

Regular inclusions of simple unital C^* -algebras

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Abstract. We prove that an inclusion $\mathcal{B} \subset \mathcal{A}$ of simple unital C^* -algebras with a finite-index conditional expectation is regular if and only if there exists a finite group G that admits a cocycle action (α, σ) on the intermediate C^* -subalgebra $\mathcal{C}$ generated by $\mathcal{B}$ and its centralizer $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$ such that $\mathcal{B}$ is outerly α -invariant and $(\mathcal{B} \subset \mathcal{A}) \cong (\mathcal{B} \subset \mathcal{C} \rtimes_{(\alpha, \sigma)}^r G)$.

Prior to this characterization, we prove the existence of two-sided and unitary quasi-bases for the minimal conditional expectation of any such inclusion, and also show that such an inclusion has integer Watatani index and depth at most 2.

1. INTRODUCTION

Of late, the study of inclusions of C^* -algebras has attracted a good deal of attention—see, for instance, [31, 18, 19, 17, 20, 28, 29, 5] and the references therein. Interestingly, people have focused on different perspectives of such inclusions and the project has thrived in the recent years. In fact, certain fundamental structure results have been achieved during the last 25 years or so. Among those results, Kajiwara–Watatani’s (Goldman type) characterization of index-2 inclusions of C^* -algebras as fixed-point inclusions via outer actions of the cyclic group $\mathbb{Z}_2$ ([24, Thm. 5.13]), Renault’s characterization of a Cartan pair $\mathcal{B} \subset \mathcal{A}$ of C^* -algebras ([28, Thm. 5.9]) and Izumi’s characterization of depth-2 inclusions of simple C^* -algebras as fixed-point inclusions via outer actions by finite-dimensional Kac algebras ([18, Cor. 6.4]) are noteworthy and serve as precursors to the theme of this article.

Our approach in [5] as well as here is mainly motivated by the Fourier theoretic aspects and certain structure results of the theory of subfactors, a theory which was initiated by Vaughan Jones in the 1980s—see [21, 22] for a quick introduction. More precisely, in [5], depending heavily on the fundamental work [31] of Watatani, we made an attempt to develop a Fourier theory on the tower of relative commutants (which was augmented further in [3]) and, motivated by [2], we also introduced a notion of angle between compatible intermediate

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C^* -subalgebras of an inclusion of unital C^* -algebras with a finite-index conditional expectation. In this article, we present a structure result for *regular* inclusions of simple unital C^* -algebras.

Recall that an inclusion $\mathcal{Q} \subset \mathcal{M}$ of von Neumann algebras is said to be regular if the normalizer of $\mathcal{Q}$ in $\mathcal{M}$, $\mathcal{N}_{\mathcal{M}}(\mathcal{Q}) := \{u \in \mathcal{U}(\mathcal{M}) \mid u\mathcal{Q}u^* = \mathcal{Q}\}$, generates $\mathcal{M}$ as a von Neumann-algebra. For any irreducible regular inclusion $N \subset M$ of factors, it is well-known that its Weyl group $G := \mathcal{N}_M(N)/\mathcal{U}(N)$ admits an outer cocycle action (α, σ) on N and $(N \subset M) \cong (N \subset N \rtimes_{(\alpha, \sigma)} G)$. Moreover, if N and M are II_1 -factors, then it is known, thanks to the work of Sutherland [30], that this cocycle action can be untwisted, *i.e.*, for any regular irreducible inclusion $N \subset M$ of factors of type II_1 with finite (Jones) index, its Weyl group G admits an outer action on N and $(N \subset M) \cong (N \subset N \rtimes G)$ (see [25, 27, 16, 23]).

Analogous to the notion of regularity of an inclusion of von Neumann algebras, we say that an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras is *regular* if the normalizer of $\mathcal{B}$ in $\mathcal{A}$ generates $\mathcal{A}$ as a C^* -algebra. The crux of this article lies in proving the following, somewhat general, characterization of regular inclusions of simple unital C^* -algebras.

Theorem 3.15. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Then the inclusion $\mathcal{B} \subset \mathcal{A}$ is regular if and only if there exists a finite group G that admits a cocycle action (α, σ) on the intermediate C^* -subalgebra $\mathcal{C} := C^*(\mathcal{B} \cup \mathcal{C}_{\mathcal{A}}(\mathcal{B}))$ such that*

- (i) $\mathcal{B}$ is invariant under α ;
- (ii) for each $e \neq g \in G$, α_g is an outer automorphism of $\mathcal{B}$;
- (iii) $(\mathcal{B} \subset \mathcal{A}) \cong (\mathcal{B} \subset \mathcal{C} \rtimes_{(\alpha, \sigma)}^r G)$.

As a consequence, for any irreducible regular inclusion $\mathcal{B} \subset \mathcal{A}$ of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$, its Weyl group $G := \mathcal{N}_{\mathcal{A}}(\mathcal{B})/\mathcal{U}(\mathcal{B})$ admits an outer cocycle action (α, σ) on $\mathcal{B}$ such that $(\mathcal{B} \subset \mathcal{A}) \cong (\mathcal{B} \subset \mathcal{B} \rtimes_{(\alpha, \sigma)}^r G)$ (see Corollary 3.16).

The proof of Theorem 3.15 relies on first showing that the intermediate C^* -subalgebra $\mathcal{C}$ generated by $\mathcal{B}$ and its centralizer

$$\mathcal{C}_{\mathcal{A}}(\mathcal{B}) := \{x \in \mathcal{A} \mid xb = bx \text{ for all } b \in \mathcal{B}\}$$

is compatible in the sense of Ino–Watatani [17], *i.e.*, there exists a (finite-index) conditional expectation $F : \mathcal{A} \rightarrow \mathcal{C}$ such that $(E_0)_{|_{\mathcal{C}}} \circ F = E_0$, where $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ is the unique minimal conditional expectation as in Theorem 2.10 (see Lemma 3.8), and then, employing the conditional expectation F , we obtain a natural outer cocycle action of the generalized Weyl group

$$W(\mathcal{B} \subset \mathcal{A}) := \frac{\mathcal{N}_{\mathcal{A}}(\mathcal{B})}{\mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))}$$

on the C^* -subalgebra $\mathcal{C}$, which leads to the desired characterization.

The skeleton of this article emerged essentially from our ongoing attempts at obtaining a better understanding of regular inclusions of type II_1 -factors (see

[4, 6]). While analyzing regular inclusions of II_1 -factors, it was also established in [6, 13] that every regular inclusion of type II_1 -factors $N \subset M$ with finite Jones index is always of depth at most 2. Drawing motivation from this, prior to the characterization obtained in Theorem 3.15, we also show that a regular inclusion $\mathcal{B} \subset \mathcal{A}$ of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ is always of depth¹ at most 2 (Theorem 3.13).

Theorem 3.13 is achieved on the lines of the proof of [6, Thm. 4.3] (also see [13, Cor. 3.2]), whose main ingredients are the following two observations, which are also of independent interest and fundamental in nature.

If $\mathcal{B} \subset \mathcal{A}$ is a regular inclusion of simple unital C^ -algebras with a finite-index conditional expectation and $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ is the minimal conditional expectation as in Theorem 2.10, then*

- (i) E_0 admits a two-sided orthonormal quasi-basis in $\mathcal{A}$ (Proposition 3.4);
- (ii) E_0 admits a unitary orthonormal quasi-basis in $\mathcal{A}$ (Corollary 3.12).

The proof of Proposition 3.4 borrows heavily from the methodology employed in [4], wherein, among other results, using the notion of path algebras by Sunder (and, independently, by Ocneanu), we had provided a partial answer to a question of Jones by showing that if $N \subset M$ is a regular inclusion of type II_1 -factors with finite Jones index, then the (unique) trace-preserving conditional $E : M \rightarrow N$ admits a two-sided Pimsner–Popa basis in M .

The proof of the fact that every regular finite-index inclusion of II_1 -factors is of depth at most 2 (demonstrated in [6, 13]) depended on the existence of a unitary orthonormal basis for the unique trace-preserving conditional expectation, which was essentially achieved by reducing the problem to a finite-dimensional set-up and then employing some nice unitary matrices from the world of quantum information theory. Here as well, a suitable adaptation of those techniques provides a proof of Corollary 3.12, which eventually allows us to deduce that every regular inclusion $\mathcal{B} \subset \mathcal{A}$ of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ has depth at most 2 (Theorem 3.13).

Along the way, analogous to [4, Thm. 3.12], we also prove that the Watatani index of a regular inclusion $\mathcal{B} \subset \mathcal{A}$ of simple unital C^* -algebras is the integer $|W(\mathcal{B} \subset \mathcal{A})| \dim(\mathcal{C}_{\mathcal{B}}(\mathcal{A}))$ (see Theorem 3.10).

