homomorphisms, i.e., elements of $KK(A, B)$.

For our purposes of defining a composition of (homotopy classes of) quasi-homomorphisms it is convenient to replace $[qA, \mathcal{K} \otimes B]$ by $\lim_n [q^n A, \mathcal{K}^n \otimes B]$, where $\mathcal{K}^n = \mathcal{K} \otimes \cdots \otimes \mathcal{K}$ is the tensor product of n copies of $\mathcal{K}$, and the limit is taken with respect to the stabilization homomorphism

$$(4.2) \qquad [q^n A, \mathcal{K}^n \otimes B] \longrightarrow [q^{n+1} A, \mathcal{K}^{n+1} \otimes B] \qquad f \mapsto (e \otimes 1) \circ f \circ \pi_0.$$

By a result of Cuntz [3, Cor. 1.7(b)] this map is an *isomorphism* for $n \geq 1$. This allows us to identify from now on

$$KK(A, B) = [qA, \mathcal{K} \otimes B] \quad \text{with} \quad \lim_n [q^n A, \mathcal{K}^n \otimes B].$$

4.3. The composition product of quasi-homomorphisms. The advantage of working with the direct limit is that there is a composition product

$$(4.4) \qquad [q^m A, \mathcal{K}^m \otimes B] \times [q^n B, \mathcal{K}^n \otimes C] \xrightarrow{\circ} [q^{m+n} A, \mathcal{K}^{m+n} \otimes C],$$

compatible with the stabilization homomorphism (4.2) which induces the desired composition product (1.3) on KK-groups. This composition product is defined by sending a pair of maps $f\colon q^m A \to \mathcal{K}^m \otimes B$, $g\colon q^n B \to \mathcal{K}^n \otimes C$ to the composition

$$q^{m+n} A = q^m (q^n A) \xrightarrow{q^m f} q^m (\mathcal{K}^n \otimes B) \xrightarrow{\chi^{mn}} \mathcal{K}^n \otimes q^m B \xrightarrow{1 \otimes g} \mathcal{K}^n \otimes \mathcal{K}^m \otimes C = \mathcal{K}^{m+n} C.$$

To describe the $*$-homomorphism χ^{mn}, it is convenient to describe the iterated Cuntz algebra $q^n A$ directly in terms of A rather than just giving an iterative construction.

4.5. The iterated Cuntz algebra. We note that qA is a subalgebra of $A * A$, hence $q^2 A$ is a subalgebra of $qA * qA \subset A * A * A * A$, e.t.c. In general, $q^m A$ is an ideal in the free product of 2^m copies of A. These copies are conveniently parameterized by the 2^m subsets K of the set $M = \{1, \ldots, m\}$ (including $K = \varnothing$ and $K = M$). It turns out that the obvious action of the symmetric group Σ_m on the free product $\mathop{*}_{K \subset M} A^K$ leaves the ideal $q^m A \subset \mathop{*}_{K \subset M} A^K$ invariant thus inducing a Σ_m-action on $q^m A$; this action will play a central role in our construction of the symmetric spectrum $\mathbb{KK}(A, B)$. To keep track of this action, it will be convenient to slightly generalize the iterated Cuntz algebra $q^m A$ by constructing for any finite set M a C^*-algebra $q^M A$ such that

- $q^M A$ is isomorphic to $q^m A$ if M has cardinality m and
- $M \mapsto q^M A$ is a functor $\mathcal{M}^{op} \to C^*$ from the opposite of the category $\mathcal{M}$ of finite sets and injective maps to the category of C^*-algebras.

We will define the C^*-algebra $q^M A$ as an ideal of the C^*-algebra

$$Q^M A \stackrel{\text{def}}{=} \mathop{*}_{K \subset M} A^K$$

which is the free product of copies of A parameterized by the subsets $K \subset M$ (we use the superscript K in A^K to keep track of the various copies of A). This C^*-algebra is generated by elements a^K, where a is an element of A and K a subset of M. We define $q^M A \subset Q^m A$ to be the ideal generated by the elements

$$q^M(a) \stackrel{\text{def}}{=} \sum_{K\subset M} (-1)^{\#(K)} a^K, \qquad a \in A \tag{4.6}$$

where the sum is taken over all subsets of M (including M and the empty set), and $\#(K)$ is the cardinality of K.

If $j : N \to M$ is a morphism in $\mathcal{M}$ the corresponding map $q^j : q^M A \to q^N A$ is induced by the map

$$Q^j : Q^M A \to Q^N A \qquad a^K \longmapsto a^{j(K)} \tag{4.7}$$

Remark 4.8. It can be shown that the ideal $q^M A \subset Q^M A$ is the intersection $\bigcap_{e\in M} \ker \rho_e$, where ρ_e is the $*$-homomorphism

$$\rho_e \colon Q^M A \to Q^{M\setminus\{e\}} A \qquad a^K \mapsto a^{K\cap(M\setminus\{e\})}.$$

4.9. The $*$-homomorphisms Δ^{MN} and χ^{mn}. If C, D are C^*-algebras and $C \otimes D$ is their spatial tensor product (cp. [11, Appendix T]) and M, N are disjoint finite sets, we define a C^*-homomorphism

$$\Delta^{MN} \colon Q^{M\cup N}(C\otimes D) \to Q^M C \otimes Q^N D \qquad (c\otimes d)^K \mapsto c^{K\cap M} \otimes d^{K\cap N}. \tag{4.10}$$

We check that Δ^{MN} maps $q^{M\cup N}(C \otimes D) \subset Q^{M\cup N}(C \otimes D)$ to $q^M C \otimes q^N D \subset Q^M C \otimes Q^N D$:

(4.11)

$$\Delta^{MN}(q^{M\cup N}(c\otimes d)) = \Delta^{MN}\Big(\sum_{K\subset M\cup N} (-1)^{\#(K)} (c\otimes d)^K\Big)$$

$$= \sum_{K\subset M\cup N} (-1)^{\#(K)} c^{K\cap M} \otimes d^{K\cap N} \tag{4.12}$$

$$= \sum_{K'\subset M} \sum_{K''\subset N} (-1)^{\#(K')+\#(K'')} c^{K'} \otimes c^{K''} \tag{4.13}$$

$$= \left(\sum_{K'\subset M} (-1)^{\#(K')} c^{K'}\right) \otimes \left(\sum_{K''\subset N} (-1)^{\#(K'')} c^{K''}\right) \tag{4.14}$$

$$= q^M(c) \otimes q^N(d) \tag{4.15}$$

The $*$-homomorphism

$$q^m(\mathcal{K}^n \otimes B) \xrightarrow{\chi^{mn}} \mathcal{K}^n \otimes q^m B \tag{4.16}$$

is obtained by specializing Δ^{MN} to $M = \varnothing$, $N = \{1, \dots, m\}$, and $C = \mathcal{K}^n$, $D = B$.

5. KK as symmetric monoidal category

Let A_1, A_2, B_1, B_2 be C^*-algebras and let $A_1 \otimes A_2$, $B_1 \otimes B_2$ be their *spatial* (also called *minimal*) tensor product (cp. [11, Appendix T 5]). The tensor product of $*$-homomorphisms gives an associative product

$$(5.1)\quad C^*(A_1, B_1) \times C^*(A_2, B_2) \xrightarrow{\otimes} C^*(A_1 \otimes A_2, B_1 \otimes B_2) \qquad (f, g) \mapsto f \otimes g.$$

In this section we will extend this tensor product from $*$-homomorphisms to (homotopy classes of) quasi-homomorphisms to obtain an associative product

$$(5.2)\qquad KK(A_1, B_1) \times KK(A_2, B_2) \xrightarrow{\otimes} KK(A_1 \otimes A_2, B_1 \otimes B_2).$$

In the language of category theory, the tensor product of C^*-algebras and $*$-homomorphisms gives the category C^* an extra structure, namely that of a *symmetric monoidal category.* The axioms for a symmetric monoidal category (which we will recall below) basically encode all the compatibility conditions one might wish to impose between the composition of morphisms and their tensor product. Similarly, saying that the tensor product (5.2) makes the category KK a symmetric monoidal category expresses concisely all the compatibility conditions between the composition product in KK-theory and the tensor product in KK-theory. The compatibility between the products in C^* and the product in KK is expressed by saying that

$$C\colon C^* \to KK$$

is a functor of symmetric monoidal categories.

For the convenience of the reader we recall the definition of a symmetric monoidal category, since this will be a central notion used in the following sections. As mentioned above, this basically axiomatizes the compatibility conditions between composition and tensor product of morphisms. The technical complications come from the fact that for objects A, B, C in such a category the object $A \otimes B$ is not *equal* to, but just *isomorphic* to $B \otimes A$ (and similarly for $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$) and these isomorphisms must carefully be kept track of.

Definition 5.3. [2, 6.1.1] A *monoidal category* $\mathcal{C}$ consists of

(5.3.1) a category $\mathcal{C}$;
(5.3.2) a bifunctor $m : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(A, B) \mapsto A \otimes B$, called the tensor product;
(5.3.3) an object $I \in \mathcal{C}$, called the unit;
(5.3.4) for every triple A, B, C of objects an *associativity isomorphism*

$$a_{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C);$$

(5.3.5) for every object A a *left unit isomorphism* $l_A : I \otimes A \longrightarrow A$;
(5.3.6) for every object A a *right unit isomorphism* $r_A : A \otimes I \longrightarrow A$.

The structure isomorphisms must depend naturally on the objects involved. Moreover it is required that the following two diagrams commute for objects A, B, C, D

(5.3.A) *associativity coherence*

$$\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{a_{A\otimes B,C,D}} & (A \otimes B) \otimes (C \otimes D) \\
\Big\downarrow{\scriptstyle a_{ABC}\otimes 1} & & \\
(A \otimes (B \otimes C)) \otimes D & & \Big\downarrow{\scriptstyle a_{A,B,C\otimes D}} \\
\Big\downarrow{\scriptstyle a_{A,B\otimes C,D}} & & \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow{1\otimes a_{B,C,D}} & A \otimes (B \otimes (C \otimes D))
\end{array}$$

(5.3.U) *unit coherence*

$$\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{a_{AIB}} & A \otimes (I \otimes B) \\
{\scriptstyle r_A\otimes 1}\searrow & & \swarrow{\scriptstyle 1\otimes l_B} \\
& A \otimes B &
\end{array}$$

Definition 5.4. [2, 6.1.2] A monoidal category is called *symmetric* if in addition for every pair A, B there is a *symmetry isomorphism*

$$s_{AB} : A \otimes B \to B \otimes A$$

which depends naturally in A and B. The symmetry isomorphisms must be compatible with the other structure isomorphisms in the sense that the following two diagrams commute for any choice of objects A, B, C

(5.4.A) *associativity coherence*

$$\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{s_{AB}\otimes 1} & (B \otimes A) \otimes C \\
\Big\downarrow{\scriptstyle a_{ABC}} & & \Big\downarrow{\scriptstyle a_{BAC}} \\
A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\
\Big\downarrow{\scriptstyle s_{A,B\otimes C}} & & \Big\downarrow{\scriptstyle 1\otimes s_{AC}} \\
(B \otimes C) \otimes A & \xrightarrow{a_{BCA}} & B \otimes (C \otimes A)
\end{array}$$

(5.4.U) *unit coherence*

$$\begin{array}{ccc}
A \otimes I & \xrightarrow{s_{AI}} & I \otimes A \\
{\scriptstyle r_A}\searrow & & \swarrow{\scriptstyle l_A} \\
& A &
\end{array}$$

and in addition they must satisfy the *symmetry axiom*, i.e. for any two objects A, B the composite $s_{BA} \circ s_{AB}$ is the identity in $\mathcal{C}(A \otimes B, A \otimes B)$.

Example 5.5. The category Ab of abelian groups with the usual tensor product is a symmetric monoidal category (the unit I is the group $\mathbb{Z}$ and all the structure isomorphisms are the obvious ones).

Example 5.6. The category C^* of C^*-algebras over $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ with the spatial tensor product is a symmetric monoidal category (the unit I is the C^*-algebra $\mathbb{F}$ and again all the structure isomorphisms are the obvious ones).

Example 5.7. The category of pointed compactly generated weak Hausdorff spaces Top_* with the tensor product being the smash product and the well-known structure isomorphisms define a symmetric monoidal category. When working with the symmetric monoidal category Top_* various constructions (e.g. like taking suitable mapping spaces) yield weak Hausdorff spaces which are not compactly generated. However there is an idempotent functor which produces out of a given topology a coarsest topology which contains the given one and is compactly generated. We therefore will tactically assume that all pointed (weak Hausdorff) spaces we are considering are first hit by this functor, so we can regard them as objects in Top_*.

