Münster J. of Math. **18** (2025), 39–83
DOI 10.17879/11958513328
urn:nbn:de:hbz:6-11958532708

Münster Journal of Mathematics
© Münster J. of Math. 2025

Higher spin representations of maximal compact subalgebras of simply laced Kac–Moody algebras

Robin Lautenbacher and Ralf Köhl

(Communicated by Bertrand Rémy)

Abstract. Given the maximal compact subalgebra $\mathfrak{k}(A)$ of a split-real Kac–Moody algebra $\mathfrak{g}(A)$ of type A, we study certain finite-dimensional representations of $\mathfrak{k}(A)$ that do not lift to the maximal compact subgroup $K(A)$ of the minimal Kac–Moody group $G(A)$ associated to $\mathfrak{g}(A)$ but only to its spin cover $\mathrm{Spin}(A)$ described in [11]. Currently, four *elementary* of these so-called spin representations are known. We study their (ir)reducibility, semisimplicity, and lift to the group level. The interaction of these representations with the spin-extended Weyl group is used to derive a partial parametrization result of the representation matrices by the real roots of $\mathfrak{g}(A)$.

1. Introduction

The involutory subalgebra $\mathfrak{k}(A)$ with respect to the Chevalley involution of a split-real Kac–Moody algebra $\mathfrak{g}(A)$ (cp. [18]) is typically referred to as its *maximal compact subalgebra*. If A is a generalized Cartan matrix of finite type, $\mathfrak{g}(A)$ is a semisimple Lie algebra and $\mathfrak{k}(A)$ indeed is its maximal compact subalgebra. If A is not of finite type, then both $\mathfrak{g}(A)$ and $\mathfrak{k}(A)$ are infinite-dimensional and $\mathfrak{k}(A)$ admits an invariant, negative definite bilinear form, but it is not compact in a topological sense, *i.e.*, it is not the Lie algebra of a compact Lie group (cp. [12]). There are at least three reasons to study the representations of $\mathfrak{k}(A)$. First, its representations and among these in particular the finite-dimensional ones reveal parts of the structure theory of $\mathfrak{k}(A)$. Second, some indefinite $\mathfrak{k}(A)$ are conjectured to arise as symmetries in theories of quantum gravity and therefore a well-developed representation theory of $\mathfrak{k}(A)$ is required there. Third, the representation theory of $\mathfrak{k}(A)$ is expected to be important to the theory of Kac–Moody symmetric spaces, similar to the finite-dimensional case.

This work was supported by DFG grant KO 4323/13 and the Studienstiftung des Deutschen Volkes.

An early result concerning the structure theory of $\mathfrak{k}(A)$ is a presentation by generators and relations given in [2]. The major challenge in the study of $\mathfrak{k}(A)$ is that it is in general not graded by a root system with finite-dimensional root spaces as $\mathfrak{g}(A)$ is, but only admits a filtered structure with respect to the roots of $\mathfrak{g}(A)$. In particular, $\mathfrak{k}(A)$ is not a simple Kac–Moody algebra or sum thereof if A is not of finite type (cp. [2, 29]), and as a consequence, the standard tools of representation theory such as highest weight representations and character formulas are not applicable. It was observed in [22] that the $\mathfrak{k}(E_n)$-series can be characterized as the quotient of a generalized intersection matrix algebra (cp. [29]), but the representation theory of these is also rather poorly understood. It is not obvious that $\mathfrak{k}(A)$ even possesses finite-dimensional representations if A is not of finite type, but of course, these provide interesting ideals of $\mathfrak{k}(A)$. At some point, it may be possible to characterize $\mathfrak{k}(A)$ as the co-limit of ideals of finite-dimensional representations. For the affine case, this has been shown in [22].

Concerning the case that A is an indefinite generalized Cartan matrix, there are currently four *elementary* representations known. The basic one has been first described in the physics literature [5, 4, 6] under the name $K(E_{10})$-Dirac spinor. It has been studied in a mathematical setting and generalized to arbitrary symmetrizable types in [13], where they were referred to as generalized spin representations. Both names, Dirac spinor as well as generalized spin representation, stem from the fact that the first and most important example is the representation of $\mathfrak{k}(E_{10})$ which extends the standard spinor representation of its naturally contained $\mathfrak{so}(10)$-subalgebra.

The so-called higher spin representations $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$, and $\mathcal{S}_{\frac{7}{2}}$ of $\mathfrak{k}(A)$ with the exception of $\mathcal{S}_{\frac{3}{2}}$ were introduced first in [23], again in a physics setting. In [27], the authors of this paper derived a coordinate-free formulation of $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ in the setting of simply laced A, but a similar formulation for $\mathcal{S}_{\frac{7}{2}}$ remained elusive back then. However, it became apparent that the Weyl group plays a central role in this construction, which builds these representations on top of the generalized spin representation $\mathcal{S}_{\frac{1}{2}}$ from [6, 4]. We provide a detailed construction and description of all these representations in Section 3 (cp. Theorems 3.3, 3.18, and 3.22). The novelties in this section are that we spell out the importance of the Weyl group in this construction as clearly as possible, that we provide everything in a mathematical and coordinate-free formulation including $\mathcal{S}_{\frac{7}{2}}$, and that we provide an abstract foundation for the generalized Γ-matrices used in [23] that does not rely on the use of Clifford algebras (Section 3.4). The subsection on generalized Γ-matrices will not be needed until the proof of Proposition 4.14 and Section 5.

We analyze these representations in Section 4, where we put the Weyl group-based formulation to work. We show that $\mathcal{S}_{\frac{3}{2}}$ is irreducible if A is indecomposable, regular and simply laced and that the image of $\mathfrak{k}(A)$ under this representation is a semisimple Lie algebra (Proposition 4.4 and Corollary 4.5). Furthermore, we show that $\mathcal{S}_{\frac{5}{2}}$ is completely reducible, always contains an invariant submodule isomorphic to $\mathcal{S}_{\frac{1}{2}}$, and that its other invariant factors are controlled by the representation theory of $W(A)$, namely how the symmetric product $\mathrm{Sym}^2(\mathfrak{h}^*)$ of the dual Cartan subalgebra decomposes as a $W(A)$-module (Proposition 4.7 and Lemma 4.9). As for $\mathcal{S}_{\frac{3}{2}}$, we show that the image of $\mathfrak{k}(A)$ under this representation is semisimple (Corollary 4.10). At the end of Section 4, we show that the kernels of some of these representations are not contained in each other (Proposition 4.14) and that their tensor product has a smaller kernel than the individual representations (Proposition 4.13).

In Section 5, we study the spin representations' lift to the group level. We confirm the common belief (cp. [6, 4, 24, 23, 27]) that these representations are spinorial in the sense that they do not lift to the involutory subgroup $K(A) = G(A)^\theta$, where $G(A)$ is the minimal split-real Kac–Moody group of type A and θ is its Chevalley involution, but instead lift only to its spin cover $\mathrm{Spin}(A)$ introduced in [11]. This belief is plausible if one compares the one-parameter subgroups induced by $\exp(\phi\sigma(X_i))$ and $\exp(\phi\,\mathrm{ad}(X_i))$, where σ is a spin representation and X_i is a so-called Berman generator of $\mathfrak{k}(A)$. We show in Proposition 5.9 that it indeed suffices to look at these one-parameter subgroups. Afterwards, we demonstrate that the spin representations' lift realizes an action of the spin-extended Weyl group from [11] on the modules $\mathcal{S}_{\frac{n}{2}}$. We use that the action of $\mathrm{Spin}(A)$ on $\mathfrak{k}(A)$ factors through the adjoint action of $K(A)$ on $\mathfrak{k}(A)$ to derive the representation matrices up to sign of all elements in the $W^{\mathrm{ext}}(A)$-orbit of the Berman generators, where $W^{\mathrm{ext}}(A)$ is the extended Weyl group. This amounts to providing the representation matrices up to sign of all $x \in \mathfrak{k}_\alpha = \mathfrak{k}(A) \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ for α a positive real root (Propositions 5.19 and 5.21).

This article contains results that were obtained in the first author's PhD thesis [26] but did not appear in any journal so far. Most of these results' proofs are as in [26], but occasionally, we decided to provide a more streamlined version that skips lengthy but elementary computations. We also correct a sign error in [26, Lem. 5.6] and note that, as a consequence, [26, Lem. 6.5] and the proof of [26, Prop. 6.7] are incorrect and it is therefore unknown if [26, Prop. 6.7] is true. We provide the appropriate references to [26] for propositions and theorems but not for lemmas.

2. Preliminaries

2.1. **Kac–Moody algebras.**

We provide a constructive definition of symmetrizable Kac–Moody algebras that is equivalent to the construction in [18, Chap. 1] due to the Gabber–Kac theorem (cp. [18, Thm. 9.11], originally [9]).

Definition 2.2. A matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{Z}^{n\times n}$ is called a *generalized Cartan matrix* (GCM) if, for all $i \neq j \in \{1, \dots, n\}$,

$$a_{ii} = 2, \quad a_{ij} \le 0, \quad a_{ij} = 0 \iff a_{ji} = 0.$$

The matrix A is called *symmetrizable* if there exist a regular, diagonal matrix D and a symmetric matrix B such that $A = DB$. The pair of matrices D and B is called a *symmetrization* of A. The GCM A is called *simply laced* if $a_{ij} \in \{0, -1\}$

for all $i \neq j$. We denote the set of unordered pairs $\{i, j\}$ such that $a_{ij} \neq 0$ by $\mathcal{E}(A)$, the edges of the generalized Dynkin diagram associated to A.

Definition 2.3. Let $A \in \mathbb{Z}^{n\times n}$ be a GCM of rank $l \leq n$ and let $\mathfrak{h}$ be a $\mathbb{K}$-vector space of dimension $2n-l$. A $\mathbb{K}$-*realization* $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A consists of linearly independent subsets $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ such that

$$\alpha_j(\alpha_i^\vee) = a_{ij} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

One calls Π the *simple roots*, $\Pi^\vee$ the *simple coroots*, $Q(A) := \operatorname{span}_{\mathbb{Z}} \Pi$ the *root lattice*, and $Q^\vee(A) := \operatorname{span}_{\mathbb{Z}} \Pi^\vee$ the *coroot lattice*.

Definition 2.4. Let $A \in \mathbb{Z}^{n\times n}$ be a symmetrizable GCM with $\mathbb{K}$-realization $(\mathfrak{h}, \Pi, \Pi^\vee)$. The *split Kac–Moody algebra over* $\mathbb{K}$ *of type* A is defined as the Lie algebra on generators $\mathfrak{h} \cup \{e_1, \dots, e_n, f_1, \dots, f_n\}$ subject to the relations

$$\begin{aligned} [h, h'] &= 0, & [e_i, f_j] &= \delta_{ij}\alpha_i^\vee, \\ [h, e_i] &= \alpha_i(h)e_i, & [h, f_i] &= -\alpha_i(h)f_i, \\ 0 &= \operatorname{ad}(e_i)^{1-a_{ij}}(e_j), & 0 &= \operatorname{ad}(f_i)^{1-a_{ij}}(f_j) \end{aligned}$$

for all $h, h' \in \mathfrak{h}$ and for all $i, j \in \{1, \dots, n\}$.

Definition 2.5. Let $\mathfrak{g} = \mathfrak{g}(A)(\mathbb{K})$ be a split Kac–Moody algebra and set

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

for $\alpha \in \mathfrak{h}$. One calls $\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq \{0\}$ a *root* and $\mathfrak{g}_\alpha$ a *root space*.

Denote the set of roots and its decomposition into positive and negative roots by $\Delta = \Delta_- \cup \Delta_+ \subset Q$, where $\Delta_+ := \{\alpha \in \Delta \mid \alpha > 0\}$ and $\Delta_- = -\Delta_+$; then $\mathfrak{g}$ admits the following *root space decomposition* (as a vector space):

$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha.$$

Proposition 2.6 (This is [18, Thm. 1.2 and Sec. 1.3])**.** *On* $\mathfrak{g}(A)(\mathbb{K})$*, there exists an involutive automorphism* ω *called the* Chevalley involution *that is determined by*

$$\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h) = -h \quad \textit{for all } h \in \mathfrak{h}.$$

It satisfies $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ *for all* $\alpha \in \Delta$.

Proposition 2.7 (cp. [18, Thm. 2.2])**.** *Let* $A \in \mathbb{Z}^{n\times n}$ *be a symmetrizable GCM with symmetrization* $A = DB$*, where* $D = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n)$ *is chosen such that* $\varepsilon_i > 0$ *for all* $i = 1, \dots, n$*, and let* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *be a* $\mathbb{K}$*-realization of* A*. Set* $\mathfrak{h}' := \operatorname{span}_{\mathbb{K}}\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ *and denote by* $\mathfrak{h}''$ *a fixed complementary subspace of* $\mathfrak{h}' \subset \mathfrak{h}$*. On* $\mathfrak{g} = \mathfrak{g}(A)(\mathbb{K})$*, there exists a* $\mathbb{K}$*-bilinear form* $(\cdot\,|\,\cdot)$ *such that*

$$\begin{aligned} (h|\alpha_i^\vee) &= \alpha_i(h)\varepsilon_i && \textit{for all } h \in \mathfrak{h}, \\ (h_1|h_2) &= 0 && \textit{for all } h_1, h_2 \in \mathfrak{h}'', \\ ([x, y]|z) &= (x|[y, z]) && \textit{for all } x, y, z \in \mathfrak{g}, \\ (\mathfrak{g}_\alpha|\mathfrak{g}_\beta) &= 0 && \textit{for all } \alpha, \beta \in \Delta \textit{ such that } \alpha \neq -\beta, \end{aligned}$$

$(\cdot|\cdot)_{|\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$ *is non-degenerate for all* $\alpha \in \Delta$,

$$[x, y] = (x|y)\nu^{-1}(\alpha) \quad \textit{for all } x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha},\ \alpha \in \Delta,$$

where $\nu : \mathfrak{h} \to \mathfrak{h}^*$ *is the isomorphism induced by the bilinear form, i.e., one has* $\nu(h_1)(h_2) = (h_1|h_2)$ *for all* $h_1, h_2 \in \mathfrak{h}$. *This form is referred to as the* standard invariant bilinear form. *If* A *is indecomposable,* $(\cdot|\cdot)$ *is unique up to scalar multiples.*

The definition of the Weyl group $W(A)$ for a GCM A in the Kac–Moody context is the straight-forward generalization of the definition in the classical setting of crystallographic root systems.

Definition 2.8. Given a split Kac–Moody algebra $\mathfrak{g}(A)(\mathbb{K})$ with GCM $A \in \mathbb{Z}^{n\times n}$, define the *fundamental reflections* $s_i \in \mathrm{GL}(\mathfrak{h}^*)$ for $i = 1, \dots, n$ via

$$s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i \quad \text{for all } \lambda \in \mathfrak{h}^*$$

and the *Weyl group* $W(A)$ of $\mathfrak{g}(A)(\mathbb{K})$ as $W(A) := \langle s_1, \dots, s_n\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$.

It is a standard fact (cp. [18, Prop. 3.13]) that $W(A)$ admits a presentation as a Coxeter group. Furthermore, the action of $W(A)$ preserves the roots. One calls a root $\alpha \in \Delta$ *real* if there exists $w \in W(A)$ such that $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Pi$, and *imaginary* if this is not the case. The sets of real and imaginary roots are denoted by Δ^{re} and $\Delta^{\mathrm{im}} = \Delta \setminus \Delta^{\mathrm{re}}$.

2.9. The maximal compact subalgebra $\mathfrak{k}(A)$.

Definition 2.10. Given a split-real, symmetrizable Kac–Moody algebra $\mathfrak{g} := \mathfrak{g}(A)(\mathbb{R})$, its *maximal compact subalgebra* $\mathfrak{k}(A)$ is defined to be the Chevalley involution's fixed-point subalgebra, *i.e.*, one has $\mathfrak{k}(A) := \{x \in \mathfrak{g} \mid \omega(x) = x\}$.

If $\mathfrak{g}(A)$ is a finite-dimensional split-real Lie algebra, then $\mathfrak{k}(A)$ is maximally compact in the sense that the Killing form restricted to $\mathfrak{k}(A)$ is negative definite and that $\mathfrak{k}(A)$ is maximal with regard to this property. Correspondingly, if G is a real Lie group with Lie algebra $\mathfrak{g}(A)$, then $\mathfrak{k}(A)$ is the Lie algebra of the maximal compact subgroup K of G. Note that the notion *maximal compact subalgebra* must not be confused with the maximal compact form, which is defined for complex Lie algebras and involves an additional twist of the Chevalley involution by complex conjugation. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, one has $\mathfrak{k} = \mathfrak{so}(n, \mathbb{R})$, but the maximal compact form of $\mathfrak{sl}(n, \mathbb{C})$ is $\mathfrak{su}(n, \mathbb{R})$. One also has to note that, for GCMs of non-finite type, the term *maximal compact subalgebra* is slightly misleading, as it is in fact not the Lie algebra of a compact Lie group; its name derives from the analogy to the finite-dimensional case and the fact that the invariant bilinear form is negative definite on $\mathfrak{k}(A)$ (cp. [18, Thm. 11.7]).

Theorem 2.11 (cp. [13, Thm. 1.8], originally due to [2])**.** *Given the split-real symmetrizable Kac–Moody algebra* $\mathfrak{g}(A)(\mathbb{R})$, *the maximal compact subalgebra* $\mathfrak{k}(A)(\mathbb{R})$ *admits a presentation by generators* $X_1, \dots, X_n$ *and relations*

$$P_{-a_{ij}}(\operatorname{ad} X_i)(X_j) = 0 \quad \textit{for all } i \neq j \in \{1, \dots, n\},$$

where

$$P_m(t) := \begin{cases} \prod_{k=0}^{\frac{m-1}{2}} \left(t^2 + (m-2k)^2\right) & \text{if } m \text{ is odd,} \\ t \cdot \prod_{k=0}^{\frac{m}{2}-1} \left(t^2 + (m-2k)^2\right) & \text{if } m \text{ is even,} \end{cases}$$

and $m = 0$ spells out as $[X_i, X_j] = 0$ whenever $a_{ij} = 0$. Explicitly, one has $X_i = e_i - f_i$ for $i = 1, \dots, n$ in terms of the Chevalley generators $e_1, \dots, e_n, f_1, \dots, f_n$ of $\mathfrak{g}(A)(\mathbb{R})$. The X_i are called the Berman generators *of $\mathfrak{k}(A)(\mathbb{R})$.*

For A simply laced, the above relations take the following simple form.

Corollary 2.12 (cp. [2, Thm. 1.31] or [13, Thm. 1.8])**.** *Let A be simply laced and denote by $\mathcal{E}(A)$ the edges of the generalized Dynkin diagram, i.e., the unordered pairs $\{i, j\}$ such that $a_{ij} = -1$. Then $\mathfrak{k}(A)(\mathbb{R})$ admits a presentation by generators $X_1 = e_1 - f_1, \dots, X_n = e_n - f_n$ and relations*

$$\begin{aligned} [X_i, [X_i, X_j]] &= -X_j && \text{for all } \{i, j\} \in \mathcal{E}(A), \\ [X_i, X_j] &= 0 && \text{for all } \{i, j\} \notin \mathcal{E}(A). \end{aligned}$$

For later reference, we collect the following elementary result.

Lemma 2.13. *The maximal compact subalgebra $\mathfrak{k}(A)(\mathbb{R})$ as well as its complexification $\mathfrak{k}(A)(\mathbb{C}) := \mathfrak{k}(A)(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ are filtered by Δ_+, i.e., one has*

$$\mathfrak{k}(A) = \bigoplus_{\alpha \in \Delta_+} \mathfrak{k}_\alpha$$

as vector spaces, where $\mathfrak{k}_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$. For $x_\alpha \in \mathfrak{k}_\alpha$, $x_\beta \in \mathfrak{k}_\beta$ such that $\alpha - \beta \in Q_+$, one has $[x_\alpha, x_\beta] \in \mathfrak{k}_{\alpha+\beta} \oplus \mathfrak{k}_{\alpha-\beta}$.

