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The Baum–Connes conjecture for extensions

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Abstract. This note provides a counterexample showing that the assumptions that Chabert and Echterhoff have imposed in their permanence property of the Baum–Connes conjecture for group extensions cannot be simplified.

Chabert and Echterhoff showed in [1, Thm. 3.3] that a discrete group G with a normal subgroup N satisfies the Baum–Connes conjecture with coefficients in a G-C*-algebra B if and only if G/N satisfies the Baum–Connes conjecture with coefficients $B \rtimes_{\mathrm{r}} N$ and any subgroup $K \subseteq G$ with $N \subseteq K$ and $[K : N] < \infty$ satisfies the Baum–Connes conjecture with coefficients B. Recently, Zhang [6] claimed that it is enough here to assume the Baum–Connes conjecture for N only instead of for all the subgroups K above. This note shows by a counterexample that this cannot be the case. So the more complicated sufficient condition by Chabert and Echterhoff is really needed and cannot be made simpler. The group in our case is just a direct product of a countable discrete group by a finite cyclic group. So an intermediate claim about direct products in [6] is also wrong.

It is known that there are counterexamples to the Baum–Connes conjecture with coefficients. That is, there is a countable discrete group G and a separable G-C*-algebra such that the Baum–Connes assembly map for G with coefficients in A is not an isomorphism (see [2]). Let $D \in \mathrm{KK}^G(P, \mathbb{C})$ be a Dirac morphism for G as in [5] and let N be a mapping cone for it. As a consequence, there is a long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{K}_{*+1}\bigl((A \otimes N) \rtimes_{\mathrm{r}} G\bigr) \\ &\to \mathrm{K}_*\bigl((A \otimes P) \rtimes_{\mathrm{r}} G\bigr) \xrightarrow{D_*} \mathrm{K}_*\bigl(A \rtimes_{\mathrm{r}} G\bigr) \to \mathrm{K}_*\bigl((A \otimes N) \rtimes_{\mathrm{r}} G\bigr) \to \cdots, \end{aligned}$$

where the map D_* induced by D is the Baum–Connes assembly map. Then the $\mathbb{Z}/2$-graded group $\mathrm{K}_*((A \otimes N) \rtimes_{\mathrm{r}} G)$ cannot be trivial because this would imply that the Baum–Connes assembly map is an isomorphism.

Lemma 1. *There is a prime p so that* $\mathrm{K}_*((A \otimes N) \rtimes_{\mathrm{r}} G) \otimes \mathbb{Z}[1/p] \neq 0$.

Proof. By assumption, there is a nonzero element $x \in \mathrm{K}_*((A \otimes N) \rtimes_{\mathrm{r}} G)$. If its order is infinite, we may take p arbitrary. If the order of x is finite, then we take p to be a prime that does not divide the order of x. Then we have $p^n x \neq 0$ for all $n \in \mathbb{N}$. The tensor product commutes with inductive limits, and $\mathrm{K}_*((A \otimes N) \rtimes_{\mathrm{r}} G) \otimes \mathbb{Z}[1/p]$ is the inductive limit of the inductive system

$$\mathrm{K}_*\big((A \otimes N) \rtimes_{\mathrm{r}} G\big) \xrightarrow{p\cdot -} \mathrm{K}_*\big((A \otimes N) \rtimes_{\mathrm{r}} G\big) \xrightarrow{p\cdot -} \mathrm{K}_*\big((A \otimes N) \rtimes_{\mathrm{r}} G\big) \xrightarrow{p\cdot -} \cdots .$$

Therefore, the condition $p^n x \neq 0$ for all $n \in \mathbb{N}$ implies that the image of x in $\mathrm{K}_*((A \otimes N) \rtimes_{\mathrm{r}} G) \otimes \mathbb{Z}[1/p]$ is not a zero element. So it follows that this group is nontrivial. □

We fix a prime p as in the lemma. Let Γ be the cyclic group of order p. Actions of the group Γ on the stabilized Cuntz algebra $B := \mathcal{O}_2 \otimes \mathbb{K}(\ell^2\mathbb{N})$ that belong to the equivariant bootstrap class in KK^Γ are classified up to KK^Γ-equivalence in [4]. Since the K-theory of $\mathcal{O}_2$ vanishes, this is the rather simple special case already treated in [4, Thm. 7.23]. The main point for our purposes here is that $\mathrm{K}_*(B \rtimes_{\mathrm{r}} \Gamma)$ may be any uniquely p-divisible, $\mathbb{Z}/2$-graded module over the ring $\mathbb{Z}[\vartheta]$, where ϑ is a primitive pth root of unity. This has also been proven by Izumi (see [3, Thm. 6.4]). In particular, there is a Γ-action on B in the equivariant bootstrap class with

$$\mathrm{K}_0(B \rtimes_{\mathrm{r}} \Gamma) \cong \mathbb{Z}[1/p, \vartheta], \quad \mathrm{K}_1(B \rtimes_{\mathrm{r}} \Gamma) \cong 0.$$

It follows that $B \rtimes_{\mathrm{r}} \Gamma$ belongs to the bootstrap class and, therefore, also satisfies the Künneth formula.

Since Γ is a finite group, the Dirac morphism $D \in \mathrm{KK}^G(P, \mathbb{C})$ is also a Dirac morphism for $G \times \Gamma$ if we let Γ act trivially on everything. Therefore, the Baum–Connes assembly map for $G \times \Gamma$ with coefficients $A \otimes B$ is the map on K-theory induced by the map

$$(A \otimes P \otimes B) \rtimes_{\mathrm{r}} (\Gamma \times G) \xrightarrow{D_*} (A \otimes B) \rtimes_{\mathrm{r}} (\Gamma \times G).$$

Its mapping cone has the K-theory

$$\begin{aligned}\mathrm{K}_*\big((A \otimes N \otimes B) \rtimes_{\mathrm{r}} (\Gamma \times G)\big) &\cong \mathrm{K}_*\big((A \otimes N) \rtimes_{\mathrm{r}} G \otimes (B \rtimes_{\mathrm{r}} \Gamma)\big) \\ &\cong \mathrm{K}_*\big((A \otimes N) \rtimes_{\mathrm{r}} G\big) \otimes \mathbb{Z}[1/p, \vartheta].\end{aligned}$$

The second isomorphism uses the Künneth formula for $B \rtimes_{\mathrm{r}} \Gamma$ and that the group $\mathbb{Z}[1/p, \vartheta] \subseteq \mathbb{Q}$ is torsion-free. Since $\mathbb{Z}[1/p]$ is isomorphic to a direct summand in $\mathbb{Z}[1/p, \vartheta]$, Lemma 1 implies that $\mathrm{K}_*((A \otimes N \otimes B) \rtimes_{\mathrm{r}} (\Gamma \times G)) \neq 0$. Thus the Baum–Connes assembly map for $G \times \Gamma$ with coefficients $A \otimes B$ is not an isomorphism.

However, the Baum–Connes assembly map for G with coefficients $A \otimes B$ is an isomorphism because the tensor factor B kills all K-theory by the Künneth theorem. And the Baum–Connes assembly map for Γ with any coefficients is an isomorphism because Γ is finite. Therefore, we have got a counterexample for the statement in [6, Thm. 3.13] about the Baum–Connes conjecture for products of groups. It is also a counterexample for the statement in [6, Thm. 1.2] when we let the normal subgroup in $\Gamma \times G$ be $\{1\} \times G$. So both theorems are false.

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