

A higher index theorem on finite-volume locally symmetric spaces

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Abstract. Let G be a connected, real semisimple Lie group. Let $K < G$ be maximal compact, and let $\Gamma < G$ be discrete and such that $\Gamma \backslash G$ has finite volume. If the real rank of G is 1 and Γ is torsion-free, then Barbasch and Moscovici obtained an index theorem for Dirac operators on the locally symmetric space $\Gamma \backslash G/K$. We obtain a higher version of this, using an index of Dirac operators on G/K in the K -theory of an algebra on which the conjugation-invariant terms in Barbasch and Moscovici's index theorem define continuous traces. The resulting index theorems also apply when Γ has torsion. The cases of these index theorems for traces defined by semisimple orbital integrals extend to Song and Tang's higher orbital integrals, and yield nonzero and computable results even when $\text{rank}(G) > \text{rank}(K)$, or the real rank of G is larger than 1.

CONTENTS

1. Introduction	85
2. A higher index	87
3. Index theorems	90
4. Traces on a Schwartz algebra	100
5. Contributions from semisimple elements	106

1. INTRODUCTION

Consider a connected, real semisimple Lie group G , a maximal compact subgroup $K < G$ and a discrete subgroup $\Gamma < G$. *Locally symmetric spaces* of the form $X = \Gamma \backslash G/K$ come up in many places in mathematics, and include, for example, all hyperbolic manifolds, several moduli spaces, and the space $\text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R}) / \text{SO}(2)$ on which modular forms are defined. Spaces X that arise in this way are sometimes compact, but in many relevant cases (such as the example mentioned), they only have finite volume. Because of the

relevance of these spaces, it is a worthwhile question how to construct and compute topological invariants for them.

Index theory is a useful tool for constructing and computing invariants of manifolds. This works particularly well if Γ is torsion-free and $\Gamma \backslash G$ is compact, so that X is a compact, smooth manifold. Then elliptic differential operators on X have well-defined Atiyah–Singer indices. And by considering the action by Γ on G/K rather than just its quotient, one can construct a higher index of Γ -equivariant elliptic operators [6, 21, 38], with values in $K_*(C^*(\Gamma))$. Pairings of this index with various cyclic cocycles were computed by Connes and Moscovici [11, 12].

If X is smooth and has finite volume, but is not necessarily compact, then Moscovici [25] constructed an index of elliptic operators on X . This was computed for Dirac operators by Barbasch and Moscovici [5] in the case where the real rank of G is 1. (For groups of higher real rank, index theorems for the signature operator were obtained in [27, 33, 34].) Barbasch and Moscovici’s computation was based on a version of Selberg’s trace formula for real rank 1 groups due to Osborne and Warner [29, 36]. The result is an expression for the index of Dirac operators as a sum of various contributions, all but finitely many of which are given by semisimple orbital integrals. In fact, if X is compact, then one only needs these semisimple contributions, so the non-semisimple contributions reflect the noncompactness of X . Some of these non-semisimple contributions are defined by conjugation-invariant functionals, which can be expected to define traces on suitable convolution algebras. The sum of the other contributions is conjugation-invariant as well, even though the individual terms are not.

Our goal here is to unify these constructions and computations, and obtain a higher index theorem on finite-volume locally symmetric spaces if G has real rank 1. More precisely, we aim to construct a K -theoretic index from which

- the classical index can be recovered,
- if G has real rank 1, then the conjugation-invariant parts of Barbasch and Moscovici’s expression for the classical index on X can be extracted and computed individually, and
- the usual Γ -equivariant index can be recovered if X is compact.

To achieve this aim, we show that

- (i) every G -equivariant, selfadjoint, elliptic, first-order differential operator on G/K has a well-defined index in $K_*(\mathcal{A}(G))$, where $\mathcal{A}(G)$ is an analog of Casselman’s Schwartz algebra [10];
- (ii) this index can be represented by a standard idempotent involving heat operators [12];
- (iii) the semisimple orbital integrals and the conjugation-invariant non-semisimple components of Barbasch and Moscovici’s index theorem define continuous traces on $\mathcal{A}(G)$.

There are several commonly used convolution algebras on G besides Casselman’s Schwartz algebra on which semisimple orbital integrals define continuous traces. These include $C_c^\infty(G)$ and Harish-Chandra’s L^1 - or L^2 -Schwartz

algebras. We use Casselman's algebra because it is large enough so that the first two points above are true (the second is not true for $C_c^\infty(G)$), and small enough so that the third point is true (which is not true for some non-semisimple contributions if one uses Harish-Chandra's Schwartz algebras). See also Remark 4.4.

Because of the second point above, applying the traces from the third point to the index in the first point leads to expressions involving heat operators, which were computed in [5]. For traces not defined by semisimple orbital integrals, we then simply apply the computations in [5]. For semisimple orbital integrals, we give an independent index-theoretic computation. We then obtain the following specific results on the three aims above.

- The classical index can be recovered from the index in $K_*(\mathcal{A}(G))$ via the quasiregular representation (Corollary 3.18).
- The values of the traces on $\mathcal{A}(G)$ defined by the conjugation-invariant terms in Barbasch and Moscovici's index theorem on the index in $K_*(\mathcal{A}(G))$ are given by the expressions for these terms found by Barbasch and Moscovici (Theorems 3.23 and 3.26). The result for semisimple orbital integrals can be obtained via direct index-theoretic methods, and extends to cases where Γ may have torsion, and the real rank of G may be larger than 1 (Theorem 5.16). This also extends to the higher orbital integrals from [32] (Theorem 3.24 and Corollary 5.21), where one obtains nonzero results even when $\text{rank}(G) > \text{rank}(K)$.
- If X is compact, then the Γ -index can be recovered from the index in $K_*(\mathcal{A}(G))$ (Corollary 3.21).

Our higher index on G/K is constructed as a special case of a general higher index of elliptic operators equivariant with respect to proper, cocompact group actions, constructed in [14]. This takes values in the K -theory of an algebra of exponentially fast decaying kernels.

2. A HIGHER INDEX

We recall the construction of a higher index in the K -theory of an algebra of very rapidly decaying kernels from [14].

2.1. An algebra of exponentially decaying kernels. Consider a Riemannian manifold X , with Riemannian distance d and Riemannian density μ . We assume that X has *uniformly exponential growth*, in the sense that there are $a_0, C > 0$ such that, for all $x \in X$,

$$(1) \quad \int_X e^{-a_0 d(x, x')} d\mu(x') \leq C.$$

We consider a Hermitian vector bundle $E \rightarrow X$. A section κ of the exterior tensor product $E \boxtimes E^* \rightarrow X \times X$ has *finite propagation* if there is an $r > 0$ such that, for all $x, x' \in X$ with $d(x, x') > r$, we have $\kappa(x, x') = 0$. If two L^∞ -sections κ, λ of $E \boxtimes E^*$ have finite propagation, then their composition $\kappa\lambda$ is

well-defined as an L^∞ -section of $E \boxtimes E^*$ with finite propagation, by

$$(\kappa\lambda)(x, x') = \int_X \kappa(x, x'')\lambda(x'', x') d\mu(x'')$$

for all $x, x' \in X$.

Let H be a Lie group (possibly with infinitely many connected components, so H may be discrete), acting isometrically on X and such that X/H is compact. Suppose that E is an H -equivariant vector bundle, and that the action preserves the Hermitian metric on E . Consider the action by $H \times H$ on $\Gamma^\infty(E \boxtimes E^*)$ given by

$$(2) \quad ((h, h') \cdot \kappa)(x, x') := h\kappa(h^{-1}x, h'^{-1}x')h'^{-1}$$

for $h, h' \in H$, $x, x' \in X$, and $\kappa \in \Gamma^\infty(E \boxtimes E^*)$. Suppose that the operators in S commute with this action.

Let Y be another Riemannian manifold with uniformly exponential growth. Suppose that H acts isometrically on Y . Let $\varphi : Y \rightarrow X$ be a surjective, H -equivariant smooth map. We assume that φ is quasi-isometry in the sense that there are $A, B > 0$ such that, for all $y, y' \in Y$,

$$(3) \quad A^{-1}d_Y(y, y') - B \leq d_X(\varphi(y), \varphi(y')) \leq Ad_Y(y, y') + B,$$

where d_X is the Riemannian distance on X , and d_Y is the one on Y . We also assume that φ is an algebra homomorphism in the sense that, for all $\kappa, \lambda \in \Gamma_{\text{fp}}^\infty(E \boxtimes E^*)$,

$$(4) \quad \varphi^*(\kappa\lambda) = (\varphi^*\kappa)(\varphi^*\lambda).$$

In (4), we note that (3) implies that $(\varphi^*\kappa)$ and $(\varphi^*\lambda)$ have finite propagation, so their composition is defined.

Fix an algebra S of differential operators on $\varphi^*E \boxtimes \varphi^*E^*$ such that

- (i) S contains the identity operator;
- (ii) for all $l \in \mathbb{Z}_{\geq 0}$, the set of order l differential operators in S is a finite-dimensional linear subspace of S ;
- (iii) for all $P \in S$, there are differential operators Q and R on φ^*E such that $Q \otimes 1$ and $1 \otimes R^*$ lie in S and $P = Q \otimes R^*$. Here R^* is the differential operator on φ^*E^* dual to R .

Let $\Gamma_{S, \varphi, \text{fp}}^\infty(E \boxtimes E^*)$ be the space of smooth sections κ of $E \boxtimes E^* \rightarrow X \times X$ with finite propagation, such that $P\varphi^*\kappa$ is bounded for all $P \in S$.

Definition 2.2. For $a > 0$ and $P \in S$, the semi-norm $\|\cdot\|_{a, P}$ on $\Gamma_{S, \varphi, \text{fp}}^\infty(E \boxtimes E^*)$ is given by

$$\|\kappa\|_{a, P} = \sup_{y, y' \in Y} e^{ad_Y(y, y')} \|(P\varphi^*\kappa)(y, y')\|$$

for $\kappa \in \Gamma_{S, \text{fp}}^\infty(E \boxtimes E^*)$, where d_Y is the Riemannian distance on Y . We denote the completion of $\Gamma_{S, \varphi, \text{fp}}^\infty(E \boxtimes E^*)$ in the semi-norms $\|\cdot\|_{a, P}$, for $a > 0$ and $P \in S$, by $\mathcal{A}_{S, \varphi}(E)$. The linear subspace of elements invariant under the action (2) is denoted by $\mathcal{A}_{S, \varphi}(E)^H$.

Lemma 2.3. *The space $\mathcal{A}_{S,\varphi}(E)^H$ is a Fréchet algebra with respect to composition of kernels.*

This lemma is a combination of [14, Lem. 2.6 and eq. (2.7)].

2.4. Construction of the index. Let D be an H -equivariant, first-order, elliptic, selfadjoint differential operator on E . We make the following assumptions.

(i) There are $Q, R \in S$ such that

$$\varphi^* \circ (D^2 \otimes 1) = Q \circ \varphi^* \quad \text{and} \quad \varphi^* \circ (1 \otimes (D^2)^*) = R \circ \varphi^*.$$

(ii) For all $f \in C_c^\infty(X)$, all repeated commutators of f with D^2 are well-defined, bounded operators on $L^2(E)$. Furthermore, if Z is any smooth manifold, and $\tilde{f} \in C^\infty(Z \times X)$ is such that $\tilde{f}(z, -) \in C_c^\infty(X)$ for all $z \in Z$, then we assume that the operator norms of the repeated commutators of D^2 with $\tilde{f}(z, -)$ depend continuously on z .

(iii) For all $P \in S$, the compositions $P \circ \varphi^* \circ (D \otimes 1)$ and $P \circ \varphi^* \circ (1 \otimes D)$ are finite sums of compositions of the form $A \circ Q \circ \varphi^*$, for bounded endomorphisms A of $\varphi^*E \boxtimes \varphi^*E^*$ and $Q \in S$.

The following result is [14, Thm. 2.13].

Theorem 2.5. *The operator D lies in the multiplier algebra $\mathcal{M}(\mathcal{A}_{S,\varphi}(E)^H)$. Its class in $\mathcal{M}(\mathcal{A}_{S,\varphi}(E)^H)/\mathcal{A}_{S,\varphi}(E)^H$ is invertible. The image of the resulting class $[D] \in K_1(\mathcal{M}(\mathcal{A}_{S,\varphi}(E)^H)/\mathcal{A}_{S,\varphi}(E)^H)$ under the boundary map*

$$\partial : K_1(\mathcal{M}(\mathcal{A}_{S,\varphi}(E)^H)/\mathcal{A}_{S,\varphi}(E)^H) \rightarrow K_0(\mathcal{A}_{S,\varphi}(E)^H)$$

equals

$$(5) \quad \partial[D] = \left[\begin{pmatrix} e^{-tD^-D^+} & e^{-\frac{1}{2}D^-D^+} \frac{1 - e^{-tD^-D^+}}{D^-D^+} D^- \\ e^{-\frac{1}{2}D^+D^-} D^+ & 1 - e^{-tD^+D^-} \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \in K_0(\mathcal{A}_{S,\varphi}(E)^H).$$

Definition 2.6. The H -equivariant *higher index* of D with respect to S and φ is the class (5). It is denoted by $\text{index}_{S,H}(D) \in K_0(\mathcal{A}_{S,\varphi}(E)^H)$.

2.7. Special case: Riemannian symmetric spaces. In this paper, we restrict ourselves to the motivating case of [14, Ex. 2.3, 2.10, and 2.11]. (See also Lemma 3.5.) We summarize these examples here, and refer to [14, Ex. 2.3, 2.10, and 2.11] for the (short and straight-forward) proofs of the claims made.