Proof. The filtered structure of $\mathfrak{k}(A)$ is a direct consequence of the graded structure of $\mathfrak{g}(A)(\mathbb{K})$ (cp. [2, 13] for a more detailed exposition). □

3. Higher spin representations

Historically, the name *spin representation* is reserved for a representation of $\mathfrak{so}(p, q)$ that does not lift to the real Lie group $\mathrm{SO}(p, q)$ but only to its double cover, the so-called spin group $\mathrm{Spin}(p, q)$. The most elementary spin representations, corresponding to the missing fundamental weights not observed in representations that lift to the group level, are typically constructed by means of Clifford algebras. In signature $(n, 0)$, the representation matrices of suitably normalized generators such as the Berman generators possess only the eigenvalues $\pm\frac{i}{2}$ which is why they square to $-\frac{1}{4}\mathrm{Id}$. As it turns out, one can generalize the Clifford algebra construction to any maximal compact subalgebra, not just $\mathfrak{so}(n+1)$ which is $\mathfrak{k}(A_n)$, but in the classical cases, this provides nothing new. For GCMs of affine and indefinite type however, these representations are quite fascinating, because they show among other things that

$\mathfrak{k}(A)$ is not a Lie algebra of Kac–Moody type, since these do not admit finite-dimensional representations if they are themselves infinite-dimensional (and simple).

Definition 3.1 (cp. [13, Def. 3.6])**.** Let A be a simply laced GCM and let $\rho : \mathfrak{k}(A) \to \mathrm{End}(S)$ be a representation of $\mathfrak{k}(A)$ for S a finite-dimensional real vector space. One calls ρ a *generalized spin representation* if

$$\rho(X_i)^2 = -\frac{1}{4}\mathrm{Id} \quad \text{for all } i = 1, \ldots, n,$$

where $X_1, \ldots, X_n$ denote the Berman generators of $\mathfrak{k}(A)$.

Proposition 3.2 (cp. [13, 3.7])**.** *Let A be a simply laced GCM, $\rho : \mathfrak{k}(A) \to \mathrm{End}(S)$ a generalized spin representation, and denote by $\{A, B\} := AB + BA$ the anti-commutator of matrices. Then one has, for all $1 \leq i \neq j \leq n$,*

$$[\rho(X_i), \rho(X_j)] = 0 \quad \textit{if } a_{ij} = 0,$$
$$\{\rho(X_i), \rho(X_j)\} = 0 \quad \textit{if } a_{ij} = -1.$$

Conversely, the extension of the map $X_i \mapsto M_i$ defines a generalized spin representation, if the linear maps $M_1, \ldots, M_n \in \mathrm{End}(S)$ satisfy

$$M_i^2 = -\frac{1}{4}\mathrm{Id}_s,$$
$$[M_i, M_j] = 0 \quad \textit{if } a_{ij} = 0,$$
$$\{M_i, M_j\} = 0 \quad \textit{if } a_{ij} = -1.$$

Theorem 3.3 (This is the specialization of [13, Thms. 3.9 and 3.14] to A simply laced)**.** *Let A be a simply laced GCM; then a generalized spin representation exists. Its image is compact, hence reductive. If the Dynkin diagram of A has no isolated nodes, then the image is furthermore semisimple.*

Later on, we will refer to the representations described in the above theorem as $\frac{1}{2}$-spin representations, because the representation matrices of the Berman generators only possess the eigenvalues $\pm\frac{i}{2}$. For a given $\mathfrak{k}(A)$, we fix a representation denoted by ρ and whose carrier space we will denote by S in general and $\mathcal{S}_{\frac{1}{2}}$ if it is an irreducible $\mathfrak{k}(A)$-module.

3.4. **Generalized spin representations and Γ-matrices.** Most explicit constructions of generalized spin representations use Clifford algebras. In [23] for instance, the authors associate an element of a Clifford algebra to any root of the root system such that the simple roots map to (multiples of) the $\rho(X_i)$ and thus define a representation. They call these elements Γ-matrices in generalization of the Dirac matrices which are typically denoted by γ_μ. We will construct and study these generalized Γ-matrices abstractly without reference to an underlying Clifford algebra but note that this realization appears to be achievable at least in the simply laced situation by using explicit basis expressions for the roots as in [23]. Later, this framework will be used to derive the representation matrices for any $x \in \mathfrak{k}_\alpha$ with $\alpha \in \Delta^{\mathrm{re}}$ up to sign.

In order to properly introduce generalized Γ-matrices, we need to associate normalized 2-cocycles to a root lattice.

Definition 3.5. Given the root lattice Q of a Kac–Moody algebra $\mathfrak{g}(A)(\mathbb{K})$ with symmetrizable GCM A, we call $\varepsilon : Q \times Q \to C_2$ an *associated, normalized 2-cocycle* if

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}, \quad \varepsilon(\alpha,0) = \varepsilon(0,\alpha) = 1, \tag{1a}$$

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma) \tag{1b}$$

for all $\alpha, \beta, \gamma \in Q$, where $(\cdot\,|\,\cdot)$ denotes the invariant bilinear form.

The root lattice to any symmetrizable GCM possesses an associated, normalized 2-cocycle (cp. for instance [19, Cor. 5.5]). As this can be checked in a few lines, we provide an explicit construction below.

Lemma 3.6. *Let A be a symmetrizable GCM and fix its symmetrization $A = DB$ such that $(\alpha_i|\alpha_i) = b_{ii} \in 2\mathbb{Z}$ for all $i = 1,\dots,n$. Define the bilinear form $\underline{\varepsilon} : Q \times Q \to \mathbb{Z}$ via bilinear extension of*

$$\underline{\varepsilon}(\alpha_i,\alpha_j) := \begin{cases} b_{ij} & \text{if } i < j, \\ \frac{1}{2}b_{ii} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Then

$$\varepsilon : Q \times Q \to C_2, \quad \varepsilon(\alpha,\beta) := (-1)^{\underline{\varepsilon}(\alpha,\beta)}$$

is an associated, normalized 2-cocycle which we refer to as the standard 2-cocycle *to Q.*

Proof. One has for all $i, j = 1,\dots,n$ that $\underline{\varepsilon}(\alpha_i,\alpha_j) + \underline{\varepsilon}(\alpha_j,\alpha_i) = b_{ij}$. Spelling out $\beta = \sum_{i=1}^n b_i\alpha_i$ and $\gamma = \sum_{i=1}^n c_i\alpha_i$, this yields

$$\underline{\varepsilon}(\beta,\gamma) + \underline{\varepsilon}(\gamma,\beta) = \sum_{i,j} b_i c_j \big(\underline{\varepsilon}(\alpha_i,\alpha_j) + \underline{\varepsilon}(\alpha_j,\alpha_i)\big) = \sum_{i,j} b_i c_j b_{ij} = (\beta|\gamma).$$

Thus, for all $\beta,\gamma \in Q(A)$, one has that

$$\varepsilon(\beta,\gamma)\varepsilon(\gamma,\beta) = (-1)^{\underline{\varepsilon}(\beta,\gamma)+\underline{\varepsilon}(\gamma,\beta)} = (-1)^{(\beta|\gamma)}$$

and (1a) follows by bilinearity of $\underline{\varepsilon}$. Towards (1b), one uses

$$\underline{\varepsilon}(\alpha,\beta) + \underline{\varepsilon}(\alpha+\beta,\gamma) = \underline{\varepsilon}(\alpha,\beta) + \underline{\varepsilon}(\alpha,\gamma) + \underline{\varepsilon}(\beta,\gamma) = \underline{\varepsilon}(\alpha,\beta+\gamma) + \underline{\varepsilon}(\beta,\gamma)$$

to compute

$$\begin{aligned} \varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) &= (-1)^{\underline{\varepsilon}(\alpha,\beta)+\underline{\varepsilon}(\alpha+\beta,\gamma)} = (-1)^{\underline{\varepsilon}(\alpha,\beta+\gamma)+\underline{\varepsilon}(\beta,\gamma)} \\ &= \varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma) \quad \text{for all } \alpha,\beta,\gamma \in Q(A). \end{aligned}$$ □

Definition 3.7 (cp. [23, Eqn. 4.6])**.** A map $\Gamma : Q \to \mathrm{End}(S)$ for a finite-dimensional vector space S is called a *generalized* Γ*-matrix* if it satisfies

$$\Gamma(\alpha)\Gamma(\beta) = (-1)^{(\alpha|\beta)}\Gamma(\beta)\Gamma(\alpha), \tag{2a}$$

$$\Gamma(0) = \mathrm{Id}, \quad \Gamma(\alpha)^2 = (-1)^{\frac{1}{2}(\alpha|\alpha)}, \quad \Gamma(\alpha) = \Gamma(-\alpha), \tag{2b}$$

$$\Gamma(\alpha)\Gamma(\beta) = \varepsilon(\alpha,\beta)\Gamma(\alpha+\beta) \tag{2c}$$

for all $\alpha, \beta \in Q$ and an associated, normalized 2-cocycle ε.

The following properties of 2-cocycles follow rather directly from (1) and will be used in the following computations:

$$\varepsilon(\alpha,\alpha) = (-1)^{\frac{1}{2}(\alpha|\alpha)},$$
$$\varepsilon(\alpha,\beta) = \begin{cases} \varepsilon(\beta,\alpha) & \text{if } (\alpha|\beta) = 0 \bmod 2, \\ -\varepsilon(\beta,\alpha) & \text{if } (\alpha|\beta) = 1 \bmod 2. \end{cases}$$

Proposition 3.8 (This is [26, Prop. 3.10]). *For a simply laced GCM A, a generalized Gamma matrix $\Gamma : Q \to \operatorname{End}(S)$ defines a generalized spin representation $\rho : \mathfrak{k} \to \operatorname{End}(S)$ via $\rho(X_i) := \frac{1}{2}\Gamma(\alpha_i)$.*

Proof. For simple roots corresponding to adjacent nodes, one has

$$\Gamma(\alpha_i)\Gamma(\alpha_j) = -\Gamma(\alpha_j)\Gamma(\alpha_i)$$

by equation (2a), whereas for non-adjacent simple roots, one has

$$\Gamma(\alpha_i)\Gamma(\alpha_j) = \Gamma(\alpha_j)\Gamma(\alpha_i).$$

This shows that the $\rho(X_i)$ commute and anti-commute as required by Proposition 3.2. The correct normalization is checked via (2b). ☐

The converse statement is also true.

Proposition 3.9 (This is [26, Prop. 3.11]). *Let A be a simply laced GCM and $\rho : \mathfrak{k} \to \operatorname{End}(S)$ a generalized spin representation. Then*

$$\Gamma(\pm\alpha_i) := 2\rho(X_i), \tag{3a}$$

$$\Gamma(\alpha_{i_1} + \cdots + \alpha_{i_n}) := \Bigg(\prod_{k=1}^{n-1} \varepsilon(\alpha_{i_k}, \alpha_{i_{k+1}} + \cdots + \alpha_{i_n})\Bigg)\Gamma(\alpha_{i_1})\cdots\Gamma(\alpha_{i_n}) \tag{3b}$$

defines a generalized Γ-matrix $\Gamma : Q \to \operatorname{End}(S)$.

Proof. We begin by showing (2) for $\alpha, \beta \in Q$ of height 1 and 2. We can always assume $\alpha = \sum_{i=1}^n k_i\alpha_i$ with $k_i \geq 0$ since $\Gamma(-\alpha_i) = \Gamma(\alpha_i)$ and the standard 2-cocycle only counts modulo 2. For height 1, there is nothing to show, and for height 2, one first checks if the expression of $\Gamma(\alpha)$ for $\alpha = \alpha_i + \alpha_j$ is well-defined. Towards this, compute

$$\Gamma(\alpha_i + \alpha_j) = \varepsilon(\alpha_i, \alpha_j)\Gamma(\alpha_i)\Gamma(\alpha_j),$$
$$\Gamma(\alpha_j + \alpha_i) = \varepsilon(\alpha_j, \alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_i).$$

By the properties of generalized spin representations, the two matrices commute if $(\alpha_i|\alpha_j) = 0$ and anti-commute if $(\alpha_i|\alpha_j) = -1$. Also, in the first case, one has $\varepsilon(\alpha_i, \alpha_j) = 1 = \varepsilon(\alpha_j, \alpha_i)$, whereas in the second case, one has

$\varepsilon(\alpha_i, \alpha_j)\varepsilon(\alpha_j, \alpha_i) = -1$. Thus, $\Gamma(\alpha_i + \alpha_j) = \Gamma(\alpha_j + \alpha_i)$. The only part of (2) left to show is $\Gamma(\alpha)^2 = (-1)^{\frac{1}{2}(\alpha|\alpha)}$ for $\alpha = \alpha_i + \alpha_j$. One has

$$\begin{aligned}\Gamma(\alpha_i + \alpha_j)^2 &= \varepsilon(\alpha_i, \alpha_j)\varepsilon(\alpha_j, \alpha_i)\Gamma(\alpha_i)^2\Gamma(\alpha_j)^2 \\ &= (-1)^{(\alpha_i|\alpha_j)}(-1)^{\frac{1}{2}(\alpha_i|\alpha_i)}(-1)^{\frac{1}{2}(\alpha_j|\alpha_j)} = (-1)^{\frac{1}{2}(\alpha_i+\alpha_j|\alpha_i+\alpha_j)}.\end{aligned}$$

From here on, we continue by induction on the height n, so assume that (2) holds for all $\alpha, \beta \in Q_+$ with $\mathrm{ht}(\alpha) + \mathrm{ht}(\beta) \leq n - 1$. In particular, $\Gamma(\gamma)$ is well-defined for $\mathrm{ht}(\gamma) \leq n - 1$ and one has for all $\alpha, \beta \in Q_+$ with $\mathrm{ht}(\alpha) + \mathrm{ht}(\beta) \leq n - 1$ that

$$\varepsilon(\alpha, \beta)\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha + \beta).$$

Given $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_n}$, we need to show $\Gamma(\alpha) = \varepsilon(\beta, \gamma)\Gamma(\beta)\Gamma(\gamma)$ for any $\beta, \gamma \in Q(A)_+$ such that $\alpha = \beta + \gamma$ and $\mathrm{ht}(\alpha) + \mathrm{ht}(\beta) = n$. We assume without loss of generality that β contains α_{i_1} and multiply from the left with $\Gamma(\alpha_{i_1})$ and exploit that $\Gamma(\alpha_{i_1}) = \Gamma(-\alpha_{i_1})$:

$$\begin{aligned}\varepsilon(\beta, \gamma)\Gamma(\alpha_{i_1})\Gamma(\beta)\Gamma(\gamma) &= \varepsilon(\beta, \gamma)\varepsilon(\alpha_{i_1}, \beta)\Gamma(\beta - \alpha_{i_1})\Gamma(\gamma), \\ \Gamma(\beta - \alpha_{i_1}) &= \Gamma(-\alpha_{i_1} + \beta) = \varepsilon(-\alpha_{i_1}, \beta)\Gamma(-\alpha_{i_1})\Gamma(\beta) \\ &= \varepsilon(\alpha_{i_1}, \beta)\Gamma(\alpha_{i_1})\Gamma(\beta).\end{aligned}$$

Since $\mathrm{ht}(\beta - \alpha_{i_1}) + \mathrm{ht}(\gamma) = n - 1$, one has

$$\Gamma(\beta - \alpha_{i_1})\Gamma(\gamma) = \varepsilon(\beta - \alpha_{i_1}, \gamma) \cdot \Gamma(\beta - \alpha_{i_1} + \gamma),$$

and in combination with

$$\varepsilon(\beta - \alpha_{i_1}, \gamma) = \varepsilon(\beta + \alpha_{i_1}, \gamma) \quad \text{and} \quad \beta - \alpha_{i_1} + \gamma = \alpha_{i_2} + \cdots + \alpha_{i_n},$$

this yields

$$\varepsilon(\beta, \gamma)\Gamma(\alpha_{i_1})\Gamma(\beta)\Gamma(\gamma) = \varepsilon(\beta, \gamma)\varepsilon(\alpha_{i_1}, \beta)\varepsilon(\beta + \alpha_{i_1}, \gamma)\Gamma(\alpha_{i_2} + \cdots + \alpha_{i_n}).$$

By (1b), one has $\varepsilon(\alpha_{i_1} + \beta, \gamma) = \varepsilon(\alpha_{i_1}, \beta)\varepsilon(\alpha_{i_1}, \beta + \gamma)\varepsilon(\beta, \gamma)$, which implies $\varepsilon(\beta, \gamma)\varepsilon(\alpha_{i_1}, \beta)\varepsilon(\beta + \alpha_{i_1}, \gamma) = \varepsilon(\alpha_{i_1}, \beta + \gamma)$ so that the above equation simplifies further to

$$\begin{aligned}&\varepsilon(\beta, \gamma)\Gamma(\alpha_{i_1})\Gamma(\beta)\Gamma(\gamma) \\ &\quad = \varepsilon(\alpha_{i_1}, \beta + \gamma)\Gamma(\alpha_{i_2} + \cdots + \alpha_{i_n}) \\ &\quad = \varepsilon(\alpha_{i_1}, \alpha_{i_1} + \cdots + \alpha_{i_n})\Bigg(\prod_{k=2}^{n-1} \varepsilon(\alpha_{i_k}, \alpha_{i_{k+1}} + \cdots + \alpha_{i_n})\Bigg)\Gamma(\alpha_{i_2}) \cdots \Gamma(\alpha_{i_n}).\end{aligned}$$

Together with

$$\begin{aligned}\varepsilon(\alpha_{i_1}, \alpha_{i_1} + \cdots + \alpha_{i_n}) &= \varepsilon(\alpha_{i_1}, \alpha_{i_1})\varepsilon(2\alpha_{i_1}, \alpha_{i_2} + \cdots + \alpha_{i_n}) \\ &\qquad \cdot \varepsilon(\alpha_{i_1}, \alpha_{i_2} + \cdots + \alpha_{i_n}) \\ &= -\varepsilon(\alpha_{i_1}, \alpha_{i_2} + \cdots + \alpha_{i_n}),\end{aligned}$$

one arrives at

$$\varepsilon(\beta, \gamma)\Gamma(\alpha_{i_1})\Gamma(\beta)\Gamma(\gamma) = -\Bigg(\prod_{k=1}^{n-1} \varepsilon(\alpha_{i_k}, \alpha_{i_{k+1}} + \cdots + \alpha_{i_n})\Bigg)\Gamma(\alpha_{i_2}) \cdots \Gamma(\alpha_{i_n}).$$

One the other hand, one has

$$\begin{aligned}\Gamma(\alpha_{i_1})\Gamma(\alpha_{i_1}+\cdots+\alpha_{i_n})\\ &=\Big(\prod_{k=1}^{n-1}\varepsilon(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n})\Big)\Gamma(\alpha_{i_1})^2\Gamma(\alpha_{i_2})\cdots\Gamma(\alpha_{i_n})\\ &=-\Big(\prod_{k=1}^{n-1}\varepsilon(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n})\Big)\Gamma(\alpha_{i_2})\cdots\Gamma(\alpha_{i_n}),\end{aligned}$$

and as multiplication with $\Gamma(\alpha_{i_1})$ can be inverted, this yields

$$\varepsilon(\beta,\gamma)\Gamma(\beta)\Gamma(\gamma)=\Gamma(\alpha_{i_1}+\cdots+\alpha_{i_n})$$

for all $\beta,\gamma\in Q_+$ of height less than n such that $\beta+\gamma=\alpha_{i_1}+\cdots+\alpha_{i_n}$ and thus shows well-definedness and (2c) by induction. We can now use this to prove (2a) by using $\Gamma(\alpha+\beta)=\Gamma(\beta+\alpha)$ and (1a):

$$\varepsilon(\alpha,\beta)\Gamma(\alpha)\Gamma(\beta)=\varepsilon(\beta,\alpha)\Gamma(\beta)\Gamma(\alpha)\iff\Gamma(\alpha)\Gamma(\beta)=(-1)^{(\alpha|\beta)}\Gamma(\beta)\Gamma(\alpha).$$

In order to show the missing parts of (2b), one computes

$$\begin{aligned}\Gamma(\alpha+\beta)^2&=\varepsilon(\alpha,\beta)\Gamma(\alpha)\Gamma(\beta)\varepsilon(\beta,\alpha)\Gamma(\beta)\Gamma(\alpha)\\ &=\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)(-1)^{\frac12(\beta|\beta)}(-1)^{\frac12(\alpha|\alpha)}\\ &=(-1)^{(\alpha|\beta)+\frac12(\beta|\beta)+\frac12(\alpha|\alpha)}=(-1)^{\frac12(\alpha+\beta|\alpha+\beta)}.\end{aligned}$$ □

There are some identities of Γ-matrices that we would like to state explicitly.