5.8. Tensor product of quasi-homomorphisms. Now we define a tensor product

$$[q^m A_1, \mathcal{K}^m \otimes B_1] \times [q^n A_2, \mathcal{K}^n \otimes B_2] \xrightarrow{\otimes} [q^{m+n}(A_1 \otimes A_2), \mathcal{K}^{m+n} \otimes B_1 \otimes B_2] \tag{5.9}$$

which sends a pair (f_1, f_2) of $*$-homomorphisms to the composition

$$\begin{aligned} q^{m+n}(A_1 \otimes A_2) &\xrightarrow{\Delta^{mn}} q^m A_1 \otimes q^n A_2 \\ &\xrightarrow{f_1 \otimes f_2} (\mathcal{K}^m \otimes B_1) \otimes (\mathcal{K}^n \otimes B_2) \xrightarrow{\nabla^{mn}} \mathcal{K}^{m+n} \otimes B_1 \otimes B_2. \end{aligned}$$

Here ∇^{mn} is the obvious $*$-isomorphism involving shuffling of the factors and the canonical isomorphism $\mathcal{K}^m \otimes \mathcal{K}^n \cong \mathcal{K}^{m+n}$; the $*$-homomorphism Δ^{mn} is equal to Δ^{MN} for $M = \{1, \dots, m\}$, $N = \{m+1, \dots, m+n\}$ using the identifications $q^m A = q^M A$, $q^n A = q^N A$, $q^{m+n} A = q^{M \cup N} A$.

We observe that this product in fact agrees with the tensor product of $*$-homomorphisms for $m = n = 0$. Moreover, this product is compatible with the stabilization homomorphism (4.2) and hence induces the desired tensor product (5.2) on KK-groups. (To see the latter one can argue as in [6, 4.7]).

6. Enriched categories

In the previous two sections we have investigated the category KK. By definition of a category for any two objects A, B one has a corresponding set of morphisms from A to B. We have seen that the morphism sets $KK(A, B)$ for C^*-algebras A and B are abelian groups, and that the composition is a bilinear map. In categorical language one would say that KK is an preadditive category. Alternatively one could say that the category KK is enriched over the symmetric monoidal category of abelian groups. Conceptually an enrichment of a category $\mathcal{D}$ over a symmetric monoidal category $\mathcal{C}$ is given by identifying the morphism sets $\mathcal{D}(A, B)$ with an object in $\mathcal{C}$ in such a way that the symmetric monoidal product of the category $\mathcal{C}$ can be used to describe the composition. Below we will see that KK also can be given an enrichment over the category of pointed spaces.

Before we define the precise definition of an enrichment we need to introduce the notion of an enriched category.

Definition 6.1. ([2, 6.2.1]) Let $\mathcal{C}$ be a monoidal category as defined in Definition 5.3. A $\mathcal{C}$-category $\mathcal{D}$ (or an *enriched category*) consists of the following data:

(6.1.1) a class of *objects* $|\mathcal{D}|$;
(6.1.2) for every pair of objects $A, B \in |\mathcal{D}|$ an object $\mathcal{D}(A, B) \in |\mathcal{C}|$.
(6.1.3) for every triple of objects $A, B, C \in |\mathcal{D}|$ a *composition morphism*

$$c_{ABC} : \mathcal{D}(A, B) \otimes \mathcal{D}(B, C) \longrightarrow \mathcal{D}(A, C);$$

(6.1.4) for every object $A \in |\mathcal{D}|$ a *unit morphism* $u_A : I \to \mathcal{D}(A, A)$,

and the structure maps are required to yield the following commutative diagrams for objects $A, B, C, D \in |\mathcal{D}|$

(6.1.A) *associativity coherence*

$$\begin{array}{ccc}
(\mathcal{D}(A,B)\otimes\mathcal{D}(B,C))\otimes\mathcal{D}(C,D) & \xrightarrow{c_{ABC}\otimes 1} & \mathcal{D}(A,C)\otimes\mathcal{D}(C,D) \\
\downarrow{\scriptstyle a_{\mathcal{D}(A,B)\mathcal{D}(B,C)\mathcal{D}(C,D)}} & & \\
\mathcal{D}(A,B)\otimes(\mathcal{D}(B,C)\otimes\mathcal{D}(C,D)) & & \downarrow{\scriptstyle c_{ACD}} \\
\downarrow{\scriptstyle 1\otimes c_{BCD}} & & \\
\mathcal{D}(A,B)\otimes\mathcal{D}(B,D) & \xrightarrow{c_{ABD}} & \mathcal{D}(A,D)
\end{array}$$

(6.1.U) *unit coherence*

$$\begin{array}{ccccc}
I\otimes\mathcal{D}(A,B) & \xrightarrow{l_{\mathcal{D}(A,B)}} & \mathcal{D}(A,B) & \xleftarrow{r_{\mathcal{D}(A,B)}} & \mathcal{D}(A,B)\otimes I \\
\downarrow{\scriptstyle u_A\otimes 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1\otimes u_B} \\
\mathcal{D}(A,A)\otimes\mathcal{D}(A,B) & \xrightarrow{c_{AAB}} & \mathcal{D}(A,B) & \xleftarrow{c_{ABB}} & \mathcal{D}(A,B)\otimes\mathcal{D}(B,B)
\end{array}$$

Example 6.2. (cp. [2, 6.2.9]) Let $\mathcal{D}$ be a $\mathcal{C}$-category as in the previous definition. Assume further that the monoidal product of $\mathcal{C}$ is symmetric. Then the $\mathcal{C}$-category $\mathcal{D} \otimes \mathcal{D}$ is defined by the following data:

(6.2.1) $|\mathcal{D} \otimes \mathcal{D}| = |\mathcal{D}| \times |\mathcal{D}|$;
(6.2.2) for a pair of objects $(A, A'), (B, B') \in |\mathcal{D} \otimes \mathcal{D}|$ the morphism object is given by $(\mathcal{D} \otimes \mathcal{D})((A, A'), (B, B')) = \mathcal{D}(A, B) \otimes \mathcal{D}(A', B')$.
(6.2.3) for every triple of objects $(A, A'), (B, B'), (C, C') \in |\mathcal{D} \otimes \mathcal{D}|$ the composition morphism is given by the obvious map

$$(\mathcal{D}(A, B) \otimes \mathcal{D}(A', B')) \otimes (\mathcal{D}(B, C) \otimes \mathcal{D}(B', C')) \longrightarrow \mathcal{D}(A, C) \otimes \mathcal{D}(A', C');$$

(6.2.4) for every object $(A, A') \in |\mathcal{D} \otimes \mathcal{D}|$ the *unit morphism* is given by $u_{(A,A')} : I \xrightarrow{r_I^{-1}} I \otimes I \xrightarrow{u_A \otimes u_{A'}} \mathcal{D}(A, A) \otimes \mathcal{D}(A', A')$,

Definition 6.3. ([2, 6.2.3]) A functor $F : \mathcal{D} \to \mathcal{E}$ between $\mathcal{C}$-categories consists of the following:

(6.3.1) for every object $A \in |\mathcal{D}|$ an object $FA \in |\mathcal{E}|$ and
(6.3.2) for every pair of objects $A, A' \in |\mathcal{D}|$ a morphism

$$F_{A,A'} : \mathcal{D}(A, A') \to \mathcal{E}(FA, FA')$$

such that or all objects $A, A', A'' \in |\mathcal{D}|$ the following diagrams commute

(6.3.N) *naturality condition*

$$\begin{array}{ccc} \mathcal{D}(A,A') \otimes \mathcal{D}(A',A'') & \xrightarrow{c_{AA'A''}} & \mathcal{D}(A,A'') \\ \downarrow{\scriptstyle F_{AA'} \otimes F_{A',A''}} & & \downarrow{\scriptstyle F_{A,A''}} \\ \mathcal{E}(FA,FA') \otimes \mathcal{E}(FA',FA'') & \xrightarrow[c_{FA,FA',FA''}]{} & \mathcal{E}(FA,FA'') \end{array}$$

(6.3.U) *unit condition*

$$\begin{array}{ccc} I & \xrightarrow{u_A} & \mathcal{D}(A,A) \\ & {\scriptstyle u_{FA}} \searrow & \downarrow{\scriptstyle F_{AA}} \\ & & \mathcal{E}(FA,FA) \end{array}$$

Note that any category naturally has the structure of a *Set*-category. On the other hand, an enriched category is not a category as the morphism objects in general cannot be interpreted as sets. This however can be done after choosing a lax symmetric monoidal functor $F : \mathcal{C} \to Set$.

Definition 6.4. (cp. [2, 6.4.3]) Let $F : \mathcal{C} \to \mathcal{C}'$ be a lax monoidal functor between monoidal categories, i.e. $F : \mathcal{C} \to \mathcal{C}'$ is a functor which comes equipped with a natural transformation of bifunctors

$$F(V) \otimes F(W) \longrightarrow F(V \otimes W), \quad V, W \in |\mathcal{C}|$$

and a unit morphism $I_{\mathcal{C}'} \to F(I_{\mathcal{C}})$ such that all coherence diagrams relating associativity and unit isomorphisms of $\mathcal{C}$ and $\mathcal{C}'$ are commutative. Given a $\mathcal{C}$-category $\mathcal{D}$ the lax monoidal functor F can be used to define a $\mathcal{C}'$-category $F_*\mathcal{D}$. The latter is given by the following data:

(6.4.1) the class of objects is $|F_*\mathcal{D}| = |\mathcal{D}|$;
(6.4.2) for a pair of objects A, B the morphism object is

$$F_*\mathcal{D}(A, B) = F(\mathcal{D}(A, B)).$$

(6.4.3) for a triple of objects A, B, C the composition morphism is given by

$$F(\mathcal{D}(A,B)) \otimes F(\mathcal{D}(B,C)) \longrightarrow F(\mathcal{D}(A,B) \otimes \mathcal{D}(B,C)) \overset{F(c_{ABC})}{\longrightarrow} F(\mathcal{D}(A,C));$$

(6.4.4) for an object A the unit morphism $u_A : I_{\mathcal{C}'} \to F(I_{\mathcal{C}}) \overset{F(u_A)}{\to} F(\mathcal{D}(A, A))$.

An *enrichment* of a category $\mathcal{D}'$ over a symmetric monoidal category $\mathcal{C}$ is a triple consisting of a $\mathcal{C}$-category $\mathcal{D}$, a lax symmetric monoidal functor $F : \mathcal{C} \to Set$ and an isomorphism $F_*\mathcal{D} \cong \mathcal{D}'$. In various examples to be discussed the category $\mathcal{C}$ will be equipped with a forgetful functor to Set. In these cases it should be understood that the corresponding lax monoidal functor we are using is the forgetful functor, unless explicitly stated otherwise.

6.5. Enrichments of the category KK**.** As already mentioned the category KK has an enrichment over the monoidal category of abelian groups. It can also be enriched over the monoidal category Top_* of pointed spaces (introduced in 5.7) using the Cuntz picture. This goes as follows. First we equip the category of C^*-algebras with an enrichment over the category of pointed topological spaces. Let us define $\mathrm{Hom}(A, B)$ to be the pointed set of $*$-homomorphisms from a C^*-algebra A to a C^*-algebra B with the compact open topology, the basepoint being the zero homomorphism. The composition of $*$-homomorphisms then yields a continuous map

$$c_{ABC} : \mathrm{Hom}(A, B) \wedge \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$

for C^*-algebras A, B, C. Finally the unit morphisms $u_A : S^0 \to \mathrm{Hom}(A, A)$ are given by requiring the image of u_A to be $\{0, id_A\} \subset \mathrm{Hom}(A, A)$. One easily checks that these data define an enrichment of the category of C^*-algebras over the category Top_*. Obviously, if we apply the forgetful functor to the morphism spaces we get back the ordinary category of C^*-algebras and $*$-homomorphisms.

The enrichment of the category of C^*-algebras and $*$-homomorphisms over the category Top_* induces a corresponding enrichment of the category KK. The morphism spaces of the corresponding Top_*-category KK_{top} are given by

$$KK_{top}(A, B) = \operatorname*{colim}_m \mathrm{Hom}(q^m A, \mathcal{K}^{\otimes m} \otimes B).$$

where the structure maps for the colimit are defined as in (4.2). The composition is defined as described in 4.3. The Top_*-category KK_{top} then is an enrichment of the category KK by means of the functor $F = \pi_0 : Top_* \to Set, X \mapsto \pi_0(X)$.

Next recall that the category KK can be regarded as a symmetric monoidal category by means of the external product

$$KK(A, B) \otimes KK(A', B') \to KK(A \otimes A', B \otimes B'),$$

for C^*-algebras A, A', B, B'. The fact that these maps are bilinear means that the product is compatible with the enrichment over KK over the category of abelian groups. In category language this can be phrased by saying that the external product gives the enriched category KK the structure of an enriched symmetric monoidal category (cp. 6.9). Conceptually the definition of an enriched symmetric monoidal category is analogous to the definition of a symmetric monoidal category. Essentially one just needs to replace the role of the category by an enriched category. However, to make this explicit one needs to say what a morphism and what an isomorphism in an enriched category is. Furthermore one needs to say what it means that data depend naturally on the objects.

Definition 6.6. Let $\mathcal{C}$ be a monoidal category. Let $\mathcal{D}$ be an enriched category as defined in Definition 6.1, and let A, B, C be objects of $\mathcal{D}$. A *morphism* from A to B we define to be a map $f : I \to \mathcal{D}(A,B)$ in $\mathcal{C}$; in symbols we write $f : A \to B$. Given a morphism $f : A \to B$ and a morphism $g : B \to C$ the composite $g \circ f : A \to C$ is given by

$$I \xrightarrow{r_I^{-1}} I \otimes I \xrightarrow{f \otimes g} \mathcal{D}(A,B) \otimes \mathcal{D}(B,C) \xrightarrow{c_{ABC}} \mathcal{D}(A,C).$$

In particular this convention allows to consider commutative diagrams of morphisms in enriched categories.[1] Moreover we can define an *isomorphism* in $\mathcal{D}$ to be a morphism $f : A \to B$ for which there is a morphism $g : B \to A$ such that $g \circ f = u_A$ and $f \circ g = u_B$.