Let G be a noncompact, connected, real semisimple Lie group, and let $K < G$ be a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and consider the G -invariant metric on $X = G/K$ defined by the $\text{Ad}(K)$ -invariant inner product on \mathfrak{g} defined by the Killing form and the Cartan involution.

Let V be a finite-dimensional unitary representation of K . Consider the trivial vector bundle $G \times V \rightarrow G$. Let $E = G \times_K V \rightarrow X$ be the quotient of $G \times V$ by the K -action given by $k \cdot (g, v) = (gk^{-1}, k \cdot v)$ for $k \in K, g \in G$, and $v \in V$. Let $H < G$ be a closed subgroup such that $H \backslash G$ is compact, acting on

the given spaces and vector bundles via left multiplication. Let $q : G \rightarrow X$ be the quotient map. Then $q^*E = G \times V$, and q satisfies all the assumptions on the map φ made in Section 2.1.

Let $S := \{R(P) \mid P \in \mathcal{U}(\mathfrak{g} \times \mathfrak{g})\}$, where R is the right regular representation of $\mathcal{U}(\mathfrak{g} \times \mathfrak{g})$ in $C^\infty(G \times G)$. Then S satisfies the three conditions listed above Definition 2.2.

The adjoint representation $\text{Ad} : K \rightarrow \text{SO}(\mathfrak{p})$ has a lift

$$(6) \quad \widetilde{\text{Ad}} : \widetilde{K} \rightarrow \text{Spin}(\mathfrak{p})$$

to double covers. In this way, the standard representation $\Delta_{\mathfrak{p}}$ of $\text{Spin}(\mathfrak{p})$ may be viewed as a representation of \widetilde{K} . Let W be a finite-dimensional, unitary representation of \widetilde{K} such that $V = \Delta_{\mathfrak{p}} \otimes W$ descends to a representation of K .

Let $\{X_1, \dots, X_r\}$ be an orthonormal basis of \mathfrak{p} . Let $c : \mathfrak{p} \rightarrow \text{End}(\Delta_{\mathfrak{p}})$ be the Clifford action. Then we have the Dirac operator

$$(7) \quad D_W := \sum_{j=1}^r R(X_j) \otimes c(X_j) \otimes 1_W$$

on $(C^\infty(G) \otimes \Delta_{\mathfrak{p}} \otimes W)^K \cong \Gamma^\infty(G \times_K V)$. This operator satisfies the assumptions on D listed above Theorem 2.5. (This operator is used in many places, e.g. [3, 5, 30, 37].)

Suppose that G/K is even-dimensional. Then $\Delta_{\mathfrak{p}}$, and hence E , has a natural $\mathbb{Z}/2\mathbb{Z}$ -grading, with respect to which D_W is odd. By Theorem 2.5, we obtain

$$(8) \quad \text{index}_{S,H}(D_W) \in K_0(\mathcal{A}_{S,q}(E)^H).$$

In this paper, we study this index in the case where $H = G$.

3. INDEX THEOREMS

We continue using the notation and assumptions from Section 2.7. (Later, G will sometimes be assumed to have real rank 1.) Fix a Haar measure dg on G . Let dk be the Haar measure on K giving it unit volume. The Riemannian distance on X is denoted by d . Let d_G be the Riemannian distance on G defined by the given inner product on \mathfrak{g} .

3.1. Geometric traces. We will use some “geometric” traces on an algebra of matrix-valued functions on G , defined by integrals over subsets of G . We also use some “spectral” traces, involving the operator trace, introduced in Section 3.10. All these traces correspond to terms in Barbasch and Moscovici’s index theorem [5].

We write L and R for the left and right regular representations of G on smooth functions, respectively. We use the same notation for the corresponding representation $\mathcal{U}(\mathfrak{g})$.

Definition 3.2. For $f \in C_c^\infty(G)$, $a > 0$, and $P_1, P_2 \in \mathcal{U}(\mathfrak{g})$, we write

$$\|f\|_{a,P_1,P_2} := \sup_{g \in G} e^{ad_G(e,g)} |(L(P_1)R(P_2)f)(g)|.$$

Let $\mathcal{S}(G)$ be the completion of $C_c^\infty(G)$ in the semi-norms $\|\cdot\|_{a,P_1,P_2}$.

Remark 3.3. The algebra $\mathcal{S}(G)$ is a version of Casselman’s Schwartz algebra [10]. We use semi-norms defined by left and right derivatives, whereas only left derivatives are used in [10]. We use both left and right derivatives so that $\mathcal{S}(G)$ embeds into Harish-Chandra’s Schwartz spaces; see Lemma 4.3. It is not essential to use both left and right derivatives, because in the end we will always restrict to subalgebras of $\mathcal{S}(G)$ corresponding to a finite number of K -types. Then it is sufficient to only use left or right derivatives to obtain an embedding into Harish-Chandra’s Schwartz spaces, by (a slight extension of) [15, Lem. 18].

Lemma 3.4. *The space $\mathcal{S}(G)$ is a Fréchet algebra with respect to convolution.*

This lemma is proved in Section 4.1.

A relation between the algebra $\mathcal{S}(G)$ and the algebra from Definition 2.2 will play a central role. As in Section 2.7, we have the Fréchet algebra $\mathcal{A}_{S,q}(E)^G$. The Fréchet algebra $\mathcal{S}(G) \otimes \text{End}(V)$ is the completion of $C_c^\infty(G) \otimes \text{End}(V)$ in the semi-norms given by

$$\|f\|_{a,P_1,P_2}^V := \sup_{g \in G} e^{ad_G(e,g)} \|(L(P_1)R(P_2)f)(g)\|_{\text{End}(V)}$$

for $a > 0$, $P_1, P_2 \in \mathcal{U}(\mathfrak{g})$, and $f \in C_c^\infty(G) \otimes \text{End}(V)$. Consider the action by $K \times K$ on $\mathcal{S}(G) \otimes \text{End}(V)$ given by

$$(9) \quad ((k, k') \cdot f)(g) = \pi(k)f(k^{-1}gk')\pi(k')^{-1}$$

for $k, k' \in K$, $f \in \mathcal{S}(G) \otimes \text{End}(V)$, and $g \in G$. Analogously to [14, Lem. 2.6], elements of $K \times K$ act on $\mathcal{S}(G) \otimes \text{End}(V)$ by continuous operators, so that the fixed point set $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$ is a Fréchet algebra.

Consider the fixed-point set $(C_c^\infty(G) \otimes \text{End}(V))^{K \times K}$ of the action (9) by $K \times K$ on $C_c^\infty(G) \otimes \text{End}(V)$.

Lemma 3.5. *For all $\kappa \in \Gamma^\infty(E \boxtimes E^*)^G$, setting*

$$(10) \quad [e, \Psi(\kappa)(g)v] = \kappa(eK, gK)[g, v]$$

for $g \in G$ and $v \in V$ defines a function $\Psi(\kappa) \in (C_c^\infty(G) \otimes \text{End}(V))^{K \times K}$. This defines a linear isomorphism $\Psi : \Gamma^\infty(E \boxtimes E^*)^G \rightarrow (C_c^\infty(G) \otimes \text{End}(V))^{K \times K}$. This linear isomorphism restricts to an algebra isomorphism from $\Gamma_{\text{fp}}^\infty(E \boxtimes E^*)^G$ to $(C_c^\infty(G) \otimes \text{End}(V))^{K \times K}$, which in turn extends continuously to an isometric algebra isomorphism from $\mathcal{A}_{S,q}(E)^G$ to $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.

This lemma is proved in Section 4.1.

We will use several continuous traces on the algebra $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$. If τ is a linear functional on $\mathcal{S}(G)$, or on the subspace of $\mathcal{S}(G)$ corresponding

to the $K \times K$ -isotypical components in $\text{End}(V)$, then we write

$$\tau^V := \tau \otimes \text{tr}_V : (\mathcal{S}(G) \otimes \text{End}(V))^{K \times K} \rightarrow \mathbb{C}.$$

If H is any group, $H' < H$ a subgroup, and $h \in H$, then we write H'_h for the centralizer of h in H' .

Let $g \in G$ be such that G_g is unimodular. Fix a Haar measure dh on G_g . Because G_g is unimodular, there is a unique G -invariant measure $d(G_g h)$ on $G_g \backslash G$ such that, for all $f \in C_c(G)$,

$$\int_G f(h) dh = \int_{G_g \backslash G} \int_{G_g} f(h'h) dh' d(G_g h).$$

For $f \in C^\infty(G)$ such that the following converges, consider the orbital integral

$$\tau_g(f) := \int_{G_g \backslash G} f(h^{-1}gh) d(G_g h).$$

Proposition 3.6. *If g is semisimple, then τ_g^V defines a continuous trace on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.*

This proposition is proved in Section 4.5.

In Theorem 3.23, we will see that semisimple orbital integrals (the traces from Proposition 3.6) map the higher index of the operators we consider to zero if $\text{rank}(G) > \text{rank}(K)$. The higher index itself is generally nonzero in that case, however, and this nonvanishing can be detected by pairing the index with cyclic cocycles defined by the *higher orbital integrals* from [32].

Let $P < G$ be a cuspidal parabolic subgroup, with Langlands decomposition $P = MA'N'$. (We use primes to distinguish the groups A' and N' from the groups A and N in an Iwasawa decomposition $G = KAN$ below.) Let $g \in M$ be semisimple. In [32, Def. 3.3 and Thm. 3.5], Song and Tang constructed a cyclic cocycle $\Phi_{P,g}$ over Harish-Chandra's Schwartz algebra $\mathcal{C}^2(G)$ (see Remark 4.2), whose degree is the dimension of A' . If $\text{rank}(G) = \text{rank}(K)$, then G is a cuspidal parabolic subgroup of itself, and $\Phi_{G,g} = \tau_g$.

Proposition 3.7. *The operator trace on $\text{End}(V)$ and the cyclic cocycle $\Phi_{P,g}$ combine into a cyclic cocycle $\Phi_{P,g}^V$ on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.*

This proposition is proved in Section 4.5.

We will also use some traces that are not associated to semisimple elements of G . From now on, until the end of Section 3.17, we suppose that G has real rank 1. Consider an Iwasawa decomposition $G = KAN$, and a compatible Γ -cuspidal parabolic subgroup $P = MAN$. Then A is one-dimensional by assumption. Let $\lambda \in \mathfrak{a}^*$ be the simple restricted root of $(\mathfrak{g}, \mathfrak{a})$ that determines N . Then we have the restricted root space decomposition $\mathfrak{n} = \mathfrak{n}_\lambda \oplus \mathfrak{n}_{2\lambda}$. With

$$N_\lambda := \exp(\mathfrak{n}_\lambda) \quad \text{and} \quad N_{2\lambda} := \exp(\mathfrak{n}_{2\lambda}),$$

we have $N = N_\lambda N_{2\lambda}$, where $N_\lambda \cap N_{2\lambda} = \{e\}$.

Since $N_{2\lambda}$ lies in the center of N , we have $G_n = M_n N$ for all nontrivial $n \in N_{2\lambda}$, so G_n is unimodular. So the orbital integral $\tau_n(f)$ can be defined on suitable functions.

Lemma 3.8. *For all $n \in N_{2\lambda}$, the orbital integral τ_n^V defines a continuous trace on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.*

This lemma is proved in Section 4.5.

For $a \in A$ and $\xi \in \mathfrak{a}^*$, we write $a^\xi := e^{\langle \xi, \log a \rangle}$. Let

$$(11) \quad \rho_N := (\dim(\mathfrak{n}_\lambda)/2 + \dim(\mathfrak{n}_{2\lambda}))\lambda.$$

In a few places, we will use the fact that the Haar measure dg on G can be decomposed as

$$(12) \quad \int_G f(g) dg = \int_{KAN} f(kan) a^{2\rho_N} dk da dn$$

for all $f \in C_c(G)$, for Haar measures dk on K , da on A , and dn on N .

For $f \in C^\infty(G)$ such that the following converges, write

$$(13) \quad \tau_\lambda(f) := \int_N \int_K f(knk^{-1}) dk dn.$$

Proposition 3.9. *The integral (13) defines a continuous trace τ_λ^V on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.*

This proposition is proved in Section 4.5.

3.10. Spectral traces. Apart from the geometric traces of Section 3.1, we also use some “spectral” traces on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.

Let $\Gamma < G$ be a discrete subgroup such that $\Gamma \backslash G$ has finite volume. As in [5, 29, 36], we make the following simplifying technical assumption. Let P_1, \dots, P_l be representatives of the Γ -conjugacy classes of Γ -cuspidal parabolic subgroups of G , with Langlands decompositions $P_j = M_j A_j N_j$. Let Z_Γ be the center of Γ . Then we assume that, for all j ,

$$(14) \quad \Gamma \cap P_j = Z_\Gamma \cdot (\Gamma \cap N_j).$$

This condition implies that a noncentral element $\gamma \in \Gamma$ is semisimple if and only if its conjugacy class has empty intersection with each of the groups P_j ; see [36, Lem. 5.3]. That in turn implies that, for all semisimple $\gamma \in \Gamma$, the quotient $\Gamma_\gamma \backslash G_\gamma$ has finite volume; see [36, Lem. 5.4]. In general, a discrete subgroup $\Gamma < G$ such that $\Gamma \backslash G$ has finite volume has a finite-index normal subgroup with the above property. (See [36, p. 56] and [13, Lem. 6.5].) It is possible to work without this condition; then noncentral semisimple elements of Γ should be replaced by noncentral elements whose conjugacy classes have empty intersections with the groups P_j .