Lemma 3.10. *Let A be a symmetrizable GCM with a symmetrization as in Lemma 3.6 and denote the standard 2-cocycle by ε. Then any generalized Γ-matrix satisfies*

(4) $$\Gamma(\alpha+\beta)=\Gamma(\alpha-\beta),\quad\Gamma(\alpha+2\beta)=\Gamma(\alpha)\quad\textit{for all }\alpha,\beta\in Q.$$

Proof. With the help of (2b), one computes

$$\Gamma(\alpha+\beta)=\varepsilon(\alpha,\beta)\Gamma(\alpha)\Gamma(\beta),\quad\Gamma(\alpha-\beta)=\varepsilon(\alpha,-\beta)\Gamma(\alpha)\Gamma(-\beta),$$

and because ε is a standard 2-cocycle, one has further that

$$\varepsilon(\alpha,-\beta)=(-1)^{\underline{\varepsilon}(\alpha,-\beta)}=(-1)^{-\underline{\varepsilon}(\alpha,\beta)}=(-1)^{\underline{\varepsilon}(\alpha,\beta)}=\varepsilon(\alpha,\beta),$$

which shows the first part of the claim. Regarding the second claim, one has that $\Gamma(\alpha+2\beta)=\Gamma(\alpha+\beta+\beta)=\Gamma(\alpha+\beta-\beta)=\Gamma(\alpha)$ by the first part. □

Remark 3.11. The above lemma shows that Γ-matrices are unable to capture distinctions beyond modulo $2Q$. As a consequence, they cannot distinguish different roots in symmetrizable root systems such as $\alpha_i+2\alpha_j$. This should come as no surprise as Γ-matrices are tailored towards a more global description of spin representations for simply laced A.

There exist generalized spin representations in the symmetrizable, non-simply laced cases (cp. [21]) $\mathfrak{k}(AE_3)$, $\mathfrak{k}(G_2^{++})$, and $\mathfrak{k}(BE_{10})$ which all arise from a generalized spin representation of simply laced type of suitable quotients which are $\mathfrak{k}(A_2) \oplus \mathfrak{k}(A_1)$, $\mathfrak{k}(A_4)$, and $\mathfrak{k}(E_9)$, respectively.

In [13], the generalized spin representations for the general symmetrizable case are constructed via embedding $\mathfrak{k}(A)$ in a simply laced cover $\mathfrak{k}(\tilde{A})$ first. This cover would admit generalized Γ-matrices with respect to the root system of type $\tilde{A}$, but it is an open question how the ones needed for the representation of $\mathfrak{k}(A)$ spell out in terms of the root system of type A. For this, it might be necessary to replace the 2-cocycles by ones of higher order.

Proposition 3.12 (This is [26, Prop. 3.14])**.** *Let A be a simply laced GCM as in Lemma 3.6, denote the standard 2-cocycle by ε, and let (ρ, S) be a generalized spin representation with associated Γ-matrix $\Gamma : Q \to \mathrm{End}(S)$ according to Proposition 3.9. Then, for all $x \in \mathfrak{k}_\alpha$, there exists $c(x) \in \mathbb{R}$ such that*

$$\rho(x) = c(x) \cdot \Gamma(\alpha).$$

Proof. Given $\alpha \in \Delta_+(A)$, there exists an ordered series of simple roots

$$\beta_1, \ldots, \beta_N \in \Pi \quad \text{such that} \quad \alpha = \beta_1 + \cdots + \beta_N.$$

To such a fixed decomposition of α, we set

$$X_{\beta_1+\cdots+\beta_N} \coloneqq [X_{\beta_1}, [X_{\beta_2}, [\ldots, X_{\beta_N}] \ldots]].$$

As $\mathfrak{k}(A)$ is generated by the X_i, there exists at least one such ordered series such that $X_{\beta_1+\cdots+\beta_N}$ is nonzero, but for now, we do not need to assume this. By Proposition 3.9, one has

$$\rho(X_{\beta_1+\cdots+\beta_N}) = \frac{1}{2^N}[\Gamma(\beta_1), [\Gamma(\beta_2), [\ldots, \Gamma(\beta_N)] \ldots]].$$

From (2a), one concludes that

$$[\Gamma(\alpha), \Gamma(\beta)] = \Gamma(\alpha)\Gamma(\beta) - \Gamma(\beta)\Gamma(\alpha) = \begin{cases} 0 & \text{if } (\alpha|\beta) \in 2\mathbb{Z}, \\ 2\Gamma(\alpha)\Gamma(\beta) & \text{if } (\alpha|\beta) \in \mathbb{Z} \setminus 2\mathbb{Z}, \end{cases}$$

so that, exploiting $\Gamma(\alpha)\Gamma(\beta) = \varepsilon(\alpha, \beta)\Gamma(\alpha + \beta)$, there are two possibilities for $\rho(X_{\beta_1+\cdots+\beta_N})$:

$$\begin{aligned} &\rho(X_{\beta_1+\cdots+\beta_N}) \\ &\qquad = \frac{1}{2^N}[\Gamma(\beta_1), [\Gamma(\beta_2), [\ldots, \Gamma(\beta_n)] \ldots]] \\ &\qquad = \begin{cases} 0, \\ \frac{1}{2}\varepsilon(\beta_1, \beta_2 + \cdots + \beta_N) \cdots \varepsilon(\beta_{N-1}, \beta_N)\Gamma(\beta_1 + \beta_2 + \cdots + \beta_N). \end{cases} \end{aligned}$$

The first case occurs if there exists i such that $(\beta_i|\beta_{i+1} + \cdots + \beta_N) \in 2\mathbb{Z}$ and the second case if no such i exists. By (3b), one has that

$$\Gamma(\alpha) = \left(\prod_{k=1}^{N-1} \varepsilon(\beta_k, \beta_{k+1} + \cdots + \beta_N)\right)\Gamma(\beta_1) \cdots \Gamma(\beta_N)$$

so that one in fact obtains

$$\rho(X_{\beta_1+\cdots+\beta_N}) = \begin{cases} 0, \\ \frac{1}{2}\Gamma(\beta_1)\cdots\Gamma(\beta_N), \end{cases}$$

because any cocycle squared is 1. The claim follows for all $X_{\beta_1+\cdots+\beta_N} \in \mathfrak{k}(A)$. Note that the order of the nested commutator here is crucial and is reflected in the order of the $\Gamma(\beta_i)$ in the product. Recall that $\mathfrak{k}$ is not graded by Δ but instead filtered by Δ_+ and set $\mathfrak{k}_{<\alpha} := \bigoplus_{\beta<\alpha} \mathfrak{k}_\beta$. To $x \in \mathfrak{k}_\alpha$, there exist ordered decompositions

$$\beta_1^{(j)} + \cdots + \beta_N^{(j)} = \alpha \quad \text{for } j = 1, \ldots, k = \dim \mathfrak{k}_\alpha,$$

$c_j \in \mathbb{K}$, and a remainder $r \in \mathfrak{k}_{<\alpha}$ such that

$$\sum_{j=1}^{k} c_j X_{\beta_1^{(j)}+\cdots+\beta_N^{(j)}} = x + r.$$

For $y_1 \in \mathfrak{k}_\beta, y_2 \in \mathfrak{k}_\gamma$, one has $[y_1, y_2] \in \mathfrak{k}_{\beta+\gamma} \oplus \mathfrak{k}_{\pm(\beta-\gamma)}$ due to the filtered structure of $\mathfrak{k}$. Now r also has a decomposition $r = \bigoplus_\gamma y_\gamma$ which is furthermore such that $y_\gamma \in \mathfrak{k}_\gamma$ is nonzero only if $\gamma < \alpha$ and $\alpha - \gamma \in 2Q(A)$. Thus, by induction on $\mathrm{ht}(\alpha)$, one concludes

$$\rho(x) = \sum_{j=1}^{k} c_j \rho(X_{\beta_1^{(j)}+\cdots+\beta_N^{(j)}}) - \rho(r) = c(x) \cdot \Gamma(\alpha)$$

for all $x \in \mathfrak{k}_\alpha$ and $\alpha \in \Delta_+(A)$ because $\Gamma(\alpha) = \Gamma(\gamma)$ if $\alpha - \gamma \in 2Q(A)$ by (4). □

At the present point, we are not able to say when $\rho(x)$ for $x \in \mathfrak{k}_\alpha$ is nonzero or not or what the constant $c(x)$ is. We will pick up this thread in Section 5 once we introduced the spin cover of the maximal compact subgroup $K(A)$ and how it interacts with generalized spin representations ρ.

3.13. **The higher spin representations $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$.** In this subsection, we provide a description of the higher spin representations from [23] in the phrasing of [27], where we put more emphasis on how these representations feature representations of $W(A)$ as a crucial piece. The name stems from the fact that the weight system of these representations also includes larger eigenvalues than $\pm\frac{i}{2}$. Throughout, we will assume that A is simply laced.

Proposition 3.14. *Let A be a simply laced GCM, let ρ be a generalized spin representation of $\mathfrak{k}$ as in Definition 3.1, where we denote the module by S, and let V be a finite-dimensional vector space. Let $X_1, \ldots, X_n$ denote the Berman generators of $\mathfrak{k}$ and set $\Delta^{\mathrm{re}} \supset \tilde{\Delta} = \{\alpha_1, \ldots, \alpha_n\} \cup \{\alpha_i + \alpha_j \mid (i,j) \in \mathcal{E}(A)\}$. Consider a map $\tau : \tilde{\Delta} \to \mathrm{End}(V)$ satisfying*

$$[\tau(\alpha), \tau(\beta)] = 0 \qquad \text{if } (\alpha|\beta) = 0, \tag{5}$$

$$\{\tau(\alpha), \tau(\beta)\} = \tau(\alpha \pm \beta) \quad \text{if } (\alpha|\beta) = \mp 1 \text{ and } \alpha \pm \beta \in \tilde{\Delta} \tag{6}$$

for all $\alpha, \beta \in \tilde{\Delta}$. *Then the assignment* $\sigma(X_i) := \tau(\alpha_i) \otimes 2\rho(X_i) \in \mathrm{End}(V \otimes S)$ *extends to a finite-dimensional representation* σ *of* $\mathfrak{k}(A)$.

Proof. This is originally [23, Eqn. (5.1)], but the phrasing is as in [27]. □

If the map τ admits an extension to all real roots, one has the following identity.

Lemma 3.15. *Let* $\tau : \Delta^{\mathrm{re}} \to \mathrm{End}(V)$ *satisfy equations* (5) *and* (6) *with* $\tilde{\Delta}$ *replaced by* Δ^{re}. *Let* $\{i,j\}, \{j,k\} \in \mathcal{E}(A)$ *but* $\{i,k\} \notin \mathcal{E}(A)$*; then*

$$\sigma([X_i, X_j]) = \tau(\alpha_i + \alpha_j) \otimes 2\rho([X_i, X_j]),$$
$$\sigma([X_i, [X_j, X_k]]) = \tau(\alpha_i + \alpha_j + \alpha_k) \otimes 2\rho([X_i, [X_j, X_k]]).$$

Proof. The proof consists of elementary computations exploiting the commutation and anti-commutation relations among the $\rho(X_i)$ as well as (5) and (6). □

Given a simply laced GCM A and real roots α, β with $(\alpha, \beta) = -1$, the reflections s_α, s_β with respect to α and β generate a subgroup of $\mathfrak{S}_{\alpha,\beta} := \langle s_\alpha, s_\beta \rangle < W(A)$ which is isomorphic to $\mathfrak{S}_3$, the symmetric group on three letters. The three distinct irreducible representations of $\mathfrak{S}_3$ are the trivial representation U (dimension 1), the sign representation U' (dimension 1), and the standard representation E of dimension 2.

Proposition 3.16 (This is [27, Rem. 4.2 (iv)], based on an observation by Paul Levy)**.** *Let* A *be simply laced and let* $\eta : W(A) \to \mathrm{End}(V)$ *be a finite-dimensional representation of* $W(A)$. *Then*

$$\tau : \Delta^{\mathrm{re}}(A) \to \mathrm{End}(V), \quad \alpha \mapsto \eta(s_\alpha) - \frac{1}{2}\mathrm{Id} \tag{7}$$

satisfies equations (5) *and* (6) *if the restriction of* η *to any* $\mathfrak{S}_{\alpha_i,\alpha_j}$ *such that* α_i, α_j *are adjacent simple roots does not contain the sign representation of* $\mathfrak{S}_3$ *as an irreducible factor.*

Proof. If $(\alpha|\beta) = 0$, then s_α, s_β commute and so do $\tau(\alpha)$, $\tau(\beta)$, as required. For $(\alpha|\beta) = -1$, one has with $s_{\alpha+\beta} = s_\beta s_\alpha s_\beta$ that $\tau(\alpha+\beta) = \{\tau(\alpha), \tau(\beta)\}$ is equivalent to

$$0 \stackrel{!}{=} -\eta(s_\beta s_\alpha s_\beta) + \eta(s_\alpha)\eta(s_\beta) + \eta(s_\beta)\eta(s_\alpha) - \eta(s_\alpha) - \eta(s_\beta) + \mathrm{Id}. \tag{8}$$

For the trivial representation, this is easily seen to be true, whereas it is false for the sign representation as then

$$-\mathrm{Id} = \eta(s_\beta s_\alpha s_\beta) = \eta(s_\alpha) = \eta(s_\beta).$$

One can set up the standard representation as the subspace $\mathrm{span}_{\mathbb{R}}\{\alpha, \beta\} \subset \mathfrak{h}^*$. In this basis, one checks explicitly that (8) holds. Since any finite-dimensional representation of $\mathfrak{S}_3$ is completely reducible, one concludes that (7) provides a representation if η restricted to $\mathfrak{S}_{\alpha_i,\alpha_j}$ contains no copies of the sign representation. Note that it suffices to consider simple roots, because relations (6) and (5) need only be satisfied for $\alpha, \beta \in \tilde{\Delta}$, where $\tilde{\Delta}$ consists of simple roots and all roots $\alpha_i + \alpha_j$ where i and j are adjacent. Those are the only root spaces involved in the pairwise Berman relations $[X_i, [X_i, X_j]] = -X_j$. □

Lemma 3.17. *Let* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *be a realization of* A *and set* $V_1 := \mathfrak{h}^*$, $V_2 := \mathrm{Sym}^2(\mathfrak{h}^*)$*, the symmetric product of* $\mathfrak{h}^*$ *with itself. Then the standard representation of* $W(A)$ *on* V_1 *and the induced representation on* V_2 *contain no copies of the sign representation upon restriction to any* $\mathfrak{S}_{\alpha_i,\alpha_j}$ *for* $\{i,j\} \in \mathcal{E}(A)$.

Proof. After restriction to $\mathfrak{S}_{\alpha_i,\alpha_j}$, consider a basis $\{\alpha_i, \alpha_j\} \cup \{b_1, \ldots, b_{m-2}\}$ of $\mathfrak{h}^*$, where $m = 2n - \mathrm{rnk}(A)$, such that the b_i are orthogonal to both α_i and α_j. Then $\mathrm{span}\{\alpha_i, \alpha_j\}$ forms a copy of the standard representation of $\mathfrak{S}_{\alpha_i,\alpha_j}$, while $\mathrm{span}\{b_1, \ldots, b_{m-2}\}$ decomposes into $m-2$ copies of the trivial representation. For $\mathrm{Sym}^2(\mathfrak{h}^*)$, one can use the basis

$$\begin{aligned}\{\alpha_i\alpha_i, \alpha_j\alpha_j, (\alpha_i+\alpha_j)(\alpha_i+\alpha_j)\} &\cup \{\alpha_i b_k, \alpha_j b_k \mid k = 1, \ldots, m-2\}\\ &\cup \{b_k b_l \mid 1 \le k \le l \le m-2\}\end{aligned}$$

to see that only the trivial representation and the standard representation occur in the decomposition of V_2 restricted to $\mathfrak{S}_{\alpha_i,\alpha_j}$. □

We obtain the following result as an immediate consequence of the previous lemma and Proposition 3.14. Note that Proposition 3.14 can be applied even if A is not regular.

Theorem 3.18 (cp. [24, 27])**.** *Let* (η_1, V_1) *and* (η_2, V_2) *with* $V_1 := \mathfrak{h}^*$, $V_2 := \mathrm{Sym}^2(\mathfrak{h}^*)$ *be the standard and induced representations of* $W(A)$*, respectively. Then* $\tau_s : \tilde{\Delta} \to \mathrm{End}(V_s)$, $\tau_s(\alpha) = \eta_s(s_\alpha) - \frac{1}{2}\mathrm{Id}$ *for* $s \in \{1,2\}$ *satisfies equations* (5) *and* (6)*. Let* $X_1, \ldots, X_n$ *denote the Berman generators of* $\mathfrak{k}(A)$ *and assign*

$$\sigma_{\frac{2s+1}{2}} : X_i \mapsto \tau_s(\alpha_i) \otimes 2\rho(X_i),$$

where ρ *is a* $\frac{1}{2}$*-spin representation as in Theorem 3.3. Then* $\sigma_{\frac{2s+1}{2}}$ *extends to a representation of* $\mathfrak{k}(A)$.

The above representation is called the $\frac{2s+1}{2}$*-spin representation* in [24, 27], motivated by the fact that the eigenvalues of $\sigma_{\frac{2s+1}{2}}(X_j)$ are half-integral up to a factor of i. However, it is somewhat misleading that the largest magnitude of eigenvalues occurring is $\frac{3}{2}$ in both cases and not $\frac{2s+1}{2}$. The above ansatz fails for $V_3 := \mathrm{Sym}^3(\mathfrak{h}^*)$, because $\alpha\beta(\alpha+\beta)$ spans a sign representation of $\mathfrak{S}_{\alpha,\beta}$ for adjacent simple roots α, β. However, this sign representation is the only one that appears in the decomposition of V_3, regardless of A.

Lemma 3.19. *Let* A *be simply laced and let* $\alpha, \beta \in \Pi$ *be adjacent simple roots. Then the induced representation of* $W(A)$ *on* $\mathrm{Sym}^3(\mathfrak{h}^*)$ *restricted to* $\mathfrak{S}_{\alpha,\beta}$ *contains exactly one copy of the sign representation, spanned by* $\alpha\beta(\alpha+\beta) \in \mathrm{Sym}^3(\mathfrak{h}^*)$.