Definition 6.7. Given an enriched category $\mathcal{D}$ in the sense of Definition 6.1, two functors of enriched categories $F, G : \mathcal{D} \to \mathcal{D}$ and a family of morphisms $f_A : F(A) \to G(A)$, then we say that the family is *natural* in A if for any two objects $A, B \in |\mathcal{D}|$ we have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{D}(A,B) \otimes I & \xrightarrow{F \otimes f_B} & \mathcal{D}(F(A),F(B)) \otimes \mathcal{D}(F(B),G(B)) \\
 & \nearrow^{r^{-1}_{\mathcal{D}(A,B)}} & & & \downarrow{\scriptstyle c_{F(A),F(B),B}} \\
\mathcal{D}(A,B) & & & & \mathcal{D}(F(A),G(B)) \\
 & \searrow_{l^{-1}_{\mathcal{D}(A,B)}} & & & \uparrow{\scriptstyle c_{F(A),G(A),G(B)}} \\
 & & I \otimes \mathcal{D}(A,B) & \xrightarrow{f_A \otimes G} & \mathcal{D}(F(A),G(A)) \otimes \mathcal{D}(G(A),G(B))
\end{array}$$

It then follows that the diagrams

$$\begin{array}{ccc}
F(A) & \xrightarrow{f_A} & G(A) \\
\downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\
F(B) & \xrightarrow{f_B} & G(B)
\end{array}$$

commute for all morphisms $f : A \to B$ between objects A and B of $\mathcal{D}$. The converse is not true in general. However it is true if I is a *generator*[2] of the category $\mathcal{D}$.

[1] The composition of morphisms in the sense of Definition 6.6 is associative, which is a consequence of the identity $l_I = r_I$. The latter follows from the axioms of a monoidal category, see [9, Theorem 3].

[2] A *generator* of a category $\mathcal{D}$ is an object G such that for any pair of morphism $f, g : A \to B$ in the category $\mathcal{D}$ the following holds: $g = f$ if and only if $g \circ h = f \circ h$ for all morphism $h \in \mathcal{D}(G, A)$.

Example 6.8. Let *Set* denote the symmetric monoidal category of sets with the tensor product being the cartesian product. Any category then is a *Set*-category in a natural way. The morphisms of a category precisely correspond to morphisms (in the sense of Definition 6.6) of the corresponding *Set*-category.

Definition 6.9. Let $\mathcal{C}$ be a monoidal category with a symmetric monoidal product. A *monoidal $\mathcal{C}$-category* consists of the following data:

(6.9.1) a $\mathcal{C}$-category $\mathcal{D}$;
(6.9.2) a bifunctor of enriched categories $m : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}, (A, B) \mapsto A \otimes B$,
(6.9.3) an object $U \in |\mathcal{D}|$, called the unit;
(6.9.4) for every triple A, B, C of objects an *associativity isomorphism*

$$a_{ABC} : I \to \mathcal{D}((A \otimes B) \otimes C, A \otimes (B \otimes C));$$

(6.9.5) for every object A a *left unit isomorphism* $l_A : I \to \mathcal{D}(U \otimes A, A)$;
(6.9.6) for every object A a *right unit isomorphism* $r_A : I \to \mathcal{D}(A \otimes U, A)$.

The structure isomorphisms must naturally depend on the objects. Moreover it is required that the structure isomorphism yield commutative associativity coherence diagrams (5.3.A) as well as commutative unit coherence diagrams (5.3.U). Furthermore for all objects $A, B, C, A', B', C' \in |\mathcal{D}|$ the following diagrams (where the unlabeled maps are induced by the structure maps) have to be commutative

(6.9.A) *associativity condition*

$$\begin{array}{ccc}
(\mathcal{D}(A,A') \otimes \mathcal{D}(B,B')) \otimes \mathcal{D}(C,C') & \xrightarrow{m\otimes 1} & \mathcal{D}(A \otimes B, A' \otimes B') \otimes \mathcal{D}(C,C') \\
\big\downarrow{\scriptstyle a_{\mathcal{D}(A,A'),\mathcal{D}(B,B'),\mathcal{D}(C,C')}} & & \big\downarrow{\scriptstyle m} \\
\mathcal{D}(A,A') \otimes (\mathcal{D}(B,B') \otimes \mathcal{D}(C,C')) & & \mathcal{D}((A \otimes B) \otimes C, (A' \otimes B') \otimes C') \\
\big\downarrow{\scriptstyle 1\otimes m} & & \big\downarrow \\
\mathcal{D}(A,A') \otimes \mathcal{D}(B \otimes C, B' \otimes B') & \xrightarrow{c} & \mathcal{D}(A \otimes (B \otimes C), A' \otimes (B' \otimes C'))
\end{array}$$

(6.9.U) *unit condition*

$$\begin{array}{ccccc}
I \otimes \mathcal{D}(A,B) & \xrightarrow{l_{\mathcal{D}(A,B)}} & \mathcal{D}(A,B) & \xleftarrow{r_{\mathcal{D}(A,B)}} & \mathcal{D}(A,B) \otimes I \\
\big\downarrow{\scriptstyle u_A\otimes 1} & & & & \big\downarrow{\scriptstyle 1\otimes u_B} \\
\mathcal{D}(U,U) \otimes \mathcal{D}(A,B) & & \big\downarrow{\scriptstyle 1} & & \mathcal{D}(A,B) \otimes \mathcal{D}(U,U) \\
\big\downarrow{\scriptstyle m} & & & & \big\downarrow{\scriptstyle m} \\
\mathcal{D}(U \otimes A, U \otimes B) & \longrightarrow & \mathcal{D}(A,B) & \longleftarrow & \mathcal{D}(A \otimes U, B \otimes U)
\end{array}$$

Definition 6.10. Let $\mathcal{C}$ be a monoidal category with a symmetric monoidal product. A monoidal $\mathcal{C}$-category $\mathcal{D}$ is called *symmetric* if in addition for every pair of objects $A, B \in \mathcal{D}$ there is a *symmetry isomorphism*

$$s_{AB} : I \to \mathcal{D}(A \otimes B, B \otimes A)$$

which depends naturally on A and B. The symmetry isomorphism must be compatible with the other structure isomorphisms in the sense that all the associativity coherence diagrams (5.4.A) and all unit coherence diagrams (5.4.U) commute. Moreover they must satisfy the *symmetry axiom*, i.e. for any two objects A, B the composite $s_{BA} \circ s_{AB}$ is the unit morphism $I \to \mathcal{D}(A \otimes B, A \otimes B)$. Furthermore for all $A, A', B, B' \in |\mathcal{D}|$ the following diagram (where the vertical morphism is induced by the symmetry isomorphism) has to be commutative

(6.10.S) *symmetry condition*

$$\begin{array}{ccc}
\mathcal{D}(A,A') \otimes \mathcal{D}(B,B') & \xrightarrow{\quad m \quad} & \mathcal{D}(A \otimes B, A' \otimes B') \\
\Big\downarrow {\scriptstyle s_{\mathcal{D}(A,A'),\mathcal{D}(B,B')}} & & \Big\downarrow \\
\mathcal{D}(B,B') \otimes \mathcal{D}(A,A') & \xrightarrow{\quad m \quad} & \mathcal{C}(B \otimes A, B' \otimes A')
\end{array}$$

Let $F : \mathcal{C} \to \mathcal{C}'$ be a lax (symmetric) monoidal functor of (symmetric) monoidal categories and let $\mathcal{D}$ be a (symmetric) monoidal $\mathcal{C}$-category. We then can extend Definition 6.4 in a straightforward manner in order to obtain a (symmetric) monoidal $\mathcal{C}'$-category $F_*\mathcal{D}$. Accordingly we can define the notion of an *enrichment* of a (symmetric) monoidal category over a (symmetric) monoidal category $\mathcal{C}$.

6.11. Monoidal enrichments of the category KK. As already mentioned, from the bilinearity of the exterior Kasparov product (5.9) it follows that the symmetric monoidal category KK has an enrichment over the symmetric monoidal category of abelian groups. One may check that the maps

$$m : KK_{top}(A_1, B_1) \otimes KK_{top}(A_2, B_2) \longrightarrow KK_{top}(A_1 \otimes A_2, B_1 \otimes B_2)$$

given by the maps introduced in 5.8 turn KK_{top} into an enrichment of the monoidal category KK over the symmetric monoidal category Top_*. However the enriched monoidal Top_*-category KK_{top} is not symmetric. To see this it suffices to look at the following diagrams

$$\begin{array}{ccc}
q^2(A_1 \otimes A_2) & \longrightarrow & qA_1 \otimes qA_2 \\
\Big\downarrow {\scriptstyle q^2(s_{A_1,A_2})} & & \Big\downarrow {\scriptstyle s_{qA_1,qA_2}} \\
q^2(A_2 \otimes A_1) & \longrightarrow & qA_2 \otimes qA_1
\end{array}
\qquad
\begin{array}{ccc}
(\mathcal{K} \otimes A_1) \otimes (\mathcal{K} \otimes A_2) & \longrightarrow & \mathcal{K}^{\otimes 2} \otimes A_1 \otimes A_2 \\
\Big\downarrow {\scriptstyle s_{\mathcal{K}\otimes A_1,\mathcal{K}\otimes A_2}} & & \Big\downarrow {\scriptstyle id \otimes s_{A_1,A_2}} \\
(\mathcal{K} \otimes A_2) \otimes (\mathcal{K} \otimes A_1) & \longrightarrow & \mathcal{K}^{\otimes 2} \otimes A_2 \otimes A_1
\end{array}$$

They do not commute, and from this one easily sees that the exterior product on KK_{top} cannot be symmetric. A similar lack of symmetry also shows up quite prominently in stable homotopy theory. In stable homotopy theory the lack of symmetry was resolved by introducing the category of symmetric spaces and spectra.

7. SYMMETRIC SPACES AND SPECTRA

Let $\wp$ denote the small category whose set of objects is the set of finite subsets of the natural numbers and whose sets of morphisms $\wp(M, N)$ for two subsets $M, N \subset \mathbb{N}$ consists of the set maps from M to N, i.e. $\wp(M,N) = Set(M,N)$. Let $\mathcal{I}$ denote the subcategory consisting of the isomorphisms. A functor from $\mathcal{I}$ into a category $\mathcal{D}$ is called a *symmetric sequence* in $\mathcal{D}$. Occasionally we also call a symmetric sequence in a category $\mathcal{D}$ just a *symmetric object*. Our major interest will be in the category of symmetric spaces and spectra, defined below.

Definition 7.1. Let $\mathcal{D}$ be a monoidal category with an initial object $*$. Assume further that $\mathcal{D}$ has finite coproducts and that the tensor product preserves the finite coproducts (up to natural coherence). The category $\mathcal{D}^{\mathcal{I}}$ of functors from $\mathcal{I}$ to $\mathcal{D}$ then carries the structure of a monoidal category. The corresponding data are given by

(7.1.1) the underlying category is $\mathcal{D}^{\mathcal{I}}$, the objects in $\mathcal{D}^{\mathcal{I}}$ are called *symmetric sequences in* $\mathcal{D}$;

(7.1.2) the bifunctor $\otimes : \mathcal{D}^{\mathcal{I}} \times \mathcal{D}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ for symmetric sequences $X, Y \in \mathcal{D}^{\mathcal{I}}$ is given by

$$(X \otimes Y)(J) = \coprod_{\substack{M\cup N=J\\ M\cap N=\varnothing}} X(M) \otimes Y(N), \quad J \in |\mathcal{I}|;$$

(7.1.3) the unit E, given by $E(\varnothing) = I$ and $E(K) = *$, the initial object, for $K \neq \varnothing$;

(7.1.4) the *associativity isomorphism* $a_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$ is given by the composite

$$\begin{array}{ccc}
((X \otimes Y) \otimes Z)(J) & \xrightarrow{=} & \displaystyle\coprod_{\substack{L\cup K=J\\ K\cap L=\varnothing}} \left(\coprod_{\substack{M\cup N=L\\ M\cap N=\varnothing}} X(M) \otimes Y(N) \right) \otimes Z(K) \\
& & \downarrow \cong \\
& & \displaystyle\coprod_{\substack{M\cup N\cup K=J\\ M\cap N=N\cap K=K\cap M=\varnothing}} (X(M) \otimes Y(N)) \otimes Z(K) \\
& & \downarrow \\
& & \displaystyle\coprod_{\substack{M\cup N\cup K=J\\ M\cap N=N\cap K=K\cap M=\varnothing}} X(M) \otimes (Y(N) \otimes Z(K)) \\
& & \downarrow \\
(X \otimes (Y \otimes Z))(J) & \xrightarrow{=} & \displaystyle\coprod_{\substack{M\cup L=J\\ M\cap L=\varnothing}} X(M) \otimes \left(\coprod_{\substack{N\cup K=L\\ N\cap K=\varnothing}} Y(N) \otimes Z(K) \right)
\end{array}$$

with the left and the right unit isomorphism induced by the left and the right unit isomorphism of the monoidal category $\mathcal{D}$. The category $\mathcal{D}^{\mathcal{I}}$ has a canonical initial object: it is given by the constant functor which sends every object in $\mathcal{I}$ to the initial object $*$.