Example 3.11. Let $G = \text{SL}(2, \mathbb{R})$ and $\Gamma = \text{SL}(2, \mathbb{Z})$. There is one Γ -conjugacy class of Γ -cuspidal parabolic subgroups, represented by the subgroup $P < G$

of upper-triangular matrices. In its Langlands decomposition $P = MA'N'$, we have

$$N' = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}, \quad \text{so} \quad \Gamma \cap N' = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z} \right\}.$$

And $Z_\Gamma = \{\pm I\}$, so

$$Z_\Gamma \cdot (\Gamma \cap N') = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \{\pm 1\}, b \in \mathbb{Z} \right\} = \Gamma \cap P.$$

So (14) holds.

There are $k_1, \dots, k_l \in K$ such that $P_j = k_j P k_j^{-1}$ for all j . So if $\sigma \in \hat{M}$, then via conjugation, we have the representation $k_j \cdot \sigma$ of $M_j = k_j M k_j^{-1}$. This yields the representation $\pi_\Gamma(\sigma) := \dim(\sigma) \text{Ind}_{P_j}^G(k_j \cdot \sigma \otimes 1_{AN})$ of G . There is a unitary intertwining operator $c_\Gamma(\sigma)$ for this representation that squares to the identity.

Lemma 3.12. *The map*

$$(15) \quad f \mapsto \sum_{\sigma \in \hat{M}} (\text{Tr} \otimes \text{tr}_V)(c_\Gamma(\sigma) \pi_\Gamma(\sigma)(f))$$

defines a continuous trace τ_{res} on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.

This lemma is proved in Section 4.10.

Consider the right quasiregular representation R^Γ of G in $L^2(\Gamma \backslash G)$: for all $g, h \in G$ and $\varphi \in L^2(\Gamma \backslash G)$,

$$(R^\Gamma(g)\varphi)(\Gamma h) = \varphi(\Gamma hg).$$

Write $L^2(\Gamma \backslash G) = L_d^2(\Gamma \backslash G) \oplus L_c^2(\Gamma \backslash G)$, where $L_d^2(\Gamma \backslash G)$ decomposes discretely into irreducible representations of G , and $L_c^2(\Gamma \backslash G)$ decomposes continuously. Let R_d^Γ be the restriction of R^Γ to $L_d^2(\Gamma \backslash G)$. It defines a representation of $L^1(G)$ in the usual way. We consider its extension

$$(R_d^\Gamma)^V : (L^1(G) \otimes \text{End}(V))^{K \times K} \rightarrow \mathcal{B}((L_d^2(\Gamma \backslash G) \otimes V)^K).$$

Proposition 3.13. *For all $f \in (\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$, the operator $(R_d^\Gamma)^V(f)$ is trace-class. The algebra homomorphism*

$$(R_d^\Gamma)^V : (\mathcal{S}(G) \otimes \text{End}(V))^{K \times K} \rightarrow \mathcal{L}^1((L_d^2(\Gamma \backslash G) \otimes V)^K)$$

is continuous.

Corollary 3.14. *The map*

$$f \mapsto \text{Tr}(R_d^\Gamma(f))$$

defines a continuous trace $(\text{Tr} \circ R_d^\Gamma)^V$ on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.

Proposition 3.13 and Corollary 3.14 are proved in Section 4.10.

Let $A(\mathfrak{n}_\lambda)$ be the area of the unit sphere in \mathfrak{n}_λ , and let $C_\lambda(\Gamma)$ and $C_{2\lambda}(\Gamma)$ be as in [29, p. 299]. Fix $n_0 \in N_{2\lambda}$ such that $\|\log(n_0)\| = 1$. For $f \in C^\infty(G)$

such that $R_d^\Gamma(f)$ is trace-class and the other terms converge, we consider the “remainder”

$$(16) \quad \tau_{\text{rem}}(f) := \text{Tr}(R_d^\Gamma(f)) + \frac{1}{4}\tau_{\text{res}}(f) - \sum_{(\gamma) \text{ semisimple}} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \tau_\gamma(f) - \frac{C_{2\lambda}(\Gamma)}{2}(\tau_{n_0}(f) + \tau_{n_0^{-1}}(f)) - \frac{C_\lambda(\Gamma)}{A(\mathbf{n}_\lambda)|\lambda|} \tau_\lambda(f),$$

Here the sum on the right-hand side is over the Γ -conjugacy classes (γ) of semisimple elements of Γ .

Remark 3.15. An explicit form of the functional τ_{rem} is given in [29]: see the theorems in [29, pp. 293 and 299].

Proposition 3.16. *The functional τ_{rem} defines a continuous trace τ_{rem}^V on $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$.*

This proposition is proved at the end of Section 4.10.

3.17. Relation with the classical index and the Γ -index. We first note that the index of elliptic operators on real rank one, finite-volume locally symmetric spaces studied in [5, 25] can be recovered from the higher index of Definition 2.6.

Consider the setting of Section 2.7. We consider the case of the index (8) where $H = G$:

$$(17) \quad \text{index}_{\mathcal{S},G}(D_W) \in K_0(\mathcal{A}_{S,q}(E)^G).$$

Using the isomorphism

$$\Psi : \mathcal{A}_{S,q}(E)^G \rightarrow (\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$$

from Lemma 3.5, we can map this index to $K_0((\mathcal{S}(G) \otimes \text{End}(V))^{K \times K})$. Then applying the map induced by the algebra homomorphism $(R_d^\Gamma)^V$ from Proposition 3.13 yields an element of

$$(18) \quad K_0(\mathcal{L}^1(L_d^2(\Gamma \backslash G) \otimes V)) = \mathbb{Z}.$$

Let D_W^Γ be the elliptic differential operator on $\Gamma \backslash G \times_K V \rightarrow \Gamma \backslash G / K$ induced by D_W . It has a well-defined index by [25, Thm. 2.1].

Corollary 3.18. *If G has real rank 1, then under the isomorphism (18), we have*

$$(19) \quad (R_d^\Gamma)_*^V \circ \Psi_*(\text{index}_{\mathcal{S},G}(D_W)) = \text{index}(D_W^\Gamma).$$

Proof. The isomorphism (18) is given by the operator trace. So, by Theorem 2.5, the left-hand side of (19) equals

$$\text{Tr}(R_d^\Gamma(\Psi(e^{-tD_w^- D_w^+}))) - \text{Tr}(R_d^\Gamma(\Psi(e^{-tD_w^+ D_w^-}))).$$

By [5, Prop. 3.1], this equals the right-hand side of (19). □

Remark 3.19. The number (19) is the value of the continuous trace

$$\text{Tr} \circ (R_d^\Gamma)^V \circ \Psi$$

on $\text{index}_{S,G}(D_W)$.

Remark 3.20. By [5, (1.2.5)], the index (19) vanishes if $\text{rank}(G) > \text{rank}(K)$. The higher index (17) will generally be nonzero, however. Indeed, [14, Cor. 4.4] states that the higher index (17) maps to the standard G -index $\text{index}_G(D_W) \in K_0(C_r^*(G))$ under the maps induced on K -theory by Ψ and the inclusion $\mathcal{S}(G) \hookrightarrow C_r^*(G)$. Bradd [8, 9] showed that the inclusion $\mathcal{S}(G) \hookrightarrow C_r^*(G)$ induces an isomorphism on K -theory if one restricts to isotypical components corresponding to certain finite sets of K -types. Because $\text{index}_G(D_W)$ is nonzero by the Connes–Kasparov conjecture, so is the higher index (17).

Suppose that Γ is torsion-free. By [14, Lem. 2.7], we have a continuous inclusion

$$(20) \quad \mathcal{A}_{S,q}(E)^\Gamma \hookrightarrow C^*(G/K)^\Gamma,$$

where $C^*(G/K)^\Gamma$ is the Γ -equivariant Roe algebra of G/K .

Corollary 3.21. *If X is compact and Γ is torsion-free, then*

$$\text{index}_{S,\Gamma}(D_W) \in K_0(\mathcal{A}_{S,q}(E)^\Gamma)$$

is mapped to the usual Γ -index

$$\text{index}_\Gamma(D_W) \in K_0(C_r^*(\Gamma)) = K_0(C^*(G/K)^\Gamma)$$

by the map induced by the inclusion (20).

Proof. This follows from Theorem 2.5 by [14, Rem. 2.16] and [31]. □

3.22. Index theorems: Semisimple and non-semisimple cases. Next, we use the algebra isomorphism from Lemma 3.5 to apply the traces from Sections 3.1 and 3.10 to the higher index (17), and obtain a higher version of the index theorem in [5]. For now, we drop the assumption that G has real rank 1; this will be assumed again in the index theorem for non-semisimple contributions, Theorem 3.26.

The index theorems are stated for Dirac operators (7) on G/K . Consider the traces $\tau_\gamma^{\Delta_p \otimes W}$ from Proposition 3.6, for semisimple $\gamma \in \Gamma$, $(\tau_{n_0} + \tau_{n_0^{-1}})^{\Delta_p \otimes W}$ from Lemma 3.8, $\tau_\lambda^{\Delta_p \otimes W}$ from Proposition 3.9, and $T_\lambda^{\Delta_p \otimes W}$ from Proposition 3.16 on the algebra $(\mathcal{S}(G) \otimes \text{End}(\Delta_p \otimes W))^{K \times K}$. Via the isomorphism Ψ from Lemma 3.5, these define continuous traces on $\mathcal{A}_{S,q}(E)^\Gamma$. These can be applied to (17). In fact, higher cyclic cocycles $\Phi_{P,\gamma}^{\Delta_p \otimes W}$ as in Proposition 3.7, for γ semisimple, can also be applied to the index (17) and computed, yielding possibly nonzero results in cases where the trace $\tau_\gamma^{\Delta_p \otimes W}$ maps (17) to zero.

To state index theorems for the resulting numbers, we introduce some notation in the various cases.

Index theorem for semisimple contributions. Let $T < K$ be a maximal torus. If T is a Cartan subgroup of G , so $\text{rank}(G) = \text{rank}(K)$, then we use the following notation. Fix a positive root system $R^+(G, T)$ for (G, T) , and consider the Weyl group $W_K = N_K(T)/T$. Let ρ be half the sum of the roots in $R^+(G, T)$, let ρ_c be half the sum of the roots in $R^+(K, T) = R^+(G, T) \cap R(K, T)$, and let $\rho_n = \rho - \rho_c$.

If $g \in T$, then we write

$$R^+(G_g, T) = R^+(G, T) \cap R(G_g, T) \quad \text{and} \quad W_{K_g} = N_{K_g}(T)/T.$$

Let ρ_g be half the sum of the roots in $R^+(G_g, T)$.

Let λ be the highest weight of W with respect to the set of compact positive roots $R^+(K, T) \subset R^+(G, T)$. For $\mu \in i\mathfrak{t}^*$, we write

$$(21) \quad d_\mu^{G_g} = \prod_{\alpha \in R^+(G_g, T)} \frac{(\mu, \alpha)}{(\rho_g, \alpha)}.$$

(This includes the special case where $g = e$, so $G_g = G$.) For a discrete series representation of G_g with Harish-Chandra parameter μ , this is its formal degree.

Theorem 3.23 (Index theorem, semisimple case). *Suppose that $\text{rank}(G) = \text{rank}(K)$.*

(a) *If $\gamma \in \Gamma$ is central in G , then*

$$\tau_\gamma^{\Delta_p \otimes W} \circ \Psi_*(\text{index}_{S,G}(D_W)) = e^{\lambda - \rho_n}(\gamma) d_{\lambda + \rho_c}^G.$$

(b) *If $\gamma \in \Gamma$ is hyperbolic, then*

$$\tau_\gamma^{\Delta_p \otimes W} \circ \Psi_*(\text{index}_{S,G}(D_W)) = 0.$$

(c) *If $\gamma \in \Gamma$ is elliptic, then with respect to a compact Cartan subgroup T containing γ ,*

$$\begin{aligned} & \tau_\gamma^{\Delta_p \otimes W} \circ \Psi_*(\text{index}_{S,G}(D_W)) \\ &= (-1)^{(\dim(G/K) + \dim(G_\gamma/K_\gamma))/2} \cdot \frac{\sum_{w \in W_{K_\gamma} \setminus W_K} \det(w) e^{w(\lambda + \rho_c - \rho_\gamma)}(\gamma) d_{w(\lambda + \rho_\gamma - \rho_n)}^{G_\gamma}}{e^{\rho - \rho_\gamma}(\gamma) \prod_{\alpha \in R^+(G,T) \setminus R^+(G_\gamma,T)} (1 - e^\alpha(\gamma))}. \end{aligned}$$

Index theorem for semisimple contributions from higher cocycles. As in Proposition 3.7, consider a cuspidal parabolic subgroup $P = MA'N' < G$, and a semisimple element $g \in M$. If P is a maximal cuspidal parabolic subgroup, then $T < M$, after conjugation if necessary. Then T is a compact Cartan subgroup of M . Fix a positive root system $R^+(M, T) \subset R(M, T)$. If $g \in T$, we write

$$R^+(M_g, T) := R(M_g, T) \cap R^+(M, T).$$

Let ρ^M be half the sum of the elements of $R^+(M, T)$, let ρ_c^M be half the sum of the elements of $R^+(K \cap M, T) := R^+(M, T) \cap R(K \cap M, T)$, and let $\rho_n^M = \rho^M - \rho_c^M$. Let ρ_g^M be half the sum of the roots in $R^+(M_g, T)$.

As in [16], we now assume that the map (6) descends to K . By [16, Lem. 4.3], there is a natural $K \cap M$ -equivariant embedding of $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})$ into \mathfrak{p} , via an identification of $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})$ with the orthogonal complement of $(\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a}$. Hence we obtain the representation $K \cap M \rightarrow \text{Spin}(\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})) \rightarrow \text{GL}(\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})})$. The spaces $\mathfrak{k}/\mathfrak{t}$ and $(\mathfrak{k} \cap \mathfrak{m})/\mathfrak{t}$ are both even-dimensional, and hence so is $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})$. So the representation $\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ of $K \cap M$ has a natural $\mathbb{Z}/2\mathbb{Z}$ grading $\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} = \Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}^+ \oplus \Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}^-$. If we view $\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ as the virtual representation $[\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}^+] - [\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}^-] \in R(K \cap M)$, then some of the multiplicities $m_U \in \mathbb{Z}$ in

$$(22) \quad \Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} \otimes W|_{K \cap M} = \bigoplus_{U \in \widehat{K \cap M}} m_U U$$

may be negative.