Proof. Consider a basis $\mathcal{B}_1 = \{\alpha, \beta, b_1, \ldots, b_{m-2}\}$ for $\mathfrak{h}^*$ such that $b_1, \ldots, b_{m-2}$ are orthogonal to α and β. This provides a basis

$$\mathcal{B}_3 = \{e_1e_2e_3 \mid e_1, e_2, e_3 \in \mathcal{B}_1,\ e_1 \le e_2 \le e_3\}$$

for V_3 with respect to some ordering of $\mathcal{B}_1$. Then elements $x \in \mathcal{B}_3$ which contain b_k for some k behave like copies of V or $\mathrm{Sym}^2(V)$ and therefore span trivial or standard representations according to Proposition 3.17. The remaining subspace of products containing only α and/or β is 4-dimensional and computation of its character reveals that it contains one of each irreducible representations of $\mathfrak{S}_{\alpha,\beta}$, *i.e.*, the trivial, the sign, and the standard representation. One also checks by direct computation that the sign representation is spanned by $\alpha\beta(\alpha+\beta)$. $\square$

3.20. **The higher spin representation $\mathcal{S}_{\frac{7}{2}}$.** In this section's Theorem 3.22, we provide a coordinate-free version of the so-called $\frac{7}{2}$-spin representation described in [23]. The main idea is a suitable extension of the previous Weyl group type ansatz on the module $V_3 \otimes S$, where $V_3 := \mathrm{Sym}^3 V$ with $V = \mathfrak{h}^*$ and S is the carrier space of the $\frac{1}{2}$-spin representation from Theorem 3.3. For better readability, we will denote the standard invariant form on $\mathfrak{h}^*$ by $b(\,\cdot\,,\cdot\,)$ in this section. According to Proposition 3.14, we need to find maps $\tau : \Delta^{\mathrm{re}} \to \mathrm{End}(V_3)$ satisfying

$$[\tau(\alpha), \tau(\beta)] = 0 \qquad \text{if } b(\alpha,\beta) = 0, \tag{9a}$$

$$\{\tau(\alpha), \tau(\beta)\} = \tau(\alpha \pm \beta) \quad \text{if } b(\alpha,\beta) = \mp 1. \tag{9b}$$

We will need some standard results on the structure of symmetric products of vector spaces. Fix the normalization on $\mathrm{Sym}^3 V$ with respect to $V^{\otimes 3}$ via

$$\mathrm{Sym}^3 V \ni v_1 \cdot v_2 \cdot v_3 := \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}.$$

The induced $W(A)$-invariant, non-degenerate bilinear form on $\mathrm{Sym}^3 V$ is given by

$$b(v_1 \cdot v_2 \cdot v_3, u_1 \cdot u_2 \cdot u_3) = \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} b(v_{\sigma(1)}, u_1) \dots b(v_{\sigma(3)}, u_3).$$

Let $e_1, \dots, e_m$ be a basis of V and set

$$\omega_{ij} := b(e_i, e_j), \quad (\omega^{ij}) := (\omega_{ij})^{-1} \iff \sum_l \omega^{kl} \omega_{ln} = \delta^k_n. \tag{10}$$

Define the *symmetric insertion* $\psi : V \to \mathrm{Sym}^3 V$ via

$$\psi(v) = \frac{1}{3!} \cdot \sum_{k,l=1}^{m} \omega^{kl} (v \otimes e_k \otimes e_l + e_k \otimes v \otimes e_l + e_k \otimes e_l \otimes v).$$

Symmetric insertions play an important role in invariant theory, and the above definition of ψ does not depend on the chosen basis (cp. [8, Secs. 17.3 and 19.5]). In the language of theoretical physics used in [23], the coordinates of the element $\psi(\alpha)$ with respect to some basis would be given as $\alpha_{(a} G_{bc)}$, where $a, b, c = 1, \dots, \dim(V)$, G_{bc} denote the elements of the bilinear form in the chosen basis (corresponding directly to the ω_{ij} above) and the parentheses denote complete symmetrization.

We consider the ansatz

$$\tau(\alpha) = s_\alpha - \frac{1}{2}\mathrm{Id} + f(\alpha) \quad \text{for all } \alpha \in \Delta^{\mathrm{re}}(A), \tag{11a}$$

$$f(\alpha) := v(\alpha) \cdot b(v(\alpha)|\,\cdot\,) \in \mathrm{End}(\mathrm{Sym}^3 V), \tag{11b}$$

$$v(\alpha) = p \cdot \alpha\alpha\alpha + q \cdot \psi(\alpha) \ \in \mathrm{Sym}^3 V, \tag{11c}$$

where s_α, by slight abuse of notation, denotes the action of the Weyl reflection s_α on $\mathrm{Sym}^3 V$ that is induced by its action on $V = \mathfrak{h}^*$, and $p, q \in \mathbb{R}$ are unknowns to be determined. It may seem natural to assume $f(\alpha)$ to be of rank 1, given that there is only a single copy of the sign representation. The above assumption on the shape of $f(\alpha)$ is even more restrictive and can only be justified afterwards: the above ansatz will suffice to solve (9), and replacing one of the $v(\alpha)$ above by an independent vector $w(\alpha)$ would lead to the requirement $v(\alpha) = w(\alpha)$ after a tedious and lengthy computation. There are only two vectors in $\mathrm{Sym}^3(V)$ which are related to α in a natural way: $\alpha\alpha\alpha$ and $\psi(\alpha)$, which motivates (11c).

Lemma 3.21. *Let A be simply laced, $m := \dim V$, and $\alpha, \beta \in \Delta^{\mathrm{re}}(A)$. Then*

$$s_\alpha(\psi(\beta)) = \psi(s_\alpha\beta),$$
$$b(\psi(\alpha), \psi(\beta)) = \frac{m+2}{12} b(\alpha, \beta), \quad b(\alpha\alpha\alpha, \psi(\beta)) = b(\alpha, \beta).$$

Proof. The first statement reduces to checking that

$$\Bigg(\sum_{k,l=1}^{m} \omega^{kl} e_k \otimes e_l\Bigg) \in \mathrm{Sym}^2(V)$$

is invariant under action of s_α, which is a result of b being $W(A)$-invariant. Therefore, s_α intertwines with ψ, *i.e.*, $s_\alpha \circ \psi = \psi \circ s_\alpha$. One computes directly

$$b(\psi(\alpha), v_1v_2v_3) = \frac{1}{6}[b(\alpha, v_1)b(v_2, v_3) + b(\alpha, v_2)b(v_1, v_3) + b(\alpha, v_3)b(v_1, v_2)].$$

For $v_1v_2v_3 = \beta\beta\beta$, this specializes to

$$b(\psi(\alpha), \beta\beta\beta) = \frac{1}{2} b(\alpha, \beta)b(\beta, \beta) = b(\alpha, \beta),$$

since $b(\beta, \beta) = 2$ as $\beta \in \Delta^{\mathrm{re}}(A)$ and A is simply laced. The computation of

$$b(\psi(\alpha), \psi(\beta)) = \frac{m+2}{12} b(\alpha, \beta)$$

is straight-forward but a bit lengthy, and one uses the identity

$$\sum_{k,l,i,j} \omega^{kl}\omega^{ij}\omega_{ki}\omega_{lj} = \sum_{k,i,j} \delta^k_j \omega^{ij}\omega_{ki} = \sum_{k,i} \omega^{ik}\omega_{ki} = \sum_i \delta^i_i = m. \qquad \square$$

Theorem 3.22 (This is [26, Thm. 3.23 and Lem. 3.24])**.** *Let A be a simply laced GCM, let ρ be a $\frac{1}{2}$-spin representation of $\mathfrak{k}(A)$ as in Theorem 3.3 and denote by $X_1, \ldots, X_n$ the Berman generators of $\mathfrak{k}(A)$. Ansatz* (11) *satisfies*

(9a) *and* (9b) *if one fixes* p *and* q *to be* ($\varepsilon = \pm 1$, $m \coloneqq \dim \mathfrak{h}$, *and the signs can be chosen independently)*

$$q_{\pm,\varepsilon} = -\varepsilon \frac{12 \mp 2\sqrt{6(m+8)}}{(m+2)\sqrt{3}}, \quad p_\varepsilon = \varepsilon \frac{1}{\sqrt{3}}.$$

One has further that $f(\alpha)$ *in* (11) *satisfies*

$$f(\alpha)^2 = 4f(\alpha) \quad \textit{for all } \alpha \in \Delta^{\mathrm{re}}.$$

With τ *as in* (11),

$$\sigma(X_i) = \tau(\alpha_i) \otimes 2\rho(X_i) \quad \textit{for all } i = 1, \dots, n$$

extends to a representation of $\mathfrak{k}(A)$.

Remark 3.23. The above theorem in fact describes two representations as there are two signs to chose but only the relative sign matters, because $v(\alpha)$ appears twice in $f(\alpha)$. In the remainder, we pick both signs as $+$ when referring to *the* higher spin representation $\mathcal{S}_{\frac{7}{2}}$. The preferred $\mathfrak{k}(A)$-module is denoted by $\mathcal{S}_{\frac{7}{2}}$ in the remainder of the text.

Proof. Plugging ansatz (11) into the left-hand side of (9a) yields

$$[\tau(\alpha), \tau(\beta)] = [s_\alpha, s_\beta] + [s_\alpha, f(\beta)] + [f(\alpha), s_\beta] + [f(\alpha), f(\beta)]$$

and is required to vanish for $b(\alpha, \beta) = 0$. For $\alpha, \beta \in \Delta^{\mathrm{re}}$ such that $b(\alpha, \beta) = 0$, one has $s_\beta(v(\alpha)) = v(\alpha)$ and $b(v(\alpha), v(\beta)) = 0$ by Lemma 3.21. Thus, it follows that $[s_\alpha, f(\beta)] = 0$ by invariance of b under $W(A)$ and $[f(\alpha), f(\beta)]$ because of $b(v(\alpha), v(\beta)) = 0$. Since s_α and s_β commute, (9a) is satisfied.

For $b(\alpha, \beta) = \mp 1$, plugging ansatz (11) into (9b) yields

$$\begin{aligned} &\Big\{s_\alpha - \frac{1}{2}\mathrm{Id}, s_\beta - \frac{1}{2}\mathrm{Id}\Big\} + \{s_\alpha, f(\beta)\} + \{f(\alpha), s_\beta\} + \{f(\alpha), f(\beta)\} \\ &\qquad - f(\alpha) - f(\beta) \overset{!}{=} s_{\alpha\pm\beta} - \frac{1}{2}\mathrm{Id} + f(\alpha \pm \beta). \end{aligned}$$

This equation is satisfied if and only if

$$\Big\{s_\alpha - \frac{1}{2}\mathrm{Id}, s_\beta - \frac{1}{2}\mathrm{Id}\Big\} + T(\alpha, \beta) = s_{\alpha\pm\beta} - \frac{1}{2}\mathrm{Id}, \tag{12a}$$

with

$$\begin{aligned} T(\alpha, \beta) = \{s_\alpha, f(\beta)\} + \{f(\alpha), s_\beta\} + \{f(\alpha), f(\beta)\} \\ - f(\alpha) - f(\beta) - f(\alpha \pm \beta). \end{aligned} \tag{12b}$$

One has by Proposition 3.16 that $\{s_\alpha - \frac{1}{2}\mathrm{Id}, s_\beta - \frac{1}{2}\mathrm{Id}\} = s_{\alpha\pm\beta} - \frac{1}{2}\mathrm{Id}$ holds on all $\mathfrak{S}_3 = \langle s_\alpha, s_\beta \rangle$-representations whose decomposition does not contain the sign representation. Thus, the support of $T(\alpha, \beta)$ needs to be the span of $\alpha\beta(\alpha + \beta)$, as this is the only occurring sign representation of $\mathfrak{S}_3 = \langle s_\alpha, s_\beta \rangle$ by Lemma 3.19, because (12a) yields $T(\alpha, \beta)u = 0$ for all u in an $\mathfrak{S}_3$-representation that is not the sign representation. Furthermore, the image of $T(\alpha, \beta)$ must also be the span of $\alpha\beta(\alpha + \beta)$, because the rest of (12a) acts diagonally on

$\mathbb{R}\alpha\beta(\alpha+\beta)$. In conclusion, support and image of $T(\alpha,\beta)$ need to be the span of $\alpha\beta(\alpha+\beta)$.

One computes, for $b(\alpha,\beta)=\mp 1$,

$$s_\alpha v(\beta) = -b(\alpha,\beta)\cdot v(\alpha\pm\beta),$$
$$v(\beta)\cdot b\big(v(\beta), s_\alpha(u_1u_2u_3)\big) = -b(\alpha,\beta)\cdot v(\beta)\cdot b\big(v(\alpha\pm\beta), u_1u_2u_3\big)$$

for all $u_1u_2u_3\in \operatorname{Sym}^3 V$. Thus,

$$\begin{aligned}\{s_\alpha, f(\beta)\}+\{f(\alpha), s_\beta\} = {} & v(\alpha)b(v(\alpha\pm\beta),\cdot\,)+v(\alpha\pm\beta)b(v(\alpha),\cdot\,)\\ & - b(\alpha,\beta)v(\beta)b(v(\alpha\pm\beta),\cdot\,)\\ & - b(\alpha,\beta)v(\alpha\pm\beta)b(v(\beta),\cdot\,).\end{aligned}$$

Setting $X(\alpha,\beta):=b(v(\alpha),v(\beta))$, one has furthermore

$$\{f(\alpha),f(\beta)\} = X(\alpha,\beta)[v(\alpha)b(v(\beta),\cdot\,)+v(\beta)b(v(\alpha),\cdot\,)].$$

With this, one computes

$$\begin{aligned}T(\alpha,\beta) = {} & v(\alpha)b\big(v(\alpha\pm\beta)+X(\alpha,\beta)v(\beta)-v(\alpha),\cdot\big)\\ & + v(\beta)b\big(X(\alpha,\beta)v(\alpha)-b(\alpha,\beta)v(\alpha\pm\beta)-v(\beta),\cdot\big)\\ & + v(\alpha\pm\beta)b\big(v(\alpha)-b(\alpha,\beta)\cdot v(\beta)-v(\alpha\pm\beta),\cdot\big).\end{aligned}$$

Now the demand that $T(\alpha,\beta)$ may only be supported on the span of $V_{\alpha,\beta}:=\alpha\cdot\beta\cdot(\alpha\pm\beta)\in\operatorname{Sym}^3 V$ leads, after rearrangements of order and signs, to three equations ($k_1,k_2,k_3\in\mathbb{R}$):

$$\begin{aligned} v(\alpha\pm\beta)-v(\alpha)+X(\alpha,\beta)v(\beta) &= k_1\cdot V_{\alpha,\beta},\\ -b(\alpha,\beta)[v(\alpha\pm\beta)-b(\alpha,\beta)^{-1}X(\alpha,\beta)v(\alpha)+b(\alpha,\beta)^{-1}v(\beta)] &= k_2\cdot V_{\alpha,\beta},\\ -[v(\alpha\pm\beta)-v(\alpha)+b(\alpha,\beta)\cdot v(\beta)] &= k_3\cdot V_{\alpha,\beta}.\end{aligned}$$

By the definition of $v(\alpha)$ and linearity of ψ, one has

$$v(\alpha\pm\beta)-v(\alpha)+\underbrace{b(\alpha,\beta)}_{=\mp 1}\cdot v(\beta) = -3pb(\alpha,\beta)V_{\alpha,\beta}.$$

As $v(\beta)$ is linearly independent from $V_{\alpha,\beta}$, all three equations concerning the support are satisfied if $X(\alpha,\beta)=b(\alpha,\beta)$. Miraculously, this satisfies the constraint concerning the image as well, since $X(\alpha,\beta)=b(\alpha,\beta)$ implies

$$T(\alpha,\beta) = -9p^2V_{\alpha,\beta}b(V_{\alpha,\beta},\cdot\,).$$

By use of Lemma 3.21, one computes

$$b(v(\alpha),v(\beta)) = p^2b(\alpha,\beta)^3+2pqb(\alpha,\beta)+q^2\frac{m+2}{12}b(\alpha,\beta), \tag{13}$$

so that $X(\alpha,\beta)=b(v(\alpha),v(\beta))=b(\alpha,\beta)$ is equivalent to

$$p^2+2pq+\frac{m+2}{12}q^2=1. \tag{14}$$

This determines p and q in the ansatz for $v(\alpha)$ such that $T(\alpha,\beta)$ has correct support and image. Evaluation of the left-hand side of (12a) on $V_{\alpha,\beta}$ fixes the scale of $v(\alpha)$ (note that $s_\alpha, s_\beta, s_{\alpha\pm\beta}$ act on $V_{\alpha,\beta}$ as $-\mathrm{Id}$):

$$\Big[\Big\{s_\alpha - \frac{1}{2}\mathrm{Id}, s_\beta - \frac{1}{2}\mathrm{Id}\Big\} + T(\alpha,\beta)\Big]V_{\alpha,\beta} = \Big[s_{\alpha\pm\beta} - \frac{1}{2}\mathrm{Id}\Big]V_{\alpha,\beta} \iff p^2 = \frac{1}{3}.$$

Together with (14), one has, with $\varepsilon = \pm 1$,

$$p_\varepsilon = \varepsilon\frac{1}{\sqrt{3}}, \tag{15a}$$

$$q_{\pm,\varepsilon} = -\varepsilon\frac{12 \mp 2\sqrt{6(m+8)}}{(m+2)\sqrt{3}}. \tag{15b}$$

Thus, by Propositions 3.14 and 3.16, one concludes that

$$\sigma(X_i) = \tau(\alpha_i) \otimes 2\rho(X_i) \quad \text{for all } i = 1,\dots,n,$$

with τ as in (11) and p, q as in (15) indeed extends to a representation of $\mathfrak{k}(A)$.

Concerning $f(\alpha)^2$, one has with (13) that

$$b(v(\alpha), v(\alpha)) = 8p^2 + 4pq + q^2\frac{m+2}{6} = \frac{8}{3} + \frac{24 + 4m + 32 - 48}{3(m+2)} = 4$$

and therefore

$$f(\alpha)^2 = v(\alpha)\cdot b(v(\alpha)|v(\alpha))\cdot b(v(\alpha)|\,\cdot\,) = 4\cdot f(\alpha). \qquad \square$$

Remark 3.24. As of now, all attempts at extending ansatz (11) to higher powers $\mathrm{Sym}^n(\mathfrak{h}^*)$ or other Schur modules $\mathcal{S}_\lambda(\mathfrak{h}^*)$ were unsuccessful. Typically, the number of sign representations exceeds the number of free parameters that is determined by the elements naturally associated to a root α, which we tested for all Schur modules associated to partitions of n up to $n = 5$. For almost all Schur modules over $\mathfrak{h}^*$, the number of sign representations depends on the dimension of $\mathfrak{h}^*$. In that regard, $\mathcal{S}_{\frac{7}{2}}$ is somewhat special because the occurrence of exactly one sign representation is universal in that it does not depend on $\mathfrak{h}^*$.

4. Reducibility and irreducibility

In this section, we derive some results concerning the irreducibility of higher spin representations and properties of their tensor products. These results have in full generality only appeared in [26] and are genuinely new except for choice examples such as $\mathcal{S}_{\frac{3}{2}}$ of $\mathfrak{k}(E_{10})$ where the image is known to be isomorphic to $\mathfrak{so}(32, 288)$ by [24].

On the technical level, the most important tool will be that the action factorizes according to the tensor product structure.

Lemma 4.1 (This is the corrected[1] version of [26, 5.6])**.** *Let $(\sigma, V\otimes S)$ denote the representation $\mathcal{S}_{\frac{3}{2}}$ or $\mathcal{S}_{\frac{5}{2}}$ of $\mathfrak{k} := \mathfrak{k}(A)(\mathbb{K})$ for A simply laced and let $X_1,\dots,X_n$ denote the Berman generators of $\mathfrak{k}$. Then one has for all $i = 1,\dots,n$*

[1] In [26, 5.6], the right-hand side of (16c) misses an overall minus sign.

that

$$\text{Id} \otimes \rho(X_i) = \frac{2}{3}\sigma(X_i)^3 + \frac{7}{6}\sigma(X_i), \tag{16a}$$

$$\eta(s_i) \otimes \text{Id} = -\frac{20}{9}\sigma(X_i)^4 - \frac{41}{9}\sigma(X_i)^2, \tag{16b}$$

$$\text{Id} \otimes \text{Id} = -\frac{16}{9}\sigma(X_i)^4 - \frac{40}{9}\sigma(X_i)^2, \tag{16c}$$

where η denotes the representation of the Weyl group $W(A)$ on V. Hence, for all $w \in W(A)$, there exists $y_1 \in \mathcal{U}(\mathfrak{k})$ such that $\sigma(y_1) = \eta(w) \otimes \text{Id}$, and for all $x \in \mathfrak{k}$, there exists $y_2 \in \mathcal{U}(\mathfrak{k})$ such that $\sigma(y_2) = \text{Id} \otimes \rho(x)$.

Remark 4.2. The above formulas also hold for $\alpha \in \Delta_+^{\text{re}}$ and $X_\alpha \in \mathfrak{k}_\alpha$ such that $(X_\alpha|X_\alpha) = (X_i|X_i)$ for some $i = 1, \ldots, n$, due to the conjugation result (Proposition 5.21) that we are going to prove later.