If $\mathcal{D}$ is a symmetric monoidal category the category $\mathcal{D}^{\mathcal{I}}$ also can be given a symmetric monoidal structure; the symmetry isomorphism s_{XY} for symmetric sequences X, Y in $\mathcal{D}$ is the map $X \otimes Y \to Y \otimes X$ whose restriction to the factors $X(M) \otimes Y(N) \subset (X \otimes Y)(K)$ is given by the composites

$$X(M) \otimes Y(N) \xrightarrow{s_{X(M)Y(N)}} Y(N) \otimes X(M) \subset (Y \otimes X)(K).$$

If $\mathcal{D}$ is an enriched symmetric monoidal category enriched over a symmetric monoidal category $\mathcal{C}$ which contains all small limits then $\mathcal{D}^{\mathcal{I}}$ also is enriched over $\mathcal{C}$: the morphism object $\mathcal{D}^{\mathcal{I}}(X, Y)$ for two symmetric objects $X, Y \in |\mathcal{D}^{\mathcal{I}}|$ is

$$\mathcal{D}^{\mathcal{I}}(X, Y) = \lim_{M \in \mathcal{I}} \mathcal{D}(X(M), Y(M))$$

In any symmetric monoidal category there is the notion of a monoid and the notion of modules over a monoid.

Definition 7.2. A *monoid* in a monoidal category $\mathcal{D}$ is an object $R \in |\mathcal{D}|$ together with a *multiplication map* $\mu : R \otimes R \to R$ and a *unit map* $\eta : I \to R$ for which the following diagrams are commutative

(7.2.A) *associativity coherence*

$$\begin{array}{ccccc}
(R \otimes R) \otimes R & & \xrightarrow{a_{RRR}} & & R \otimes (R \otimes R) \\
\downarrow{\scriptstyle \mu\otimes 1} & & & & \downarrow{\scriptstyle \mu} \\
R \otimes R & \xrightarrow{\mu} & R & \xleftarrow{\mu} & R \otimes R
\end{array}$$

(7.2.U) *unit coherence*

$$\begin{array}{ccccc}
R \otimes I & \xrightarrow{\eta\otimes 1} & R \otimes R & \xleftarrow{1\otimes\eta} & R \otimes I \\
& {\scriptstyle l_R}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle r_R} & \\
& & R & &
\end{array}$$

If $\mathcal{D}$ is symmetric monoidal category a monoid is called *commutative* if $\mu = s_{RR}\mu$.

Definition 7.3. A *left module* over a monoid R in a monoidal category $\mathcal{D}$ is an object $M \in \mathcal{D}$ together with a map $\nu : R \otimes M \to M$ such that the following diagram is commutative

$$\begin{array}{ccccccc}
(R \otimes R) \otimes M & \xrightarrow{a_{RRM}} & R \otimes (R \otimes M) & \xrightarrow{1\otimes\nu} & R \otimes M & \xleftarrow{\eta\otimes M} & I \otimes M \\
\downarrow{\scriptstyle \mu\otimes 1} & & & & \downarrow{\scriptstyle \nu} & \swarrow{\scriptstyle l_M} & \\
R \otimes M & & \xrightarrow{\nu} & & M & &
\end{array}$$

Similarly one can define a *right module* over R. If $\mathcal{D}$ is a symmetric monoidal category and R is a commutative monoid in $\mathcal{D}$ any left module over R can be given the structure of a right module over R by defining the right action to be the composite $\nu s_{MR} : M \otimes R \to M$.

Lemma 7.4 (Hovey-Shipley-Smith, Lemma 2.2.2 & Theorem 2.2.10)**.** *Let $\mathcal{D}$ be a symmetric monoidal category that is cocomplete and let R be a commutative monoid in $\mathcal{D}$ such that the functor $R \otimes \_\_ : \mathcal{D} \to \mathcal{D}$ preserves coequalizers. Then there is a symmetric monoidal product $\otimes_R$ on the category of left R-modules with R as unit. For two left R-modules X, Y the product is given by the coequalizer described by the diagram*

$$X \otimes R \otimes Y \underset{1\otimes\mu}{\overset{\mu s_{XR}\otimes 1}{\rightrightarrows}} X \otimes Y \longrightarrow X \otimes_R Y.$$

7.5. Symmetric spectra. From now on we will specify to the special case of our major interest which is the case where $\mathcal{D}$ is the category Top_* of pointed spaces.

Definition 7.6. Let S denote the monoid in pointed symmetric spaces which is given by the functor $M \mapsto (S^1)^{\wedge M}$ and the obvious structure maps. A *symmetric spectrum* is a left S-module.

Definition 7.7. A *commutative symmetric ring spectrum* is a map of commutative monoids $S \to R$. A (left) R-module canonically inherits the structure of a symmetric spectrum by means of the monoid map. By the previous lemma the category of (left) R-module spectra has a symmetric monoidal product $\wedge_R$ with unit R.

The definition of a symmetric spectrum given above is not the standard definition (cp. [7]). However the category of symmetric spectra as we defined it is equivalent to category of symmetric spectra defined via the standard definition. In both approaches one uses a diagram category of "symmetric sequences"; the difference between the standard and our approach is that we use a bigger but equivalent diagram category. More precisely, in the standard set-up one uses the full subcategory $\Sigma \subset \mathcal{I}$ with objects the sets $n = \{1, 2, \dots, n\},\ n \in \mathbb{N}$.

Definition 7.8. Let E be a symmetric spectrum. For a natural number $n \in \mathbb{N}$ let n also denote the subset $n = \{1, 2, 3, \dots, n\} \subset \mathbb{N}$. The 0*-th homotopy group* of E is defined by

$$\pi_0(E) = \operatorname*{colim}_n \pi_n(E(n)), \tag{7.9}$$

where the structure maps are induced by the natural inclusions $n \subset n + 1$.

8. Enrichments over symmetric spaces

In this section we will define two categories which are enriched over the category of symmetric spaces, $\mathcal{KK}$ and $\mathbb{KK}$ (Theorem 8.7 and Theorem 8.13). In Section 10 we will see that $\mathbb{KK}$ has an enrichment over the category of symmetric spectra.

8.1. The enriched category $\mathcal{KK}$. In 4.5 we have seen that the iterated Cuntz construction $q^\bullet A$ for a C^*-algebra A defines a contravariant functor from the category $\mathcal{I}$ of finite subsets of $\mathbb{N}$ and isomorphisms to the category of C^*-algebras. On the other hand given a C^*-algebra B we have the symmetric C^*-algebra

$$\mathcal{K}^\bullet B : M \longmapsto \mathcal{K}^M \otimes B.$$

These functors define a symmetric space

$$\mathcal{KK}(A,B) : M \longrightarrow \mathcal{KK}^M(A,B) = \operatorname{Hom}(q^M A, \mathcal{K}^M B), \tag{8.2}$$

where we use the compact-open topology to topologize the sets of $*$-homomorphisms. We thus have a bivariant functor from the category of C^*-algebras to the category of symmetric spaces. We shall see that they are the morphism objects of an enriched category. We need to define the composition morphisms.

Let $M, N \in \wp$ be subsets with $M \cap N = \varnothing$. Recall from Section 4.9 the definition of the map Δ^{MN}. For a C^*-algebra B we used $\Delta^{\varnothing M}$ to define

$$\chi^{MN} : q^M(\mathcal{K}^N \otimes B) \xrightarrow{\Delta^{\varnothing M}} \mathcal{K}^N \otimes q^M B. \tag{8.3}$$

For $f \in \operatorname{Hom}(q^M A, \mathcal{K}^M \otimes B)$ and $g \in \operatorname{Hom}(q^N B, \mathcal{K}^N \otimes C)$ define $c_{ABC}(f,g) \in \operatorname{Hom}(q^{M\cup N} A, \mathcal{K}^{M\cup N} C)$ as the composition

$$q^{M\cup N} A \xrightarrow{\cong} q^N q^M A \xrightarrow{q^N f} q^N(\mathcal{K}^M \otimes B) \xrightarrow{\chi^{MN}} \mathcal{K}^M \otimes q^N B \xrightarrow{id\otimes g} \mathcal{K}^{\otimes M\cup N} \otimes C.$$

This defines a map

$$c_{ABC} : \operatorname{Hom}(q^M A, \mathcal{K}^M B) \wedge \operatorname{Hom}(q^N B, \mathcal{K}^N C) \to \operatorname{Hom}(q^{M\cup N} A, \mathcal{K}^{M\cup N} C),$$

and varying the subsets M, N yields a corresponding map

$$c_{ABC} : \mathcal{KK}(A,B) \otimes \mathcal{KK}(B,C) \longrightarrow \mathcal{KK}(A,C). \tag{8.4}$$

We claim that these maps define an enriched category $\mathcal{KK}$. To check this statement involves quite a bit of combinatorics and we will derive it from a general recipe.

8.5. Composition data. Let $\mathcal{C}$ be a symmetric monoidal category with an initial object $*$. Assume further that $\mathcal{C}$ has finite coproducts and that the tensor product preserves the finite coproducts (up to natural coherence). Let $\mathcal{D}$ be a $\mathcal{C}$-category which is an enrichment of an ordinary category by means of a faithful forgetful functor to *Set*. This means that we can regard the morphism objects of $\mathcal{D}$ as sets.[3] For a functor F into the category $\mathcal{D}^{\mathcal{I}}$ let F^M denote the evaluation of the functor on an object $M \in |\mathcal{I}|$.

Let $F, G : \mathcal{D} \to \mathcal{D}^{\mathcal{I}}$ be two functors of enriched categories. Assume that the functors $F, G : \mathcal{D} \to \mathcal{D}^{\mathcal{I}}$ are *augmented* in the sense that $F^\varnothing = id = G^\varnothing$. A set of *composition data* for the pair (F, G) consists of natural transformations of functors $\mathcal{D} \to \mathcal{D}$

$$\varrho_{MN} : F^{M\cup N} \to F^M F^N; \;\; \iota_{MN} : G^M G^N \to G^{M\cup N}; \;\; \chi_{MN} : F^M G^N \to G^N F^M$$

[3]This is a technical assumption which leads to a simplification of the statement and proof of the following proposition. An analogous statement also holds without this assumption.

for any pair of finite subsets $M, N \subset \mathbb{N}$ with $M \cap N = \varnothing$. These functors must naturally depend on M and N. Moreover for any triple of pairwise disjoint finite subsets $L, M, N \subset \mathbb{N}$ the natural transformations must yield commutative diagrams

$$\begin{array}{ccc} F^{L\cup M\cup N} & \xrightarrow{\varrho_{(L\cup M)N}} & F^{L\cup M}F^{N} \\ \downarrow{\scriptstyle \varrho_{L(M\cup N)}} & & \downarrow{\scriptstyle \varrho_{LM}\circ 1} \\ F^{L}F^{(M\cup N)} & \xrightarrow{1\circ\varrho_{MN}} & F^{L}F^{M}F^{N} \end{array} \qquad \begin{array}{ccc} G^{L}G^{M}G^{N} & \xrightarrow{1\circ\iota_{MN}} & G^{L}G^{M\cup N} \\ \downarrow{\scriptstyle \iota_{LM}\circ 1} & & \downarrow{\scriptstyle \iota_{L(M\cup N)}} \\ G^{L\cup M}G^{N} & \xrightarrow{\iota_{(L\cup M)N}} & G^{L\cup M\cup N} \end{array}$$

$$\begin{array}{ccc} F^{L\cup M}G^{N} & \xrightarrow{\varrho_{LM}\circ 1} & F^{L}F^{M}G^{N} \\ \Big\downarrow{\scriptstyle \chi_{(L\cup M)N}} & & \downarrow{\scriptstyle 1\circ\chi_{MN}} \\ & & F^{L}G^{N}F^{M} \\ & & \downarrow{\scriptstyle \chi_{LN}\circ 1} \\ G^{N}F^{L\cup M} & \xrightarrow{1\circ\varrho_{L\cup M}} & G^{N}F^{L}F^{M} \end{array} \qquad \begin{array}{ccc} F^{L}G^{M}G^{N} & \xrightarrow{1\circ\iota_{MN}} & F^{L}G^{M\cup N} \\ \downarrow{\scriptstyle \chi_{ML}\circ 1} & & \Big\downarrow{\scriptstyle \chi_{L(M\cup N)}} \\ G^{M}F^{L}G^{N} & & \\ \downarrow{\scriptstyle 1\circ\chi_{LN}} & & \\ G^{M}G^{N}F^{L} & \xrightarrow{\iota_{MN}\circ 1} & G^{M\cup N}F^{L} \end{array}$$

In addition the maps ϱ^{MN}, χ^{MN} and ι^{MN} must be the identity if either of M or N is the empty set.