For each $U \in \widehat{K \cap M}$, let $\lambda_U \in i(\mathfrak{t} \cap \mathfrak{m})^*$ be its highest weight with respect to the positive root system $R^+(K \cap M, T)$. Consider the Weyl groups $W_{K \cap M} = N_{K \cap M}(T)/T$ and $W_{K \cap M_g} = N_{K \cap M_g}(T)/T$. For $\mu \in i(\mathfrak{t} \cap \mathfrak{m})^*$, define $d_\mu^{M_g}$ analogously to (21), where the product now runs over $\alpha \in R^+(M_g, T)$. For semisimple $g \in M$, let $\Phi_{P,g}^{\Delta_{\mathfrak{p}} \otimes W}$ be as in Proposition 3.7.

Theorem 3.24 (Index theorem, semisimple case for higher cocycles). *If P is not a maximal cuspidal parabolic subgroup of G , then*

$$\langle \Phi_{P,\gamma}^{\Delta_{\mathfrak{p}} \otimes W}, \Psi_*(\text{index}_{S,G}(DW)) \rangle = 0$$

for all semisimple $\gamma \in \Gamma \cap M$. Suppose now that P is a maximal cuspidal parabolic subgroup of G .

(a) *If $\gamma \in \Gamma \cap M$ is central in M , then*

$$\langle \Phi_{P,\gamma}^{\Delta_{\mathfrak{p}} \otimes W}, \Psi_*(\text{index}_{S,G}(DW)) \rangle = \sum_{U \in \widehat{K \cap M}} m_U e^{\lambda_U - \rho_n^M}(\gamma) d_{\lambda_U + \rho_c^M}^M.$$

(b) *If $\gamma \in \Gamma \cap M$ is hyperbolic in M , then*

$$\langle \Phi_{P,\gamma}^{\Delta_{\mathfrak{p}} \otimes W}, \Psi_*(\text{index}_{S,G}(DW)) \rangle = 0.$$

(c) *If $\gamma \in \Gamma \cap M$ is elliptic in M , then with respect to a compact Cartan subgroup $T < M$ containing γ ,*

$$\begin{aligned} & \langle \Phi_{P,\gamma}^{\Delta_{\mathfrak{p}} \otimes W}, \Psi_*(\text{index}_{S,G}(DW)) \rangle \\ &= (-1)^{(\dim(M/(K \cap M)) + \dim(M_\gamma/(K \cap M_\gamma)))/2} \\ & \cdot \sum_{U \in \widehat{K \cap M}} m_U \frac{\sum_{w \in W_{K \cap M_\gamma} \setminus W_{K \cap M}} \det(w) e^{w(\lambda_U + \rho_c^M - \rho_\gamma^M)}(\gamma) d_{w(\lambda_U + \rho_\gamma^M - \rho_n^M)}^{M_\gamma}}{e^{\rho^M - \rho_\gamma^M}(\gamma) \prod_{\alpha \in R^+(M,T) \setminus R^+(M_\gamma,T)} (1 - e^\alpha(\gamma))}. \end{aligned}$$

Remark 3.25. If $\text{rank}(G) = \text{rank}(K)$, then G is a maximal cuspidal parabolic subgroup of itself, and $\Phi_{G,g} = \tau_g$ for all semisimple $g \in G$. Hence Theorem 3.24 reduces to Theorem 3.23 in this case.

Index theorem for non-semisimple contributions. To state an index theorem for traces not associated to semisimple elements of Γ , we introduce some further notation. We now assume that G has real rank 1.

Let l be the number of Γ -conjugacy classes of Γ -cuspidal parabolic subgroups of G . Choose representatives $P_j = M_j A_j N_j$ of these conjugacy classes. If $G = \text{SU}(2n, 1)$, then we write

$$C_2(\Gamma) := \sum_{j=1}^l \frac{\text{vol}(\Gamma \cap N_j \backslash N_j)}{\text{vol}(\Gamma \cap (N_j)_{2\lambda} \backslash (N_j)_{2\lambda})} \frac{(2\pi)^{2n}}{(2n)!} B_n,$$

where B_n is the n th Bernoulli number. Furthermore, let $\mathfrak{t} \subset \mathfrak{su}(2n, 1)$ be the diagonal Cartan. Let $Z_0 \in \mathfrak{t}$ be the diagonal matrix with diagonal elements $i, i, \dots, i, -2ni$. Let

$$\varepsilon(R^+(G, T)) = \text{sign} \prod_{\alpha \in R^+(G, T) \setminus R^+(K, T)} \langle \alpha, Z_0 \rangle.$$

Let $\Omega \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the Casimir element. For $\sigma \in \hat{M}$, consider the operators

$$c_{\lambda}(\sigma) := -\text{Ind}_P^G(\sigma \otimes 1_{AN})(\Omega) + \|\lambda + \rho_c\|^2 - \|\rho\|^2$$

on the representation space $\mathcal{H}(\sigma)$ of $\text{Ind}_P^G(\sigma \otimes 1_{AN})$. Let $c_{\Gamma, \lambda}^{\pm}(\sigma)$ be the restriction of $c_{\Gamma}(\sigma) \otimes 1_{\Delta_{\mathbb{P}}^{\otimes W}}$ to K -invariant vectors. (Here $c_{\Gamma}(\sigma)$ is as in Lemma 3.12.)

For general G , if $R_0^+(G, T)$ is a positive root system chosen as in [1, Sec. 2.a], then we write

$$\varepsilon(\lambda + \rho_c) := \text{sign} \prod_{\alpha \in R_0^+(G, T)} (\lambda + \rho_c, \alpha).$$

If $N_{2\lambda}$ is trivial, then we choose a vector $\tilde{X} \in \mathfrak{n}_{\lambda}$ of norm $\sqrt{2}/\|\lambda\|$. If $N_{2\lambda}$ is nontrivial, then we choose a vector $\tilde{X} \in \mathfrak{n}_{2\lambda}$ of norm $1/(\sqrt{2}\|\lambda\|)$. Write $\tilde{Y} := \theta(\tilde{X})$, where θ is the Cartan involution. For integral $\mu \in i\mathfrak{t}^*$, define $k(\mu) \in \mathbb{Z}$ by the relation $e^{\langle \mu, t(\tilde{X} - \tilde{Y}) \rangle} = e^{ik(\mu)t}$ for all $t \in \mathbb{R}$. Recall the choice of $n_0 \in N_{2\lambda}$ above (16). Consider the Weyl group $W_M = N_M(T \cap M)/(T \cap M)$.

Theorem 3.26 (Index theorem, non-semisimple case). *Suppose that G has real rank 1 and that $\text{rank}(G) = \text{rank}(K)$.*

(a) *If $G \neq \text{SU}(2n, 1)$, then*

$$(\tau_{n_0} + \tau_{n_0^{-1}})^{\Delta_{\mathbb{P}}^{\otimes W}} \circ \Psi_*(\text{index}_{S, G}(D_W)) = 0.$$

If $G = \text{SU}(2n, 1)$, then

$$(23) \quad (\tau_{n_0} + \tau_{n_0^{-1}})^{\Delta_{\mathbb{P}}^{\otimes W}} \circ \Psi_*(\text{index}_{S, G}(D_W)) \\ = \|\lambda\| \frac{2^{3n-3}(2n+1)^n}{\text{area}(S^{4n-1})} C_2(\Gamma) \varepsilon(R^+(G, T)) \dim(W).$$

(b) *We have*

$$\tau_{\lambda}^{\Delta_{\mathbb{P}}^{\otimes W}} \circ \Psi_*(\text{index}_{S, G}(D_W)) = 0.$$

(c) If $\lambda + \rho_c$ is regular, then

$$\tau_{\text{res}} \circ \Psi_*(\text{index}_{S,G}(D_W)) = 0.$$

If $\lambda + \rho_c$ is singular, then

$$\tau_{\text{res}} \circ \Psi_*(\text{index}_{S,G}(D_W)) = 2 \sum_{\sigma \in \tilde{M}; c_\lambda(\sigma)=0} \text{Tr}(c_{\Gamma, \lambda}^+(\sigma)).$$

(d) If $\lambda + \rho_c$ is regular, then

$$\begin{aligned} (24) \quad & \tau_{\text{rem}}^{\Delta_{\mathbb{P}} \otimes W} \circ \Psi_*(\text{index}_{S,G}(D_W)) \\ &= \frac{(-1)^{\dim(G/K)/2}}{2} l\varepsilon(\lambda + \rho_c) \\ & \cdot \sum_{w \in W_M \setminus W_K} \det(w) \text{sgn}(k(w(\lambda + \rho_c))) \\ & \cdot \dim(\sigma_{w_M w(\lambda + \rho_c)|_{\mathfrak{t} \cap \mathfrak{m}} - \rho^M}), \end{aligned}$$

where (for every $w \in W_K$) an element $w_M \in W_M$ is chosen such that $w_M w(\lambda + \rho_c)|_{\mathfrak{t} \cap \mathfrak{m}}$ is strictly dominant for M , and $\sigma_{w_M w(\lambda + \rho_c)|_{\mathfrak{t} \cap \mathfrak{m}} - \rho^M} \in \tilde{M}$ has highest weight $w_M w(\lambda - \rho_c)|_{\mathfrak{t} \cap \mathfrak{m}} - \rho^M$.

If G/K is real hyperbolic space of dimension at least 4, then

$$\tau_{\text{rem}}^{\Delta_{\mathbb{P}} \otimes W} \circ \Psi_*(\text{index}_{S,G}(D_W)) = 0.$$

Remark 3.27. The condition that $\lambda + \rho_c$ is regular in Theorem 3.26 (d) may be weakened to the more general but less explicit condition that $c_\lambda(\sigma) > 0$ for all $\sigma \in \tilde{M}$ such that $[\Delta_{\mathbb{P}} \otimes W : \sigma^*] > 0$. See [5, Prop. 6.5].

Theorems 3.23, 3.24, and 3.26 are proved in Section 5.25. The proof of Theorem 3.26 is taken from [5]. We give an independent, index-theoretic proof of Theorem 3.23 in Section 5. In the setting of Theorem 3.24, the computations from [5] do not apply directly, and we use results from [16] combined with arguments from the proof of Theorem 3.23.

4. TRACES ON A SCHWARTZ ALGEBRA

We consider the setting of Section 2.7, and initially do not assume that G has real rank 1.

4.1. Schwartz spaces.

Proof of Lemma 3.4. Let a_0 and C be as in (1), for $X = G$. Let $f_1, f_2 \in C_c^\infty(G)$, $a \geq a_0$, and $P_1, P_2 \in \mathcal{U}(\mathfrak{g})$. Then, as in the proof of [14, Lem. 2.5],

$$\|f_1 * f_2\|_{a, P_1, P_2} = \|L(P_1)f_1 * R(P_2)f_2\|_{a, 1} \leq C \|f_1\|_{2a, P_1, 1} \|f_2\|_{a, 1, P_2}.$$

If $a < a_0$, then

$$\|f_1 * f_2\|_{a, P_1, P_2} \leq \|f_1 * f_2\|_{a_0, P_1, P_2} \leq C \|f_1\|_{2a_0, P_1, 1} \|f_2\|_{a_0, 1, P_2}.$$

So $\mathcal{S}(G)$ is closed under convolution, and convolution is a continuous operation on $\mathcal{S}(G)$. □

Proof of Lemma 3.5. The following facts can be verified by writing out the various definitions.

- (i) For all $\kappa \in \Gamma^\infty(E \boxtimes E^*)^G$, relation (10) indeed yields a well-defined function $\Psi(\kappa) \in (C^\infty(G) \otimes \text{End}(V))^{K \times K}$.
- (ii) The map Ψ is linear.
- (iii) For $f \in (C^\infty(G) \otimes \text{End}(V))^{K \times K}$, define $\kappa_f \in \Gamma^\infty(E \boxtimes E^*)$ by

$$\kappa_f(gK, g'K)[g, v] = [g, f(g^{-1}g')v]$$

for $g, g' \in G, v \in V$. This yields a well-defined section $\kappa_f \in \Gamma^\infty(E \boxtimes E^*)^G$.

- (iv) The map $f \mapsto \kappa_f$ from the previous point is a two-sided inverse of Ψ .
- (v) The map Ψ restricts to a bijection from $\Gamma_{\text{fp}}^\infty(E \boxtimes E^*)^G$ to $(C_c^\infty(G) \otimes \text{End}(V))^{K \times K}$.
- (vi) For all $\kappa, \lambda \in \Gamma_{\text{fp}}^\infty(E \boxtimes E^*)^G$, we have $\Psi(\kappa\lambda) = \Psi(\kappa)\Psi(\lambda)$.
- (vii) For all $f \in (C^\infty(G) \otimes \text{End}(V))^{K \times K}$, all $P_1, P_2 \in \mathcal{U}(\mathfrak{g})$, and all $g, g' \in G$,

$$(R(P_1) \otimes R(P_2)q^*\kappa_f)(g, g') = (L(P_1)R(P_2)f)(g^{-1}g').$$

By left G -invariance of the Riemannian distance on G , the last point implies that, for all $a > 0$ and $P_1, P_2 \in \mathcal{U}(\mathfrak{g})$, $\|\kappa_f\|_{a, R(P_1) \otimes R(P_2)} = \|f\|_{a, P_1, P_2}$. So Ψ indeed defines an isometric algebra isomorphism

$$\Psi : \mathcal{A}_{S, q}(E)^G \xrightarrow{\cong} (\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}. \quad \square$$

We will use a relation between $\mathcal{S}(G)$ and Harish-Chandra’s L^p -Schwartz spaces. Let Ξ be the matrix coefficient for a unit vector in the representation induced from the trivial representation of a minimal parabolic. Fix $p > 0$. For $f \in C_c^\infty(G)$, $a > 0$, and $P_1, P_2 \in \mathcal{U}(\mathfrak{g})$, we write

$$\|f\|_{C^{p, a, P_1, P_2}} := \sup_{g \in G} (1 + d_{G/K}(eK, gK))^a \Xi(x)^{-2/p} |(L(P_1)R(P_2)f)(g)|.$$

Let $\mathcal{C}^p(G)$ be the completion of $C_c^\infty(G)$ in the semi-norms $\|\cdot\|_{C^{p, a, P, Q}}$.