2. PRELIMINARIES

2.1. Watatani index and C^* -basic construction.

2.1.1. Watatani index. Consider an inclusion $\mathcal{B} \subseteq \mathcal{A}$ of unital C^* -algebras with a conditional expectation $E : \mathcal{A} \rightarrow \mathcal{B}$. A finite set $\{\lambda_i \mid 1 \leq i \leq n\} \subset \mathcal{A}$ is said to be a *right* (resp., *left*) *quasi-basis* for E if

$$x = \sum_{i=1}^n E(x\lambda_i)\lambda_i^* \quad (\text{resp., } x = \sum_{i=1}^n E(x\lambda_i^*)\lambda_i)$$

¹Analogous to the notion of depth for inclusions of II_1 -factors, the notion of depth for inclusions of unital C^* -algebras was formalized in [20].

for every $x \in \mathcal{A}$. A finite collection $\{\lambda_i\} \subset \mathcal{A}$ is said to be a *two-sided quasi-basis* for E if it is both a left and a right quasi-basis for E . For example, if a right (or a left) quasi-basis for E is a selfadjoint set, i.e., $\{\lambda_i^*\} = \{\lambda_i\}$, then it is a two-sided quasi-basis for E . Note that a finite set $\{\lambda_i \mid 1 \leq i \leq n\}$ is a right quasi-basis for E if and only if $\{\lambda_i^* \mid 1 \leq i \leq n\}$ is a left quasi-basis for E .

A conditional expectation $E: \mathcal{A} \rightarrow \mathcal{B}$ is said to have *finite index* (in the sense of Watatani) if it admits a right (equivalently, a left) quasi-basis $\{\lambda_i \mid 1 \leq i \leq n\}$ in $\mathcal{A}$. Then the right Watatani index of E is defined as

$$\text{Ind}_W^r(E) = \sum_{i=1}^n \lambda_i \lambda_i^*$$

and is independent of the right quasi-basis. Likewise, the left Watatani index of E is defined as $\text{Ind}_W^l(E) = \sum_{i=1}^n \lambda_i^* \lambda_i$, where $\{\lambda_i\}$ is a left quasi-basis. Clearly,

$$\text{Ind}_W^r(E) = \text{Ind}_W^l(E)$$

and this common quantity is called the Watatani index of E and is denoted by $\text{Ind}_W(E)$. In general, $\text{Ind}_W(E)$ is not a scalar but it is an invertible positive element of $\mathcal{Z}(\mathcal{A})$.

Further, a right (resp., a left) quasi-basis $\{\lambda_i \mid 1 \leq i \leq n\}$ of E is said to be *orthonormal* if $E(\lambda_i^* \lambda_i) = \delta_{i,j}$ (resp., $E(\lambda_i \lambda_j^*) = \delta_{i,j}$) for all $1 \leq i, j \leq n$. And a right (or a left) quasi-basis of E is said to be *unitary* if it consists of unitary elements.

Note. Following Watatani [31], throughout this article, by a quasi-basis, we shall simply mean a right quasi-basis, and not a two-sided quasi-basis!

Remark 2.2. (i) A finite-index conditional expectation $E: \mathcal{A} \rightarrow \mathcal{B}$ is faithful and $1_{\mathcal{B}} = E(1_{\mathcal{A}}) = 1_{\mathcal{A}}$.

(ii) If $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ is an inclusion of unital C^* -algebras, and $E: \mathcal{A} \rightarrow \mathcal{C}$ and $F: \mathcal{C} \rightarrow \mathcal{B}$ are finite-index conditional expectations with right (resp., left) quasi-bases $\{\lambda_i\}$ and $\{\mu_j\}$, respectively, then $\{\lambda_i \mu_j\}_{i,j}$ (resp., $\{\mu_j \lambda_i\}_{j,i}$) is a right (resp., a left) quasi-basis for $F \circ E$ (see [31, Prop. 1.7.1]).

In particular, the composition of two finite-index conditional expectations

$$\mathcal{A} \xrightarrow{E} \mathcal{C} \quad \text{and} \quad \mathcal{C} \xrightarrow{F} \mathcal{B}$$

is also a finite-index conditional expectation and, moreover, if $\text{Ind}_W(F) \in \mathcal{Z}(\mathcal{A})$, then it follows readily that $\text{Ind}_W(F \circ E) = \text{Ind}_W(F) \text{Ind}_W(E)$.

2.2.1. Watatani's C^* -basic construction. We now briefly recall the theory of C^* -basic construction introduced by Watatani in [31].

Consider an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras with common unit and a faithful conditional expectation $E: \mathcal{A} \rightarrow \mathcal{B}$. Then $\mathcal{A}$ becomes a pre-Hilbert $\mathcal{B}$ -module with respect to the $\mathcal{B}$ -valued inner product given by

$$\langle x, y \rangle_{\mathcal{B}} = E(x^* y), \quad x, y \in \mathcal{A}.$$

Let $\mathfrak{A}$ denote the Hilbert $\mathcal{B}$ -module completion of $\mathcal{A}$ and let $\iota: \mathcal{A} \rightarrow \mathfrak{A}$ denote the isometric inclusion map with respect to the norm $\|x\|_{\mathfrak{A}} := \|E(x^* x)\|^{1/2}$,

$x \in \mathcal{A}$. The space $\mathcal{L}_{\mathcal{B}}(\mathfrak{A})$ consisting of adjointable $\mathcal{B}$ -linear maps on $\mathfrak{A}$ is a unital C^* -algebra.

For each $a \in \mathcal{A}$, consider $\lambda(a) \in \mathcal{L}_{\mathcal{B}}(\mathfrak{A})$ given by $\lambda(a)(\iota(x)) = \iota(ax)$ for $x \in \mathcal{A}$. The map $\iota(\mathcal{A}) \ni \iota(x) \mapsto \iota(E(x)) \in \iota(\mathcal{A})$ extends to an adjointable projection on $\mathfrak{A}$, and is denoted by $e_1 \in \mathcal{L}_{\mathcal{B}}(\mathfrak{A})$. The projection e_1 is called the Jones projection for the inclusion $\mathcal{B} \subset \mathcal{A}$; thus $e_1(\iota(x)) = \iota(E(x))$ for all $x \in \mathcal{A}$. Moreover, $e_1 \in \lambda(\mathcal{B})'$ and it satisfies the relation $e_1 \lambda(x) e_1 = \lambda(E(x)) e_1$ for all $x \in \mathcal{A}$.

The Watatani C^* -basic construction for the inclusion $\mathcal{B} \subset \mathcal{A}$ with respect to the conditional expectation E is defined as the C^* -subalgebra

$$\mathcal{A}_1 = \overline{\text{span}}\{\lambda(x)e_1\lambda(y) \mid x, y \in \mathcal{A}\} \text{ of } \mathcal{L}_{\mathcal{B}}(\mathfrak{A}).$$

Also, λ is an injective $*$ -homomorphism and thus we can consider $\mathcal{A}$ as a C^* -subalgebra of $\mathcal{A}_1$.

Remark 2.3 ([31]). With running notation and the identification mentioned in the preceding paragraph, if E is a finite-index conditional expectation, then the following hold.

- (i) $\mathcal{A}$ is complete with respect to the norm $\|\cdot\|_{\mathcal{A}}$ (see [5, Lem. 2.11]).
- (ii) $\mathcal{A}_1 = \text{span}\{xe_1y \mid x, y \in \mathcal{A}\} = C^*(\mathcal{A}, e_1)$ (see [31, Prop. 1.3.3]).
- (iii) There exists a unique finite-index conditional expectation $\tilde{E} : \mathcal{A}_1 \rightarrow \mathcal{A}$ satisfying

$$\tilde{E}(xe_1y) = x(\text{Ind}_W(E)^{-1})y = (\text{Ind}_W(E)^{-1}xy)$$

for all $x, y \in \mathcal{A}$ (see [31, Prop. 1.6.1]). ($\tilde{E}$ is called the dual conditional expectation of E .)

The following characterization of a (right) quasi-basis has proved to be extremely handy in the theory of subfactors and we have slowly started to realize that it is quite useful in the C^* -context as well. Its proof is most likely a folklore. For instance, its necessity is well-known—see [31, Lem. 2.1.6] (also see [5, Prop. 2.12])—and its sufficiency follows on the lines of the proof of $((2) \Rightarrow (3))$ of [1, Thm. 2.2] by applying [20, Lem. 3.7] (also see [5, Lem. 2.15]), which is the C^* -analog of the so-called “Pushdown Lemma” by Pimsner and Popa.

Proposition 2.4. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^* -algebras, let $E : \mathcal{A} \rightarrow \mathcal{B}$ be a finite-index conditional expectation, let $\mathcal{A}_1$ be the Watatani C^* -basic construction of $\mathcal{B} \subset \mathcal{A}$ with respect to E , and let e_1 be the corresponding Jones projection. Then a finite set $\{\lambda_i \mid 1 \leq i \leq n\} \subset \mathcal{A}$ is a (right) quasi-basis for E if and only if $\sum_{i=1}^n \lambda_i e_1 \lambda_i^* = 1$.*

The following result by Watatani is fundamental in nature and was used extensively in developing the Fourier theory for inclusions of simple unital C^* -algebras in [5, 3].