Proposition 8.6. *Let us be given composition data as defined in 8.5. Then we can define a $\mathcal{C}^{\mathcal{I}}$-category $\mathbb{D}$ as follows*

(8.6.1) $|\mathbb{D}| = |\mathcal{D}|$;

(8.6.2) *for each pair of objects $A, B \in |\mathbb{D}|$ the object $\mathbb{D}(A, B) \in |\mathcal{C}^{\mathcal{I}}|$ is given by*

$$\mathbb{D}(A,B) : M \mapsto \mathbb{D}^{M}(A,B) = \mathcal{D}(F^{M}(A), G^{M}(B)), M \in |\mathcal{I}|;$$

(8.6.3) *for every triple of objects $A, B, C \in |\mathcal{D}|$ the composition morphism $c_{ABC} : \mathbb{D}(A,B) \otimes \mathbb{D}(B,C) \to \mathbb{D}(A,C)$ is the unique morphism determined by the maps*

$$\mathcal{D}(F^{N}(A), G^{N}(B)) \otimes \mathcal{D}(F^{M}(B), G^{M}(C)) \longrightarrow \mathcal{D}(F^{M\cup N}(A), G^{M\cup N}(C)),$$

for disjoint finite subsets $M, N \subset \mathbb{N}$, which sends an element $f \otimes g$ to the map

$$c_{ABC}(f\otimes g) : F^{M\cup N}(A) \overset{\varrho_{MN}}{\to} F^{M}F^{N}(A) \overset{F^{M}(f)}{\to} F^{M}G^{N}(B)$$
$$\overset{\chi^{MN}}{\longrightarrow} G^{N}F^{M}(B) \overset{G^{N}(g)}{\to} G^{N}G^{M}(C) \overset{\iota_{MN}}{\to} G^{M\cup N}(C);$$

(8.6.4) *for an object $A \in |\mathcal{D}|$ the unit morphism $u_A^{\mathbb{D}} : E \to \mathbb{D}(A,A)$ is determined by $(u_A^{\mathbb{D}})(\varnothing) = u_A^{\mathcal{D}} : I \to \mathcal{D}(A,A) = \mathbb{D}^{\varnothing}(A,A)$.*

The category $\mathbb{D}$ is an enrichment over the category $\mathcal{D}$ by means of the forgetful functor which associates to a symmetric $\mathcal{D}$-object X the $\mathcal{D}$-object $X(\varnothing)$.

Proof. Let L, M, N be pairwise disjoint finite subsets of $\mathbb{N}$, and let us be given $f \in \mathcal{D}(F^N(A), G^N(B))$, $g \in \mathcal{D}(F^M(B), G^M(C))$ and $h \in \mathcal{D}(F^L(C), G^L(D))$. The following diagram (with the obvious maps) commutes and thereby shows that the associativity coherence conditions (6.1.A) hold in $\mathbb{D}$.

$$
\begin{array}{ccccccc}
F^{L\cup M\cup N}(A) & & & \xrightarrow{\quad = \quad} & & & F^{L\cup M\cup N}(A) \\
\Big\downarrow & & & & & & \Big\downarrow \\
F^L F^{M\cup N}(A) & \longrightarrow & F^L F^M F^N(A) & & \longleftarrow & & F^{L\cup M} F^N(A) \\
\Big| & & \Big\downarrow & & & & \Big\downarrow \\
\Big| & & F^L F^M G^N(B) & & \longleftarrow & & F^{L\cup M} G^N(B) \\
\Big| & & \Big\downarrow & & & & \Big\downarrow \\
\Big| & & F^L G^N F^M(B) & \longrightarrow & G^N F^L F^M(B) & \longleftarrow & G^N F^{L\cup M}(B) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big| \\
F^L G^{M\cup N}(C) & \longleftarrow & F^L G^N G^M(C) & \longrightarrow & G^N F^L G^M(C) & & \Big| \\
\Big\downarrow & & & & \Big\downarrow & & \Big| \\
G^{M\cup N} F^L(C) & & \longleftarrow & & G^N G^M F^L(C) & & \Big| \\
\Big\downarrow & & & & \Big\downarrow & & \Big\downarrow \\
G^{M\cup N} G^L(D) & & \longleftarrow & & G^N G^M G^L(D) & \longrightarrow & G^N G^{L\cup M}(D) \\
\Big\downarrow & & & & & & \Big\downarrow \\
G^{L\cup M\cup N}(D) & & & \xrightarrow{\quad = \quad} & & & G^{L\cup M\cup N}(D)
\end{array}
$$

The unit coherence conditions (6.1.U) for $\mathbb{D}$ are fulfilled if and only if the following equations hold

$$c_{AAB}(1_A \otimes g) = g \qquad \text{for all } g \in \mathcal{D}^M(A,B),\ M \in |\mathcal{I}|, A, B \in |\mathbb{D}|;$$
$$c_{ABB}(f \otimes 1_B) = f \qquad \text{for all } f \in \mathcal{D}^M(A,B),\ M \in |\mathcal{I}|, A, B \in |\mathbb{D}|.$$

These equations follow immediately from the assumption that the functors $F^\bullet$ and $G^\bullet$ are augmented and the last property mentioned in 8.5. □

Theorem 8.7. *The following data define an enriched category $\mathcal{KK}$:*

(8.7.1) *$|\mathcal{KK}|$ is the class of C^*-algebras;*
(8.7.2) *for a pair of C^*-algebras A, B the morphism object is the symmetric space $\mathcal{KK}(A,B)$ defined in* (8.2)
(8.7.3) *for every triple of C^*-algebras A, B, C the composition morphism c_{ABC} is given by* (8.4).

(8.7.4) *for all C^*-algebras A the unit morphism $u_A : E \to \mathcal{KK}(A,A)$ is the one determined by $((u_A)(\varnothing))(S^0) = 0 \cup 1_A \subset \mathrm{Hom}(A,A) = \mathcal{KK}^{\varnothing}(A,A)$.*

The enriched category $\mathcal{KK}$ is an enrichment of the category of C^*-algebras over the category of symmetric spaces by means of the forgetful functor which associates to a symmetric space $X^\bullet$ the space $X^{\varnothing}$.

Proof. Put $F^\bullet A = q^\bullet A, G^\bullet B = \mathcal{K}^\bullet B$. For any pair (M,N) of disjoint finite subsets of $\mathbb{N}$ we then have canonical isomorphisms $\varrho^{MN} : q^{M\cup N}A \cong q^M q^N A$ and $\iota^{MN} : \mathcal{K}^M\mathcal{K}^N B \cong \mathcal{K}^{M\cup N}B$, as well as the natural transformation χ^{MN} defined in (8.3). It now is straight forward to check the compatibility conditions introduced in 8.5. The assertion then follows from the previous proposition. □

8.8. (Co)associative functors. The notion of composition data introduced in 8.5 can be regarded as a collection of several pieces of information which also can be looked at separately. For example it makes sense to consider the pair $F = (F^\bullet, \varrho)$ consisting of the functor $F^\bullet$ and the natural transformation ϱ and just require the upper right diagram in the definition to be commutative. Such a structure we might call an *coassociative functor*. Dually the pair $G = (G^\bullet, \iota)$ consisting of the functor $G^\bullet$ and the natural transformation ι subject to the commutativity of the upper right diagram we might call an *associative functor*. The natural transformation χ then is a sort of intertwining operator between F and G and the two diagrams in the lower row correspond to the compatibility of the intertwining operator χ with the natural transformations ϱ and ι which define the coassociative and the associative structure respectively. Completely analogously one could define composition data for two coassociative functors F_1 and F_2 or for two associative functors G_1 and G_2. Corresponding composition data would imply that one can define functors $(F_1F_2)^\bullet, (G_1G_2)^\bullet : \mathcal{D} \to \mathcal{D}^{\mathcal{I}}$ given by $M \mapsto F_1^M F_2^M(A)$ and $M \mapsto G_1^M G_2^M(B)$ which then would come equipped with the structure of a (co-)associative functor. Furthermore, if there are given composition data for two coassociative functors F_1 and F_2 with a natural transformation $\chi^{MN} : F_1^M F_2^N \to F_2^N F_1^M$ as well as composition data for F_1 and G and for F_2 and G for an associative functor G then this data define composition data for (F_1F_2) and G. Similarly given two associative functors G_1 and G_2 and a coassociative functor F together with corresponding composition data then one obtains from this composition data for the functors F and (G_1G_2). This recipe sometimes simplifies checking the commutativity of the relevant diagrams for specific composition data.

We now give the definition of the second enrichment which eventually will give the desired enrichment over the category of symmetric spectra. To motivate the construction we recall the following variant of the Bott periodicity theorem.

Theorem 8.9. (Bott periodicity) *For any $n \in \mathbb{N}$ and all C^*-algebras A, B the exterior multiplication with the element $1_{C_0(\mathbb{R}^n)} \in KK(C_0(\mathbb{R}^n), C_0(\mathbb{R}^n))$*

yields an isomorphism

$$KK(A,B) \longrightarrow KK(C_0(\mathbb{R}^n)\otimes A, C_0(\mathbb{R}^n)\otimes B). \tag{8.10}$$

Via the canonical isomorphism $\mathrm{Hom}(A, C_0(X)\otimes B) \cong map_*(X, \mathrm{Hom}(A,B))$ for locally compact spaces X and C^*-algebras A and B we obtain from the Bott periodicity theorem for all $n \in \mathbb{N}$ an isomorphism

$$\begin{aligned} &\pi_n(\mathrm{Hom}(q^n(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^n B)) \\ &\quad\cong \pi_n(\Omega^n \mathrm{Hom}(q^n(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^n C_0(\mathbb{R}^n)\otimes B)) \\ &\quad\cong \pi_0(\Omega^n \mathrm{Hom}(q^n(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^n B)) \\ &\quad\cong \pi_0(\mathrm{Hom}(q^n(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^n C_0(\mathbb{R}^n)\otimes B)) \end{aligned}$$

Thus $\mathrm{Hom}(q^n(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^n B)$ qualifies as the n-th space of a spectrum which represents $KK(A,B)$. In view of this observation we introduce

8.11. The enriched category $\mathbb{KK}$. Let $(qC)^\bullet A$ for a C^*-algebra A denote the symmetric C^*-algebra with

$$(qC)^M A = q^M(C_0(\mathbb{R}^M)\otimes A), \quad M \in |\mathcal{I}|.$$

For C^*-algebras A, B define the symmetric pointed space

$$\mathbb{KK}(A,B) : M \longmapsto \mathbb{KK}^M(A,B) = \mathrm{Hom}((qC)^M A, \mathcal{K}^M B).$$

For disjoint subsets $M, N \subset \mathbb{N}$ and C^*-algebras A, B, C we have the map

$$\mathbb{KK}^M(A,B)\wedge \mathbb{KK}^N(B,C) \to \mathbb{KK}^{M\cup N}(A,C)$$

which sends $f \wedge g \in \mathrm{Hom}((qC)^M A, \mathcal{K}^M B)\wedge \mathrm{Hom}((qC)^N A, \mathcal{K}^N B)$ to the following composition[4] in $\mathrm{Hom}((qC)^{M\cup N}A, \mathcal{K}^{M\cup N}B)$

$$\begin{aligned} &q^{M\cup N}(C_0(\mathbb{R}^{M\cup N})\otimes A) \overset{\varrho^{MN}}{\to} q^M q^N(C_0(\mathbb{R}^M)\otimes C_0(\mathbb{R}^N)\otimes A) \overset{q^M(\Delta^{\varnothing N})}{\longrightarrow} \\ &q^M(C_0(\mathbb{R}^M)\otimes q^N(C_0(\mathbb{R}^N)\otimes A)) \overset{q^M(id_{C_0(\mathbb{R}^M)}\otimes f)}{\longrightarrow} q^M(C_0(\mathbb{R}^M)\otimes \mathcal{K}^N B) \overset{\cong}{\longrightarrow} \\ &q^M(\mathcal{K}^N C_0(\mathbb{R}^M)\otimes B) \overset{\chi^{MN}}{\longrightarrow} \mathcal{K}^N q^M(C_0(\mathbb{R}^M)\otimes B) \overset{\mathcal{K}^N g}{\longrightarrow} \mathcal{K}^N\mathcal{K}^M C \overset{\iota^{MN}}{\longrightarrow} \mathcal{K}^{N\cup M}C; \end{aligned}$$

where the $*$-homomorphisms $\varrho^{MN}, \chi^{MN}, \iota^{MN}$ are as in the proof of Theorem 8.7; $\Delta^{\varnothing N}$ has been defined in 4.9. These maps define a map

$$c_{ABC} : \mathbb{KK}(A,B)\wedge \mathbb{KK}(B,C) \longrightarrow \mathbb{KK}(A,C). \tag{8.12}$$

Theorem 8.13. *The following data define an enriched category $\mathbb{KK}$:*

(8.13.1) $|\mathbb{KK}|$ *is the class of C^*-algebras;*
(8.13.2) *for a pair of C^*-algebras A, B the morphism object is $\mathbb{KK}(A,B)$*
(8.13.3) *for every triple of C^*-algebras A, B, C the composition morphism is the map* (8.12)
(8.13.4) *for a C^*-algebras A the unit morphism $u_A : E \to \mathbb{KK}(A,A)$ is the one determined by $(u_A)_\varnothing(S^0) = 0 \cup 1_A \subset \mathrm{Hom}(A,A) = \mathbb{KK}^\varnothing(A,A)$.*

[4]The composition can be written down more compactly using the $*$-homomorphism Δ^{MN}; this presentation however fits better with the strategy of the proof Theorem 8.13.