Remark 4.2. The space $\mathcal{C}^p(G)$ is the L^p -version of Harish-Chandra’s Schwartz space. See e.g. [5, pp.161–162]. For $p = 1$, this reduces to the definition in [36, p.34]; for equivalence of the semi-norms used, see [22, Prop. 7.15 and (7.49)–(7.51)]. The space $\mathcal{C}_\varepsilon^1(G)$ in [36, p.98] equals $\mathcal{C}^{2/(2+\varepsilon)}(G)$.

Lemma 4.3. *The space $\mathcal{S}(G)$ is contained in $\mathcal{C}^p(G)$ for all $p > 0$, and the inclusion map is continuous.*

Proof. The weight functions in the definition of the semi-norms on $\mathcal{C}^p(G)$ can be bounded by functions that increase exponentially in the distance to the identity element. □

Remark 4.4. At this point, let us comment on why we use the algebra $\mathcal{S}(G)$ in the construction of the higher index on G/K . The spaces $\mathcal{C}^1(G)$ and $\mathcal{C}^2(G)$ are in fact Fréchet algebras as well; see [36, pp.49–51] for the former and [22, Prop. 12.16] for the latter. These have two disadvantages for our purposes, however.

- (i) While the geometric traces from Section 3.1 extend continuously to

$$(\mathcal{C}^2(G) \otimes \text{End}(V))^{K \times K},$$

the trace (16) is only continuous on $(\mathcal{C}^p(G) \otimes \text{End}(V))^{K \times K}$ for $0 < p < 1$, hence not necessarily on the algebras

$$(\mathcal{C}^1(G) \otimes \text{End}(V))^{K \times K} \quad \text{and} \quad (\mathcal{C}^2(G) \otimes \text{End}(V))^{K \times K}.$$

- (ii) It was shown in [16, Lem. 3.8] that the image of the idempotent in (5) under the map Ψ from Lemma 3.5 lies in the unitization of

$$(\mathcal{C}^2(G) \otimes \text{End}(V))^{K \times K},$$

but the Fourier transform techniques used there do not seem to generalize easily to an argument that it lies in $(\mathcal{C}^p(G) \otimes \text{End}(V))^{K \times K}$ for smaller p . The heat operators on the diagonal in (5) were shown to map into $(\mathcal{C}^p(G) \otimes \text{End}(V))^{K \times K}$ for all $p > 0$ in [5, Prop. 2.4]. An extension of these arguments to the off-diagonal elements in (5) may be possible, but in view of the proof of [5, Lem. 2.3], we expect this to be as involved as the proof that they map into $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$ in this paper and [14].

So two key advantages of the algebra $\mathcal{S}(G)$ over $\mathcal{C}^1(G)$ and $\mathcal{C}^2(G)$ are that

- (i) the functionals in Section 3.10 are continuous traces on

$$(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K},$$

- (ii) the idempotent in (5) maps into $(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$, as shown in the present paper.

Furthermore, the algebra $\mathcal{A}_{S,\varphi}(E)^H$, and hence the higher index of Definition 2.6, can be defined for more general spaces than G/K ; see [14].

4.5. Geometric traces on $\mathcal{S}(G)$. In this subsection, we prove Propositions 3.6, 3.7, and 3.9 and Lemma 3.8. We start with a precise form of the statement that a conjugation-invariant functional is a trace.

Lemma 4.6. *Let H be a unimodular Lie group. Let $\mathcal{A}(H)$ be a locally convex convolution algebra of functions on H , containing $C_c^\infty(H)$ as a dense subalgebra. Suppose that convolution is continuous as a map from $\mathcal{A}(H) \times \mathcal{A}(H)$ to $\mathcal{A}(H)$. Let $\tau : \mathcal{A}(H) \rightarrow \mathbb{C}$ be a conjugation-invariant, continuous linear functional. Suppose that, for all $f_1, f_2 \in C_c^\infty(H)$,*

$$(25) \quad \tau(f_1 * f_2) = \int_H \tau(x \mapsto f_1(xy^{-1})f_2(y)) \, dy.$$

Then τ is a trace on $\mathcal{A}(H)$.

Proof. Let $f_1, f_2 \in C_c^\infty(H)$. Then, by (25) and conjugation-invariance of τ ,

$$\tau(f_1 * f_2) = \int_H \tau(x \mapsto f_1(y^{-1}x))f_2(y) \, dy.$$

By a substitution $y \mapsto y^{-1}x$ (where we use unimodularity of H) and (25), this equals

$$\int_H \tau(x \mapsto f_2(xy^{-1})f_1(y)) \, dy = \tau(f_2 * f_1).$$

So τ is a trace on $C_c^\infty(H)$. By continuity of convolution and of τ , this implies that τ is a trace on $\mathcal{A}(H)$. \square

Proposition 4.7. *If g is semisimple, then τ_g defines a continuous trace on $\mathcal{S}(G)$.*

Proof. If g is semisimple, then τ_g is a continuous functional on $\mathcal{C}^2(G)$; see [17, Thm. 2.1], which is a consequence of [15, Thm. 6]. By Lemma 4.3, τ_g restricts to a continuous functional on $\mathcal{S}(G)$. (For a more direct argument, τ_g is a continuous functional on $\mathcal{S}(G)$ by [7, (3.4.4), (3.4.36), and (3.4.44)].) It is clearly conjugation-invariant, and it has property (25) by the Fubini–Tonelli theorem. Convolution is continuous on $\mathcal{S}(G)$ by Lemma 3.4. So the claim follows by Lemma 4.6. \square

Proposition 4.7 implies Proposition 3.6.

Proof of Proposition 3.7. By [32, Thm. 3.5], $\Phi_{P,g}$ is a continuous cyclic cocycle on $\mathcal{C}^2(G)$. So the claim follows from Lemma 4.3. (Analogously to the proof of Proposition 4.7, we expect a more direct proof based on inequalities from [7] to be possible.) \square

For the rest of this section, we assume that G has real rank 1. It is noted in [29, p. 299] that $\tau_{n_0} + \tau_{n_0^{-1}}$ is a continuous linear functional on $\mathcal{C}^2(G)$. In fact, the two terms separately define continuous traces on $\mathcal{S}(G)$; we write out the proof here to show that this is particularly straight-forward for the algebra $\mathcal{S}(G)$.

Lemma 4.8. *For all $n \in N_{2\lambda}$, the orbital integral τ_n defines a continuous trace on $\mathcal{S}(G)$.*

Proof. We first show that τ_n is a continuous linear functional on $\mathcal{S}(G)$. Note that $N < G_n$, because $n \in N_{2\lambda}$. So, for all $f \in \mathcal{S}(G)$, (12) implies that

$$|\tau_n(f)| \leq \int_{G/N} |f(xnx^{-1})| d(xN) = \int_{KA} |f(kana^{-1}k^{-1})| a^{2\rho_N} dk da.$$

For all $c > 0$, this is smaller than or equal to

$$(26) \quad \begin{aligned} & \|f\|_{c,0} \int_{KA} e^{-cd(e,kana^{-1}k^{-1})} a^{2\rho_N} da \\ & \leq \|f\|_{c,0} e^{c \operatorname{diam}(K)} \int_A e^{-cd(e,ana^{-1})} a^{2\rho_N} da \end{aligned}$$

by the triangle inequality. If $n = \exp(Y)$ for $Y \in \mathfrak{n}_{2\lambda}$, then the distance from e to ana^{-1} in the abelian group $N_{2\lambda}$ is

$$d_{N_{2\lambda}}(1, \exp(\operatorname{Ad}(a)Y)) = \|\operatorname{Ad}(a)Y\| = a^{2\lambda}\|Y\|.$$

This is the same as the distance from e to ana^{-1} in G , because $t \mapsto \exp(tY)$ is a geodesic. So the right-hand side of (26) is smaller than or equal to

$$\|f\|_{c,0} e^{c \operatorname{diam}(K)} \|Y\| \int_A a^{-2c\lambda} a^{2\rho_N} da.$$

By (11), the integral converges if $c > \dim(\mathfrak{n}_\lambda)/2 + \dim(\mathfrak{n}_{2\lambda})$. So τ_n is indeed continuous.

It is immediate that τ_n is conjugation-invariant. And it satisfies (25) for compactly supported functions, by the Fubini–Tonelli theorem. So Lemma 4.6 implies the claim. \square

Lemma 4.8 implies Lemma 3.8.

Proposition 4.9. *The integral (13) converges for all $f \in \mathcal{S}(G)$, and defines a continuous trace on this algebra.*

Proof. The integral defines a continuous functional on $\mathcal{C}^2(G)$, as noted in [29, p. 299]. So, by Lemma 4.3, it is a continuous functional on $\mathcal{S}(G)$.

For the trace property, we use Lemma 4.6. Let us show that τ_λ is conjugation-invariant.¹ Because the action by AN on G by right multiplication is proper and cocompact, there exists a function $\varphi \in C_c(G)$ such that, for all $x \in G$,

$$(27) \quad \int_{AN} \varphi(xan) \, da \, dn = 1.$$

The group AN normalizes N , and for all $f \in C_c(G)$, and all $a \in A$ and $n' \in N$,

$$a^{2\rho_N} \int_N f(an'n(an')^{-1}) \, dn = \int_N f(n) \, dn,$$

where $a^{2\rho_N}$ is a Jacobian. It follows that, for any function φ as in (27), and all $f \in C_c^\infty(G)$,

$$(28) \quad \begin{aligned} \tau_\lambda(f) &= \int_N \int_{KAN} \varphi(kan') f(knk^{-1}) \, dk \, da \, dn' \, dn \\ &= \int_N \int_{KAN} \varphi(kan') f(kan'n(kan')^{-1}) a^{2\rho_N} \, dk \, da \, dn' \, dn \\ &= \int_N \int_G \varphi(g) f(gng^{-1}) \, dg \, dn, \end{aligned}$$

by (12). For $h \in G$, let $f_h \in C_c^\infty(G)$ be given by $f_h(g) = f(hgh^{-1})$. Then, by (28) and a substitution $hg \mapsto g$,

$$(29) \quad \tau_\lambda(f_h) = \int_N \int_G (L_h\varphi)(g) f(gng^{-1}) \, dg \, dn.$$

The function $L_h\varphi$ has the same property (27) as φ , so (29) equals $\tau_\lambda(f)$.

We have seen that τ_λ is continuous and conjugation-invariant. It satisfies (25) by the Fubini–Tonelli theorem, so it is a trace on $\mathcal{S}(G)$ by Lemma 4.6. \square

Proposition 4.9 implies Proposition 3.9.

¹This seems to be known; the functional τ_λ is called an “invariant term” in [5] (and denoted by T_1 there). We include a proof here because we could not immediately find a reference.

4.10. **Spectral traces on $\mathcal{S}(G)$.** In this subsection, we prove Lemma 3.12, Propositions 3.13 and 3.16, and Corollary 3.14.

Proof of Lemma 3.12. The functional (15) is continuous on

$$(\mathcal{C}^p(G) \otimes \text{End}(V))^{K \times K} \quad \text{for } 0 < p < 1,$$

as pointed out below [36, Thm. 8.4]. Hence it is continuous on

$$(\mathcal{S}(G) \otimes \text{End}(V))^{K \times K}$$

by Lemma 4.3. It is immediate from the definition that τ_{rem} is conjugation-invariant, so the claim follows by Lemma 4.6. \square

Proposition 4.11. *Let $F \subset \hat{K}$ be a finite set, and let $\mathcal{C}_F^1(G) \subset \mathcal{C}^1(G)$ be the subspace of isotypical components corresponding to F for the right K -action. For all $f \in \mathcal{C}_F^1(G)$, the operator $R_d^\Gamma(f)$ is trace-class. The linear map*

$$(30) \quad R_d^\Gamma : \mathcal{C}_F^1(G) \rightarrow \mathcal{L}^1(L_d^2(\Gamma \backslash G))$$

is continuous.

Proof. The first claim is [36, Thm. 4.6].

Let $L_0^2(\Gamma \backslash G) \subset L_d^2(\Gamma \backslash G)$ be the G -invariant subspace of cusp forms. By [36, Thm. 4.3], the representation R^Γ defines a linear map

$$(31) \quad R_0^\Gamma : \mathcal{C}^1(G) \rightarrow \mathcal{L}^1(L_0^2(\Gamma \backslash G)).$$

In [36, pp. 39–40], it is shown that $f \mapsto \text{Tr}(R_0^\Gamma(f))$ is a continuous functional on $\mathcal{C}^1(G)$. In fact, the arguments given there can be strengthened slightly, to show that (31) is continuous. Indeed, apart from replacing the left quasiregular representation used in [36] by the right quasiregular representation we use here, the only changes that need to be made are that, in the second-last set of displayed equations in [36, p. 40], one should take absolute values of $L_{G/\Gamma}^0(\alpha)$, $(U(\alpha)e_{U,i}, e_{U,i})$ and the integral over G in the third line. Then the bottom equation in [36, p. 40] becomes $\text{tr}(|L_{G/\Gamma}^0(\alpha)|) \leq C \|\Delta_L^{2N} \alpha\|_1$, and continuity of (31) follows.