Theorem 2.5 ([31, Cor. 2.2.14]). *If $\mathcal{B}$ is a simple C^* -subalgebra of a unital C^* -algebra $\mathcal{A}$ and $E : \mathcal{A} \rightarrow \mathcal{B}$ is a finite-index conditional expectation, then the C^* -basic construction $\mathcal{A}_1$ of $\mathcal{B} \subset \mathcal{A}$ with respect to E is also simple.*

2.6. Markov trace and two-sided quasi-basis. Recall that, for a unital inclusion $Q \subset P$ of finite-dimensional C^* -algebras with inclusion matrix Λ , a tracial state τ on P with (column) trace vector $\bar{t}$ is said to be a Markov trace for the inclusion $Q \subset P$ with modulus $\beta > 0$ if $\Lambda^t \Lambda \bar{t} = \beta \bar{t}$. If $Q \subset P$ is a connected unital inclusion, then there exists a unique Markov trace for the inclusion $Q \subset P$ with modulus $\|\Lambda\|^2$. For more on Markov traces, see [22, 14].

Remark 2.7. If P is a finite-dimensional C^* -algebra with dimension vector $[n_1, \dots, n_k]$, then the inclusion matrix of any unital inclusion $\mathbb{C} \subset P$ is given by $\Lambda = [n_1, \dots, n_k] \in M_{1,k}(\mathbb{N})$ and the unique Markov trace for this inclusion with modulus $\|\Lambda\|^2$ has trace vector $\bar{t} = (n_1/d, \dots, n_k/d)^t$, where $d := \|\Lambda\|^2 = \dim(P)$.

The first part of the following observation was essentially made in [4] and was derived by employing the notion of path algebras associated to an inclusion of finite-dimensional C^* -algebras (introduced independently by Ocneanu and Sunder). We rephrase and reproduce some portion of it to suit the requirements of this article. (We must mention that the approach of path algebras was employed by Watatani as well to provide an example of a (selfadjoint and hence two-sided) quasi-basis for the trace-preserving conditional expectation from a finite-dimensional C^* -algebra onto a subalgebra—see [31, Lem. 2.4].)

Proposition 2.8 ([4]). *Let P be a finite-dimensional C^* -algebra and tr a faithful tracial state on P . Then there exists a two-sided orthonormal quasi-basis for tr and*

$$\text{Ind}_W(\text{tr}) = \sum_{i=1}^k \frac{n_i^2}{\text{tr}(p_i)} p_i,$$

where $\{p_i \mid 1 \leq i \leq k\}$ is the set of minimal central projections of P with $n_i^2 = \dim(p_i \mathcal{A})$ for all $1 \leq i \leq k$.

Moreover, $\text{Ind}_W(\text{tr})$ is a scalar if and only if tr is the Markov trace for the inclusion $\mathbb{C} \subset P$ with modulus $\dim(P)$. And in that case, $\text{Ind}_W(\text{tr}) = \dim(P)$.

Proof. That tr admits a two-sided orthonormal quasi-basis follows verbatim on the lines of the proof of [4, Prop. 3.3] (see also [6, Lem. 3.11]).

Moreover, from the proof of [4, Prop. 3.3], it also follows that there exists a system of matrix units

$$\{e_{(\kappa, \beta)}^{(i)} \mid 1 \leq \kappa, \beta \leq n_i, 1 \leq i \leq k\}$$

for P such that

$$\left\{ \sqrt{\frac{n_i}{\text{tr}(p_i)}} e_{(\kappa, \beta)}^{(i)} \mid 1 \leq \kappa, \beta \leq n_i, 1 \leq i \leq k \right\}$$

is a two-sided orthonormal quasi-basis for tr . Hence

$$\text{Ind}_W(\text{tr}) = \sum_{i=1}^k \sum_{\{1 \leq \kappa, \beta \leq n_i\}} \frac{\sqrt{n_i}}{\sqrt{\text{tr}(p_i)}} e_{(\kappa, \beta)}^{(i)} \frac{\sqrt{n_i}}{\sqrt{\text{tr}(p_i)}} (e_{(\kappa, \beta)}^{(i)})^* = \sum_{i=1}^k \frac{n_i^2}{\text{tr}(p_i)} p_i.$$

Next, suppose that tr is the Markov trace for the inclusion $\mathbb{C} \subset P$ with modulus $d = \dim(P)$; then its trace vector is given by $\bar{t} = (n_1/d, \dots, n_k/d)^t$. Thus $\text{tr}(p_i) = n_i^2/d$ for all i . Hence

$$\text{Ind}_W(\text{tr}) = \sum_{i=1}^k \frac{n_i^2}{\text{tr}(p_i)} p_i = \sum_{i=1}^k dp_i = d.$$

Conversely, suppose that $\text{Ind}_W(\text{tr})$ is a scalar, say, $c > 0$. Then

$$\sum_{i=1}^k \frac{n_i^2}{\text{tr}(p_i)} p_i = c = \sum_{i=1}^k cp_i,$$

which implies that $n_i^2/\text{tr}(p_i) = c$ for all i , so that $\text{tr}(p_i) = n_i^2/c$ for all i . So $\bar{s} := (n_1/c, \dots, n_k/c)^t$ is the trace vector of tr . Also, the inclusion matrix for $\mathbb{C} \subset P$ is given by $\Lambda = [n_1, \dots, n_k]$ so that $\|\Lambda\|^2 = \sum_{i=1}^k n_i^2 = \dim(P)$. Further, we observe that $\Lambda^t \Lambda \bar{s} = \|\Lambda\|^2 \bar{s}$, i.e., tr is the Markov trace for the inclusion $\mathbb{C} \subset P$ with modulus $\dim(P)$. (Further, by its uniqueness, it follows that $\bar{s} = \bar{t}$ and hence that $c = d$.) $\square$

The preceding observation is relatable to [31, Prop. 2.4.2 and Cor. 2.4.3] as well.

2.9. Minimal conditional expectation. Note that, for an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras, if $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, then every finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ has a scalar Watatani index. Recall from [31] that a conditional expectation is said to be minimal if it has the smallest Watatani index. In general, a minimal conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ need not exist. However, when it comes to inclusions of simple unital C^* -algebras, a minimal conditional expectation exists and is, in fact, unique.

Theorem 2.10 ([31, Thm. 2.12.3]). *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Then there exists a unique minimal conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ (which is usually denoted by E_0).*

Definition 2.11 ([31]). Let $\mathcal{B} \subset \mathcal{A}$ be as in the preceding theorem. Then the Watatani index of the inclusion $\mathcal{B} \subset \mathcal{A}$ is defined as $[\mathcal{A} : \mathcal{B}]_0 = \text{Ind}_W(E_0)$.

Remark 2.12. For $\mathcal{A}$, $\mathcal{B}$ and E_0 as in Theorem 2.10,

- (a) the C^* -basic construction $\mathcal{A}_1$ is also simple—see [31, Cor. 2.2.14];
- (b) the dual conditional expectation $\tilde{E}_0 : \mathcal{A}_1 \rightarrow \mathcal{A}$ is also minimal—see [24, Cor. 3.4];
- (c) $\text{Ind}_W(E_0) = \text{Ind}_W(\tilde{E}_0)$ (see [31, Prop. 2.3.4]).

For any inclusion $\mathcal{B} \subset \mathcal{A}$ of algebras, recall that the centralizer of $\mathcal{B}$ in $\mathcal{A}$ is given by $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) = \{x \in \mathcal{A} \mid xb = bx \text{ for all } b \in \mathcal{B}\}$, which is also called the relative commutant of $\mathcal{B}$ in $\mathcal{A}$ and denoted by $\mathcal{B}' \cap \mathcal{A}$.

Proposition 2.13. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ and let $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ denote the unique minimal conditional expectation as in Theorem 2.10. Then the following hold.*

(i) [31] *For any quasi-basis $\{\lambda_i \mid 1 \leq i \leq n\}$ of E_0 ,*

$$E_0(x) = \frac{1}{[\mathcal{A} : \mathcal{B}]_0} \sum_{i=1}^n \lambda_i x \lambda_i^* \quad \text{for all } x \in \mathcal{C}_{\mathcal{A}}(\mathcal{B}),$$

$E_0(\mathcal{C}_{\mathcal{A}}(\mathcal{B})) = \mathbb{C}$ and $\tau_0 := E_0|_{\mathcal{C}_{\mathcal{A}}(\mathcal{B})}$ is a faithful tracial state on $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$.

(ii) *Let $\mathcal{C} := C^*(\mathcal{B} \cup \mathcal{C}_{\mathcal{A}}(\mathcal{B}))$. Then the conditional expectation $E_0|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{B}$ admits a two-sided orthonormal quasi-basis contained in $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$.*

Proof. (i) Let $\{\lambda_i \mid 1 \leq i \leq n\}$ be a quasi-basis for E_0 . Since E_0 is minimal, it follows from [31, Prop. 1.2.9 and Thm. 2.12.3] that

(a) $E_0(x) = (1/[\mathcal{A} : \mathcal{B}]_0) \sum_{i=1}^n \lambda_i x \lambda_i^*$ for all $x \in \mathcal{C}_{\mathcal{A}}(\mathcal{B})$ (in particular, this expression is independent of the quasi-basis $\{\lambda_i\}$);

(b) $E_0(\mathcal{C}_{\mathcal{A}}(\mathcal{B})) = \mathcal{Z}(\mathcal{B}) = \mathbb{C}$;

(c) $\tau_0 := E_0|_{\mathcal{C}_{\mathcal{A}}(\mathcal{B})} : \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \rightarrow \mathbb{C}$ is a faithful tracial state.