Proof. One has that

$$\sigma(X_i) = \Big(\eta(s_i) - \frac{1}{2}\Big) \otimes (2\rho(X_i)), \tag{17}$$

and from this, one computes

$$\begin{aligned}
\sigma(X_i)^2 &= \Big(\eta(s_i) - \frac{5}{4}\Big) \otimes \text{Id},\\
\sigma(X_i)^3 &= \Big(-\frac{7}{4}\eta(s_i) + \frac{13}{8}\Big) \otimes 2\rho(X_i) = -\frac{7}{4}\sigma(X_i) + \frac{3}{4}\text{Id} \otimes 2\rho(X_i),\\
\sigma(X_i)^4 &= -\frac{5}{2}\Big(\eta(s_i) - \frac{41}{40}\Big) \otimes \text{Id},
\end{aligned}$$

from which (16) follows. As $W(A)$ is generated by the s_i and $\mathfrak{k}$ is generated by the X_i, it is always possible to find $y_1, y_2 \in \mathcal{U}(\mathfrak{k})$ such that $\sigma(y_1) = \eta(w) \otimes \text{Id}$ and $\sigma(y_2) = \text{Id} \otimes \rho(x)$.

Concerning the remark: by Proposition 5.21, one has that

$$\sigma(X_\alpha) = \pm\Big(\eta(s_\alpha) - \frac{1}{2}\Big) \otimes (2\rho(X_\alpha))$$

for $\alpha \in \Delta^{\text{re}}$ and the sign $\pm$ does not change the computation. □

We need to collect a few basic results concerning the action of $W(A)$ on $\mathfrak{h}^*$.

Lemma 4.3. *Let $A \in \mathbb{Z}^{n\times n}$ be an indecomposable, regular, not necessarily simply laced GCM and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of A. Then $\mathfrak{h}^*$ is an irreducible $W(A)$-module.*

Proof. Since A is regular, $\mathfrak{h}^*$ is spanned by $\alpha_1, \ldots, \alpha_n$ which are by definition linearly independent and therefore, to any $0 \neq \lambda \in U$ in an invariant submodule $U \subset \mathfrak{h}^*$, there exists $i \in \{1, \ldots, n\}$ such that $\lambda(\alpha_i^\vee) \neq 0$. But then $s_i\lambda = \lambda - \lambda(\alpha_i^\vee)\alpha_i \in U$ implies $\alpha_i \in U$. Now $s_j(\alpha_i) - \alpha_i = a_{ji}\alpha_j \in U$ shows by successive application that all simple roots are contained in U because A is indecomposable. Thus, $\mathfrak{h}^*$ is an irreducible $W(A)$-module. □

We remark that the assumption of regularity is necessary. If A is an affine GCM, then $\mathbb{K} \cdot \delta$, where δ denotes the null root, is an invariant subspace of $\mathfrak{h}^*$ with respect to the action of $W(A)$. Also, indecomposability is necessary because otherwise the span of simple roots corresponding to each sub-block of A are an invariant subspace of $\mathfrak{h}^*$.

Proposition 4.4 (This is [26, Prop. 5.8])**.** *Let $A \in \mathbb{Z}^{n\times n}$ be an indecomposable, simply laced, regular GCM and $n \geq 2$. Then there exists an irreducible generalized spin representation $\mathcal{S}_{\frac{1}{2}}$ of $\mathfrak{k}(A)$. Furthermore, the representation $\mathcal{S}_{\frac{3}{2}}$ built on this $\mathcal{S}_{\frac{1}{2}}$ is irreducible as well.*

Proof. Since the generalized spin representations from Theorem 3.3 are finite-dimensional, restriction to a smallest invariant submodule is always possible and yields an irreducible generalized spin representation $\mathcal{S}_{\frac{1}{2}}$.

By Lemma 4.3, $\mathfrak{h}^*$ is an irreducible $W(A)$-module because A is regular, and by Lemma 4.1, one can act independently on the factors of $\mathcal{S}_{\frac{3}{2}} = \mathfrak{h}^* \otimes \mathcal{S}_{\frac{1}{2}}$. Irreducibility of $\mathcal{S}_{\frac{3}{2}}$ follows if one can show that any invariant submodule contains an elementary tensor $\lambda \otimes u \in \mathcal{S}_{\frac{3}{2}}$. In order to do so, introduce the projections $p_i := \frac{1}{2}(\mathrm{Id} - s_i) \in \mathrm{End}(\mathfrak{h}^*)$, where $i = 1, \ldots, n$, to the simple roots. Since $(\,\cdot\,,\cdot\,)$ is non-degenerate $(\alpha_i, \lambda) = 0$ for all $i = 1, \ldots, n$ and $\lambda \in \mathfrak{h}^*$ is equivalent to $\lambda = 0$ and in consequence to any $\lambda \neq 0$, there exists i such that $p_i(\lambda) \neq 0$. Thus, to $0 \neq v \in \mathfrak{h}^* \otimes \mathcal{S}_{\frac{1}{2}}$, there exists i such that $(p_i \otimes \mathrm{Id})(v) \neq 0$. But $(p_i \otimes \mathrm{Id})(v) = \alpha_i \otimes u$ for some $0 \neq u \in \mathcal{S}_{\frac{1}{2}}$. This shows that any invariant submodule of $\mathcal{S}_{\frac{3}{2}}$ contains an elementary tensor and therefore that $\mathcal{S}_{\frac{3}{2}}$ is irreducible. □

Corollary 4.5. *Let $A \in \mathbb{Z}^{n\times n}$ and $\mathcal{S}_{\frac{3}{2}}$ be as in Proposition 4.4 and denote the representation by $\sigma : \mathfrak{k}(A) \to \mathrm{End}(\mathcal{S}_{\frac{3}{2}})$. Then $\mathrm{im}(\sigma)$ is semisimple.*

Proof. According to [3, I.6.4, Prop. 5], one of seven equivalent characterizations for a Lie algebra $\mathfrak{g}$ to be reductive is to possess a faithful, finite-dimensional, completely reducible[2] representation. By definition, $\mathrm{im}(\sigma) \subset \mathrm{End}(\mathcal{S}_{\frac{3}{2}})$ possesses a faithful finite-dimensional representation, because $\mathcal{S}_{\frac{3}{2}}$ is a finite-dimensional vector space. By Proposition 4.4, $\mathcal{S}_{\frac{3}{2}}$ is an irreducible $\mathfrak{k}(A)$-module and therefore an irreducible, hence completely reducible, $\mathrm{im}(\sigma)$-module. Thus, $\mathrm{im}(\sigma)$ is reductive. Since A is indecomposable and simply laced, $\mathfrak{k}(A)$ is perfect, *i.e.*, $[\mathfrak{k}(A), \mathfrak{k}(A)] = \mathfrak{k}(A)$ (this is essentially a consequence of the fact that each Berman generator X_j can be written as the commutator $-[X_i, [X_i, X_j]]$ for i adjacent to j). One concludes that $\mathrm{im}(\sigma)$ is semisimple. □

Lemma 4.6. *Let A be a simply laced GCM. Then the higher spin representations $(\sigma, \mathcal{S}_{\frac{2s+1}{2}})$ of $\mathfrak{k}(A)(\mathbb{R})$ for $s \in \{1, 2, 3\}$ admit a non-degenerate bilinear form $\langle\,\cdot\,,\cdot\,\rangle$ with respect to which the representation matrices are skew-adjoint.*

[2]The exact wording is *semisimple*, which is used synonymously for *completely reducible* in [3] when applied to $\mathfrak{g}$-modules (see [3, I.3.1, Def. 2]). Note also that the reference to [3, I.6.4, Prop. 5] is sensitive to publisher and edition.

This so-called contravariant bilinear form is given explicitly by

(18) $$\langle a \otimes s, b \otimes t\rangle = (a|b)_V \cdot (s|t)_S \quad \text{for all } a, b \in V,\ s, t \in S,$$

with $V \in \{\mathfrak{h}^*, \mathrm{Sym}^2(\mathfrak{h}^*), \mathrm{Sym}^3(\mathfrak{h}^*)\}$ *and* S *a generalized spin representation. Here,* $(\cdot\,|\,\cdot)_V$ *denotes the symmetric bilinear form invariant under the action of* $W(A)$ *induced by the standard invariant form on* $\mathfrak{h}^*$*, and* $(\cdot\,|\,\cdot)_S$ *denotes an inner product on* S*.*

Proof. The generalized spin representation S is compact according to Proposition 3.2. Hence, S admits an inner product $(\cdot\,|\,\cdot)_S$ with respect to which the $\rho(X_i)$ are skew-adjoint. Furthermore, $V \in \{\mathfrak{h}^*, \mathrm{Sym}^2(\mathfrak{h}^*), \mathrm{Sym}^3(\mathfrak{h}^*)\}$ carries the invariant bilinear form induced by $(\cdot\,|\,\cdot)$ on $\mathfrak{h}^*$. The (induced) Weyl reflection s_α as well as the projection $f(\alpha)$ from 11 for $\alpha \in \Pi$ are symmetric with respect to $(\cdot\,|\,\cdot)_V$. The bilinear form (18) is non-degenerate because $(\cdot\,|\,\cdot)_V$ and the inner product on S are. Skew-adjointness of $\sigma(X_i)$ extends to the image of σ as it is generated as a Lie algebra by the $\sigma(X_i)$. □

We would like to remark that, for the existence of a contravariant bilinear form, compactness of the underlying generalized spin representation appears to be crucial. One can also construct spin representations for other involutory subalgebras where the involution has a sign twist for some of the Berman generators, which typically results in a noncompact representation (cp. [31, 20]). In this case, a contravariant bilinear form needs not exist (cp. [7, Sec. 2.3]).

Proposition 4.7 (This is [26, Prop. 5.10])**.** *Let* A *be an indecomposable, simply laced, regular GCM and let* $\mathcal{S}_{\frac{5}{2}}$ *be the representation from Theorem 3.18 built on an irreducible generalized spin representation. Then* $\mathcal{S}_{\frac{5}{2}}$ *decomposes into an orthogonal sum of invariant submodules*

$$\mathcal{S}_{\frac{5}{2}} \cong \tilde{\mathcal{S}}_{\frac{5}{2}} \oplus \mathcal{S}_{\frac{1}{2}}$$

with respect to the contravariant form (18)*. The module* $\tilde{\mathcal{S}}_{\frac{5}{2}}$*, called the* trace-free *part of* $\mathcal{S}_{\frac{5}{2}}$*, is irreducible if the* $W(A)$*-module* $\mathrm{Sym}^2(\mathfrak{h}^*)$ *decomposes into exactly two irreducible factors, where one of them is always the trivial representation.*

Proof. One again uses that the action of $\mathfrak{k}(A)$ on $V \otimes S$ can be split into the action on $V = \mathrm{Sym}^2(\mathfrak{h}^*)$ and $S = \mathcal{S}_{\frac{1}{2}}$, where the action on V is essentially that of $W(A)$ on V. The symmetric element

(19) $$\Psi := \sum_{k,l} \omega^{kl} e_k \otimes e_l$$

with ω from (10) is $W(A)$-invariant as it is invariant under any $A \in \mathrm{End}(V)$ that is induced by a $g \in O(\mathfrak{h}^*)$ (cp. [8, Secs. 17.3 and 19.5]). Since S admits an inner product and $(\Psi|\Psi) = \dim \mathfrak{h}^*$, one has that $\mathbb{K} \cdot \Psi \otimes S$ is anisotropic with respect to the contravariant form (18). Thus, its orthogonal complement is an invariant submodule which we call the trace-free part.

In order to describe irreducibility of its complement in terms of the action of $W(A)$, we proceed as before and try to show that there exists an elementary tensor in $\tilde{\mathcal{S}}_{\frac{5}{2}}$. It may be possible to derive this by the induced action of $W(A)$ on $\Psi^\perp$ similar to the proof of Proposition 4.4, but this time, it is in fact more economic to work with S instead. One has from Theorem 3.3 that $\mathrm{im}(\rho) \subset \mathrm{End}(S)$ is semisimple. The irreducible $\mathrm{im}(\rho)$-module S becomes a finite-dimensional highest weight module of some semisimple complex Lie algebra $\mathring{\mathfrak{g}} = \mathrm{im}(\rho)_\mathbb{C}$ after complexification. This implies that S decomposes into weight spaces with respect to to a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$ whose triangular decomposition we denote by $\mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$; we also denote the Chevalley generators of $\mathring{\mathfrak{n}}_+$ by $\mathring{e}_1, \ldots, \mathring{e}_m$ and the weights of $S_\mathbb{C}$ by $P(S)$. Any invariant submodule $U_\mathbb{C}$ of $\tilde{\mathcal{S}}_{\frac{5}{2},\mathbb{C}}$ therefore admits a basis whose elements are linear combinations of elementary tensors $\alpha\beta \otimes s^j_\lambda$ with $\alpha, \beta \in \mathfrak{h}^*$ and s^j_λ for $j = 1, \ldots, \mathrm{mult}(\lambda)$ weight vectors of weight $\lambda \in P(S) \subset \mathring{\mathfrak{h}}^*$. As S is a highest weight module, each λ can be written uniquely as $\lambda = \Lambda - \sum_{i=1}^m k_i \mathring{\beta}_i$, where the $\mathring{\beta}_i$ denote the simple roots of $\mathring{\mathfrak{h}}^*$ and the k_i are nonnegative. As the basis of $U_\mathbb{C}$ is finite, there exist weights λ of maximal depth (defined as $\sum k_i$) k such that s_λ occurs in the basis of $U_\mathbb{C}$. To each such s^i_λ, there exists an element $e_+(\lambda, i) = \mathring{e}_{i_1} \cdots \mathring{e}_{i_k} \in \mathcal{U}(\mathring{\mathfrak{n}}_+)$ such that $e_+(\lambda, i)s^i_\lambda \in S_\Lambda$ and is nonzero. This is because to each nonzero $x \in S_\lambda$, there exists an $i = 1, \ldots, m$ such that $\mathring{e}_i x \neq 0$ unless $\lambda = \Lambda$ by uniqueness of the highest weight vector. The same applies to any nonzero linear combination of the s^i_λ. Furthermore, the $e_+(\lambda, i)$ map any s^j_μ with $\mu \neq \lambda$ but of the same or smaller depth to 0. Thus, with $\{b_1, \ldots, b_N\}$ a basis of $\mathrm{Sym}^2(\mathfrak{h}^*)$ and given

$$u = \sum_{i=1}^N \sum_{\mu \in P(S)} \sum_{j=1}^{\mathrm{mult}(\mu)} c^{(\mu)}_{ij} b_i \otimes s^j_{(\mu)} \in U_\mathbb{C},$$

there exists $e_+(\lambda)$ such that

$$\begin{aligned}(\mathrm{Id} \otimes e_+(\lambda))u &= \sum_{i=1}^N \sum_{j=1}^{\mathrm{mult}(\lambda)} c^{(\lambda)}_{ij} b_i \otimes e_+(\lambda) s^j_{(\lambda)} \\ &= \sum_{i=1}^N \Bigg(b_i \otimes \Bigg[\sum_{j=1}^{\mathrm{mult}(\lambda)} c^{(\lambda)}_{ij} e_+(\lambda) s^j_{(\lambda)} \Bigg] \Bigg) \\ &= \sum_{i=1}^N b_i \otimes k_i s_\Lambda = \Bigg(\sum_{i=1}^N k_i b_i \Bigg) \otimes s_\Lambda\end{aligned}$$

with at least one $k_i \neq 0$. Therefore, an invariant submodule $U_\mathbb{C}$ of $\tilde{\mathcal{S}}_{\frac{5}{2},\mathbb{C}}$ always contains an elementary tensor. As in the proof of Proposition 4.4, one now deduces from Lemma 4.1 that $\tilde{\mathcal{S}}_{\frac{5}{2},\mathbb{C}}$ is irreducible if the action of $W(A)$ on $\Psi^\perp \subset \mathrm{Sym}^2(\mathfrak{h}^*)$ is. This in turn implies irreducibility of the real module $\tilde{\mathcal{S}}_{\frac{5}{2}}$ because if U were an invariant submodule of $\tilde{\mathcal{S}}_{\frac{5}{2}}$, then $U_\mathbb{C} = U + iU$ would be an invariant submodule of $\tilde{\mathcal{S}}_{\frac{5}{2},\mathbb{C}}$. □

Remark 4.8. Note that $\Psi^\perp \subset \mathrm{Sym}^2(\mathfrak{h}^*)$ can be irreducible or not; an example for the latter case is $A = A_{n-1}$. In this case, $W(A_{n-1})$ is isomorphic to the symmetric group $\mathfrak{S}_n$ and $\mathfrak{h}^*$ is isomorphic to its standard representation. According to [8, Ex. 4.19], $\mathrm{Sym}^2 V \cong U \oplus V \oplus V_{(n-2,2)}$, where U denotes the trivial representation, V denotes the standard representation, and $V_{(n-2,2)}$ is the irreducible representation associated to the partition $(n-2,2)$ of n. The exceptional diagrams E_6, E_7, and E_8 provide an example for $\Psi^\perp$ being irreducible: according to [10, Tabs. C.4–C.6], $W(E_n)$ for $n = 6,7,8$ admits an irreducible character of degree $\binom{n+1}{2} - 1$ that occurs[3] in $\mathrm{Sym}^2(V)$, where V denotes the standard representation of $W(E_n)$ as before.

We would like to show that the image of $\mathfrak{k}(A)$ under the representation $(\sigma, \mathcal{S}_{\frac{5}{2}})$ is a semisimple Lie algebra by showing that its image is reductive as we did in Corollary 4.5 and then exploit perfectness of $\mathfrak{k}(A)$. Since even the module $\tilde{\mathcal{S}}_{\frac{5}{2}}$ is not always irreducible, we need to show more generally that the module is completely reducible. By the factorization of the actions, this essentially requires to show that $\mathrm{Sym}^2(V)$ is a completely reducible $W(A)$-module, which is not obvious if $W(A)$ is not finite.

Lemma 4.9. *Let A be an indecomposable, simply laced, regular GCM and denote by $V = \mathrm{Sym}^2(\mathfrak{h}^*)$ the $W(A)$-module induced by the natural action of $W(A)$ on $\mathfrak{h}^*$. The module V is completely reducible.*

Proof. The module V comes with a natural $W(A)$-invariant bilinear form $(\cdot|\cdot)$. In order to show complete reducibility, we need to show that any invariant submodule U has an invariant complement. The submodule $U^\perp$ orthogonal to U with respect to $(\cdot|\cdot)$ is a natural candidate by $W(A)$-invariance. It is an invariant complement if and only if $U^\perp \cap U = \{0\}$. If A is of finite type, there is nothing to prove since $(\cdot|\cdot)$ is positive definite in this case. For A indefinite however, this requires some consideration. We will show first that $\mathbb{K}\Psi$ with Ψ from (19) is the only $W(A)$-invariant subspace of dimension 1. We then show that $s_i.u \neq u$ for $u \in U^\perp \cap U$ leads to a contradiction, so that $U^\perp \cap U$ can only be trivial because Ψ is not isotropic.