Let $L_0^2(\Gamma \backslash G)^\perp \subset L_d^2(\Gamma \backslash G)$ be the orthogonal complement to $L_0^2(\Gamma \backslash G)$. It is shown in [36, p. 32] that the representation of G on this space is K -finite. Hence the image of the compression

$$(32) \quad (R_d^\Gamma)^\perp : \mathcal{C}_F^1(G) \rightarrow \mathcal{L}^1(L_0^2(\Gamma \backslash G)^\perp)$$

of (30) is finite-dimensional. Let n_F be its dimension. Because $\mathcal{C}^1(G)$ embeds continuously into $L^1(G)$, we find that, for all $f \in \mathcal{C}_F^1(G)$,

$$\text{Tr}(|(R_d^\Gamma)^\perp(f)|) \leq n_F \|(R_d^\Gamma)^\perp(f)\|_{\mathcal{B}(L_0^2(\Gamma \backslash G)^\perp)} \leq n_F \|f\|_{L^1(G)},$$

and (32) is also continuous.

Continuity of (31) and (32) implies continuity of (30). \square

Proposition 4.11 and Lemma 4.3 imply Proposition 3.13.

Remark 4.12. In the proof of Proposition 4.11, we used [36, Thm. 4.6], which states that $R_d^\Gamma(f)$ is trace-class for all K -finite $f \in \mathcal{C}^1(G)$. This was in fact generalized to G of arbitrary real rank by Müller [28]. This was further generalized to $f \in \mathcal{C}^1(G)$ that are not necessarily K -finite by Müller [26] and Ji [20].

Now consider a finite subset $F \subset \widehat{K \times K}$. Let $\mathcal{S}_F(G) \subset \mathcal{S}(G)$ be the subalgebra of functions in isotypical components corresponding to F , with respect to the left and right regular representations. Lemma 4.3 and Proposition 4.11 have the following consequence. This implies Corollary 3.14.

Corollary 4.13. *The map $f \mapsto \text{Tr}(R_d^\Gamma(f))$ is a continuous trace on $\mathcal{S}_F(G)$.*

Remark 4.14. To prove Corollary 4.13, we could have used the result from [36] that $f \mapsto \text{Tr}(R_0^\Gamma(f))$ is a continuous functional on $\mathcal{C}^1(G)$, rather than the stronger Proposition 4.11. We have included this proposition, because a consequence, Proposition 3.13, is also used in Corollary 3.18.

The following result is a consequence of [36, Thm. 8.4] and the theorem in [29, p. 299] and the comments below it.

Theorem 4.15 (Osborne–Warner). *For all $0 < p < 1$, expression (16) defines a continuous linear functional τ_{rem} on $\mathcal{C}^p(G)$.*

Corollary 4.16. *The functional τ_{rem} from Theorem 4.15 restricts to a continuous trace on $\mathcal{S}(G)$.*

Proof. The restriction of τ_{rem} to $\mathcal{S}(G)$ is continuous by Lemma 3.4 and Theorem 4.15. It is a trace by Propositions 3.6 and 3.9 and Lemma 3.8. \square

Corollary 4.16 implies Proposition 3.16.

5. CONTRIBUTIONS FROM SEMISIMPLE ELEMENTS

To prove Theorem 3.23, one can use the computations in [5, Sec. 4.1]. Those computations are based on some nontrivial results from representation theory. We give an independent, index-theoretic proof here, based on a generalization of the main result in [19]. We hope that this approach and some of the intermediate results may be of independent interest. The index-theoretic arguments also extend to a proof of Theorem 3.24.

For a locally compact group H acting properly and cocompactly on a manifold, we will use the standard equivariant index $\text{index}_H(D) \in K_0(C_r^*(H))$ of H -equivariant elliptic operators D . There are many equivalent definitions; see *e.g.* [6, 21, 38].

5.1. A holomorphic fixed-point formula. The first step is a version of the fixed-point formula in [17] for Dolbeault–Dirac operators. Let X be a complex manifold. Let G be a connected, real semisimple Lie group acting properly and cocompactly on X by holomorphic maps. Let $E \rightarrow M$ be a holomorphic, Hermitian, G -equivariant vector bundle. Let $\bar{\partial}_E$ be the Dolbeault operator

on antiholomorphic differential forms twisted by E , and let $\bar{\partial}_E^*$ be its formal adjoint. Then we have the Dolbeault–Dirac operator $\bar{\partial}_E + \bar{\partial}_E^*$.

Let $g \in G$ be semisimple. Because G acts holomorphically on X , the fixed point set $X^g \subset X$ of g is a complex submanifold, and the normal bundle $N \rightarrow X^g$ is a complex vector bundle. (To be precise, the connected components of X^g are complex submanifolds of possibly different dimensions, so the rank of N may jump between these components.) Its complexification decomposes into holomorphic and antiholomorphic parts as $N \otimes \mathbb{C} = N^{1,0} \oplus N^{0,1}$. Fix a Haar measure dz on the centralizer G_g . Let $\chi^g \in C_c^\infty(X^g)$ be such that, for all $x \in X^g$,

$$\int_{G_g} \chi^g(zx) dz = 1.$$

(Here we use compactness of X^g/Z ; see [17, Lem. 2.4].)

The restriction $E^g \rightarrow X^g$ of E is a holomorphic vector bundle over the complex manifold X^g , so we have the Dolbeault–Dirac operator $\bar{\partial}_{E^g} + \bar{\partial}_{E^g}^*$ on X^g . For every connected component X_j^g of X^g , we write $E_j^g := E|_{X_j^g}$. We similarly have the Dolbeault–Dirac operator $\bar{\partial}_{E_j^g} + \bar{\partial}_{E_j^g}^*$ on X_j^g .

Consider the equivariant indices

$$\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*) \in K_0(C_r^*(G)), \quad \text{index}_{G_g}(\bar{\partial}_{E_j^g} + \bar{\partial}_{E_j^g}^*) \in K_0(C_r^*(G_g)).$$

The orbital integral trace τ_g is a well-defined map on $K_0(C_r^*(G))$; see [17, Sec. 2.1]. The von Neumann trace τ_e defines a map on $K_0(C_r^*(G_g))$. We write $g^{\mathbb{Z}}$ for the closure of the cyclic subgroup of G generated by g .

Theorem 5.2. *The following statements hold.*

- (a) *If g does not lie in a compact subgroup of G , then $\tau_g(\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)) = 0$.*
- (b) *If g lies in a compact subgroup of G , then*

$$\tau_g(\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)) = \int_{T(X^g)} \chi^g \frac{\text{ch}([\sigma_{\bar{\partial}_{E^g} + \bar{\partial}_{E^g}^*}](g)) \text{Todd}(T(X^g) \otimes \mathbb{C})}{\text{ch}([\wedge N^{0,1}](g))}.$$

- (c) *Suppose that g lies in a compact subgroup of G . Suppose, furthermore, that, on each connected component X_j^g of X^g , the group element g acts on every fiber of E_j^g as the same scalar $g|_{E_j^g}$. Suppose also that the bundle $N^{0,1}$ is trivial as a $g^{\mathbb{Z}}$ -equivariant vector bundle. Then*

$$\tau_g(\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)) = \sum_j \frac{g|_{E_j^g}}{\det_{\mathbb{C}}(1 - g|_{N^{0,1}})|_{X_j^g}} \tau_e(\text{index}_{G_g}(\bar{\partial}_{E_j^g} + \bar{\partial}_{E_j^g}^*)).$$

Proof. Part (a) follows from the second statement in [17, Thm. 2.5]. (This in fact holds for all semisimple $g \in G$ not contained in compact subgroups. The “finite Gaussian orbital integral” condition used there holds for general semisimple group elements, by the equalities from [7] also mentioned in the proof of Proposition 4.7.)

To prove part (b), we again apply [17, Thm. 2.5]. This now yields

$$\tau_g(\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)) = \int_{T(X^g)} \chi^g \frac{\text{ch}([\sigma_{\bar{\partial}_E + \bar{\partial}_E^*}](g)) \text{Todd}(T(X^g) \otimes \mathbb{C})}{\text{ch}([\wedge N \otimes \mathbb{C}](g))}.$$

To be precise, in [17, Thm. 2.5], it is stated that this equality holds for almost all elliptic elements g not contained in compact subgroups. The result applies to “almost all” such g because of the “finite Gaussian orbital integrals” condition. But this turns out to hold in general; see [7, (3.3.36), (3.4.4), and (4.2.9)].

We use the decomposition

$$\begin{aligned} [\sigma_{\bar{\partial}_E + \bar{\partial}_E^*}|_{X^g}](g) &= [\wedge(N^{0,1})^*](g) \otimes [\sigma_{\bar{\partial}_{E^g} + \bar{\partial}_{E^g}^*}](g) \\ &= [\wedge N^{1,0}](g) \otimes [\sigma_{\bar{\partial}_{E^g} + \bar{\partial}_{E^g}^*}](g). \end{aligned}$$

By multiplicativity of the Chern character,

$$\frac{\text{ch}([\wedge N^{1,0}](g))}{\text{ch}([\wedge N \otimes \mathbb{C}](g))} = \frac{1}{\text{ch}([\wedge N^{0,1}](g))},$$

so (b) follows.

To deduce (c) from (b), we first note that the assumption on the way g acts on $E|_{X^g}$ implies that

$$[\sigma_{\bar{\partial}_{E^g} + \bar{\partial}_{E^g}^*}|_{X_j^g}](g) = g|_{E_j^g}[\sigma_{\bar{\partial}_{E_j^g} + \bar{\partial}_{E_j^g}^*}],$$

for every connected component X_j^g of X^g . And if $N^{0,1}$ is trivial, then for all j , we have $N^{0,1}|_{X_j^g} = X_j^g \times W_j$ for some representation W_j of the compact abelian group $\overline{g^{\mathbb{Z}}}$. (The dimension of X_j^g , and hence of W_j , may depend on j .) Then

$$[\wedge N^{0,1}|_{X_j^g}] = [X_j^g \times \mathbb{C}] \otimes [\wedge W_j] \in K_{\overline{g^{\mathbb{Z}}}}^0(X_j^g) \cong K^0(X_j^g) \otimes R(\overline{g^{\mathbb{Z}}}).$$

Here $\wedge W_j$ should be interpreted as a graded representation, *i.e.*

$$[\wedge W_j] = [\wedge^{\text{even}} W_j] - [\wedge^{\text{odd}} W_j] \in R(\overline{g^{\mathbb{Z}}}).$$

So

$$\begin{aligned} \text{ch}([\wedge N^{0,1}](g)) &= \text{tr}(g|_{\wedge^{\text{even}} W_j}) - \text{tr}(g|_{\wedge^{\text{odd}} W_j}) \\ &= \det_{\mathbb{C}}(1 - g|_{W_j}) = \det_{\mathbb{C}}(1 - g|_{N^{0,1}})|_{X_j^g}. \end{aligned}$$

We find that

$$\begin{aligned} \tau_g(\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)) &= \sum_j \frac{\text{tr}(g|_E)}{\det_{\mathbb{C}}(1 - g|_{N^{0,1}})} \Big|_{X_j^g} \\ &\quad \cdot \int_{T(X_j^g)} \chi^g|_{X_j^g} \text{ch}([\sigma_{\bar{\partial}_{E_j^g} + \bar{\partial}_{E_j^g}^*}]) \text{Todd}(T(X_j^g) \otimes \mathbb{C}). \end{aligned}$$

By [35, Prop. 4.4 and Thm. 6.12], the integral on the right equals

$$\tau_e(\text{index}_{G_g}(\bar{\partial}_{E_j^g} + \bar{\partial}_{E_j^g}^*)). \quad \square$$

Example 5.3. If X and G are compact, and X^g is finite, then Theorem 5.2 (c) reduces to

$$\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)(g) = \sum_{x \in X^g} \frac{g|_{E_x}}{\det_{\mathbb{C}}(1 - g|_{T_x X^{0,1}})} = \sum_{x \in X^g} \frac{g|_{E_x}}{\det_{\mathbb{C}}(1 - g^{-1}|_{T_x X^{1,0}})}.$$

This is consistent with Atiyah and Bott’s holomorphic fixed-point theorem [2, Thm. 4.12] because, in the notation of that paper,

$$\text{index}_G(\bar{\partial}_E + \bar{\partial}_E^*)(g) = L(g^{-1}, g).$$

5.4. A fixed-point set. Until the end of Section 5.13, we suppose that $T < K$ is a compact Cartan subgroup of G , so $\text{rank}(G) = \text{rank}(K)$. We fix a positive root system $R^+(G, T)$ for (G, T) . This determines a G -invariant complex structure on G/T such that

$$T_{eT}^{1,0}(G/T) = \bigoplus_{\alpha \in R^+(G,T)} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

Let $g \in T$. We will apply Theorem 5.2 to the space $X = G/T$, and we start by analyzing the fixed-point set $(G/T)^g$.