(ii) Let $\{\mu_j\}$ be a two-sided orthonormal quasi-basis for τ_0 in $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$ as in Proposition 2.8. We assert that $\{\mu_j\}$ is a two-sided quasi-basis for $E_0|_{\mathcal{C}}$ as well. Note that, for any $z \in \mathcal{C}_{\mathcal{A}}(\mathcal{B})$ and $b \in \mathcal{B}$, $\{\mu_j\}$ being a right quasi-basis for τ_0 , we have

$$zb = \sum_j \tau_0(z\mu_j)\mu_j^*b = \sum_j E_0(z\mu_j)\mu_j^*b = \sum_j E_0(zb\mu_j)\mu_j^*,$$

and $\{\mu_j\}$ being a left quasi-basis for τ_0 , we have

$$zb = \sum_j \tau_0(z\mu_j^*)\mu_jb = \sum_j E_0(z\mu_j^*)\mu_jb = \sum_j E_0(zb\mu_j^*)\mu_j.$$

Since $\mathcal{C} = \overline{\text{span}}\{zb \mid z \in \mathcal{C}_{\mathcal{A}}(\mathcal{B}), b \in \mathcal{B}\}$, it follows that

$$\sum_j E_0(w\mu_j)\mu_j^* = w = \sum_j (w\mu_j^*)\mu_j \quad \text{for all } w \in \mathcal{C}.$$

Hence $\{\mu_j\}$ is a two-sided orthonormal quasi-basis for $E_0|_{\mathcal{C}}$. □

Watatani had realized in [31] itself that the minimal conditional expectation is characterized by the tracial property on the centralizer algebra $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$. This observation allowed us in [5] to obtain a sequence of consistent tracial states on the tower of finite-dimensional C^* -algebras

$$\mathbb{C} \cong \mathcal{Z}(\mathcal{B}) \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}_{\mathcal{A}_1}(\mathcal{B}) \subseteq \cdots \subseteq \mathcal{C}_{\mathcal{A}_k}(\mathcal{B}) \subseteq \cdots$$

which then paved way for a Fourier theory on this tower of centralizer algebras; see [5, 3].

2.14. Compatible intermediate C^* -subalgebras. As in [17] (also see [5, 15]), for an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras with a finite-index conditional expectation $E : \mathcal{B} \rightarrow \mathcal{A}$, let $\text{IMS}(\mathcal{B}, \mathcal{A}, E)$ denote the set of intermediate C^* -subalgebras $\mathcal{C}$ of $\mathcal{B} \subset \mathcal{A}$ with a compatible conditional expectation $F : \mathcal{A} \rightarrow \mathcal{C}$ satisfying the compatibility condition $E = E|_{\mathcal{C}} \circ F$.

We shall need the following well-known elementary observations.

Remark 2.15. With notation as in the preceding paragraph, let $\mathcal{A}_1$ denote the Watatani C^* -basic construction of $\mathcal{B} \subset \mathcal{A}$ with respect to E and let e_1 denote the corresponding Jones projection.

- (i) If $\mathcal{C} \in \text{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to two compatible conditional expectations F and F' , then $F = F'$. (See [17, p. 3].)
- (ii) If $\mathcal{C} \in \text{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the compatible conditional expectation $F : \mathcal{A} \rightarrow \mathcal{C}$, then F has finite index. (See [15, Rem. 2.4].)
- (iii) Let $\mathcal{C} \in \text{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the compatible conditional expectation $F : \mathcal{A} \rightarrow \mathcal{C}$, let $\mathcal{C}_1$ denote the Watatani C^* -basic construction of $\mathcal{C} \subset \mathcal{A}$ with respect to F (with Jones projection $e_{\mathcal{C}}$), and let $\{\lambda_i\}$ be a quasi-basis for $E|_{\mathcal{C}}$. Then $\mathcal{C}_1 \subset \mathcal{A}_1$ and $\sum_i \lambda_i^* e_1 \lambda_i = e_{\mathcal{C}}$. (See [15, Prop. 2.7].)

2.16. Reduced twisted crossed product. Recall that a discrete twisted C^* -dynamical system is a quadruple $(\mathcal{A}, G, \alpha, \sigma)$ consisting of a unital C^* -algebra $\mathcal{A}$, a discrete group G , a map $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ and a map $\sigma : G \times G \rightarrow \mathcal{U}(\mathcal{A})$ satisfying the following identities:

$$\begin{aligned}\alpha_g \circ \alpha_h &= \sigma(g, h) \alpha_{gh} \sigma(g, h)^*, \\ \sigma(g, h) \sigma(gh, k) &= \alpha_g(\sigma(h, k)) \sigma(g, hk), \\ \sigma(g, e) &= \sigma(e, g) = 1\end{aligned}$$

for all $g, h, k \in G$. Such a σ is called a normalized $\mathcal{U}(\mathcal{A})$ -valued 2-cocycle on G , and an α as above is called a twisted action of G on $\mathcal{A}$ with respect to the cocycle σ .

Note that if σ is the trivial map, i.e., $\sigma(g, h) = 1$ for all $g, h \in G$, then α is a homomorphism and $(\mathcal{A}, G, \alpha)$ is a usual C^* -dynamical system.

We shall work with the following working definition of the reduced twisted crossed product: for a (discrete) twisted C^* -dynamical system $(\mathcal{A}, G, \alpha, \sigma)$, there exist a representation $\mathcal{A} \subset B(\mathcal{H})$ and an injective map $u : G \rightarrow \mathcal{U}(B(\mathcal{H}))$ such that $u_g u_h = \sigma(g, h) u_{gh}$, $\alpha_g(a) = u_g a u_g^*$ for all $g, h \in G$ and $a \in \mathcal{A}$, and the reduced twisted crossed product $\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ (also denoted by $C_r^*(\mathcal{A}, G, \alpha, \sigma)$) can be identified with $C^*(\mathcal{A} \cup u(G)) \subset B(\mathcal{H})$. For more on reduced twisted crossed product, we refer the reader to [7, 8].

Remark 2.17. With notation as in the preceding paragraph, the following aspects of the reduced twisted crossed product will be relevant for this article.

- (i) $\{\sum_{\text{finite}} x_g u_g \mid x_g \in \mathcal{A}\}$ is a unital $*$ -subalgebra of $\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ as

$$x u_g y u_h = x \alpha_g(y) u_g u_h = x \alpha_g(y) \sigma(g, h) u_{gh}$$

for all $x, y \in \mathcal{A}$, $g, h \in G$, and thus $\{\sum_{\text{finite}} x_g u_g \mid x_g \in \mathcal{A}\}$ is dense in $\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$.

- (ii) $(u_g)^* = u_{g^{-1}}\sigma(g, g^{-1})^* = \sigma(g, g^{-1})^*u_{g^{-1}}$ for all $g \in G$ (see [7, p. 5]).
- (iii) There exists a faithful conditional expectation $E : \mathcal{A} \rtimes_{(\alpha, \sigma)}^r G \rightarrow \mathcal{A}$ such that $E(u_g) = 0$ for all $e \neq g \in G$ (see [7, p. 7]).
- (iv) The canonical conditional expectation is G -equivariant, i.e., $E(u_g x u_g^*) = \alpha_g(E(x))$ for all $x \in \mathcal{A}$ and $g \in G$ (see [7, p. 8]).
- (v) If G is finite, then the unital $*$ -subalgebra $\{\sum_{\text{finite}} x_g u_g \mid x_g \in \mathcal{A}\}$ is closed and hence

$$\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G = \left\{ \sum_{g \in G} x_g u_g \mid x_g \in \mathcal{A} \right\}.$$

Moreover, $\{u_g \mid g \in G\}$ is a quasi-basis for E because if

$$x = \sum_g x_g u_g \in \mathcal{A} \rtimes_{(\alpha, \sigma)}^r G,$$

then $E(x u_g^*) = x_g$ for all $g \in G$, so that

$$x = \sum_g E(x u_g^*) u_g$$

for all $x \in \mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$. In particular, $\text{Ind}_W(E) = |G|$.

- (vi) When $\mathcal{A}$ is simple and (α, σ) is outer, then there is a Galois correspondence between subgroups of G and intermediate C^* -subalgebras of $\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ (see [8, Thm. 5.2]).

2.18. Some generalities.

2.18.1. Outer automorphisms and free automorphisms. Recall that an automorphism θ of a unital C^* -algebra $\mathcal{A}$ is said to be free if, for a given $y \in \mathcal{A}$, $y\theta(x)y$ for every $x \in \mathcal{A}$ if and only if $y = 0$.

It is easily seen that a free automorphism is outer (i.e., not inner) and it is well-known that an automorphism of a II_1 -factor is free if and only if it is outer. Analogous to this, it can be deduced easily from [10] that the same equivalence holds for any automorphism of a unital C^* -algebra with trivial center. We derive it here for the sake of convenience.