First of all, $U^\perp \cap U$ is a $W(A)$-invariant submodule, too. As a consequence, $(u|u) = 0$ for all $u \in U^\perp \cap U$. Consider a basis $\omega_1, \dots, \omega_n$ of $\mathfrak{h}^*$ defined by $(\alpha_i|\omega_j) = \delta_{ij}$ (since A is simply laced, these are the fundamental weights) and spell out $u = \sum_{i,j} c_{ij}\omega_i\omega_j$ and $\alpha_i = \sum_j B_j^{(i)}\omega_j$. Now $s_i u = u$ for all i is equivalent to

$$
\begin{aligned}
0 &= s_i u - u \\
&= c_{ii}(\omega_i - \alpha_i)(\omega_i - \alpha_i) - c_{ii}\omega_i\omega_i + (\omega_i - \alpha_i)\sum_{j\neq i} c_{ij}\omega_j - \omega_i \sum_{j\neq i} c_{ij}\omega_j \\
&= c_{ii}\alpha_i\alpha_i - 2c_{ii}\alpha_i\omega_i - \alpha_i \sum_{j\neq i} c_{ij}\omega_j \\
&= \alpha_i\Big[c_{ii}\sum_j B_j^{(i)}\omega_j - 2c_{ii}\omega_i - \sum_{j\neq i} c_{ij}\omega_j\Big],
\end{aligned}
$$

[3]This can be seen from the value of b_χ in this table. For an irreducible character χ, a value of $b_\chi = d$ means that $\mathrm{Sym}^d(V)$ is the smallest symmetric product of V that affords χ as an irreducible component.

which is only zero if the second factor is. This leads to the system of equations

$$c_{ii}B_i^{(i)} = 2c_{ii}, \quad c_{ii}B_j^{(i)} = c_{ij} \quad \text{for all } j \neq i.$$

The first equation is satisfied for all i because $s_i\alpha_i = -\alpha_i$ yields that $B_i^{(i)} = 2$ so that the second equation fixes all c_{ij} for $j \neq i$. Thus, if all c_{ii} were 0, then $u = 0$ so that we can from now on assume otherwise. Let i be such that $c_{ii} \neq 0$ and pick j such that $\{i,j\} \in \mathcal{E}(A)$, which is possible since A is indecomposable. One has

$$0 \neq (\alpha_i|\alpha_j) = B_j^{(i)}(\omega_j|\alpha_j) = B_j^{(i)}$$

but also

$$0 \neq (\alpha_i|\alpha_j) = B_i^{(j)}(\alpha_i|\omega_i) = B_i^{(j)}$$

so that $B_i^{(j)} = B_j^{(i)}$ for all i,j. Since $c_{ij} = c_{ji}$ because $u \in \mathrm{Sym}^2(\mathfrak{h}^*)$, one has

$$c_{ii}B_j^{(i)} = c_{ij} = c_{ji} = c_{jj}B_i^{(j)}$$

and therefore $c_{jj} = c_{ii}$ for all j. Thus, the space of all $W(A)$-invariant vectors is of dimension 1 and therefore equal to $\mathbb{K}\Psi$. We had already seen that the norm of Ψ is nonzero so that $\Psi^\perp \cap \mathbb{K}\Psi = \{0\}$.

Now assume $u \in \Psi^\perp \cap U \cap U^\perp$ are nonzero. Then there exists i such that $s_iu \neq u$. Spelling out u in the slightly altered basis $\{\alpha_i\} \cup \{\omega_j \mid j \neq i\}$ as $u = c_{ii}\alpha_i\alpha_i + \alpha_i \sum_{j\neq i} c_{ij}\omega_j + \sum_{j\neq i\neq k} c_{jk}\omega_j\omega_k$ yields

$$s_iu - u = -2\alpha_i \sum_{j\neq i} c_{ij}\omega_j =: \alpha_i\beta \in \Psi^\perp \cap U \cap U^\perp.$$

One has that $(\beta|\alpha_i) = 0$, and so $\alpha_i\beta \in \Psi^\perp \cap U \cap U^\perp$ implies

$$0 = (\alpha_i\beta|\alpha_i\beta) = \frac{1}{2}(\alpha_i|\alpha_i)(\beta|\beta) + \frac{1}{2}(\alpha_i|\beta)(\beta|\alpha_i) = (\beta|\beta).$$

Since A is regular and β is nonzero, there exists $j \neq i$ such that $(\alpha_j|\beta) =: c \neq 0$.

Case 1: $(\alpha_i|\alpha_j) = 0$. One has $-s_j(\alpha_i\beta) + \alpha_i\beta = c\alpha_i\alpha_j \in \Psi^\perp \cap U \cap U^\perp$, but $(c\alpha_i\alpha_j|c\alpha_i\alpha_j) = \frac{1}{2}c^2(\alpha_i|\alpha_i)(\alpha_j|\alpha_j) \neq 0$ contradicts this.

Case 2: $(\alpha_i|\alpha_j) = -1$.

$$\begin{aligned} s_j(\alpha_i\beta) - \alpha_i\beta &= c\alpha_i\alpha_j + \beta\alpha_j + c\alpha_j\alpha_j =: w, \\ (w|w) &= \frac{5}{2}c^2 - 2c \cdot \frac{1}{2}c - 4c^2 + \frac{1}{2}c^2 + 4c^2 + 4c^2 = 6c^2 \neq 0, \end{aligned}$$

which again contradicts $w \in \Psi^\perp \cap U \cap U^\perp$. Therefore, $U \cap U^\perp = \{0\}$ and invariance of $(\,\cdot\,|\,\cdot\,)$ show that V is a completely reducible $W(A)$-module. $\square$

Corollary 4.10. *Let A and $\mathcal{S}_{\frac{5}{2}}$ be as in Proposition 4.7; then the image of $\mathfrak{k}(A)$ under this representation is semisimple.*

Proof. If $\mathrm{Sym}^2(\mathfrak{h}^*)$ is a completely reducible $W(A)$-module, then $\mathcal{S}_{\frac{5}{2}}$ is a completely reducible $\mathfrak{k}(A)$-module. This implies the claim by the same reasoning as in Corollary 4.5. $\square$

Lemma 4.11. *Let $\rho_i : \mathfrak{k}(A)(\mathbb{K}) \to \mathrm{End}(V_i)$ for $i = 1, 2$ be finite-dimensional representations and let $x \in \ker(\rho_1 \otimes \rho_2)$. Then either $x \in \ker(\rho_1) \cap \ker(\rho_2)$ or there exists $0 \neq \lambda \in \mathbb{C}$ such that*

$$\rho_1(x) = \lambda \cdot \mathrm{Id}_{V_1} \quad \text{and} \quad \rho_2(x) = -\lambda \cdot \mathrm{Id}_{V_2}.$$

Proof. One has $\ker \rho_1 \cap \ker \rho_2 \subset \ker(\rho_1 \otimes \rho_2)$ because of

$$\begin{aligned}(\rho_1 \otimes \rho_2)(x)(v \otimes w) &= (\rho_1(x)v) \otimes w + v \otimes (\rho_2(x)w) \\ &= 0 \otimes w + v \otimes 0 = 0 \quad \text{for all } v \in V_1,\ w \in V_2.\end{aligned}$$

Now assume $x \in \ker(\rho_1 \otimes \rho_2)$ but $x \notin \ker(\rho_1)$; then $(\rho_1 \otimes \rho_2)(x)$ needs to vanish everywhere and so in particular on every element of a basis of $V_1 \otimes V_2$. Take bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ of V_1 and V_2, respectively, and compute

$$\begin{aligned} & 0 = \rho_1(x)e_i \otimes f_j + e_i \otimes \rho_2(x) f_j \\ \iff & 0 = \sum_{k \neq i}^{n} \rho_1(x)_{ki} e_k \otimes f_j + \rho_1(x)_{ii} e_i \otimes f_j \\ & \qquad + \sum_{l \neq j}^{m} \rho_2(x)_{lj} e_i \otimes f_l + \rho_2(x)_{jj} e_i \otimes f_j \end{aligned}$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$. This holds if and only if $\rho_1(x)$ and $\rho_2(x)$ are diagonal and such that $\rho_1(x)_{ii} = -\rho_2(x)_{jj} \neq 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. □

Lemma 4.12. *Let $\rho : \mathfrak{k}(A)(\mathbb{K}) \to \mathrm{End}(V)$ be a finite-dimensional representation; then*

$$\ker(\rho \otimes \rho) = \ker(\rho), \quad \ker(\mathrm{Sym}^n(\rho)) = \ker(\rho), \quad \ker(\wedge^n \rho) = \ker \rho,$$

as long as $n < \dim(V)$.

Proof. For $\rho \otimes \rho$, one applies Lemma 4.11. For $\mathrm{Sym}^n(\rho)$, consider a basis $\{b_1, \dots, b_m\}$ of V; then $x.b_i \cdot b_i \cdot \dots \cdot b_i = 0$ is equivalent to $\rho(x)_{ji} = 0$ for all $i, j = 1, \dots, m$. Similarly, one computes on $\bigwedge^n V$ that $x.b_{i_1} \wedge \dots \wedge b_{i_n} = 0$ is equivalent to $\sum_{k=1}^n \rho(x)_{i_k i_k} = 0$ and $\rho(x)_{ij} = 0$ for all $j \neq i$ (it may be instructional to look at $\bigwedge^2 V$ first). □

Proposition 4.13 (This is [26, Prop. 6.3])**.** *Let A be a simply laced and indecomposable GCM. Then all the higher spin representations $(\rho_{\frac{2s+1}{2}}, \mathcal{S}_{\frac{2s+1}{2}})$ of $\mathfrak{k}(A)$ for $s = 1, 2, 3$ (cp. Theorems 3.3, 3.18, and 3.22) satisfy*

$$\ker \rho_{\frac{2s_1+1}{2}} \otimes \rho_{\frac{2s_2+1}{2}} = \ker \rho_{\frac{2s_1+1}{2}} \cap \ker \rho_{\frac{2s_2+1}{2}}.$$

Proof. There exists a non-degenerate bilinear form on each of the mentioned modules and the action of $\mathfrak{k}(A)$ is skew (cp. Lemma 4.6) with respect to this bilinear form. The representation matrices must therefore be traceless, which excludes the second case of Lemma 4.11. □

Proposition 4.14. *Let A be a regular, indefinite, simply laced GCM and denote by $\mathcal{I}_{\frac{1}{2}}$, $\mathcal{I}_{\frac{3}{2}}$ and $\mathcal{I}_{\frac{5}{2}}$ the kernels of the representations $(\rho, \mathcal{S}_{\frac{1}{2}})$, $(\sigma, \mathcal{S}_{\frac{3}{2}})$, and $(\sigma, \tilde{\mathcal{S}}_{\frac{5}{2}})$, respectively. If the image of σ does not contain a compact ideal, then $\mathcal{I}_{\frac{1}{2}} \cap \mathcal{I}_{\frac{m}{2}} \subsetneq \mathcal{I}_{\frac{k}{2}}$ for $m \in \{3,5\}$ and $k \in \{1, m\}$.*

Proof. We start by showing $\mathcal{I}_{\frac{1}{2}} \not\subseteq \mathcal{I}_{\frac{m}{2}}$. Let $x_\alpha \in \mathfrak{k}_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$, $x_\beta \in \mathfrak{k}_\beta$ with $\alpha, \beta \in \Delta_+^{\mathrm{re}}$ be such that (recall the definition of Γ-matrices, Definition 3.7, and their relation to generalized spin representations, Proposition 3.9)

$$\rho(x_\alpha) = \frac{1}{2}\Gamma(\alpha), \qquad \rho(x_\beta) = \frac{1}{2}\Gamma(\beta),$$
$$\sigma(x_\alpha) = \Big(s_\alpha - \frac{1}{2}\mathrm{Id}\Big) \otimes \Gamma(\alpha), \quad \sigma(x_\beta) = \Big(s_\beta - \frac{1}{2}\mathrm{Id}\Big) \otimes \Gamma(\beta).$$

Such elements exist due to Lemma 5.19 and Proposition 5.21. We can furthermore choose $\alpha, \beta \in \Delta_+^{\mathrm{re}}$ such that $\alpha - \beta \in 2Q$, which implies $\Gamma(\alpha) = \Gamma(\beta)$ and therefore $x_\alpha - x_\beta \in \mathcal{I}_{\frac{1}{2}}$. However, $\sigma(x_\alpha) - \sigma(x_\beta) = (s_\alpha - s_\beta) \otimes \Gamma(\alpha) \neq 0$.

In order to show $\mathcal{I}_{\frac{m}{2}} \not\subseteq \mathcal{I}_{\frac{1}{2}}$, we are going to exploit that the image of the generalized spin representation ρ is compact but that $\mathcal{S}_{\frac{3}{2}}$ and $\tilde{\mathcal{S}}_{\frac{5}{2}}$ have an invariant bilinear form of mixed signature. We had excluded the case $\mathcal{I}_{\frac{m}{2}} = \mathcal{I}_{\frac{1}{2}}$ and therefore it is the same to assume $\mathcal{I}_{\frac{m}{2}} \subsetneq \mathcal{I}_{\frac{1}{2}}$ and produce a contradiction. We know by Corollaries 4.5 and 4.10 that $\mathrm{im}\,\sigma$ is semisimple. If $\mathcal{I}_{\frac{m}{2}} \subsetneq \mathcal{I}_{\frac{1}{2}}$, then there exists a nontrivial homomorphism $\phi : \mathrm{im}\,\sigma \to \mathrm{im}\,\rho$ that factors through ρ and σ, *i.e.*, $\phi \circ \sigma = \rho$. Since $\mathrm{im}\,\rho$ is compact, there exists an invariant inner product on $\mathcal{S}_{\frac{1}{2}}$ so that ϕ provides a nontrivial finite-dimensional unitary representation of $\mathrm{im}\,\sigma$. As noncompact simple Lie algebras do not admit nontrivial finite-dimensional unitary representations, all noncompact simple factors of $\mathrm{im}\,\sigma$ must act trivially on $\mathcal{S}_{\frac{1}{2}}$. Thus, if $\mathcal{I}_{\frac{m}{2}} \subsetneq \mathcal{I}_{\frac{1}{2}}$, then $\mathrm{im}\,\sigma$ has a nontrivial semisimple compact factor $\mathfrak{k}_0$ which we excluded by hypothesis. □

Remark 4.15. The hypothesis on A to be indefinite is crucial for the above proposition. For $\mathfrak{k}(E_9)$, it is shown in [25] that the kernels of $\mathcal{S}_{\frac{2s+1}{2}}$ form an ascending chain $\mathcal{I}_{\frac{7}{2}} \subsetneq \mathcal{I}_{\frac{5}{2}} \subsetneq \mathcal{I}_{\frac{3}{2}} \subsetneq \mathcal{I}_{\frac{1}{2}}$. This chain of inclusions is due to affine Kac–Moody algebras containing the loop algebra $\mathbb{K}[t^{-1}, t] \otimes \mathring{\mathfrak{g}}$ and the fact that $\mathfrak{k}(A)$ is contained in the loop algebra. This makes it possible to spell out $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ as generalized evaluation maps of the loop algebra involving also derivatives of order up to 2. This leads to all but the first inclusions, which is more subtle.

By the above proposition, the kernels of tensor products of representations can provide truly smaller ideals. In [26], it was shown for $\mathfrak{k}(E_{10})$ by a computer-based analysis that the tensor products $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}} \otimes \bigwedge^2 \mathcal{S}_{\frac{1}{2}}$ are both irreducible. We believe that this is generally true for A regular, indefinite, simply laced, where $\mathcal{S}_{\frac{3}{2}}$ can also be replaced by $\tilde{\mathcal{S}}_{\frac{5}{2}}$, as long as each factor in the tensor product is irreducible. The proof given in [26] however (cp. [26, Lem. 6.5 and Prop. 6.7]) contains an error that cannot be easily fixed: the applied strategy is similar to that from Propositions 4.4 and 4.7, *i.e.*, one tries to exploit a polynomial identity of the representation matrices that enables to act on each factor of the tensor product individually. The issue is that the identity derived in [26, Lem. 6.5] does not hold if the sign error in [26, Lem. 5.6] is fixed as we did in Lemma 4.1. With the correct sign, the matrices $\sigma(X_i) \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes \rho(X_i)$ are not contained in the span of $\mu(X_i)^n$ for $n \in \mathbb{N}$ with $\mu(X_i) = \sigma(X_i) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho(X_i)$. Instead, one has that $\mu(X_i)$ satisfies the identity $\mu(X_i)^4 = -\frac{5}{2}\mu(X_i)^2 - \frac{9}{16}$.

5. Lift to $\mathrm{Spin}(A)$ and compatibility with action of $W^{\mathrm{spin}}(A)$

In this section, we show that the higher spin representations do not lift to the maximal compact subgroup $K(A)$ of the minimal split-real Kac–Moody group $G(A)$ but only to its spin cover $\mathrm{Spin}(A)$. We also analyze the interaction of the spin representations with the action of the spin-extended Weyl group introduced in [11], which we use to derive a parametrization result for the representation matrices. Our exposition follows [26, Chap. 4] and we only briefly review spin covers, referring to [11] for more details.

5.1. **Maximal compact subgroups, spin covers, and a lift criterion.** There are different groups that can be associated to a Kac–Moody algebra (cp. [28]). The minimal simply connected, split-real Kac–Moody group $G(A)$ (cp. [28, Chap. 7]) possesses an involution θ that is the group analog of the Chevalley involution ω. The fixed-point subgroup $K(A) \coloneqq G(A)^\theta$ is commonly referred to as the maximal compact subgroup and its Lie algebra is $\mathfrak{k}(A)$. We are mostly concerned with the spin cover $\mathrm{Spin}(A)$ of $K(A)$ introduced in [11] for simply laced A and are therefore able to avoid the machinery involving the constructive Tits functor. Instead, we follow [11] and construct $K(A)$ as an $\mathrm{SO}(2)$-amalgam (defined below) without constructing $G(A)$ first. We merely note that the amalgamation approach to $K(A)$ as well as for $G(A)$ is equivalent to the constructive Tits functor for two-spherical diagrams, *i.e.*, diagrams whose rank-2 subdiagrams are all of finite type.

Definition 5.2 (cp. [11, 3.1 and 3.4])**.** For $I = \{1, \dots, n\}$ and $i \neq j \in I$, let G_i, G_{ij} be groups with monomorphisms $\psi^i_{ij} : G_i \to G_{ij}$. One calls

$$\mathcal{A} := \{G_i, G_{ij}, \psi^i_{ij} \mid i \neq j \in I\}$$

and the ψ^i_{ij} an *amalgam of groups* and *connecting homomorphisms*, respectively. If $G_i \cong U$ for all $i \in I$, one calls $\mathcal{A}$ an U-amalgam. It is called *continuous* if all G_i, G_{ij} are topological groups with continuous connecting homomorphisms ψ^i_{ij}.

Definition 5.3 (cp. [11, 3.5–3.6])**.** Let $\mathcal{A} = \{G_i, G_{ij}, \psi^i_{ij} \mid i \neq j \in I\}$ be an amalgam of groups. A group G together with homomorphisms $\tau := \{\tau_{ij} : G_{ij} \to G\}$ such that $\tau_{ij} \circ \psi^i_{ij} = \tau_{ik} \circ \psi^i_{ik}$ for all $j \neq i \neq k \in I$ is called an *enveloping group* of $\mathcal{A}$ with *enveloping homomorphisms* τ_{ij}. One calls (G, τ) *faithful* if all τ_{ij} are injective, and *universal* if, to any enveloping group $(H, \tilde{\tau})$ of $\mathcal{A}$, there exists a unique epimorphism $\pi : G \to H$ such that $\pi \circ \tau_{ij} = \tilde{\tau}_{ij}$.

Given a fixed amalgam $\mathcal{A}$, two universal enveloping groups are uniquely isomorphic by universality. The canonical universal enveloping group (CUEG)

$$G(\mathcal{A}) := \Big\langle \bigcup_{i \neq j \in I} G_{ij} \;\Big|\; \text{all relations in } G_{ij},\ \psi^j_{ij}(x) = \psi^j_{kj}(x) \text{ for all } i \neq j \neq k \text{ and all } x \in G_j \Big\rangle.$$

is a universal enveloping group [15, 1.3.2]; the above phrasing is as in [11]. For general GCMs A and field $\mathbb{K}$, the split minimal Kac–Moody group $G(A)$ over $\mathbb{K}$ associated to $\mathfrak{g}(A)(\mathbb{K})$ is defined via the constructive Tits functor (cp. [30]). For A two-spherical however, one has from the main result of [1] that the split minimal Kac–Moody group $G(A)$ over $\mathbb{R}$ is the universal enveloping group of the amalgam $\mathcal{A} = \{G_i, G_{ij}, \phi^i_{ij}\}$, where $G_i = \mathrm{SL}(2,\mathbb{R})$ and G_{ij} is the split-real algebraic group of type $A_{\{i,j\}}$. The $\phi^i_{ij} : G_i \hookrightarrow G_{ij}$ are the canonical inclusion maps induced from G_{ij} being generated by its fundamental rank-1 subgroups G_i and G_j. The restriction $\theta|_{G_{ij}}$ of θ to any fundamental rank-2 subgroup yields the classical Cartan–Chevalley involution on the split-real Lie group G_{ij} because A is two-spherical. In [11], it is shown that $K(A)$ is an amalgam of the G^θ_{ij} which are the classical maximal-compact subgroups. In order to state their result precisely, we need to introduce a few more objects first and we restrict ourselves to the simply laced situation.