Lemma 5.5. *For all $x \in G$ such that $xgx^{-1} \in T$, there exists $y \in N_G(T)$ such that $xgx^{-1} = ygy^{-1}$.*

Proof. Because $xTx^{-1} \subset G_{xgx^{-1}}$, we have two maximal tori T and xTx^{-1} in the connected component $(G_{xgx^{-1}})_0$. These are conjugate in $(G_{xgx^{-1}})_0$. (See e.g. [23, Props. 6.59 and 6.61], or use the facts that any two maximal compact subgroups of $(G_{xgx^{-1}})_0$ are conjugate, and so are any two maximal tori inside such a compact subgroup.) Let $h \in (G_{xgx^{-1}})_0$ be such that $hxTx^{-1}h^{-1} = T$. Then $y := hx$ has the desired properties. \square

Recall that we used the Weyl groups

$$W_K = N_K(T)/T \quad \text{and} \quad W_{K_g} = N_{K_g}(T)/T.$$

Note that $G_g w$ is a right T -invariant subset of G for all $w \in N_G(T)$.

Lemma 5.6. *The fixed-point set $(G/T)^g$ decomposes as*

$$(33) \quad (G/T)^g = \coprod_{w \in W_{K_g} \setminus W_K} G_g w/T.$$

The disjoint union is over representatives $w \in N_K(T)$ of classes in $W_{K_g} \setminus W_K$.

Proof. We first show that

$$(34) \quad (G/T)^g = G_g N_G(T)/T.$$

It is immediate that the right-hand side is contained in the left-hand side. Now let $xT \in (G/T)^g$. Then $x^{-1}gx \in T$. By Lemma 5.5, there is a $y \in N_G(T)$ such that $x^{-1}gx = y^{-1}gy$. So $xy^{-1} \in G_g$, and $xT = (xy^{-1})yT$ lies in the right-hand side of (34).

Next, we show that

$$(35) \quad (G/T)^g = \coprod_{w \in N_{G_g}(T) \setminus N_G(T)} G_g w/T,$$

where the disjoint union is over a set of representatives $w \in N_G(T)$. Indeed, if $y \in N_{G_g}(T) \subset G_g$ and $w \in N_G(T)$, then we have $G_g yw/T = G_g w/T$. And

if $w, w' \in N_G(T)$ and $G_g w/T = G_g w'/T$, then let $y := w'w^{-1} \in N_G(T)$. Then $G_g w/T = G_g yw/T$, so there are $a \in G_g$ and $b \in T$ such that $awb = yw$. Hence $a^{-1}y = bw^{-1} \in T \subset G_g$. So $y \in G_g$, and therefore $y \in N_{G_g}(T)$. We find that $G_g w/T = G_g w'/T$ if and only if $w'w^{-1} \in N_{G_g}(T)$, which implies that the right-hand side of (34) equals the right-hand side of (35).

Finally, we have $N_G(T) = N_K(T)$ and $N_{G_g}(T) = N_{K_g}(T)$; see for example [18, Lem. 6.15]. So the decomposition (33) follows from (35) and the equality

$$N_{K_g}(T) \setminus N_K(T) = (N_{K_g}(T)/T) \setminus (N_K(T)/T). \quad \square$$

Example 5.7. If $g = e$, then Lemma 5.6 reduces to the equality $G/T = G/T$. At the other extreme, if the powers of g are dense in T , then Lemma 5.6 becomes $(G/T)^T = W_K$; this is [18, Lem. 6.15].

5.8. A normal bundle. For $w \in N_G(T)$, let J_w be the G -invariant complex structure on G/T such that

$$T_{eT}^{1,0}(G/T) \cong \bigoplus_{\alpha \in R^+(G,T)} \mathfrak{g}_{w\alpha}^{\mathbb{C}}.$$

Then J_e is the complex structure we have considered so far.

Lemma 5.9. *Let $w \in N_G(T)$. The map $R_w : G/T \rightarrow G/T$, given by $R_w(xT) = xwT$ for $x \in G$, is a well-defined, G -equivariant diffeomorphism. It is holomorphic as a map*

$$(36) \quad R_w : (G/T, J_w) \rightarrow (G/T, J_e).$$

Proof. The first claim follows directly from the definitions. We show that (36) is holomorphic.

Note that $R_w = L_w \circ C_{w^{-1}}$, where L is the standard action by G on G/T , and $C_{w^{-1}} = L_{w^{-1}} \circ R_w$ is induced by conjugation by w^{-1} . The map L_w is holomorphic for any of the complex structures $J_{w'}$, because they are G -invariant. And

$$T_{eT} C_{w^{-1}} = \text{Ad}(w^{-1}) : \mathfrak{g}/\mathfrak{t} \rightarrow \mathfrak{g}/\mathfrak{t}.$$

Because $\text{Ad}(w^{-1})\mathfrak{g}_{\alpha}^{\mathbb{C}} = \mathfrak{g}_{w^{-1}\alpha}^{\mathbb{C}}$, this map induces a complex linear map from $(T_{eT}(G/T), J_w)$ to $(T_{eT}(G/T), J_e)$. For general $x \in G$, we have

$$C_{w^{-1}} = L_{w^{-1}xw} \circ C_{w^{-1}} \circ L_{x^{-1}}.$$

So

$$T_{xT} C_{w^{-1}} = T_{eT} L_{w^{-1}xw} \circ T_{eT} C_{w^{-1}} \circ T_{xT} L_{x^{-1}}.$$

The three maps on the right-hand side are complex-linear, hence so is the map on the left. We find that the map (36) is indeed holomorphic. \square

The complex structure J_w for $w \in N_G(T)$ restricts to the normal bundle $N \subset T(G/T)|_{(G/T)^g}$ to $(G/T)^g$ in G/T . We denote this restriction by J_w as well.

Lemma 5.10. *For every $w \in N_G(T)$, the map $R_w : G_g/T \rightarrow G_g w/T$ given by $R_w(xT) = xwT$ for $x \in G_g$ is a well-defined, G_g -equivariant diffeomorphism. The vector bundles*

$$(37) \quad (N|_{G_g/T}, J_w) \rightarrow G_g/T, \quad R_w^*(N|_{G_g w/T}, J_e) \rightarrow G_g/T$$

are isomorphic as G_g -equivariant, holomorphic vector bundles.

Proof. Let $w \in N_G(T)$. The first claim follows from the first claim in Lemma 5.9.

We write $N(S)$ for the normal bundle of a submanifold $S \subset G/T$. Because the sets $G_g w/T$ are connected components of $(G/T)^g$, we have

$$\begin{aligned} TR_w(N|_{G_g/T}) &= TR_w(N(G_g/T)) = N(R_w(G_g/T)) \\ &= N(G_g w/T) = N|_{G_g w/T}. \end{aligned}$$

This fact, together with G -equivariance of R_w and the fact that the map (36) is holomorphic by Lemma 5.9, implies that the bundles (37) are equivariantly isomorphic as holomorphic vector bundles. \square

Lemma 5.11. *The bundle $N^{0,1} \rightarrow (G/T)^g$ is trivial as a G_g -vector bundle.*

Proof. Let $w \in N_K(T)$. We first prove that $(N^{0,1}|_{G_g/T}, J_w) \rightarrow G_g/T$ is G_g -equivariantly trivial. Indeed,

$$N|_{G_g/T} = G_g \times_T \mathfrak{g}/\mathfrak{g}_g,$$

where \mathfrak{g}_g is the Lie algebra of G_g . So

$$(38) \quad (N^{0,1}|_{G_g/T}, J_w) = G_g \times_T ((\mathfrak{g}/\mathfrak{g}_g)^{0,1}, J_w),$$

where

$$(39) \quad ((\mathfrak{g}/\mathfrak{g}_g)^{0,1}, J_w) = \bigoplus_{\alpha \in R^+(G,T) \setminus R^+(G_g,T)} \mathfrak{g}_{-\alpha}^{\mathbb{C}}.$$

The bundle (38) is isomorphic as a G_g -equivariant, complex vector bundle to $G_g/T \times ((\mathfrak{g}/\mathfrak{g}_g)^{0,1}, J_w)$ via the map given by $[x, Y] \mapsto (xY, \text{Ad}(x)Y)$, for $x \in G_g$ and $Y \in ((\mathfrak{g}/\mathfrak{g}_g)^{0,1}, J_w)$. So $(N^{0,1}|_{G_g/T}, J_w) \rightarrow G_g/T$ is equivariantly trivial as claimed.

The claim now follows from Lemmas 5.6 and 5.10. \square

As in Theorem 3.23, let ρ be half the sum of the positive roots in $R^+(G, T)$, and let ρ_g be half the sum of the positive roots in $R^+(G_g, T)$.

Lemma 5.12. *For all $w \in W_K$,*

$$(40) \quad \begin{aligned} \det_{\mathbb{C}}(1 - g|_{N^{0,1}})|_{G_g w/T} \\ = e^{w(\rho_g - \rho)}(g) \det(w)e^{\rho - \rho_g}(g) \prod_{\alpha \in R^+(G,T) \setminus R^+(G_g,T)} (1 - e^{-\alpha}(g)). \end{aligned}$$

Proof. Let $w \in N_K(T)$. Then, by (38) and (39),

$$\begin{aligned} & \det_{\mathbb{C}}(1 - g|_{(N^{0,1}, J_w)})|_{G_g/T} \\ &= \prod_{\alpha \in R^+(G,T) \setminus R^+(G_g,T)} (1 - e^{-w\alpha}(g)) \\ &= e^{w(\rho_g - \rho)}(g) e^{w(\rho - \rho_g)}(g) \prod_{\alpha \in R^+(G,T) \setminus R^+(G_g,T)} (1 - e^{-w\alpha}(g)). \end{aligned}$$

The right-hand side with the first exponential factor removed is a ratio of Weyl denominators. This is alternating with respect to w . So the last expression equals the right-hand side of (40). The claim now follows by Lemma 5.10. \square

5.13. A fixed-point formula on G/T . For analytically integral $\nu \in \mathfrak{it}^*$, we write \mathbb{C}_ν for the complex numbers equipped with the unitary representation of T with weight ν .

Lemma 5.14. *Let $\mu \in \mathfrak{it}^*$ be integral, and let $w \in N_K(T)$ be such that $w \cdot \mu = \mu \circ \text{Ad}(w)^{-1}$ is integral. Then there is a G_g -equivariant isomorphism of complex line bundles*

$$R_w^*(G_g w/T \times_T \mathbb{C}_\mu) \cong G_g \times_T \mathbb{C}_{w \cdot \mu} \rightarrow G_g/T.$$

Proof. We have

$$R_w^*(G_g w/T \times_T \mathbb{C}_\mu) = \{(xT, [xwt, z]) \mid x \in G_g, t \in T, z \in \mathbb{C}_\mu\}.$$

The map from the right-hand side to $G_g \times_T \mathbb{C}_{w \cdot \mu}$ mapping $(xT, [xwt, z])$ in this set to $[x, e^\mu(t)z]$ is a well-defined, G_g -equivariant isomorphism of complex line bundles. \square

We consider a Dirac operator D_W of the form (7). Let $\lambda \in \mathfrak{it}^*$ be the highest weight of W with respect to the set of compact positive roots $R^+(K, T) \subset R^+(G, T)$. As in Theorem 3.23, let ρ_c be half the sum of the positive compact roots in $R^+(K, T)$. Let $\rho_n = \rho - \rho_c$. For analytically integral $\nu \in \mathfrak{it}^*$, consider the holomorphic line bundle $L_\nu := G \times_T \mathbb{C}_\nu \rightarrow G/T$. Let $d_\mu^{G_g}$ be as in (21).

Lemma 5.15. *The element $\lambda - \rho_n \in \mathfrak{it}^*$ is analytically integral. We have*

$$\begin{aligned} (41) \quad & \tau_e(\text{index}_{G_g}(\bar{\partial}_{L_{\lambda - \rho_n}|_{G_g w/T}} + \bar{\partial}_{L_{\lambda - \rho_n}|_{G_g w/T}}^*)) \\ &= (-1)^{\dim(G_g/K_g)/2} d_{w(\lambda + \rho_g - \rho_n)}^{G_g} \end{aligned}$$

for all $w \in N_K(T)$, where the Dolbeault operator on the left is defined with respect to the restriction of the complex structure J_e on G/T to the connected component $G_g w/T$ of the complex submanifold $(G/T)^g \subset G/T$.

Proof. The element $\lambda - \rho_n \in \mathfrak{it}^*$ is analytically integral, because $\Delta_{\mathfrak{p}} \otimes W$ defines a representation of K with highest weight $\lambda + \rho_n$.

For $w \in N_K(T)$, let $J_w^{G_g}$ be the G_g -invariant complex structure on G_g/T such that

$$T_{eT}^{1,0}(G_g/T) = \bigoplus_{\alpha \in R^+(G_g,T)} \mathfrak{g}_{w\alpha}^{\mathbb{C}}.$$

Let $w \in N_K(T)$. Then Lemma 5.9 implies that the map R_w restricts to a G_g -equivariant, holomorphic diffeomorphism

$$R_w : (G_g/T, J_w^{G_g}) = (G_g/T, J_w|_{G_g/T}) \rightarrow (G_g w/T, J_e|_{G_g w/T}).$$

And Lemma 5.14 implies that

$$R_w^*(L_{\lambda-\rho_n}|_{G_g w/T}) \cong L_{w(\lambda-\rho_n)}^{G_g/T} := G_g \times_T \mathbb{C}_{w(\lambda-\rho_n)} \rightarrow G_g/T$$

as G_g -equivariant, complex line bundles. We find that

$$(42) \quad \begin{aligned} \text{index}_{G_g}(\bar{\partial}_{L_{\lambda-\rho_n}|_{G_g w/T}} + \bar{\partial}_{L_{\lambda-\rho_n}|_{G_g w/T}}^*) \\ = \text{index}_{G_g}(\bar{\partial}_{L_{w(\lambda-\rho_n)}^{G_g/T}} + \bar{\partial}_{L_{w(\lambda-\rho_n)}^{G_g/T}}^*), \end{aligned}$$

where the Dolbeault operator on the right is defined with respect to $J_w^{G_g}$.