Lemma 2.19 ([10]). *Let θ be an automorphism of a unital C^* -algebra $\mathcal{A}$ with $\mathcal{Z}(\mathcal{A}) \cong \mathbb{C}$. Then θ is outer if and only if it is free.*

Proof. We just need to show the necessity. So let θ be outer.

Suppose that there exists an $a \in \mathcal{A}$ such that $ax = \theta(x)a$ for all $x \in \mathcal{A}$. Then, by [10, Thm. 1], $aa^* = a^*a \in \mathcal{Z}(\mathcal{A}) \cong \mathbb{C}$; thus $aa^* = a^*a = \|a\|^2$. So if $a \neq 0$, then $u := \|a\|^{-1}a$ is a unitary in $\mathcal{A}$ and $\theta(x) = uxu^*$ for all $x \in \mathcal{A}$, which contradicts the outerness of θ . Hence $a = 0$, i.e., θ is free. $\square$

2.19.1. Regular inclusions. Recall that, for an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras with common identity, the normalizer of $\mathcal{B}$ in $\mathcal{A}$ is the group of unitaries given by $\mathcal{N}_{\mathcal{A}}(\mathcal{B}) = \{u \in \mathcal{U}(\mathcal{A}) \mid u\mathcal{B}u^* = \mathcal{B}\}$, and we say that the inclusion $\mathcal{B} \subset \mathcal{A}$ is regular if $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$ generates the C^* -algebra $\mathcal{A}$.

Remark 2.20. It must be mentioned here that Kumjian and Renault use a different definition for the normalizer, namely, the set

$$\{x \in \mathcal{A} \mid x\mathcal{B} \subset \mathcal{B}, \mathcal{B}x \subset \mathcal{B}\}.$$

And Renault [28] calls an inclusion $\mathcal{B} \subset \mathcal{A}$ regular if $\{x \in \mathcal{A} \mid x\mathcal{B} \subset \mathcal{B}, \mathcal{B}x \subset \mathcal{B}\}$ generates $\mathcal{A}$ as a C^* -algebra. It has been kindly pointed out to us by Renault (in a private communication) that both definitions are equivalent when $\mathcal{B} \subset \mathcal{A}$ is an inclusion of W^* -algebras and $\mathcal{B}$ is a MASA in $\mathcal{A}$. It will be interesting to know whether the two notions of regularity are equivalent for a general inclusion of unital C^* -algebras.

Example 2.21. Consider a reduced twisted crossed product $\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ as in Remark 2.17. Since $u_g x u_g^* = \alpha_g(x)$ for all $x \in \mathcal{A}$, $g \in G$, it follows that

$$\{u_g \mid g \in G\} \subset \mathcal{N}_{(\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G)(\mathcal{A})}$$

and hence that $\mathcal{A} \subset \mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ is a regular inclusion.

Theorem 2.22 ([7, Thm. 3.2] and [8, Thm. 5.1]). *With notation as in Remark 2.17, if $\mathcal{A}$ is simple and α_g is outer for every $e \neq g \in G$, then $\mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ is simple and the inclusion $\mathcal{A} \subset \mathcal{A} \rtimes_{(\alpha, \sigma)}^r G$ is irreducible.*

As a consequence of the main result of this article, we shall see in Corollary 3.16 that every irreducible regular inclusion of simple unital C^* -algebras with a finite-index conditional expectation arises only in this fashion.

As mentioned in the introduction, the essence of this article lies in Theorem 3.15, wherein we establish that every finite-index regular inclusion of simple unital C^* -algebras can be realized as a cocycle crossed product via an outer action of a finite group.

2.22.1. Finite-depth C^* -inclusions. The notion of depth is well established in the theory of subfactors (see [26]). Recently, its analog in the theory of C^* -algebras has been developed and studied in good detail in [20]. We shall only need the definition.

Consider an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^* -algebras with a finite-index conditional expectation $E : \mathcal{A} \rightarrow \mathcal{B}$. Then consider the Watatani C^* -basic construction $\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_1$ with respect to the conditional expectation E . We know that the dual conditional expectation $E_1 : \mathcal{A}_1 \rightarrow \mathcal{A}$ also has finite index (Remark 2.3). Thus one can iterate the C^* -basic construction to obtain a tower of unital C^* -algebras

$$\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_k \subset \cdots,$$

where, for each $k \geq 0$, $\mathcal{A}_{k+1} = C^*(\mathcal{A}_k \cup \{e_{k+1}\})$ denotes the C^* -basic construction of the inclusion $\mathcal{A}_{k-1} \subset \mathcal{A}_k$ with respect to the finite-index conditional expectation $E_k : \mathcal{A}_k \rightarrow \mathcal{A}_{k-1}$, with $\mathcal{A}_0 := \mathcal{A}$ and $\mathcal{A}_{-1} := \mathcal{B}$.

The inclusion $\mathcal{B} \subset \mathcal{A}$ is said to have finite depth if

$$(\mathcal{B}' \cap \mathcal{A}_k) = (\mathcal{B}' \cap \mathcal{A}_{k-1})e_k(\mathcal{B}' \cap \mathcal{A}_{k-1})$$

for some $k \geq 1$. The least such k is called the depth of $\mathcal{B} \subset \mathcal{A}$. We shall show in Theorem 3.13 that every regular inclusion of simple unital C^* -algebras with a finite-index conditional expectation has depth at most 2. We refer the reader to [20] for more on finite-depth inclusions of C^* -algebras.

3. STRUCTURE OF REGULAR INCLUSIONS OF SIMPLE C^* -ALGEBRAS

3.1. Generalized Weyl group of an inclusion of C^* -algebras. Note that if $\mathcal{B} \subset \mathcal{A}$ is an inclusion of unital C^* -algebras with common unit, then $\mathcal{U}(\mathcal{B})$ and $\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$ are both normal subgroups of $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$ and hence $\mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$ ($= \mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))\mathcal{U}(\mathcal{B})$) is also a normal subgroup of $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$.

Analogous to the notions of the Weyl group and the generalized Weyl group of an inclusion of von Neumann algebras (see [4, Def. 2.11]), we make the following definitions.

Definition 3.2. Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^* -algebras with common unit. Then

- (i) the Weyl group of the inclusion $\mathcal{B} \subset \mathcal{A}$ is defined as the quotient group $\mathcal{N}_{\mathcal{A}}(\mathcal{B})/\mathcal{U}(\mathcal{B})$ and will be denoted by $W_0(\mathcal{B} \subset \mathcal{A})$;
- (ii) the generalized Weyl group of the inclusion $\mathcal{B} \subset \mathcal{A}$ is defined as the quotient group $\mathcal{N}_{\mathcal{A}}(\mathcal{B})/(\mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B})))$ and will be denoted by $W(\mathcal{B} \subset \mathcal{A})$.

Clearly, there exists a canonical surjective homomorphism from the Weyl group onto the generalized Weyl group. And if $\mathcal{B} \subset \mathcal{A}$ is an irreducible inclusion, i.e., $\mathcal{B}' \cap \mathcal{A} = \mathbb{C}$, then $W(\mathcal{B} \subset \mathcal{A}) = W_0(\mathcal{B} \subset \mathcal{A})$. In this article, we shall focus only on the generalized Weyl group.

The following elementary observation will be used ahead.

Lemma 3.3. Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^* -algebras and let w be a unitary in $\mathcal{N}_{\mathcal{A}}(\mathcal{B}) \setminus \mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$. Then Ad_w is an outer automorphism of $\mathcal{B}$.

Moreover, if $\mathcal{Z}(\mathcal{B}) \cong \mathbb{C}$, then Ad_w is a free automorphism of $\mathcal{B}$ and, for any conditional expectation $E: \mathcal{A} \rightarrow \mathcal{B}$, $E(w) = 0$. In particular, for any two elements $u, v \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$, $E(vu^*) = 0 = E(v^*u)$ if $[u] \neq [v]$ in $W(\mathcal{B} \subset \mathcal{A})$.

Proof. We have $w\mathcal{B}w^* = \mathcal{B}$. Let $\varphi = \text{Ad}_w$ and suppose, on contrary, that φ is not an outer automorphism of $\mathcal{B}$. Then there exists a $v \in \mathcal{U}(\mathcal{B})$ such that $wxw^* = vxv^*$ for all $x \in \mathcal{B}$. This implies that $v^*w \in \mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$, so that $w \in \mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$, which is a contradiction. Thus Ad_w must be an outer automorphism of $\mathcal{B}$.

If $\mathcal{Z}(\mathcal{B}) \cong \mathbb{C}$, then by Lemma 2.19, it follows that φ is free as well. Further, let $E: \mathcal{A} \rightarrow \mathcal{B}$ be a conditional expectation. Since $\varphi(x)w = wx$ for all $x \in \mathcal{B}$, it follows that

$$\varphi(x)E(w) = E(w)x \quad \text{for all } x \in \mathcal{B}.$$

So, by freeness of φ , we must have $E(w) = 0$. □

A priori, it is not clear whether the generalized Weyl group of an inclusion $\mathcal{B} \subset \mathcal{A}$ is finite or not. However, if $\mathcal{B}$ has trivial center and there exists a finite-

index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$, we can show that $W(\mathcal{A} \subset \mathcal{B})$ is finite and provide a bound for its cardinality.