For A simply laced, one has $G^\theta_i \cong \mathrm{SO}(2)$ and G^θ_{ij} isomorphic to $\mathrm{SO}(3)$ or $\mathrm{SO}(2) \times \mathrm{SO}(2)$. We set

$$K_{ij} := \begin{cases} \mathrm{SO}(3) & \text{if } \{i,j\} \in \mathcal{E}, \\ \mathrm{SO}(2) \times \mathrm{SO}(2) & \text{if } \{i,j\} \notin \mathcal{E}. \end{cases} \tag{20}$$

For a group H, set

$$i_1 : H \to H \times H, \quad h \mapsto (h,e), \qquad i_2 : H \to H \times H, \quad h \mapsto (e,h).$$

Furthermore, denote by $\varepsilon_{12} : \mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3)$ the embedding via the upper-left $\mathrm{SO}(2)$-subgroup and by $\varepsilon_{23} : \mathrm{SO}(2) \hookrightarrow \mathrm{SO}(3)$ that via the lower-right $\mathrm{SO}(2)$-subgroup.

Definition 5.4 (cp. [11, Def. 9.1])**.** Let $A \in \mathbb{Z}^{n \times n}$ be a simply laced GCM, $\mathcal{E}$ its Dynkin diagram's edges, and $I = \{1, \dots, n\}$. The *standard* $\mathrm{SO}(2)$*-amalgam* of type A is defined as

$$\mathcal{A}(A, \mathrm{SO}(2)) := \{K_i \cong \mathrm{SO}(2), K_{ij}, \phi^i_{ij} \mid i \neq j \in I\}$$

with K_{ij} as in (20) and, for all $i < j \in I$,

$$\phi^i_{ij} = \begin{cases} \varepsilon_{12} & \text{if } \{i,j\} \in \mathcal{E}, \\ i_1 & \text{if } \{i,j\} \notin \mathcal{E}, \end{cases} \qquad \phi^j_{ij} = \begin{cases} \varepsilon_{23} & \text{if } \{i,j\} \in \mathcal{E}, \\ i_2 & \text{if } \{i,j\} \notin \mathcal{E}. \end{cases}$$

According to [11, 9.5], changing the Dynkin diagram's labeling I does not affect the isomorphism type of $\mathcal{A}$ (we have not defined this term; cp. [11, 3.2]). It is possible to concatenate the ϕ^i_{ij} by an isomorphism of $\mathrm{SO}(2)$, which [11] calls simply an $\mathrm{SO}(2)$-amalgam and such an $\mathrm{SO}(2)$-amalgam is only guaranteed to be isomorphic to the standard one if the connecting homomorphisms are continuous. Split-real Kac–Moody groups of 2-spherical type naturally carry the so-called Kac–Peterson topology that induces the Lie topology on their spherical subgroups (cp. [16, 14]). Thus, it is natural in our setting to consider only continuous connecting homomorphisms.

The block embeddings $\varepsilon_{12}, \varepsilon_{23}$ have a canonical lift $\tilde{\varepsilon}_{12}, \tilde{\varepsilon}_{23} : \mathrm{Spin}(2) \hookrightarrow \mathrm{Spin}(3)$ (cp. [11, Lem. 6.10]) which enables one to define the standard Spin(2)-amalgam of type A in the same manner as the standard SO(2)-amalgam.

Definition 5.5 (cp. [11, Def. 10.1])**.** Let $A \in \mathbb{Z}^{n\times n}$ be a simply laced GCM, $\mathcal{E}$ its Dynkin diagram's edges, and $I = \{1, \dots, n\}$. Set

$$K_{ij} := \begin{cases} \mathrm{Spin}(3) & \text{if } \{i,j\} \in \mathcal{E}, \\ \mathrm{Spin}(2) \times \mathrm{Spin}(2) \big/ \langle(-1,-1)\rangle & \text{if } \{i,j\} \notin \mathcal{E}. \end{cases} \tag{21}$$

The *standard* Spin(2)*-amalgam* of type A is defined as

$$\mathcal{A}(A, \mathrm{Spin}(2)) := \{K_i \cong \mathrm{Spin}(2), K_{ij}, \phi^i_{ij} \mid i \neq j \in I\}$$

with K_{ij} as in (21) and, for all $i < j \in I$,

$$\phi^i_{ij} = \begin{cases} \tilde{\varepsilon}_{12} & \text{if } \{i,j\} \in \mathcal{E}, \\ i_1 & \text{if } \{i,j\} \notin \mathcal{E}, \end{cases} \qquad \phi^j_{ij} = \begin{cases} \tilde{\varepsilon}_{23} & \text{if } \{i,j\} \in \mathcal{E}, \\ i_2 & \text{if } \{i,j\} \notin \mathcal{E}. \end{cases}$$

As before, the Dynkin diagram's labeling does not matter [11, Cor. 10.7] and any continuous Spin(2)-amalgam of type A is isomorphic to $\mathcal{A}(A, \mathrm{Spin}(2))$ (cp. [11, Thm. 10.9]).

Definition 5.6 (cp. [11, Def. 11.5])**.** Let A be a simply laced GCM. Define $\mathrm{Spin}(A)$ to be the CUEG of the standard Spin(2)-amalgam $\mathcal{A}(A, \mathrm{Spin}(2))$ of type A.

Theorem 5.7 (cp. [11, Thm. 11.2])**.** *Let A be a simply laced GCM and $G(A)$ the minimal, simply connected split-real Kac–Moody group of type A. Denote its Cartan–Chevalley involution by θ; then the maximal compact subgroup $K(A) := G(A)^\theta$ is a faithful universal enveloping group of the standard* SO(2)*-amalgam $\mathcal{A}(A, \mathrm{SO}(2))$.*

Theorem 5.8 (cp. [11, Thm. 11.17])**.** *Let A be a simply laced GCM; then* $\mathrm{Spin}(A)$ *is a 2^s-fold central extension of $K(A)$, where s is the number of connected components of the Dynkin diagram of type A.*

We are now in the position to formulate a criterion that decides whether or not a given representation of $\mathfrak{k}(A)$ lifts to $K(A)$ or to $\mathrm{Spin}(A)$.

Proposition 5.9 (This is [26, Prop. 4.8])**.** *Let A be a simply laced, indecomposable GCM and let $\rho : \mathfrak{k}(A)(\mathbb{R}) \to \mathrm{End}(V)$ be a finite-dimensional representation. As usual, denote the Berman generators of $\mathfrak{k}(A)(\mathbb{R})$ by X_i; then one of the following two cases applies:*

$$\exp(2\pi\rho(X_i)) = \begin{cases} -\mathrm{Id}_V & \textit{for all } i \in I, \\ \mathrm{Id}_V & \textit{for all } i \in I. \end{cases} \tag{22}$$

$$\begin{array}{ccc} \mathfrak{k}_J & \xrightarrow{\phi} & \mathrm{End(U)} \\ {\scriptstyle \exp_0}\downarrow & \circlearrowright & \downarrow{\scriptstyle \exp} \\ \tilde{K}_J & \xrightarrow[\Phi]{} & \mathrm{GL}(U) \end{array} \qquad \begin{array}{ccc} \mathfrak{k}_i & \xrightarrow{\widetilde{\exp}_i} & \tilde{K}_i \\ {\scriptstyle (\psi^i_{ij})_*}\downarrow & \circlearrowright & \downarrow{\scriptstyle \psi^i_{ij}} \\ \mathfrak{k}_{ij} & \xrightarrow[\widetilde{\exp}_{ij}]{} & \tilde{K}_{ij} \end{array}$$

FIGURE 1. Two important commutative diagrams for a spherical subdiagram J, where $\exp_0$ denotes the abstract exponential map of a finite-dimensional Lie algebra to the identity component of a Lie group with that Lie algebra.

In both cases, ρ lifts to a representation Ω of $\mathrm{Spin}(A)$*, but it lifts to $K(A)$ only in the second case.*

Proof. We denote the canonical subalgebras generated by the subdiagram $J \subset I$ by $\mathfrak{k}_J := \langle X_j \mid j \in J \rangle$. Assume J is such that A_J is of finite type and let (ϕ, U) be a finite-dimensional irreducible representation of $\mathfrak{k}_J$. The representation always lifts to the simply connected Lie group $\tilde{K}_J$ with Lie algebra $\mathfrak{k}_J$ and the first diagram in Figure 1 commutes. For J such that A_J is of finite type, in particular for $|J| \leq 2$, one has that ρ restricted to $\mathfrak{k}_J$ is completely reducible. In the rank-1 case, there is no distinction in the lift of ρ since the involved groups are $\mathrm{SO}(2) \cong U(1) \cong \mathrm{Spin}(2)$. For $J = \{i,j\}$ such that $\{i,j\} \in \mathcal{E}$, $\mathfrak{k}_{\{i,j\}} \cong \mathfrak{so}(3)$ and all finite-dimensional representations lift to the universal cover $\mathrm{Spin}(3)$, but not all lift to $\mathrm{SO}(3)$. By comparison with the adjoint action of $\mathfrak{so}(3)$ on itself, one can determine if a given representation lifts to $\mathrm{SO}(3)$. Due to $[X_i, [X_i, X_j]] = -X_j$, one has

$$\begin{aligned} \exp(\phi \cdot \mathrm{ad}_{X_i})(X_j) &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} X_j + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} [X_i, X_j] \\ &= \cos(\phi) X_j + \sin(\phi) [X_i, X_j] \end{aligned}$$

for all i, j such that $\{i,j\} \in \mathcal{E}$. Since the exponential of the ad_{X_i} is 2π-periodic in the sense that $\exp(2\pi \cdot \mathrm{ad}_{X_i}) = \mathrm{Id}$, a representation that lifts to $\mathrm{SO}(3)$ has to satisfy the second case of (22). Also, the first case of (22) is incompatible with a lift to $\mathrm{SO}(3)$ and therefore such representations only lift to $\mathrm{Spin}(3)$.

We now need to extend this lifting behavior from the $\mathfrak{k}_J$ to all of $\mathfrak{k}$. We have already seen by the above that lifts exist locally for fixed spherical subalgebras and that the normalization of a single Berman generator tells us all we need to know for the others. The extension to $\mathrm{Spin}(A)$ and $K(A)$ respectively works similar to the proof of [11, Thm. 11.14]. For every i, j, the local lift of $\mathfrak{k}_{\{i,j\}}$ to the group level induces an enveloping homomorphism $\tau_{\{i,j\}} : \tilde{K}_{ij} \to \mathrm{GL}(V)$ in case one and $\tau_{ij} : K_{ij} \to \mathrm{GL}(V)$ in case two of (22). In both cases, we collect them in the set τ. One obtains these lifts as follows. In the first case, one identifies $\tilde{K}_{\{i,j\}}$ with the exponential image of $\mathfrak{k}_{\{i,j\}}$

under $\widetilde{\exp}_{\{i,j\}} : \mathfrak{k}_{\{i,j\}} \to \tilde{K}_{\{i,j\}}$, which is possible as the exponential map is surjective for compact Lie groups. This naturally induces connecting monomorphisms $\psi^i_{ij} : \tilde{K}_i \to \tilde{K}_{ij}$ that are compatible with exponentiation in the sense that the second diagram in Figure 1 commutes. This enables one to define $\tau_{ij} : \tilde{K}_{ij} \to \mathrm{GL}(V)$ by $\tau_{ij}(\widetilde{\exp}_{ij}(x)) = \exp(\rho(x))$ for all $x \in \mathfrak{k}_{\{i,j\}}$. Since ρ is globally defined on $\mathfrak{k}$, one has

$$\tau_{ij} \circ \psi^i_{ij}(\widetilde{\exp}_i(x)) = \exp(\rho(x)) = \tau_{ik} \circ \psi^i_{ik}(\widetilde{\exp}_i(x)) \quad \text{for all } x \in \mathfrak{k}_{\{i\}}.$$

This shows that the exponential of $\mathrm{im}(\rho)$ is an enveloping group of the standard Spin(2)-amalgam of type A so that there exists a unique homomorphism $\Omega : \mathrm{Spin}(A) \to \mathrm{GL}(V)$. If ρ restricted to the rank-2 subalgebras isomorphic to $\mathfrak{so}(3)$ lifts to SO(3), then the enveloping homomorphisms are $\tau_{\{i,j\}} : K_{ij} \to \mathrm{GL}(V)$ and the above argument shows that the exponential of $\mathrm{im}(\rho)$ is an enveloping group of the standard SO(2)-amalgam of type A. □

5.10. **Lifting and interaction with spin-extended Weyl group.** Let

$$\sigma(X_i) := \tau(\alpha_i) \otimes \Gamma(\alpha_i)$$

be a higher spin representation as in (3.18) and (3.22) and denote the induced one-parameter subgroups by

$$\Sigma_i(\phi) := \exp(\phi \cdot \sigma(X_i)).$$

We would now like to compute an explicit formula for the above representation matrix. This has been done in [23, Sec. 5] in a second-quantized form and without explicit reference to the Weyl group action, which is why we repeat the computation in our terminology.

Proposition 5.11. *Let A be simply laced and denote by (σ, V) the representation $\mathcal{S}_{\frac{3}{2}}$ or $\mathcal{S}_{\frac{5}{2}}$ of $\mathfrak{k}(A)(\mathbb{R})$ from Theorem 3.18. Then*

$$\Sigma_i(\phi) = \Big[\cos(\phi)\cos\Big(\frac{\phi}{2}\Big) \cdot \mathrm{Id} \otimes \mathrm{Id} - \cos(\phi)\sin\Big(\frac{\phi}{2}\Big) \cdot \mathrm{Id} \otimes \Gamma(\alpha_i) + \sin(\phi)\sin\Big(\frac{\phi}{2}\Big) \cdot \eta(s_i) \otimes \mathrm{Id} + \sin(\phi)\cos\Big(\frac{\phi}{2}\Big) \cdot \eta(s_i) \otimes \Gamma(\alpha_i)\Big], \tag{23}$$

and (σ, V) lifts to a representation (Σ, V) of Spin(A) *but not to $K(A)$. The restriction of (Σ, V) to the fundamental one-parameter subgroups are given by the Σ_i.*

Proof. One computes with $\Gamma(\alpha_i)^{2n} = (-1)^n$ and the shorthand

$$\tau(\alpha_i)^n = a(n) + b(n)\eta(s_i)$$

for the representations $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ that

$$\Sigma_i(\phi) = A_1(\phi) \cdot \mathrm{Id} \otimes \mathrm{Id} + A_2(\phi) \cdot \mathrm{Id} \otimes \Gamma(\alpha_i) + A_3(\phi) \cdot \eta(s_i) \otimes \mathrm{Id} + A_4(\phi) \cdot \eta(s_i) \otimes \Gamma(\alpha_i),$$

with

$$A_1(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} a(2n), \quad A_2(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} a(2n+1),$$
$$A_3(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} b(2n), \quad A_4(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} b(2n+1).$$

Because $\binom{n}{k} = 0$ for $k > n$ and $s_i^2 = e$, one has

$$\tau(\alpha_i)^n = \eta(s_i)^n \cdot \left[\sum_{k=0}^{\infty} \binom{n}{2k} (-2)^{-2k} + \eta(s_i) \sum_{k=0}^{\infty} \binom{n}{2k+1} (-2)^{-2k-1} \right].$$

The coefficients of a holomorphic function's Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

satisfy

$$\sum_{m=0}^{\infty} a_{2m} z^{2m} = \frac{1}{2}[f(z) + f(-z)] \quad \text{and} \quad \sum_{m=0}^{\infty} a_{2m+1} z^{2m+1} = \frac{1}{2}[f(z) - f(-z)].$$

Specialized to $f(z) := (1+z)^n$, this yields

$$\sum_{k=0}^{\infty} \binom{n}{2k} (-2)^{-2k} = \frac{1}{2}[f(z) + f(-z)]_{z=-\frac{1}{2}} = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n + \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n,$$
$$\sum_{k=0}^{\infty} \binom{n}{2k+1} (-2)^{-2k-1} = \frac{1}{2}[f(z) - f(-z)]_{z=-\frac{1}{2}} = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n - \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n,$$

and therefore

$$a(2n) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n} + \left(\frac{3}{2}\right)^{2n} \right], \quad a(2n+1) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n+1} - \left(\frac{3}{2}\right)^{2n+1} \right],$$
$$b(2n) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n} - \left(\frac{3}{2}\right)^{2n} \right], \quad b(2n+1) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n+1} + \left(\frac{3}{2}\right)^{2n+1} \right].$$

From this, one has with some trigonometric identities that

$$A_1(\phi) = \frac{1}{2} \cos\Big(\frac{\phi}{2}\Big) + \frac{1}{2} \cos\Big(\frac{3\phi}{2}\Big) = \cos(\phi) \cos\Big(\frac{\phi}{2}\Big),$$
$$A_2(\phi) = \frac{1}{2} \sin\Big(\frac{\phi}{2}\Big) - \frac{1}{2} \sin\Big(\frac{3\phi}{2}\Big) = -\cos(\phi) \sin\Big(\frac{\phi}{2}\Big),$$
$$A_3(\phi) = \frac{1}{2} \cos\Big(\frac{\phi}{2}\Big) - \frac{1}{2} \cos\Big(\frac{3\phi}{2}\Big) = \sin(\phi) \sin\Big(\frac{\phi}{2}\Big),$$
$$A_4(\phi) = \frac{1}{2} \sin\Big(\frac{\phi}{2}\Big) + \frac{1}{2} \sin\Big(\frac{3\phi}{2}\Big) = \sin(\phi) \cos\Big(\frac{\phi}{2}\Big).$$

Finally, one evaluates this for $\phi = 2\pi$ and obtains $A_1 = -1$ and $A_2 = A_3 = A_4 = 0$ so that $\exp(2\pi \cdot \sigma(X_i)) = -\mathrm{Id} \otimes \mathrm{Id}$. Now Proposition 5.9 implies that σ lifts only to $\mathrm{Spin}(A)$. □

Proposition 5.12. *Let A be simply laced and denote by (σ, V) the representation $\mathcal{S}_{\frac{7}{2}}$ of $\mathfrak{k}(A)(\mathbb{R})$ from Proposition 3.22. Denote the image of the fundamental one-parameter subgroups by $\tilde{\Sigma}_i$; then*

$$\tilde{\Sigma}_i(\phi) = \Sigma_i(\phi) + \frac{1}{4}\Big[\cos\Big(\frac{5}{2}\phi\Big) - \cos\Big(\frac{3}{2}\phi\Big)\Big] f(\alpha_i) \otimes \mathrm{Id} + \frac{1}{4}\Big[\sin\Big(\frac{5}{2}\phi\Big) + \sin\Big(\frac{3}{2}\phi\Big)\Big] f(\alpha_i) \otimes \Gamma(\alpha_i),$$

with $\Sigma_i(\phi)$ as in (23) *and $(\sigma, \mathcal{S}_{\frac{7}{2}})$ lifts to* $\mathrm{Spin}(A)$ *but not to $K(A)$.*

Proof. Recall from (11) that

$$\sigma(X_i) = \tau(\alpha_i) \otimes \Gamma(\alpha_i), \quad \tau(\alpha) \coloneqq \eta(s_\alpha) - \frac{1}{2}\mathrm{Id} + f(\alpha) \quad \text{for all } \alpha \in \Delta_+^{\mathrm{re}}$$

with $\eta : W \to \mathrm{GL}(V)$ denoting the induced action of the Weyl group on $V = \mathrm{Sym}^3(\mathfrak{h}^*)$ and $f(\alpha) = v(\alpha)(v(\alpha)| \cdot)$ with $v(\alpha) \in \mathrm{Sym}^3(\mathfrak{h}^*)$ defined in Theorem 3.22 and equation (11) satisfying

$$f(\alpha)^2 = 4 \cdot f(\alpha), \quad \eta(s_\alpha) f(\alpha) = f(\alpha)\eta(s_\alpha) = -f(\alpha).$$