In the proof of [17, Prop. 5.2], it is shown that the right-hand side of (42) equals

$$(-1)^{\dim(G_g/K_g)/2} \text{D-Ind}_{K_g}^{G_g}[W_{w(\lambda-\rho_n+\rho_{n,g})}^{K_g}],$$

where $\text{D-Ind}_{K_g}^{G_g}$ denotes Dirac induction, $\rho_{n,g}$ is half the sum of the noncompact roots in $R^+(G_g, T)$, and $W_{w(\lambda-\rho_n+\rho_{n,g})}^{K_g}$ is the irreducible representation of K_g with highest weight $w(\lambda - \rho_n + \rho_{n,g})$ with respect to the positive root system $wR^+(K, T) \cap R(K_g, T)$. If $\lambda - \rho_n + \rho_g$ is regular for G_g , then [17, Lem. 5.4] implies that

$$\text{D-Ind}_{K_g}^{G_g}[W_{w(\lambda-\rho_n+\rho_{n,g})}^{K_g}] = [\pi_w^{G_g}(\lambda+\rho_g-\rho_n)],$$

the K -theory class of the discrete series representation of G_g with Harish-Chandra parameter $w(\lambda + \rho_g - \rho_n)$. Hence (41) follows, because the value of τ_e on a K -theory class of a discrete series representation is the formal degree of that representation [3, 11, 24].

If $\lambda - \rho_n + \rho_g$ is singular for G_g , then $d_{w(\lambda+\rho_g-\rho_n)}^{G_g} = 0$. Then the Dirac operator whose G_g -equivariant index is

$$\text{D-Ind}_{K_g}^{G_g}[W_{w(\lambda-\rho_n+\rho_{n,g})}^{K_g}]$$

has trivial kernel by [3, Thm. 9.3]. And

$$\tau_e(\text{D-Ind}_{K_g}^{G_g}[W_{w(\lambda-\rho_n+\rho_{n,g})}^{K_g}])$$

equals the L^2 -index of this Dirac operator (see [35, Prop. 4.4]), which is defined in terms of the projection onto its kernel, and hence zero. So now the left-hand side of (41) is zero as well, and (41) holds. \square

Theorem 5.16. *Let $g \in G$ be semisimple. Then $\tau_g(\text{index}_G(D_W)) = 0$ if g is hyperbolic. If $\text{rank}(G) = \text{rank}(K)$ and g is elliptic, then*

$$(43) \quad \tau_g(\text{index}_G(D_W)) = (-1)^{(\dim(G/K)+\dim(G_g/K_g))/2} \cdot \frac{\sum_{w \in W_{K_g} \setminus W_K} \det(w) e^{w(\lambda+\rho_c-\rho_g)}(g) d_{w(\lambda+\rho_g-\rho_n)}^{G_g}}{e^{\rho-\rho_g}(g) \prod_{\alpha \in R^+(G, T) \setminus R^+(G_g, T)} (1 - e^\alpha(g))}.$$

Proof. The case for hyperbolic g follows from Theorem 5.2 (a).

Suppose that $\text{rank}(G) = \text{rank}(K)$ and that g is elliptic. The element $\lambda - \rho_n \in \mathfrak{it}^*$ is analytically integral as noted in Lemma 5.15. In the proof of [17, Prop. 5.2], it is shown that

$$\text{index}_G(D_W) = (-1)^{\dim(G/K)/2} \text{index}_G(\bar{\partial}_{L_{\lambda-\rho_n}} + \bar{\partial}_{L_{\lambda-\rho_n}}^*).$$

(This is a place where we use the assumption that $\text{rank}(G) = \text{rank}(K)$.) By this equality and Lemmas 5.6, 5.11, and 5.12, Theorem 5.2 (c) implies that

$$\begin{aligned} &\tau_g(\text{index}_G(D_W)) \\ &= (-1)^{\dim(G/K)/2} \left(e^{\rho-\rho_g}(g) \prod_{\alpha \in R^+(G,T) \setminus R^+(G_g,T)} (1 - e^\alpha(g)) \right)^{-1} \\ &\quad \cdot \sum_{w \in W_{K_g} \setminus W_K} \det(w) e^{w(\rho-\rho_g)}(g) (\text{tr}(g|_{L_{\lambda-\rho_n}})|_{G_g w/T}) \\ &\quad \cdot \tau_e(\text{index}_{G_g}(\bar{\partial}_{L_{\lambda-\rho_n}|_{G_g w/T}} + \bar{\partial}_{L_{\lambda-\rho_n}|_{G_g w/T}}^*)). \end{aligned}$$

For all $w \in N_K(T)$, Lemma 5.14 implies that

$$\text{tr}(g|_{L_{\lambda-\rho_n}})|_{G_g w/T} = e^{w(\lambda-\rho_n)}(g).$$

So (43) follows by Lemma 5.15. □

Example 5.17. If g is central, then Theorem 5.16 reduces to

$$\tau_g(\text{index}_G(D_W)) = e^{\lambda-\rho_n}(g) d_{\lambda+\rho_c}^G.$$

This is [5, Prop. 4.2]. If $g = e$, then versions of this equality were obtained in [11, 24]. This case was used in the proof of Theorem 5.16.

Example 5.18. If the powers of g are dense in T , then $G_g = K_g = T$, and for all $w \in W_K$,

$$\begin{aligned} &\text{tr}(g|_{L_{\lambda-\rho_n}})|_{G_g w/T} = e^{w(\lambda-\rho_n)}(g), \\ &\tau_e(\text{index}_{G_g}(\bar{\partial}_{L_{\lambda-\rho_n}|_{G_g w/T}} + \bar{\partial}_{L_{\lambda-\rho_n}|_{G_g w/T}}^*)) = 1. \end{aligned}$$

So Theorem 5.16 reduces to

$$\tau_g(\text{index}_G(D_W)) = (-1)^{\dim(G/K)/2} \frac{\sum_{w \in W_K} \det(w) e^{w(\lambda+\rho_c)}(g)}{e^\rho(g) \prod_{\alpha \in R^+(G,T)} (1 - e^{-\alpha}(g))}.$$

This is [19, Thm. 3.1 (a)].

Remark 5.19. If $\text{rank}(G) > \text{rank}(K)$, then [19, Thm. 3.2 (a)] implies that the number $\tau_g(\text{index}_G(D_W))$ equals zero for all hyperbolic g and all elliptic g with powers dense in a maximal torus. Then also $\tau_e(\text{index}_G(D_W)) = 0$, as in the proof of Lemma 5.15. We expect that $\tau_g(\text{index}_G(D_W)) = 0$ for all semisimple g in this case, but we have not worked out the details. That would imply that all numbers in Theorem 3.23 equal zero if $\text{rank}(G) > \text{rank}(K)$.

5.20. Higher orbital integrals. We no longer assume that G has a compact Cartan subgroup. We saw in Theorem 5.16 that $\tau_g(\text{index}_G(D_W)) = 0$ if $\text{rank}(G) > \text{rank}(K)$. However, $\text{index}_G(D_W) \in K_0(C_r^*(G))$ is nonzero, and one way to detect its nonvanishing is to pair it with the higher orbital integrals from [32]. We can compute the resulting pairings using the results in [16] to generalize Theorem 5.16. We use notation and assumptions as in Theorem 3.24. As in [16] and in Theorem 3.24, we assume that the map (6) descends to K .

Corollary 5.21. *If P is not a maximal cuspidal parabolic subgroup, or $g \in M$ is hyperbolic, then $\langle \Phi_{P,g}, \text{index}_G(D_W) \rangle = 0$. If P is a maximal cuspidal parabolic subgroup and $g \in T \subset M$, then*

$$\begin{aligned}
 (44) \quad & \langle \Phi_{P,g}, \text{index}_G(D_W) \rangle \\
 &= (-1)^{(\dim(M/(K \cap M)) + \dim(M_g/(K \cap M_g)))/2} \\
 &\quad \cdot \left(e^{\rho^M - \rho_g^M}(g) \prod_{\alpha \in R^+(M,T) \setminus R^+(M_g,T)} (1 - e^\alpha(g)) \right)^{-1} \\
 &\quad \cdot \sum_{U \in \widehat{K \cap M}} m_U \sum_{w \in W_{K \cap M_g} \setminus W_{K \cap M}} \det(w) e^{w(\lambda_U + \rho_c^M - \rho_g^M)}(g) \\
 &\quad \quad \cdot d_{w(\lambda_U + \rho_g^M - \rho_n^M)}^{M_g}.
 \end{aligned}$$

Proof. The first statement is one conclusion of [16, Thm. 2.1].

Now suppose that P is a maximal cuspidal parabolic subgroup and $g \in T$. Let $\{X_1, \dots, X_s\}$ be an orthonormal basis of $\mathfrak{p} \cap \mathfrak{m}$. For every finite-dimensional representation Z of $K \cap M$, consider the Dirac operator

$$D_Z^M = \sum_{j=1}^s R(X_j) \otimes c(X_j) \otimes 1_Z$$

on $(C^\infty(M) \otimes \Delta_{\mathfrak{p} \cap \mathfrak{m}} \otimes Z)^{K \cap M}$, analogous to (7). Let τ_g^M be the orbital integral with respect to g as an element of M . Then [16, (5.4) and (5.5)] imply that

$$\begin{aligned}
 \langle \Phi_{P,g}, \text{index}_G(D_W) \rangle &= \tau_g^M \left(\text{index}_M(D_{\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}^+}^M \otimes W|_{K \cap M}) \right) \\
 &\quad - \tau_g^M \left(\text{index}_M(D_{\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}^-}^M \otimes W|_{K \cap M}) \right).
 \end{aligned}$$

Hence (44) follows by (22) and Theorem 5.16. □

Example 5.22. If $\text{rank}(G) = \text{rank}(K)$, then G is a (maximal) cuspidal parabolic subgroup of itself, and $\Phi_{G,g} = \tau_g$. So (44) reduces to (43).

Example 5.23. If P is a maximal cuspidal parabolic subgroup and g lies in the center of M , then analogously to Example 5.17, equality (44) reduces to

$$\langle \Phi_{P,g}, \text{index}_G(D_W) \rangle = \sum_{U \in \widehat{K \cap M}} m_U e^{\lambda_U - \rho_n^M}(g) d_{\lambda_U + \rho_c^M}^M.$$

Example 5.24. If P is a maximal cuspidal parabolic subgroup, $g \in T$, and the powers of g are dense in T , then analogously to Example 5.18, equality (44) reduces to

$$\begin{aligned} & \langle \Phi_{P,g}, \text{index}_G(D_W) \rangle \\ &= (-1)^{\dim(M/(K \cap M))/2} \sum_{U \in \widehat{K \cap M}} m_U \frac{\sum_{w \in W_{K \cap M}} \det(w) e^{w(\lambda_U + \rho_c^M)}(g)}{e^{\rho^M}(g) \prod_{\alpha \in R^+(M,T)} (1 - e^{-\alpha}(g))} \\ &= (-1)^{\dim(M/(K \cap M))/2} \frac{\chi_{\Delta_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}}(g) \chi_W(g)}{\chi_{\Delta_{\mathfrak{p} \cap \mathfrak{m}}}(g)}, \end{aligned}$$

where χ denotes the character of a finite-dimensional representation. This is [16, (6.3)].

5.25. Proofs of the index theorems. Let $F \subset \widehat{K \times K}$ be the set of irreducible subrepresentations of $\text{End}(\Delta_{\mathfrak{p}} \otimes W)$. For any continuous trace τ on $\mathcal{S}_F(G)$ (with notation as above Corollary 4.13), Theorem 2.5 implies that

$$(45) \quad \begin{aligned} \tau^{\Delta_{\mathfrak{p}} \otimes W} \circ \Psi_*(\text{index}_{S,G}(D_W)) &= \tau^{\Delta_{\mathfrak{p}} \otimes W} (\Psi(e^{-tD_W^- D_W^+}) \\ &\quad - \tau^{\Delta_{\mathfrak{p}} \otimes W} (\Psi(e^{-tD_W^+ D_W^-})). \end{aligned}$$

If $g \in G$ is semisimple and $\tau = \tau_g$, then [17, Prop. 3.6] implies that the right-hand side equals $\tau_g(\text{index}_G(D_W))$. So Theorem 3.23 (b) and (c) follow from Theorem 5.16. Part (a) follows by Example 5.17.

Similarly, in the setting of Theorem 3.24, we have

$$\langle \Phi_{P,g}^{\Delta_{\mathfrak{p}} \otimes W}, \text{index}_{S,G}(D_W) \rangle = \langle \Phi_{P,g}, \text{index}_G(D_W) \rangle.$$

So Theorem 3.24 follows from Corollary 5.21 and Example 5.23.

To prove Theorem 3.26, we just apply the computations in [5] to the right-hand side of (45). The case of Theorem 3.26 (a) where $G \neq \text{SU}(2n, 1)$, and case (b) of this theorem, are [5, (4.5.3)], which comes from [4]. In the setting of Theorem 3.26 (a) for $G = \text{SU}(2n, 1)$, [5, eq. (4.5.6)] states that the right-hand side of (45) equals the right-hand side of (23). Theorem 3.26 (c) is [5, Prop. 5.6]. (Note that there is a typo in [5, (7.1.5)]: $c_{\mu}(\sigma) > 0$ should be $c_{\mu}(\sigma) = 0$.)

In the setting of Theorem 3.26 (d) with $\lambda + \rho_c$ regular, the right-hand side of (45), with $\tau = \tau_{\text{rem}}$, is computed in [5, p. 194]. There is a slight error here, however, as pointed out in [34]: the computation of the term involving W_{Γ} in [5] should be modified as in [34, (6.5)]. This leads to (24). The case of Theorem 3.26 (d) for real hyperbolic space of dimension at least 4 is proved in [5, p. 194].

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