Proposition 3.4. *Suppose that $\mathcal{B} \subset \mathcal{A}$ is an inclusion of unital C^* -algebras with $\mathcal{Z}(\mathcal{B}) \cong \mathbb{C}$. If there exists a finite-index conditional expectation $E : \mathcal{A} \rightarrow \mathcal{B}$, then $W(\mathcal{B} \subset \mathcal{A})$ is finite and $|W(\mathcal{B} \subset \mathcal{A})| \leq \dim(\mathcal{B}' \cap \mathcal{A}_1)$, where $\mathcal{A}_1$ is the C^* -basic construction of $\mathcal{B} \subset \mathcal{A}$ with respect to the conditional expectation E .*

Proof. Let $G := W(\mathcal{B} \subset \mathcal{A})$ and let $\{u_g \mid g \in G\}$ denote a set of coset representatives of G in $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$. Also, let e_1 denote the Jones projection corresponding to E .

We first assert that $\{u_g e_1 u_g^* \mid g \in G\}$ is a collection of mutually orthogonal projections in the algebra $\mathcal{B}' \cap \mathcal{A}_1$.

Note that, for each $g \in G$, since $e_1 \in \mathcal{B}'$ and $u_g^* \mathcal{B} u_g = \mathcal{B}$, we have

$$(u_g e_1 u_g^*)x = u_g e_1 (u_g^* x u_g) u_g^* = u_g (u_g^* x u_g) e_1 u_g^* = x (u_g e_1 u_g^*)$$

for all $x \in \mathcal{B}$. Hence $u_g e_1 u_g^* \in \mathcal{B}' \cap \mathcal{A}_1$ for all $g \in G$. Further, by Lemma 3.3, we observe that

$$(u_g e_1 u_g^*)(u_h e_1 u_h^*) = u_g E(u_g^* u_h) e_1 u_h^* = \delta_{g,h} u_g e_1 u_g^*$$

for all $g, h \in G$. This proves our assertion (for which we did not require E to have finite index).

Next, let $\tilde{E} : \mathcal{A}_1 \rightarrow \mathcal{A}$ denote the dual conditional expectation of E . Then $\tilde{E}$ and (hence) $\tilde{E} \circ E : \mathcal{A}_1 \rightarrow \mathcal{B}$ are finite-index conditional expectations—see Remark 2.2. Thus it follows from [31, Prop. 2.7.3] (also see [5, Prop. 2.16]) that $\mathcal{B}' \cap \mathcal{A}_1$ is finite-dimensional. Hence we have that G must be finite and $|G| \leq \dim(\mathcal{B}' \cap \mathcal{A}_1)$. $\square$

3.5. Two-sided and unitary bases for regular inclusions. The following is an obvious adaptation of [4, Lem. 3.5]. We skip the proof.

Lemma 3.6 ([4]). *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^* -algebras with a finite-index conditional expectation $E : \mathcal{A} \rightarrow \mathcal{B}$ and let $\mathcal{C}$ denote the intermediate C^* -subalgebra generated by $\mathcal{B}$ and its centralizer $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$. If θ is an automorphism of $\mathcal{C}$ whose restriction to $\mathcal{B}$ is an outer automorphism of $\mathcal{B}$, then θ is a free automorphism of $\mathcal{C}$.*

Corollary 3.7. *Let the notation be as in Lemma 3.6. Then*

$$\mathcal{N}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{C}_{\mathcal{A}}(\mathcal{B})) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C})$$

and, for each $w \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \setminus \mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$, $w\mathcal{C}w^* = \mathcal{C}$, Ad_w is a free automorphism of $\mathcal{C}$ and $E(w) = 0$.

In particular, for any two elements $u, v \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$, $E(vu^*) = 0 = E(v^*u)$ if $[u] \neq [v]$ in the generalized Weyl group $W(\mathcal{B} \subset \mathcal{A})$.

Proof. That $\mathcal{N}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{C}_{\mathcal{A}}(\mathcal{B})) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C})$ follows on the lines of the proof of [4, Lem. 3.2]. Further, it readily follows that $w\mathcal{C}w^* = \mathcal{C}$. Then, because of the preceding lemma, the rest follows on the lines of Lemma 3.3. $\square$

Lemma 3.8. *Let the notation be as in Lemma 3.6 and denote by $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ the minimal conditional expectation as in Theorem 2.10. Then $\mathcal{C} \in \text{IMS}(\mathcal{B}, \mathcal{A}, E_0)$.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are simple and $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ has finite index, it follows from [18, Prop. 6.1] that there exists a (finite-index) conditional expectation $F : \mathcal{A} \rightarrow \mathcal{C}$. We assert that F is compatible with respect to E_0 .

Note that $(E_0)|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{B}$ has finite index, by Proposition 2.13. Thus, in view of [31, Lem. 2.12.2], it is enough to show that the restrictions of E_0 and $(E_0)|_{\mathcal{C}} \circ F$ to $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$ are the same. Clearly, $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) = \mathcal{B}' \cap \mathcal{C}$ because $\mathcal{C}$ contains $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$, and for any $z \in \mathcal{B}' \cap \mathcal{C}$, we have

$$((E_0)|_{\mathcal{C}} \circ F)(z) = (E_0)|_{\mathcal{C}}(z) = E_0(z).$$

Hence F is a compatible conditional expectation and $\mathcal{C} \in \text{IMS}(\mathcal{B}, \mathcal{A}, E_0)$. $\square$

We shall now focus only on regular inclusions of simple unital C^* -algebras. Thus, from here on, $\mathcal{B} \subset \mathcal{A}$ denotes a fixed regular inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$.

Let $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ denote the (unique) minimal conditional expectation as in Theorem 2.10 and let G denote the generalized Weyl group of the inclusion $\mathcal{B} \subset \mathcal{A}$ with a fixed set of left coset representatives $\{u_g \mid g \in G\}$ in $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$, with $u_e = 1$. Further, let $\mathcal{C} := C^*(\mathcal{B} \cup (\mathcal{C}_{\mathcal{A}}(\mathcal{B})))$, let $F : \mathcal{A} \rightarrow \mathcal{C}$ be the compatible conditional expectation as in Lemma 3.8 and let $\mathcal{A}_1$ denote the Watatani C^* -basic construction of $\mathcal{B} \subset \mathcal{A}$ with respect to the minimal conditional expectation E_0 (and Jones projection $e_1 \in \mathcal{A}_1$).

Proposition 3.9. *The set $\{u_g \mid g \in G\}$ is a two-sided orthonormal quasi-basis for F . In particular, $\text{Ind}_W(F) = |G|$.*

Proof. We first assert that $\sum_{g \in G} u_g \mathcal{C} = \mathcal{A}$.

Let $\mathcal{L} := \sum_{g \in G} u_g \mathcal{C}$. Note that, for any two $g, h \in G$, $(u_g \mathcal{C})(u_h \mathcal{C}) = u_k \mathcal{C}$ for some $k \in G$. Also, for any $g \in G$, we have $(u_g \mathcal{C})^* = \mathcal{C} u_g^* = u_g^* u_g \mathcal{C} u_g^* = u_g^* \mathcal{C}$ because $u_g \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{C})$ (by Corollary 3.7), so that $(u_g \mathcal{C})^* = u_k \mathcal{C}$ for some $k \in G$. Hence $\mathcal{L}$ is a unital $*$ -subalgebra of $\mathcal{A}$. Further, since $\mathcal{B} \subset \mathcal{A}$ is regular, it follows that $\mathcal{L}$ is dense in $\mathcal{A}$ (because $\mathcal{N}_{\mathcal{A}}(\mathcal{B}) = \cup_{g \in G} u_g \mathcal{C}$). So it just remains to show that $\mathcal{L}$ is closed.

Let $a \in \overline{\mathcal{L}}$. Then $\sum_{g \in G} u_g c_g^{(n)} \rightarrow a$ for some sequence $\{\sum_g u_g c_g^{(n)}\} \subset \mathcal{L}$. Thus

$$F\left(u_h^* \sum_{g \in G} u_g c_g^{(n)}\right) \rightarrow F(u_h^* a)$$

for all $h \in G$. Note that, by Corollary 3.7, we have $F(u_s^* u_t) = \delta_{s,t}$ for all $s, t \in G$, so

$$F\left(u_h^* \sum_{g \in G} u_g c_g^{(n)}\right) = \sum_g F(u_h^* u_g) c_g^{(n)} = c_h^{(n)}$$

for all $h \in G$, $n \in \mathbb{N}$. Thus $\sum_g u_g c_g^{(n)} \rightarrow \sum_g u_g F(u_g^* a) \in \mathcal{L}$, so $a \in \mathcal{L}$ and $\mathcal{L}$ is closed. This proves our assertion.

Now, every $x \in \mathcal{A}$ can be written as $x = \sum_g u_g c_g$, $c_g \in \mathcal{C}$. Thus

$$F(u_h^* x) = \sum_g F(u_h^* u_g) c_g = c_h$$

for all $h \in G$, so that

$$x = \sum_g u_g F(u_g^* x)$$

for all $x \in \mathcal{A}$. Also, by Corollary 3.7 again, we have $F(u_g^* u_h) = \delta_{g,h}$ for all $g, h \in G$. Hence $\{u_g \mid g \in G\}$ is an orthonormal right quasi-basis for F .