Proof. For disjoint finite subsets $M, N \subset \mathbb{N}$ define a map $\widetilde{\varrho}^{MN}$ by

$$(qC)^{M\cup N}A \overset{\varrho^{MN}}{\to} q^M q^N(C_0(\mathbb{R}^{M\cup N})\otimes A) \cong q^M q^N(C_0(\mathbb{R}^M)\otimes C_0(\mathbb{R}^N)\otimes A) \overset{q^M(\Delta^{\varnothing N})}{\to} q^M(C_0(\mathbb{R}^M)\otimes q^N(C_0(\mathbb{R}^N)\otimes A)) = (qC)^M(qC)^N A.$$

Furthermore define maps $\widetilde{\chi}^{MN}$ by

$$(qC)^M\mathcal{K}^N A = q^M(C_0(\mathbb{R}^M)\otimes \mathcal{K}^N A) \cong q^M(\mathcal{K}^N C_0(\mathbb{R}^M)\otimes A) \overset{\chi^{MN}}{\to} \mathcal{K}^N q^M(C_0(\mathbb{R}^M)\otimes A) = \mathcal{K}^N(qC)^M A,$$

and let the maps $\widetilde{\iota}^{MN} = \iota^{MN} : \mathcal{K}^M\mathcal{K}^N A \to \mathcal{K}^{M\cup N}A$ be the canonical isomorphisms. These maps define composition data for the functors $(qC)^\bullet$ and $\mathcal{K}^\bullet$ in the sense of 8.5. The commutativity of the relevant diagrams can be checked directly, or one can use the recipe that we introduced in 8.8. Proposition 8.6 then yields an enrichment of the category of C^*-algebras. It is straightforward to check that the composition morphism that one obtains from Proposition 8.6 coincides with the definition of c_{ABC} as given by (8.12). □

9. Symmetric monoidal enrichments over symmetric spaces

Next we want to show that the exterior Kasparov product gives $\mathcal{KK}$ and $\mathbb{KK}$ (introduced in the previous section) the structure of an enriched symmetric monoidal category. As the category $\mathbb{KK}$ is the category of preferred interest we only give the details for this case. The main result we are after is Theorem 9.5.

9.1. The bifunctor $\otimes : \mathbb{KK}\wedge\mathbb{KK} \to \mathbb{KK}$. To put a symmetric monoidal structure on $\mathbb{KK}$ requires the definition of a bifunctor $\otimes : \mathbb{KK}\wedge\mathbb{KK} \to \mathbb{KK}$. Recall from Definition 6.3 that a functor between enriched categories is given by two pieces of data. In the situation at hand these are given by the following

(9.1.1) To a pair of C^*-algebras (A, B) we certainly associate its (spatial) tensor product $A\otimes B$. After all we want to have an enrichment of the symmetric monoidal category of C^*-algebras;

(9.1.2) For two pairs of C^*-algebras (A, B), (A', B') we define the corresponding morphism $\mathbb{KK}(A, B)\wedge\mathbb{KK}(A', B') \to \mathbb{KK}(A\otimes A', B\otimes B')$ through the individual maps

$$\hom(q^M(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^M B)\wedge \hom(q^N(C_0(\mathbb{R}^N)\otimes A', \mathcal{K}^N B') \longrightarrow \hom(q^{M\cup N}(C_0(\mathbb{R}^{M\cup N})\otimes A), \mathcal{K}^{M\cup N}B\otimes B')$$

given by sending a pair of $*$-homomorphisms $f \wedge g$ to the following composition

$$\begin{aligned} q^{M\cup N}(C_0(\mathbb{R}^{M\cup N})A\otimes A') &\cong q^{M\cup N}((C_0(\mathbb{R}^M)\otimes A)\otimes(C_0(\mathbb{R}^N)\otimes A'))) \\ &\xrightarrow{\Delta^{MN}} q^M((C_0(\mathbb{R}^M)\otimes A)\otimes q^N(C_0(\mathbb{R}^N)\otimes A')) \\ &\xrightarrow{f\otimes g} \mathcal{K}^M B\otimes \mathcal{K}^N B' \cong \mathcal{K}^{M\cup N}B\otimes B'. \end{aligned}$$

One of course needs to check that this defines a bifunctor of enriched categories. We shall derive this and the main assertion we are after, which is Theorem 9.5, from a general recipe.

9.2. Monoidal product data. Let us be given composition data as in 8.5. Assume further that the $\mathcal{C}$-category $\mathcal{D}$ has a symmetric monoidal product in the sense of Definition 6.9 and Definition 6.10 respectively. A set of *monoidal product data* for $\mathcal{D}$ and the composition data consists of natural transformations of bifunctors

$$\Delta^{MN}: F^{M\cup N}(A\otimes B)\to F^M(A)\otimes F^N(B);$$
$$\nabla^{MN}: G^M(A)\otimes G^N(B)\to G^{M\cup N}(A\otimes B)$$

for any pair of finite subsets $M,N\subset\mathbb{N}$ with $M\cap N=\varnothing$. These functors must naturally depend on M and N. Moreover they have to satisfy the following conditions

(9.2.N) For any triple of pairwise disjoint finite subsets $K,L,M,N\subset\mathbb{N}$ the natural transformations must yield commutative diagrams

$$\begin{array}{ccc} F^{K\cup L\cup M\cup N}(A\otimes B) & \longrightarrow & F^{K\cup L}(A)\otimes F^{M\cup N}(B) \\ \downarrow & & \\ F^{K\cup M}F^{L\cup N}(A\otimes B) & & \downarrow \\ \downarrow & & \\ F^{K\cup M}(F^L(A)\otimes F^N(B)) & \longrightarrow & F^KF^L(A)\otimes F^MF^N(B); \end{array}$$

$$\begin{array}{ccc} G^LG^K(A)\otimes G^NG^M(B) & \longrightarrow & G^{L\cup N}(G^K(A)\otimes G^M(B)) \\ & & \downarrow \\ \downarrow & & G^{L\cup N}G^{K\cup M}(A\otimes B) \\ & & \downarrow \\ G^{K\cup L}(A)\otimes G^{M\cup N}(B) & \longrightarrow & G^{K\cup M\cup L\cup N}(A\otimes B); \end{array}$$

$$\begin{array}{ccc}
F^{K\cup M}G^{L\cup N}(A\otimes B) & \longrightarrow & G^{L\cup N}F^{K\cup M}(A\otimes B) \\
\uparrow & & \downarrow \\
F^{K\cup M}(G^{L}(A)\otimes G^{N}(B)) & & G^{L\cup N}(F^{K}(A)\otimes F^{M}(B)) \\
\downarrow & & \uparrow \\
F^{K}G^{L}(A)\otimes F^{M}G^{N}(B) & \longrightarrow & G^{L}F^{K}(A)\otimes G^{N}F^{M}(B)
\end{array}$$

(9.2.A) For any three disjoint finite subsets $L, M, N \subset \mathbb{N}$ and all objects $A, B, C \in |\mathcal{D}|$ the following diagram commutes

$$\begin{array}{ccc}
F^{L\cup M\cup N}((A\otimes B)\otimes C) & \xrightarrow{\Delta^{(L\cup M)N}} & F^{L\cup M}(A\otimes B)\otimes F^{N}(C) \\
\big\downarrow{\scriptstyle F^{L\cup M\cup N}(a_{ABC})} & & \\
F^{L\cup M\cup N}(A\otimes (B\otimes C)) & & \big\downarrow{\scriptstyle \Delta^{LM}\otimes 1} \\
\big\downarrow{\scriptstyle \Delta^{L(M\cup N)}} & & \\
F^{L}(A)\otimes F^{M\cup N}(B\otimes C) & \xrightarrow{1\otimes\Delta^{MN}} & F^{L}(A)\otimes F^{M}(B)\otimes F^{N}(C)
\end{array}$$

$$\begin{array}{ccc}
G^{L}(A)\otimes G^{M}(B)\otimes G^{N}(C) & \xrightarrow{1\otimes\nabla^{MN}} & G^{L}(A)\otimes G^{M\cup N}(B\otimes C) \\
 & & \big\downarrow{\scriptstyle \nabla^{L(M\cup N)}} \\
\big\downarrow{\scriptstyle \nabla^{LM}\otimes 1} & & G^{L\cup M\cup N}(A\otimes (B\otimes C)) \\
 & & \big\downarrow{\scriptstyle G^{L\cup M\cup N}(a_{ABC})} \\
G^{L\cup M}(A\otimes B)\otimes G^{N}(C) & \xrightarrow{\nabla^{(L\cup M)N}} & G^{L\cup M\cup N}((A\otimes B)\otimes C)
\end{array}$$

(9.2.S) For any pair of disjoint finite sets $M, N \subset \mathbb{N}$ and all objects $A, B \in |\mathcal{D}|$ the following diagram commutes

$$\begin{array}{ccc}
F^{M\cup N}(A\otimes B) & \xrightarrow{\Delta^{MN}} & F^{M}(A)\otimes F^{N}(B) \\
\big\downarrow{\scriptstyle F^{M\cup N}(s_{AB})} & & \big\downarrow{\scriptstyle s_{F^{M}(A),F^{N}(B)}} \\
F^{N\cup M}(B\otimes A) & \xrightarrow{\Delta^{NM}} & F^{N}(B)\otimes F^{M}(A)
\end{array}
\qquad
\begin{array}{ccc}
G^{M}A\otimes G^{N}B & \xrightarrow{\nabla^{MN}} & G^{M\cup N}(A\otimes B) \\
\big\downarrow{\scriptstyle s_{G^{M}A,G^{N}B}} & & \big\downarrow{\scriptstyle G^{M\cup N}(s_{AB})} \\
G^{N}B\otimes G^{M}A & \xrightarrow{\nabla^{NM}} & G^{N\cup M}(B\otimes A)
\end{array}$$

(9.2.U) For all $M \in |\mathcal{I}|$ and $A \in |\mathcal{D}|$ the following diagrams commute

$$\begin{array}{ccc} F^M(U\otimes A) & \xrightarrow{\Delta^{\varnothing M}} & U\otimes F^M(A) \\ \downarrow{\scriptstyle F^M(l_A)} & & \downarrow{\scriptstyle l_{F^M(A)}} \\ F^M(A) & \xrightarrow{=} & F^M(A) \\ \uparrow{\scriptstyle F^M(r_A)} & & \uparrow{\scriptstyle r_{F^M(A)}} \\ F^M(A\otimes U) & \xrightarrow{\Delta^{M\varnothing}} & F^M(A)\otimes U \end{array} \qquad \begin{array}{ccc} U\otimes G^M(A) & \xrightarrow{\nabla^{\varnothing M}} & G^M(U\otimes A) \\ \downarrow{\scriptstyle l_{G^M(A)}} & & \downarrow{\scriptstyle G^M(l_A)} \\ G^M(A) & \xrightarrow{=} & G^M(A) \\ \uparrow{\scriptstyle r_{G^M(A)}} & & \uparrow{\scriptstyle G^M(r_A)} \\ G^M(A)\otimes U & \xrightarrow{\nabla M\varnothing} & G^M(A\otimes U) \end{array}$$

Proposition 9.3. *Let us be given composition data as in 8.5. Assume further that the $\mathcal{C}$-category $\mathcal{D}$ has a symmetric monoidal product, and that we are given associated monoidal product data in the sense of 9.2. The following data then define a symmetric monoidal $\mathcal{C}^{\mathcal{I}}$-category which is an enrichment of $\mathcal{D}$:*

(9.3.1) *The $\mathcal{C}$-category is $\mathbb{D}$, defined as in Proposition 8.6;*

(9.3.2) *the bifunctor $m : \mathbb{D}\otimes\mathbb{D} \to \mathbb{D}$ is given on objects by $(A, A') \mapsto A \otimes A'$; the corresponding morphisms $\mathbb{D}(A, B)\otimes\mathbb{D}(A', B') \to \mathbb{D}(A\otimes A', B\otimes B')$ for pairs of objects (A, A') and (B, B') are given by the maps*

$$\mathcal{D}(F^M A, G^M B) \otimes \mathcal{D}(F^N A', G^N B') \to \mathcal{D}(F^{M\cup N}(A \otimes A'), G^{M\cup N}(B \otimes B'))$$

for disjoint finite sets $M, N \subset \mathbb{N}$ which send a pair $f \wedge g$ to the composition

$$F^{M\cup N}(A \otimes A') \xrightarrow{\Delta^{MN}} F^M(A) \otimes F^N(A')$$
$$\xrightarrow{f\otimes g} G^M(B) \otimes G^N(B') \xrightarrow{\nabla^{MN}} G^{M\cup N}(B \otimes B').$$