Again, since $\{u_g\} \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{C})$, we have $u_g \mathcal{C} = \mathcal{C} u_g$ for all $g \in G$. So $\sum_g \mathcal{C} u_g = \mathcal{A}$. And as above, it is easily seen that $x = \sum_g F(x u_g^*) u_g$ for all $x \in \mathcal{A}$. Hence $\{u_g\}$ is an orthonormal left quasi-basis for F as well.

Thus $\{u_g \mid g \in G\}$ is a two-sided orthonormal quasi-basis for F consisting of unitaries in $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$. Finally, we have $\text{Ind}_W(F) = \sum_{g \in G} u_g u_g^* = |G|$. $\square$

Theorem 3.10. *With running notation, the following hold:*

- (i) $E_0 : \mathcal{A} \rightarrow \mathcal{B}$ admits a two-sided orthonormal quasi-basis;
- (ii) $\tau_0 := (E_0)_{\upharpoonright_{\mathcal{C}_{\mathcal{A}}(\mathcal{B})}} : \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \rightarrow \mathbb{C}$ is the (unique) Markov trace for the (connected) inclusion $\mathbb{C} \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{B})$ with modulus $\dim(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$;
- (iii) $[\mathcal{A} : \mathcal{B}]_0 = |W(\mathcal{B} \subset \mathcal{A})| \dim(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$.

In particular, $[\mathcal{A} : \mathcal{B}]_0$ is an integer, $|W(\mathcal{B} \subset \mathcal{A})| \leq [\mathcal{A} : \mathcal{B}]_0$ and if, in addition, $\mathcal{B} \subset \mathcal{A}$ is irreducible, then $[\mathcal{A} : \mathcal{B}]_0 = |W_0(\mathcal{B} \subset \mathcal{A})|$.

Proof. (i) From Proposition 3.9, $\{u_g \mid g \in G\}$ is a two-sided orthonormal quasi-basis for F contained in $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$. And from Proposition 2.13, there exists a two-sided orthonormal quasi-basis for $E_0|_{\mathcal{C}}$ contained in $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$, say, $\{\lambda_i \mid 1 \leq i \leq n\}$. We assert that $\{u_g \lambda_i \mid g \in G, 1 \leq i \leq n\}$ is a two-sided orthonormal quasi-basis for E_0 .

Since $E_0 = E_0|_{\mathcal{C}} \circ F$, it follows easily that $\{u_g \lambda_i \mid 1 \leq i \leq n, g \in G\}$ is a right orthonormal quasi-basis for E_0 (see Remark 2.2). So we just need to show that it is a left orthonormal quasi-basis as well, equivalently, $\{\lambda_i^* u_g^*\}$ is a right orthonormal quasi-basis for E_0 . Clearly,

$$E_0((\lambda_i^* u_g^*)^* \lambda_j^* u_h^*) = E_0(u_g F(\lambda_i \lambda_j^*) u_h^*) = \delta_{i,j} \delta_{g,h}.$$

So, in view of Proposition 2.4, it suffices to show that $\sum_{g,i} \lambda_i^* u_g^* e_1 u_g \lambda_i = 1$.

For each $g \in G$, it readily follows that $\{u_g \lambda_i u_g^* \mid i\}$ is also a two-sided quasi-basis for $E_0|_{\mathcal{C}}$ (see [4, Lem. 3.8]). Let $e_{\mathcal{C}}$ denote the Jones projection for the inclusion $\mathcal{C} \subset \mathcal{A}$ with respect to the finite-index conditional expectation F . Then $e_{\mathcal{C}} \in \mathcal{A}_1$ and $\sum_i (u_g \lambda_i u_g^*)^* e_1 \lambda_i (u_g \lambda_i u_g^*) = e_{\mathcal{C}}$ (by Remark 2.15). Thus

$$\sum_{g,i} \lambda_i^* u_g^* e_1 u_g \lambda_i = \sum_{g,i} u_g^* (u_g \lambda_i u_g^*)^* e_1 (u_g \lambda_i u_g^*) u_g = \sum_g u_g^* e_{\mathcal{C}} u_g = 1,$$

where the last equality follows from Proposition 2.4. This proves (i).

(ii) and (iii) Since $E_0 = E_0|_{\mathcal{C}} \circ F$, it follows that

$$\text{Ind}_W(E_0) = \text{Ind}_W(F) \text{Ind}_W(E_0|_{\mathcal{C}})$$

(see Remark 2.2). Hence, from Proposition 3.9 and Proposition 2.8, we obtain

$$|G| \dim(\mathcal{C}_{\mathcal{A}}(\mathcal{B})) = [\mathcal{A} : \mathcal{B}]_0 = \sum g, i\lambda_i^* u_g^* u_g \lambda_i = \text{Ind}_W(\tau_0).$$

In particular, τ_0 has scalar Watatani index. Thus, in view of Proposition 2.8, it follows that τ_0 is the Markov trace for the inclusion $\mathbb{C} \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{B})$ with modulus $\dim(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$. $\square$

Corollary 3.11. *Let $\mathcal{B}$ be a simple unital C^* -algebra and suppose a finite group K admits a $\mathcal{U}(\mathcal{B})$ -valued outer cocycle action (α, σ) on $\mathcal{B}$. Then the Weyl group of the inclusion $\mathcal{B} \subset \mathcal{B} \rtimes_{(\alpha, \sigma)}^r K$ is isomorphic to K .*

Proof. We write $\mathcal{B} \rtimes K$ for the reduced twisted crossed product $\mathcal{B} \rtimes_{(\alpha, \sigma)}^r K$.

Note that, by the universality of the reduced twisted crossed product, we can assume that there exists a Hilbert space $\mathcal{H}$ such that $\mathcal{B} \subseteq B(\mathcal{H})$ and there is a map $w : K \rightarrow \mathcal{U}(B(\mathcal{H}))$ such that $w_s \notin \mathcal{B}$ (for $s \neq e$),

$$w_s w_t = \sigma(s, t) w_{st}, \quad w_e = 1, \quad (w_s)^* = \sigma(s^{-1}, s)^* w_{s^{-1}}, \quad \alpha_s(x) = w_s x w_s^*$$

for all $s, t \in K$ and $x \in \mathcal{B}$, and $\mathcal{B} \rtimes K = C^*(\mathcal{B}, w(K)) \subset B(\mathcal{H})$.

Since the (twisted) action is outer, $\mathcal{B} \rtimes K$ is simple and $\mathcal{B}' \cap (\mathcal{B} \rtimes K) = \mathbb{C}$, i.e., $\mathcal{B} \subset \mathcal{B} \rtimes K$ is irreducible (see Theorem 2.22). Also, $\mathcal{B} \subset \mathcal{B} \rtimes K$ is regular (see Example 2.21), so $|W_0(\mathcal{B} \subset \mathcal{B} \rtimes K)| = [\mathcal{B} \rtimes K : \mathcal{B}]_0$, by Theorem 3.10.

Further, since $\mathcal{B} \subset \mathcal{B} \rtimes K$ is irreducible, the canonical conditional expectation $E : \mathcal{B} \rtimes K \rightarrow \mathcal{B}$ (as in Remark 2.17 (iii)) is unique (by [31, Cor. 1.4.3]) and hence minimal, which then implies that $[\mathcal{B} \rtimes K : \mathcal{B}]_0 = \text{Ind}_W(E) = |K|$, by Remark 2.17 (v). Thus $|W_0(\mathcal{B} \subset \mathcal{B} \rtimes K)| = |K|$.

Finally, $\{w_s \mid s \in K\} \subset \mathcal{N}_{\mathcal{B} \rtimes K}(\mathcal{B})$ and the map

$$K \ni s \mapsto [w_s] \in W_0(\mathcal{B} \subset \mathcal{B} \rtimes K)$$

is an injective group homomorphism. Hence $W_0(\mathcal{B} \subset \mathcal{B} \rtimes K) \cong K$. $\square$

It will be interesting to answer the following natural question.

Question. Suppose a (countable) discrete group G admits a cocycle action (α, σ) on a unital C^* -algebra $\mathcal{B}$. Is the generalized Weyl group of the inclusion $\mathcal{B} \subset \mathcal{B} \rtimes_{(\alpha, \sigma)}^r G$ isomorphic to G ?

The first part of the following observation now follows from [13], which is based on a beautiful application of the so-called “circulant matrices” from quantum information theory.

Corollary 3.12. *With running notation, the following hold.*

- (i) *There exists a unitary orthonormal quasi-basis for τ_0 .*
- (ii) *The conditional expectation E_0 admits a unitary orthonormal quasi-basis.*

Proof. (i) Let $(n_1, \dots, n_k)$ be the dimension vector of $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$, so

$$\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C}).$$

By Theorem 3.10, τ_0 is the Markov trace for the inclusion $\mathbb{C} \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{B})$ with modulus $d = \dim(\mathcal{C}_{\mathcal{A}}(\mathcal{B}))$. So the trace vector of τ_0 is given by $\vec{t} = (n_1/d, \dots, n_k/d)^t$.