(9.3.3) *the unit is U, the unit of $\mathcal{D}$;*

(9.3.4) *for every triple A, B, C of objects the associativity isomorphism $a^{\mathbb{D}}_{ABC}$ is the one determined by $a^{\mathbb{D}}_{ABC}(\varnothing) = a^{\mathcal{D}}_{ABC} : I \to \mathcal{D}((A \otimes B) \otimes C, A \otimes (B \otimes C)) = \mathbb{D}^{\varnothing}((A \otimes B) \otimes C, A \otimes (B \otimes C))$.*

(9.3.5) *for every object A the left unit isomorphism $l^{\mathbb{D}}_A$ is the one determined by $l^{\mathbb{D}}_A(\varnothing) = l^{\mathcal{D}}_A : I \to \mathcal{D}(U \otimes A, A) = \mathbb{D}^{\varnothing}(U \otimes A, A)$.*

(9.3.6) *for every object A the right unit isomorphism $r^{\mathbb{D}}_A$ is the one determined by $r^{\mathbb{D}}_A(\varnothing) = r^{\mathcal{D}}_A : I \to \mathcal{D}(A \otimes U, A) = \mathbb{D}^{\varnothing}(A \otimes U, A)$.*

Proof. The commutative diagram displayed in Figure 1 (on the next page) shows the naturality condition (6.3.N) for the bifunctor m; to see commutativity one needs condition (9.2.N). From (9.2.U) it follows that the bifunctor m respects the unit condition (6.3.U). Similar diagrams show how to verify the associativity condition (6.9.A) from (9.2.A), the unit condition (6.9.U) from (9.2.U), and the symmetry condition (6.10.S) from (9.2.S). □

9.4. $\mathbb{KK}$ as an enriched symmetric monoidal category. Before we state the following theorem recall that the unit of the symmetric monoidal category of symmetric pointed spaces is the symmetric pointed space E with $E(\varnothing) = S^0 = \{0, +\}$ and $E(M) = \{+\}$ for all nonempty finite subsets $M \subset \mathbb{N}$. A map

$$
\begin{array}{ccccccc}
F^{K\cup L\cup M\cup N}(A\otimes A') & & & \xrightarrow{=} & & & F^{K\cup L\cup M\cup N}(A\otimes A') \\
\downarrow & & & & & & \downarrow \\
F^{K\cup M}(A)\otimes F^{L\cup N}(A') & \longrightarrow & F^{K}F^{M}(A)\otimes F^{L}F^{N}(A') & \longleftarrow & F^{K\cup L}(F^{M}(A)\otimes F^{N}(A')) & \longleftarrow & F^{K\cup L}F^{M\cup N}(A\otimes A') \\
\Big\downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & F^{K}G^{M}(B)\otimes F^{L}G^{N}(B') & \longleftarrow & F^{K\cup L}(G^{M}(A)\otimes G^{N}(A')) & \longrightarrow & F^{K\cup L}G^{M\cup N}(B\otimes B') \\
\Big\downarrow & & \downarrow & & & & \downarrow \\
 & & G^{M}F^{K}(B)\otimes G^{N}F^{L}(B') & \longrightarrow & G^{M\cup N}(F^{K}(B)\otimes F^{L}(B')) & \longleftarrow & G^{M\cup N}F^{K\cup L}(B\otimes B') \\
\Big\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G^{K\cup M}(C)\otimes G^{L\cup N}(C') & \longleftarrow & G^{K}G^{M}(C)\otimes G^{L}G^{N}(C') & \longrightarrow & G^{K\cup L}(G^{M}(C)\otimes G^{N}(C')) & \longrightarrow & G^{K\cup L}G^{M\cup N}(C\otimes C') \\
\downarrow & & & & & & \downarrow \\
G^{K\cup L\cup M\cup N}(C\otimes C') & & & \xrightarrow{=} & & & G^{K\cup L\cup M\cup N}(C\otimes C')
\end{array}
$$

Figure 1. The commutative diagram needed to verify the naturality condition in the proof of Proposition 9.3.

$f : E \to X$ from E to a symmetric pointed space X therefore is determined through the point $(f(\varnothing))(0) \in X(\varnothing)$.

Theorem 9.5. *Let $\mathbb{KK}$ be the enrichment of the category C^*-algebras over symmetric spaces that we introduced in Theorem 8.13. The following data put a symmetric monoidal structure on $\mathbb{KK}$ which is an enrichment of symmetric monoidal structure on the category of C^*-algebras given by the (spatial) tensor product*

(9.5.2) *the bifunctor $m : \mathbb{KK} \wedge \mathbb{KK} \to \mathbb{KK}$ is the one introduced in 9.1;*
(9.5.3) *the unit is $\mathbb{F}$;*
(9.5.4) *for every triple of C^*-algebras A, B, C the associativity isomorphism $a^{\mathbb{KK}}_{ABC}$ is the one determined by $a^{\mathbb{KK}}_{ABC}(\varnothing)(0) = a_{ABC}$, the associativity isomorphism for the spatial tensor product of C^*-algebras;*
(9.5.5) *for every object A the left unit isomorphism $l^{\mathbb{KK}}_A$ is the one determined by $l^{\mathbb{KK}}_A(\varnothing)(0) = l_A$, the left unit isomorphism for the spatial tensor product;*
(9.5.6) *for every object A the right unit isomorphism $r^{\mathbb{KK}}_A$ is the one determined by $r^{\mathbb{KK}}_A(\varnothing)(0) = r_A$, the right unit isomorphism for the spatial tensor product.*

Proof. Let A, B be C^*-algebras, and let M, N be disjoint finite subsets of the natural numbers $\mathbb{N}$. For these data define Δ^{MN} as in 4.9. On the other hand let $\nabla^{MN} : \mathcal{K}^M A \otimes \mathcal{K}^B \to \mathcal{K}^{M \cup N}(A \otimes B)$ be the canonical isomorphism which comes from the coherence isomorphism of the spatial tensor product. It is straightforward to check that the $*$-homomorphisms Δ^{MN}, ∇^{MN} define monoidal product data in the sense of 9.2 for the composition data $(q^\bullet, \mathcal{K}^\bullet, \widetilde{\varrho}^{MN}, \widetilde{\chi}^{MN}, \widetilde{\iota}^{MN})$ that we have used in the proof of Theorem 8.13 to define $\mathbb{KK}$. It then follows from Proposition 9.3 that the data given in the theorem define an enrichment of the symmetric monoidal category of C^*-algebras. ☐

10. The enrichment of the category *KK* over symmetric spectra

After we have seen (Theorem 9.5) that the category $\mathbb{KK}$ is a symmetric monoidal category which is enriched over the category of symmetric spaces it follows that the endomorphism object of the unit $\mathbb{KK}(\mathbb{F}, \mathbb{F})$ is a monoid in the category of symmetric spaces and $\mathbb{KK}$ inherits the structure of a symmetric monoidal category which is enriched over the category of $\mathbb{KK}(\mathbb{F}, \mathbb{F})$-modules. To obtain an enrichment over the category of spectra it suffices to turn $\mathbb{KK}(\mathbb{F}, \mathbb{F})$ into a symmetric ring spectrum, i.e. we need to define a monoid map from the sphere spectrum S into $\mathbb{KK}(\mathbb{F}, \mathbb{F})$.

10.1. The ring spectrum $\mathbb{K} = \mathbb{KK}(\mathbb{F}, \mathbb{F})$. Recall that a map of symmetric sequences $\eta : S \to \mathbb{KK}(\mathbb{F}, \mathbb{F})$ is determined by maps $\eta^M : S^M \to \operatorname{Hom}(q^M(C_0(\mathbb{R}^M) \otimes \mathbb{F}), \mathcal{K}^M \mathbb{F})$. We define these maps through their adjoints

$$\widehat{\eta}^M : S^0 \to \operatorname{Hom}(q^M(C_0(\mathbb{R}^M) \otimes \mathbb{F}), C_0(\mathbb{R}^M) \otimes \mathcal{K}^M \mathbb{F}).$$

To specify the map $\widehat{\eta}^M$ we just need to determine the image of the point which is different from the basepoint in S^0. This image we define to be the $*$-homomorphism

$$q^M(C_0(\mathbb{R}^M)\otimes \mathbb{F}) \xrightarrow{q^j} C_0(\mathbb{R}^M)\otimes \mathbb{F} \xrightarrow{\mathcal{K}^j} C_0(\mathbb{R}^M)\otimes \mathcal{K}^M\mathbb{F},$$

where j is the map $j : \varnothing \to M$, q^j is as defined in (4.7), and $\mathcal{K}^j$ is the composition

$$\mathcal{K}^j : \mathbb{F} \cong \mathbb{F}^{\otimes M} \otimes \mathbb{F} \xrightarrow{e^{\otimes M}\otimes id} \mathcal{K}^M\mathbb{F},$$

with $e^{\otimes M}$ the M-fold tensor product of the $*$-homomorphism $e : \mathbb{F} \to \mathcal{K}$, which is given by a fixed choice of a rank one projection (cp. 2.1). It is straightforward to check that these maps define a map of commutative monoids $S \to \mathbb{K} = \mathbb{KK}(\mathbb{F},\mathbb{F})$.

Theorem 10.2. *The symmetric monoidal category* $\mathbb{KK}$ *(defined by Theorem 8.13 and 9.5) is enriched over the category Sp of symmetric spectra (in fact* $\mathbb{K}$*-module spectra). The symmetric monoidal Sp-category* $\mathbb{KK}$ *is an enrichment of the symmetric monoidal category KK by means of the lax-monoidal functor* π_0 *which associates to a symmetric spectrum its* 0*-th homotopy group, i.e. there is a canonical isomorphism*

$$\pi_0(\mathbb{KK}(A,B)) \cong KK(A,B),$$

which is compatible with composition and the symmetric monoidal structure induced by the tensor product.

Proof. The first assertion is a consequence of 10.1. It remains to check that $\pi_0(\mathbb{KK}(A,B)) \cong KK(A,B)$ for all C^*-algebras A, B. We have $\pi_0\mathbb{KK}(A,B) = \operatorname{colim}_n \pi_n(\mathbb{KK}^n(A,B))$ where the structure maps

$$s_n : \pi_n(\mathbb{KK}^n(A,B)) \to \pi_{n+1}(\mathbb{KK}^{n+1}(A,B))$$

are induced by the structure maps of the spectrum. By construction they fit into the following diagram

$$\begin{array}{ccc}
\pi_n\mathbb{KK}^n(A,B) & \xrightarrow{\quad s_n \quad} & \pi_{n+1}\mathbb{KK}^{n+1}(A,B) \\
\downarrow\cong & & \downarrow\cong \\
\pi_0\Omega^n\mathbb{KK}^n(A,B) & \longrightarrow & \pi_0\Omega^{n+1}\mathbb{KK}^{n+1}(A,B) \\
\downarrow\cong & & \downarrow\cong \\
[q^n(C_0(\mathbb{R}^n)\otimes A), \mathcal{K}^nC_0(\mathbb{R}^n)\otimes B] & & [q^{n+1}(C_0(\mathbb{R}^{n+1})\otimes A), \mathcal{K}^{n+1}C_0(\mathbb{R}^{n+1})\otimes B] \\
\uparrow c_n & & \uparrow c_{n+1} \\
[q^nA, \mathcal{K}^nB] & \longrightarrow & [q^{n+1}A, \mathcal{K}^{n+1}B]
\end{array}$$

Here the maps between the first, second, and third row are induced by adjunction isomorphisms. The homomorphism c_n is induced by the map

$$Hom(q^n A, \mathcal{K}^n B) \to Hom(q^n(C_0(\mathbb{R}^n) \otimes A), \mathcal{K}^n C_0(\mathbb{R}^n) \otimes B)$$

which maps a $*$-homomorphism f to the composition $(id_{C_0(\mathbb{R}^n)} \otimes f) \circ \Delta^{\varnothing N}$ (see 4.9 for the definition of $\Delta^{\varnothing N}$). Via the isomorphisms $KK(A,B) \cong [q^n A, \mathcal{K}^n B]$ and $KK(C_0(\mathbb{R}^n) \otimes A, C_0(\mathbb{R}^n) \otimes B) \cong [q^n(C_0(\mathbb{R}^n) \otimes A), \mathcal{K}^n C_0(\mathbb{R}^n) \otimes B]$ the map c_n corresponds to the Bott periodicity isomorphism (8.10). The map $[q^n A, \mathcal{K}^n B] \to [q^{n+1} A, \mathcal{K}^{n+1} B]$ is the stabilization map (2.2). It follows that $\pi_0 \mathbb{KK}(A,B) \cong \operatorname{colim}_n [q^n A, \mathcal{K}^n B] = KK(A,B)$ (cp. Section 2). $\square$

11. $\mathbb{Z}/2$-graded C^*-algebras

The results of the previous sections deal with *ungraded* C^*-algebras. In this sections we want to discuss the modifications necessary to deal with $\mathbb{Z}/2$-graded C^*-algebras.

We recall that a $\mathbb{Z}/2$-grading on a C^*-algebra is a vector space decomposition $A = A_0 \oplus A_1$ such that the anti-involution $*$ preserves the A_is, and $A_i \cdot A_j \subset A_{i+j}$, $i, j \in \mathbb{Z}/2$. Equivalently, a $\mathbb{Z}/2$-grading on A is just an involution of the C^*-algebra A (the $+1$-eigenspace (resp. -1-eigenspace) of this "grading involution" is the subspace A_0 (resp. A_1)). Of course every ungraded C^*-algebra A can be interpreted as $\mathbb{Z}/2$-graded C^*-algebra by equipping it with the trivial grading involution (so that $A_0 = A$, $A_1 = 0$).

Kasparov defined his bivariant KK-groups $KK(A,B)$ not just for C^*-algebras A, B, but for $\mathbb{Z}/2$-graded C^*-algebras. It should be emphasized that $KK(A,B)$ does depend on the gradings of A, B; in other words: in general $KK(A,B)$ is *not* isomorphic to $KK(A^{ung}, B^{ung})$, where A^{ung}, B^{ung} are the C^*-algebras A, B equipped with the trivial grading.

Due to the functoriality of the construction of the Cuntz algebra an involution on the C^*-algebra A induces an involution on the Cuntz algebra qA. Hence we may consider the *equivariant Cuntz group* $[qA, \mathcal{K} \otimes B]_{\mathbb{Z}/2}$ of homotopy classes of grading preserving $*$-homomorphisms from qA to $\mathcal{K} \otimes B$, where $\mathcal{K}$ is the $\mathbb{Z}/2$-graded C^*-algebra of compact operators on a graded Hilbert space, and $\mathcal{K} \otimes B$ is the *graded* tensor product (which affects the definition of the product by setting $(a_1 \otimes b_1) \cdot (a_2 \otimes b_1) = (-1)^{|b_1||a_2|} a_1 b_1 \otimes a_2 b_2$). Cuntz' arguments showing that his group $[qA, \mathcal{K} \otimes B]$ is isomorphic to Kasparov's group $KK(A,B)$ generalize to show that there is an isomorphism (natural in both arguments) (cp. [4, (2)])

$$[qA, \mathcal{K} \otimes B]_{\mathbb{Z}/2} \cong KK_{\mathbb{Z}/2}(A,B), \tag{11.1}$$

where $KK_{\mathbb{Z}/2}(A,B)$ is the $\mathbb{Z}/2$-equivariant Kasparov group. Here $\mathbb{Z}/2$ acts on A, B via the grading involutions.

If the C^*-algebras A, B are trivially graded, then the group $KK_{\mathbb{Z}/2}(A,B)$ is isomorphic to $KK(A,B)$, but this is not the case in general for $\mathbb{Z}/2$-graded C^*-algebras. However, these groups are closely related; there is an isomorphism (natural in both arguments) (cp. [5, Def. 2.3, Prop. 3.8])

$$[q(\widehat{S} \otimes A), \mathcal{K} \otimes B]_{\mathbb{Z}/2} \cong KK(A, B). \tag{11.2}$$

Here $\widehat{S}$ is the $\mathbb{Z}/2$-graded C^*-algebra of continuous functions on the real line which vanish at ∞. The grading involution is induced by the involution $\mathbb{R} \to \mathbb{R}$, $x \mapsto -x$; in other words, $\widehat{S}_0$ consists of the even functions and $\widehat{S}_1$ consists of the odd functions.

As in the other sections of this paper we will exclusively work with the Cuntz picture; consequently, we will take the above isomorphisms as the *definitions* of $KK_{\mathbb{Z}/2}(A, B)$ resp. $KK(A, B)$.

Next we want to extend the discussion of the previous sections of the functorial properties of the KK-groups (the axiomatic characterization a à la Higson, the composition product and the tensor product) from ungraded C^*-algebras to $\mathbb{Z}/2$-graded C^*-algebras. First we will discuss $KK_{\mathbb{Z}/2}(A, B)$, which will turn out to be a straightforward extension of the corresponding results for ungraded algebras, then we will discuss how to adapt the setup to deal with $KK(A, B)$.

Let $C^*_{\mathbb{Z}/2}$ be the category of $\mathbb{Z}/2$-graded C^*-algebras (i.e., the objects are C^*-algebras equipped with involutions, and the morphisms are equivariant $*$-homomorphisms). Furnishing $KK_{\mathbb{Z}/2}(A, B) \overset{\text{def}}{=} [qA, \mathcal{K} \otimes B]_{\mathbb{Z}/2}$ with the structure of an abelian group as in 2.6, after fixing a $\mathbb{Z}/2$-graded C^*-algebra A we obtain a functor

$$KK_{\mathbb{Z}/2}(A, -)\colon C^*_{\mathbb{Z}/2} \to \mathrm{Ab}\,.$$

The results of Section 3 generalize to give the following theorem (cp. [5, Thm. 1]).

Theorem 11.3. *Let A be a $\mathbb{Z}/2$-graded C^*-algebra.*

(1) *The functor $KK_{\mathbb{Z}/2}(A, -)\colon C^*_{\mathbb{Z}/2} \to \mathrm{Ab}$ is homotopy invariant, stable and split exact.*

(2) *If $F\colon C^*_{\mathbb{Z}/2} \to \mathrm{Ab}$ is a homotopy invariant, stable and split exact functor, and $x \in F(A)$, then there is a unique natural transformation $\alpha\colon KK_{\mathbb{Z}/2}(A, -) \to F$ with $\alpha_A(1_A) = x$.*

We can define "composition products" and "tensor products" for the $\mathbb{Z}/2$-equivariant KK-groups, by the same formulas as in the nonequivariant case. As in the nonequivariant case, Theorem 11.3 implies that these products are unique. Moreover, we obtain a symmetric monoidal category $KK_{\mathbb{Z}/2}$ whose objects are C^*-algebras with involutions, and whose set of morphisms from A to B is $KK_{\mathbb{Z}/2}(A, B)$.

Now we will derive an axiomatic characterization of $KK(A, B)$ for $\mathbb{Z}/2$-equivariant C^*-algebras A, B which is analogous to the axiomatic characterization of $KK_{\mathbb{Z}/2}(A, B)$ in Theorem 11.3. The idea is to replace the category $C^*_{\mathbb{Z}/2}$ by the following category.

11.4. The category $\widehat{C}^*_{\mathbb{Z}/2}$. The objects of $\widehat{C}^*_{\mathbb{Z}/2}$ are $\mathbb{Z}/2$-graded C^*-algebras. A morphism from A to B is a $\mathbb{Z}/2$-equivariant $*$-homomorphism from $\widehat{S} \otimes A$

to B. The composition

$$\widehat{C}^*_{\mathbb{Z}/2}(A,B) \times \widehat{C}^*_{\mathbb{Z}/2}(B,C) \to \widehat{C}^*_{\mathbb{Z}/2}(A,C)$$

sends a pair of morphisms $f\colon \widehat{S} \otimes A \to B$, $g\colon \widehat{S} \otimes B \to C$ to the morphism given by

$$\widehat{S} \otimes A \stackrel{\Delta \otimes 1_A}{\longrightarrow} \widehat{S} \otimes \widehat{S} \otimes A \stackrel{1_{\widehat{S}} \otimes f}{\longrightarrow} \widehat{S} \otimes B \stackrel{g}{\to} C. \tag{11.5}$$

Here $\Delta\colon \widehat{S} \otimes \widehat{S} \to \widehat{S}$ is the $\mathbb{Z}/2$-equivariant $*$-homomorphism dual to the addition map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if we identify $\widehat{S} \otimes \widehat{S}$ with the space of functions of two *anti-commuting* variables x, y. To illustrate what is meant, consider the function $f(x) = e^{-x^2} \in \widehat{S}$. Then $\Delta(f)$ is a function of two variables, say x and y given by $\Delta(f) = f(x+y)$. To interpret $f(x+y)$ as an element of $\widehat{S} \otimes \widehat{S}$, we expand $f(x+y)$ in terms of x and y.

$$\Delta(e^{x^2}) = e^{(x+y)^2} = e^{x^2+xy+yx+y^2} = e^{x^2}e^{y^2} \in \widehat{S} \otimes \widehat{S}.$$

Here the third equality holds, since the variables x, y are assumed to *anti-commute*. Similarly, we have:

$$\Delta(xe^{x^2}) = (x+y)e^{(x+y)^2} = (x+y)e^{x^2}e^{y^2} = (xe^{x^2})e^{y^2} + e^{x^2}(ye^{y^2}) \in \widehat{S} \otimes \widehat{S}.$$

We note that the C^*-algebra $\widehat{S}$ is generated by e^{x^2} and xe^{x^2}. So we could have *defined* Δ by the above equations. Thinking of Δ as induced by addition makes it obvious that the "coproduct" Δ is $\mathbb{Z}/2$-equivariant (since $+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is) and the Δ is coassociative, while from the other point of view this needs a little calculation to check. Coassociativity of Δ implies associativity of the composition defined above.

11.6. The category $\widehat{KK}_{Z/2}$. The category $\widehat{KK}_{Z/2}$ is built out of the category $KK_{\mathbb{Z}/2}$ the same way the category $\widehat{C}^*_{\mathbb{Z}/2}$ is built from the category $C^*_{\mathbb{Z}/2}$; i.e.,

- The objects of $\widehat{KK}_{\mathbb{Z}/2}$ are the $\mathbb{Z}/2$-graded C^*-algebras;
- $\widehat{KK}_{\mathbb{Z}/2}(A,B) \stackrel{\text{def}}{=} KK_{\mathbb{Z}/2}(\widehat{S} \otimes A, B)$; i.e., a morphism from A to B in the category $\widehat{KK}_{\mathbb{Z}/2}$ is just a morphism from $\widehat{S} \otimes A$ to B in the category $KK_{\mathbb{Z}/2}$.
- the composition of a morphism $f \in \widehat{KK}_{\mathbb{Z}/2}(A,B)$ with a morphism $g \in \widehat{KK}_{\mathbb{Z}/2}(B,C)$ is given by the formula 11.5 with the only difference that now these arrows have to be interpreted as morphisms in the category $KK_{\mathbb{Z}/2}$ instead of as morphisms in $C^*_{\mathbb{Z}/2}$ (Δ is interpreted as morphism in $KK_{\mathbb{Z}/2}$ by means to the obvious functor $C^*_{\mathbb{Z}/2} \to KK_{\mathbb{Z}/2}$).

Passing from $\mathbb{Z}/2$-equivariant $*$-homomorphisms $\widehat{S} \otimes A \to B$ to elements of $KK_{\mathbb{Z}/2}(\widehat{S} \otimes A, B)$ then defines a functor

$$\widehat{C}^*_{\mathbb{Z}/2} \to \widehat{KK}_{\mathbb{Z}/2},$$

and, after fixing a $\mathbb{Z}/2$-graded C^*-algebra A, a functor

$$\widehat{KK}_{\mathbb{Z}/2}(A,-)\colon \widehat{C}^*_{\mathbb{Z}/2} \to \mathrm{Ab}\,.$$

We note that according to the isomorphism 11.2, we have $\widehat{KK}_{\mathbb{Z}/2}(A,B) \cong KK(A,B)$ for C^*-algebras A, B.

Theorem 11.3 then implies the following result.

Theorem 11.7. *Let A be a $\mathbb{Z}/2$-graded C^*-algebra.*

(1) *The functor $\widehat{KK}_{\mathbb{Z}/2}(A,-)\colon \widehat{C}^*_{\mathbb{Z}/2} \to \mathrm{Ab}$ is homotopy invariant, stable and split exact.*
(2) *If $F\colon \widehat{C}^*_{\mathbb{Z}/2} \to \mathrm{Ab}$ is a homotopy invariant, stable and split exact functor, and $x \in F(A)$, then there is a unique natural transformation $\alpha\colon \widehat{KK}(A,-) \to F$ with $\alpha_A(1_A) = x$.*

11.8. Enrichments of the categories $KK_{\mathbb{Z}/2}$ and $\widehat{KK}_{\mathbb{Z}/2}$ over the category of symmetric spectra. Since the Cuntz stabilization isomorphism ([5, Theorem 2.4]) and Bott periodicity (in the sense of Theorem 8.9) also hold for the $\mathbb{Z}/2$-graded setting one obtains completely analogous to the treatment of the ungraded setting an enrichment $\mathbb{KK}_{\mathbb{Z}/2}$ of $KK_{\mathbb{Z}/2}$ over the category of symmetric spectra. For a pair A, B of $\mathbb{Z}/2$-graded C^*-algebras the corresponding symmetric space $\mathbb{KK}_{\mathbb{Z}/2}(A,B)$ is given by

$$\mathbb{KK}_{\mathbb{Z}/2}(A,B) : M \mapsto Hom_{\mathbb{Z}/2}(q^M(C_0(\mathbb{R}^M) \otimes A), \mathcal{K} \otimes B).$$

For the category $\widehat{KK}_{\mathbb{Z}/2}$ one obtains an enrichment over symmetric spectra, if we define the morphism spectra by

$$\widehat{\mathbb{KK}}_{\mathbb{Z}/2}(A,B) : M \mapsto Hom_{\mathbb{Z}/2}(q^M(\widehat{S} \otimes C_0(\mathbb{R}^M) \otimes A), \mathcal{K} \otimes B).$$

On the formal level the treatment of this case is completely analogous to one above; however one has to work with the morphisms and the tensor product of the category $\widehat{C}^*_{\mathbb{Z}/2}$ instead of honest $*$-homomorphisms and the standard tensor product.

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