Note that $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$ is unittally $*$ -isomorphic to $P := \bigoplus_{i=1}^k (I_{n_i} \otimes M_{n_i}(\mathbb{C}))$ and P is a unital C^* -subalgebra of $M_d(\mathbb{C})$. Further, if τ denotes the (unique) tracial state on M_d , then the trace vector of $\tau|_P$ is given by $(n_1/d, \dots, n_k/d)^t$. Thus $\tau|_P$ corresponds to τ_0 via the $*$ -isomorphism between P and $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$. By [13, Thm. 2.2], there exists a unitary orthonormal quasi-basis for $\tau|_P$. Hence there exists a unitary orthonormal quasi-basis for τ_0 .

(ii) Let $\{w_i \mid 1 \leq i \leq n\} \subset \mathcal{C}_{\mathcal{A}}(\mathcal{B})$ be a unitary orthonormal quasi-basis for τ_0 . Then it follows on the lines of Proposition 2.13 (ii) that $\{w_i\}$ is a unitary orthonormal quasi-basis for $E_0|_{\mathcal{C}}$ as well. From Proposition 3.9, we know that $\{u_g \mid g \in G\}$ is a unitary orthonormal quasi-basis for $F : \mathcal{A} \rightarrow \mathcal{C}$. Hence $\{u_g w_i \mid g \in G, 1 \leq i \leq n\}$ is a unitary orthonormal quasi-basis for E_0 . $\square$

With all requirements at our disposal, imitating the proof of [6, Thm. 4.3], we obtain the following.

Theorem 3.13. *Let $\mathcal{B} \subset \mathcal{A}$ be a regular inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Then the inclusion $\mathcal{B} \subset \mathcal{A}$ has finite depth and the depth is at most 2.*

3.14. Characterization of regular inclusions of simple C^* -algebras.
We are now all set to prove the main result of this article.

Theorem 3.15. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Then the inclusion $\mathcal{B} \subset \mathcal{A}$ is regular if and only if there exists a finite group G that admits a cocycle action (α, σ) on the intermediate C^* -subalgebra $\mathcal{C} := C^*(\mathcal{B} \cup \mathcal{C}_{\mathcal{A}}(\mathcal{B}))$ such that*

- (i) $\mathcal{B}$ is invariant under α ;
- (ii) for each $e \neq g \in G$, α_g is an outer automorphism of $\mathcal{B}$;
- (iii) $(\mathcal{B} \subset \mathcal{A}) \cong (\mathcal{B} \subset \mathcal{C} \rtimes_{(\alpha, \sigma)}^r G)$.

Proof. Suppose that a finite group G admits a cocycle action (α, σ) on $\mathcal{C}$ as in the statement. We can consider a representation $\mathcal{C} \subseteq B(\mathcal{H})$ such that α is implemented by a map $w : G \rightarrow \text{Aut}(B(\mathcal{H}))$, i.e., $\alpha_g = \text{Ad}(w_g)$ for all $g \in G$. So $\mathcal{C} \rtimes_{(\alpha, \sigma)}^r G = C^*(\mathcal{C} \cup w(G))$. The regularity of $\mathcal{B} \subset \mathcal{C} \rtimes_{(\alpha, \sigma)}^r G$ is clear as

$$\{w_g \mid g \in G\} \cup \mathcal{U}(\mathcal{C}) \subseteq \mathcal{N}_{\mathcal{C} \rtimes_{(\alpha, \sigma)}^r G}(\mathcal{B}).$$

Conversely, suppose that the inclusion $\mathcal{B} \subset \mathcal{A}$ is regular. Consider its generalized Weyl group G . Then G is finite by Proposition 3.4. We assert that G admits a desired cocycle action on $\mathcal{C}$.

Let $\{u_g \mid g = [u_g] \in G\}$ denote a fixed set of (left) coset representatives of G in $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$. Since $u_g \mathcal{C} u_g^* = \mathcal{C}$ (see Corollary 3.7), $\alpha_g := \text{Ad}_{u_g}$ is an automorphism of $\mathcal{C}$ for every $g \in G$. Moreover, from Corollary 3.7 again, it follows that $\alpha_g : \mathcal{C} \rightarrow \mathcal{C}$ is (free and hence) outer for every $g \neq e$. We assert that the map $\alpha : G \rightarrow \text{Aut}(\mathcal{C})$, $g \mapsto \alpha_g$ is in fact a cocycle action with respect to a $\mathcal{U}(\mathcal{C})$ -valued cocycle σ , which we describe now.

Note that $[u_g u_h] = [u_{gh}]$ for all $g, h \in G$. So there exists a function

$$\sigma : G \times G \rightarrow \mathcal{U}(\mathcal{B})\mathcal{U}(\mathcal{C}_{\mathcal{A}}(\mathcal{B})) \subset \mathcal{U}(\mathcal{C})$$

satisfying $u_g u_h = \sigma(g, h) u_{gh}$ for all $g, h \in G$. We assert that (α, σ) is a cocycle action of G on $\mathcal{C}$. First, observe that

$$\begin{aligned} \alpha_g \alpha_h(x) &= \alpha_g(u_h x u_h^*) = u_g u_h x u_h^* u_g^* \\ &= \text{Ad}(\mu(g, h)) u_{gh} x u_{gh}^* = \text{Ad}(\mu(g, h)) \alpha_{gh}(x) \end{aligned}$$

for all $g, h \in G$, $x \in \mathcal{C}$. Thus $\alpha_g \alpha_h = \text{Ad}(\sigma(g, h)) \alpha_{gh}$ for all $g, h \in G$.

Now applying the relation $u_g u_h = \sigma(g, h) u_{gh}$ twice, we see that

$$(u_g u_h) u_k = \sigma(g, h) \sigma(gh, k) u_{(gh)k},$$

and on the other hand,

$$u_g (u_h u_k) = u_g (\sigma(h, k) u_{hk}) = \text{Ad}(u_g) (\sigma(h, k)) \sigma(g, hk) u_{g(hk)}$$

for all $g, h, k \in G$. Thus

$$\sigma(g, h) \sigma(gh, k) = \alpha_g(\sigma(h, k)) \sigma(g, hk)$$

for all $g, h, k \in G$. And clearly, $\sigma(g, 1) = \sigma(1, g) = 1$. This proves our assertion that (α, σ) is a cocycle action of G on $\mathcal{C}$.

Further, note that $u_g \mathcal{B} u_g^* = \mathcal{B}$ for all $g \in G$ and $\text{Ad}_{u_g} : \mathcal{B} \rightarrow \mathcal{B}$ for each $e \neq g \in G$ is outer by Lemma 3.3.

Finally, we show there exists a $*$ -isomorphism φ from $\mathcal{A}$ onto $\mathcal{C} \rtimes_{(\alpha, \sigma)}^r G$ such that $\varphi|_{\mathcal{C}} = \text{id}_{\mathcal{C}}$. Let $x \in \mathcal{A}$. By Proposition 3.9, we see that $x = \sum_g F(x u_g^*) u_g$. Define $\varphi : \mathcal{A} \rightarrow \mathcal{C} \rtimes_{(\alpha, \sigma)}^r G$ by $\varphi(x) = \sum_g E(x g^{-1}) g$, where $E : \mathcal{C} \rtimes_{(\alpha, \sigma)}^r G \rightarrow \mathcal{C}$ is the canonical finite-index conditional expectation (as in Remark 2.17). Clearly, $\varphi|_{\mathcal{C}} = \text{id}_{\mathcal{C}}$ and it is easy to check that φ is a unital $*$ -homomorphism. Since the inclusion $\mathcal{B} \subset \mathcal{A}$ is regular, φ is surjective as well. We omit the necessary details. Since $\mathcal{A}$ is simple, φ is injective and we are done. $\square$

Corollary 3.16. *Let $\mathcal{B} \subset \mathcal{A}$ be a regular irreducible inclusion of simple unital C^* -algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Then its Weyl group G admits an outer cocycle action (α, σ) on $\mathcal{B}$ such that $(\mathcal{B} \subset \mathcal{A}) \cong (\mathcal{B} \subset \mathcal{B} \rtimes_{(\alpha, \sigma)}^r G)$.*

Remark 3.17. Two results from subfactor theory which are very relevant to the preceding theorem need to be mentioned here.

- (i) Choda [12, Thm. 4], based on one of her earlier techniques in [11, Thm. 7], had proved that, for any factor M with separable predual, for every irreducible regular subfactor $N \subset M$ with a faithful conditional expectation from M onto N , there exists a countable discrete group G which admits an outer cocycle action (σ, ω) on N such that $M \cong N \rtimes_{(\sigma, \omega)} G$.
- (ii) Later, employing the same technique of Choda [11, Thm. 7], Cameron [9, Thm. 4.6] showed that, given any regular inclusion $N \subset M$ of II_1 -factors, there exists a countable discrete group G which admits a cocycle action

(σ, ω) on Q , the von Neumann algebra generated by N and $N' \cap M$, such that $M \cong Q \rtimes_{(\sigma, \omega)} G$. However, Cameron does not mention whether the action is outer or not.

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