Setting $\tilde{\tau}(\alpha) \coloneqq \eta(s_\alpha) - \frac{1}{2}\mathrm{Id}$, one has

$$\tilde{\tau}(\alpha) f(\alpha) = f(\alpha)\tilde{\tau}(\alpha) = -\frac{3}{2} f(\alpha).$$

With this, one computes

$$\begin{aligned}
\tau(\alpha)^n &= [\tilde{\tau}(\alpha) + f(\alpha)]^n = \sum_{k=0}^{n} \binom{n}{k} \tilde{\tau}(\alpha)^{n-k} f(\alpha)^k \\
&= \tilde{\tau}(\alpha)^n + \sum_{k=1}^{n} \binom{n}{k} \Big(-\frac{3}{2}\Big)^{n-k} 4^{k-1} f(\alpha) \\
&= \tilde{\tau}(\alpha)^n + \frac{1}{4}\Big[\Big(\frac{5}{2}\Big)^n - \Big(-\frac{3}{2}\Big)^n\Big] f(\alpha), \\
\tilde{\Sigma}_i(\phi) &\coloneqq \exp(\phi\sigma(X_i)) = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \tau(\alpha_i)^n \otimes \Gamma(\alpha_i)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} \tau(\alpha_i)^{2n} \otimes \mathrm{Id} + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} \tau(\alpha_i)^{2n+1} \otimes \Gamma(\alpha_i) \\
&= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} \tilde{\tau}(\alpha_i)^{2n} \otimes \mathrm{Id} + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} \tilde{\tau}(\alpha_i)^{2n+1} \otimes \Gamma(\alpha_i)}_{=:\Sigma_i(\phi)} \\
&\quad + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} \Big[\Big(\frac{5}{2}\Big)^{2n} - \Big(-\frac{3}{2}\Big)^{2n}\Big] f(\alpha_i) \otimes \mathrm{Id} \\
&\quad + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} \Big[\Big(\frac{5}{2}\Big)^{2n+1} - \Big(-\frac{3}{2}\Big)^{2n+1}\Big] f(\alpha_i) \otimes \Gamma(\alpha_i),
\end{aligned}$$

$$\begin{aligned}\tilde{\Sigma}_i(\phi) &= \Sigma_i(\phi) + \frac{1}{4}\Big[\cos\Big(\frac{5}{2}\phi\Big) - \cos\Big(\frac{3}{2}\phi\Big)\Big] f(\alpha_i) \otimes \mathrm{Id} \\ &\qquad + \frac{1}{4}\Big[\sin\Big(\frac{5}{2}\phi\Big) + \sin\Big(\frac{3}{2}\phi\Big)\Big] f(\alpha_i) \otimes \Gamma(\alpha_i).\end{aligned}$$

Above, $\Sigma_i(\phi)$ stands for the same expression as in (23) but now with $\eta : W(A) \to \mathrm{GL}(V)$ for $V = \mathrm{Sym}^3(\mathfrak{h}^*)$. Concerning the periodicity of Σ_i, the proof of Proposition 5.11 only relies on η being a representation of $W(A)$ and the properties of Γ-matrices. Therefore, Σ_i here is also 4π-periodic and so is the remainder of $\tilde{\Sigma}_i$. As before, Proposition 5.9 now implies that σ lifts only to $\mathrm{Spin}(A)$. □

Remark 5.13. One observes that eigenvalues of largest absolute value of $\sigma(X_i)$ are $\pm\frac{5}{2}$ for $(\sigma, \mathcal{S}_{\frac{7}{2}})$ and $\pm\frac{3}{2}$ for $(\sigma, \mathcal{S}_{\frac{5}{2}})$ and $(\sigma, \mathcal{S}_{\frac{3}{2}})$, so that the notion of $\frac{n}{2}$-representation used in [24, 27] does not quite fit. Essential to the occurrence of $\frac{5}{2}$ as an eigenvalue is that $f(\alpha)^2 = 4f(\alpha)$. Consider abstractly $f(\alpha)^2 = af(\alpha)$ and compute the adjoint action on the representation side, *i.e.*, $\tilde{\Sigma}_i(\frac{\pi}{2})\sigma(X_j)\tilde{\Sigma}_i(-\frac{\pi}{2})$. This must be proportional to $\sigma([X_i, X_j])$, which it only is if $a = 0 \bmod 4$.

In the above remark, we claimed without further details that

$$\tilde{\Sigma}_i\Big(\frac{\pi}{2}\Big)\sigma(X_j)\tilde{\Sigma}_i\Big(-\frac{\pi}{2}\Big)$$

must be proportional to $\sigma([X_i, X_j])$. The reasoning behind this is how representations interact with the extended and spin-extended Weyl group and their action on $\mathfrak{k}$, which we will now analyze in detail.

Definition 5.14 (cp. [11, Def. 18.4])**.** Let A be a symmetrizable GCM and define $n(i,j)$ to be 0 if a_{ij} is even and $n(i,j) = 1$ if a_{ij} is odd. Furthermore, define the Coxeter coefficients m_{ij} for $i \neq j$ via

$a_{ij}a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

Define the *extended Weyl group* $W^{\mathrm{ext}}(A)$ as

$$W^{\mathrm{ext}}(A) = \langle t_1, \ldots, t_n \mid t_i^4 = e \text{ for all } i \in I,\ t_j^{-1}t_i^2t_j = t_i^2t_j^{2n(i,j)} \text{ for all } i \neq j \in I,\ \underbrace{t_it_jt_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{t_jt_it_j\cdots}_{m_{ij}\text{ factors}} \text{ for all } i \neq j \in I\rangle,$$

and similarly define the *spin-extended Weyl group* $W^{\mathrm{spin}}(A)$ as

$$W^{\mathrm{spin}}(A) = \langle r_1, \ldots, r_n \mid r_i^8 = e \text{ for all } i \in I,\ r_j^{-1}r_i^2r_j = r_i^2r_j^{2n(i,j)} \text{ for all } i \neq j \in I,\ \underbrace{r_ir_jr_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{r_jr_ir_j\cdots}_{m_{ij}\text{ factors}} \text{ for all } i \neq j \in I\rangle.$$

Note that $W(A)$ is not a subgroup of the minimal Kac–Moody group $G(A)$. However, to any integrable representation (π, V) of $\mathfrak{g}$, one can introduce

$$t_i := \exp \pi(f_i) \exp(-\pi(e_i)) \exp \pi(f_i)$$

which have the following effect on the weight spaces V_λ (cp. [18, Lem. 3.8]):

$$t_i(V_\lambda) = V_{s_i.\lambda}, \tag{24}$$

with the simple Weyl reflection $s_i \in W(A)$. Denote by $G^\pi \leq \mathrm{GL}(V)$ the group generated by $\exp \pi(kf_i)$, $\exp \pi(k\alpha_i^\vee)$, and $\exp \pi(ke_i)$ for $k \in \mathbb{K}$. Then the subgroup $W^\pi(A) < G^\pi$ generated by the t_i contains an abelian normal subgroup $D^\pi = \langle t_i^2 \mid i \in I\rangle$ that satisfies $W^\pi(A) / D^\pi \cong W(A)$ if $\ker \pi \subset \mathfrak{h}$ (this is [18, Rem. 3.8], originally due to [17]). Furthermore, $W^{\mathrm{ext}}(A)$ and $W^{\mathrm{spin}}(A)$ are in fact subgroups of $K(A)$ and $\mathrm{Spin}(A)$, respectively, which can be seen by the following construction.

Definition 5.15 (cp. [11, Def. 18.3])**.** Let A be simply laced and $\mathcal{A}(A, \mathrm{Spin}(2))$ the associated standard spin-amalgam whose connecting monomorphisms we denote by $\tilde{\phi}^i_{ij} : \tilde{G}_i \to \tilde{G}_{ij}$, and similarly let $\mathcal{A}(A, \mathrm{SO}(2))$ be the associated standard $\mathrm{SO}(2)$-amalgam with connecting monomorphisms $\phi^i_{ij} : G_i \to G_{ij}$. We denote the enveloping homomorphisms by

$$\tilde{\psi}_{ij} : \tilde{G}_{ij} \to \mathrm{Spin}(A) \quad \text{and} \quad \psi_{ij} : G_{ij} \to K(A),$$

respectively. The 2π-periodic covering maps $S : \mathbb{R} \to \mathrm{Spin}(2)$ and $D : \mathbb{R} \to \mathrm{SO}(2)$ are given explicitly by $S(\alpha) = \cos\alpha + \sin\alpha e_1 e_2$, where $e_1, e_2 \in \mathrm{Cl}(\mathbb{R}^2)$, the Clifford algebra over $\mathbb{R}^2$, and $D(\alpha) = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$, respectively (cp. [11, 8.1]). We set, for $i < j$,

$$\hat{r}_i := \tilde{\psi}_{ij} \circ \tilde{\phi}^i_{ij}\Big(S\Big(\frac{\pi}{4}\Big)\Big), \quad \hat{W}(A) := \langle \hat{r}_i \mid i \in I\rangle < \mathrm{Spin}(A),$$
$$\tilde{s}_i := \psi_{ij} \circ \phi^i_{ij}\Big(D\Big(\frac{\pi}{2}\Big)\Big), \quad \tilde{W}(A) := \langle \tilde{s}_i \mid i \in I\rangle < K(A).$$

Then $\tilde{W}(A) \cong W^{\mathrm{ext}}(A)$ by [17, Cor. 2.4] (and a few steps explained in [11, Rem. 18.5] in more detail). Furthermore, the map $\hat{r}_i \mapsto r_i$ for $i = 1, \ldots, n$ defines an isomorphism from $\hat{W}(A)$ to $W^{\mathrm{spin}}(A)$ (cp. [11, Thm. 18.15]). Recall from [11, Thm. 11.17] (cited here as Theorem 5.8) that $\mathrm{Spin}(A)$ is a central extension of $K(A)$.

Lemma 5.16. *The adjoint action of* $\mathrm{Spin}(A)$ *on* $\mathfrak{k}$ *factors through the natural projection* $\varphi : \mathrm{Spin}(A) \to K(A)$ *and the induced action*

$$\mathrm{Ad}_g(x) := \mathrm{Ad}_{\varphi(g)}(x) \quad \textit{for all } g \in \mathrm{Spin}(A),\ x \in \mathfrak{g}(A)$$

on $\mathfrak{g}(A)$ *satisfies, for all* $i \in I$,

$$\mathrm{Ad}_{r_i}(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i.\alpha} \quad \textit{for all } \alpha \in \Delta. \tag{25}$$

Furthermore, given $w \in W(A)$, *there exists* $\hat{w} \in W^{\mathrm{spin}}(A)$ *such that* $\mathrm{Ad}_{\hat{w}}(\mathfrak{g}_\alpha) = \mathfrak{g}_{w.\alpha}$ *for all* $\alpha \in \Delta$.

Proof. By [11, 18.11], one has that $Z = \ker\varphi \subset W^{\mathrm{spin}}(A)$ and so the adjoint action of $W^{\mathrm{spin}}(A)$ on $\mathfrak{g}(A)$ factors through the projection of $W^{\mathrm{spin}}(A)$ to $W^{\mathrm{ext}}(A)$. But $W^{\mathrm{ext}}(A)$ acts precisely like (25) as has been noted in (24). By [17, Cor. 2.3 b)], there exists a unique map from $W(A)$ to $W^{\mathrm{ext}}(A)$ satisfying

$$e \mapsto e, \quad s_i \mapsto t_i, \quad \omega\omega' \mapsto \tilde{\omega}\tilde{\omega}' \text{ if } l(\omega\omega') = l(\omega) + l(\omega').$$

Thus, to any reduced word $w \in W(A)$, one finds a word $\tilde{w}$ in $W^{\mathrm{ext}}(A)$ and one just uses the analogous word in $W^{\mathrm{spin}}(A)$,

$$w = s_{i_1} \cdots s_{i_k} \mapsto \hat{w} = r_{i_1} \cdots r_{i_k} \in W^{\mathrm{spin}}(A).$$

This works because the action factors through φ and hence through the projection from $W^{\mathrm{spin}}(A)$ to $W^{\mathrm{ext}}(A)$. □

Lemma 5.17. *Let A be simply laced and indecomposable and suppose that $\rho : \mathfrak{k}(A) \to \mathrm{End}(V)$ is a finite-dimensional representation whose lift to $K(A)$ and/or $\mathrm{Spin}(A)$ we denote by Ω. The lift satisfies*

$$\rho(\mathrm{Ad}_g(x)) = \Omega(g)\rho(x)\Omega(g)^{-1} \quad \textit{for all } g \in \mathrm{Spin}(A) \textit{ and all } x \in \mathfrak{k}(A).$$

Proof. Our proof is essentially based on the formula

$$\exp(\rho(a))\rho(x)\exp(-\rho(a)) = \rho(\exp(\mathrm{ad}\, a)(x)) \tag{26}$$

with $a, x \in \mathfrak{k}(A)$ such that $\rho(a)$ and $\mathrm{ad}(a)$ are locally finite. The above formula is shown in [18, (3.8.1)] for all $a, x \in \mathfrak{k}(A)$ such that $\rho(a)$, $\mathrm{ad}(a)$, and $\rho(x)$ are locally nilpotent, but later, in [18, Sec. 3.8], it is also shown to be correct for $\rho(a)$ locally finite and such that the span of $\mathrm{ad}(a)^n(x)$ for $n \in \mathbb{N}$ is finite-dimensional. Thus, (26) is in particular applicable if a is ad-locally finite and (ρ, V) is a finite-dimensional $\mathfrak{k}(A)$-module. Now the Berman elements $x_\alpha \coloneqq e_\alpha - \omega(e_\alpha)$ for $\alpha \in \Delta_+^{\mathrm{re}}$ are ad-locally finite because the e_α are locally nilpotent as long as $\alpha \in \Delta_+^{\mathrm{re}}$ (because real root spaces are conjugate to simple root spaces by the extended Weyl group). Since subalgebras $\mathfrak{k}_J \coloneqq \langle X_j \mid j \in J \subset I\rangle$ corresponding to $J \subset \{1, \dots, n\}$ such that A_J is spherical consist only of real root spaces, this also means that all $x \in \mathfrak{k}_J$ are ad-locally finite. Now $\mathfrak{k}_J$ possesses an exponential map $\exp_J : \mathfrak{k}_J \to K_J$. Here, K_J denotes both the maximal compact subgroup $K_J < G_J$ and its spin cover, since this detail will not matter. Although we will not need it in this strength, note that $\exp_J$ is surjective because K_J is compact. We denote the restrictions of Ω to K_J and ρ to $\mathfrak{k}_J$ by Ω_J and ρ_J, respectively. Then the diagram in Figure 2 commutes, *i.e.*, one has for $a \in \mathfrak{k}_J$ and $g = \exp_J(a)$ that

$$\Omega(g) = \Omega_J(\exp_J(a)) = \exp(\rho_J(a)) = \exp(\rho(a)) \quad \text{for all } g = \exp_J a \in K_J.$$

Further, one has for all $a \in \mathfrak{k}_J, x \in \mathfrak{k}(A)$ that

$$\begin{aligned} \Omega_J(\exp_J(a))\rho(x)\Omega_J(\exp_J(a))^{-1} &= \exp(\rho(a))\rho(x)\exp(-\rho(a)) \\ &\overset{(26)}{=} \rho(\exp(\mathrm{ad}\, a)(x)) \\ &= \rho(\mathrm{Ad}_{\exp_J a}(x)) = \rho(\mathrm{Ad}_g(x)). \end{aligned}$$

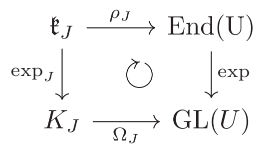

Figure 2. A commutative diagram for J spherical and U a finite-dimensional module.

The second-to-last equality uses that the involved Lie group and representation are finite-dimensional. Finally, one uses that $K(A)$ and $\mathrm{Spin}(A)$ are generated by their fundamental rank-1 subgroups (or rank-2 if one prefers). □

We are now in the position to determine how the one-parameter subgroups associated to simple roots act via conjugation on generalized Γ-matrices. As in Proposition 5.11, similar computations have been carried out in [23] in a second-quantized form.

Lemma 5.18. *Let A be simply laced and let (S, ρ) be a generalized spin representation of $\mathfrak{k}(A)$ as in Definition 3.1 with associated generalized Γ-matrix as in Proposition 3.9. With*

$$\hat{r}_i(\phi) := \exp(\phi \cdot \rho(X_i)),$$

one has for all $\alpha \in \Delta$ that

$$\hat{r}_i(\phi)\Gamma(\alpha)\hat{r}_i(\phi)^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}, \\ \cos\phi \cdot \Gamma(\alpha) + \sin\phi \cdot \Gamma(\alpha_i)\Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1, \end{cases}$$

$$r_i\Gamma(\alpha)r_i^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}, \\ \varepsilon(\alpha_i, \alpha)\Gamma(s_i.\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1, \end{cases}$$

where $\varepsilon : Q \times Q \to \{\pm 1\}$ denotes the standard normalized 2-cocycle from Lemma 3.6, and $r_i := \hat{r}_i(\frac{\pi}{2})$ are the images of the generators of $W^{\mathrm{spin}}(A)$.

Proof. Recall that $\rho(X_i) = \frac{1}{2}\Gamma(\alpha_i)$ so that

$$\hat{r}_i(2\phi) := \exp(2\phi \cdot \rho(X_i)) = \cos\phi \cdot \mathrm{Id} + \sin\phi\Gamma(\alpha_i).$$

Thus,

$$\begin{aligned} \hat{r}_i(2\phi)\Gamma(\alpha)\hat{r}_i(2\phi)^{-1} &= \exp(2\phi\rho(X_i))\Gamma(\alpha)\exp(-2\phi\rho(X_i)) \\ &= \cos^2\phi \cdot \Gamma(\alpha) - \sin^2\phi\Gamma(\alpha_i)\Gamma(\alpha)\Gamma(\alpha_i) \\ &\qquad + \sin\phi\cos\phi\big(\Gamma(\alpha_i)\Gamma(\alpha) - \Gamma(\alpha)\Gamma(\alpha_i)\big) \\ &= \cos^2\phi \cdot \Gamma(\alpha) - \sin^2\phi\Gamma(\alpha_i)^2\Gamma(\alpha) \\ &\qquad - \sin^2\phi\Gamma(\alpha_i)[\Gamma(\alpha), \Gamma(\alpha_i)] \\ &\qquad + \sin\phi\cos\phi[\Gamma(\alpha_i), \Gamma(\alpha)] \\ &= \Gamma(\alpha) - (\sin^2\phi\Gamma(\alpha_i) + \sin\phi\cos\phi)[\Gamma(\alpha), \Gamma(\alpha_i)]. \end{aligned}$$

By the use of (2a), one has

$$[\Gamma(\alpha),\Gamma(\alpha_i)]=\begin{cases}0 & \text{if } (\alpha|\alpha_i)\in 2\mathbb{Z},\\ -2\Gamma(\alpha_i)\Gamma(\alpha) & \text{if } (\alpha|\alpha_i)\in 2\mathbb{Z}+1,\end{cases}$$

which yields in combination with $2\Gamma(\alpha_i)\Gamma(\alpha_i)\Gamma(\alpha)=-2\Gamma(\alpha)$ and a few trigonometric identities that

$$\hat{r}_i(2\phi)\Gamma(\alpha)\hat{r}_i(2\phi)^{-1}=\begin{cases}\Gamma(\alpha) & \text{if } (\alpha|\alpha_i)\in 2\mathbb{Z},\\ \cos 2\phi\Gamma(\alpha)+\sin 2\phi\Gamma(\alpha_i)\Gamma(\alpha) & \text{if } (\alpha|\alpha_i)\in 2\mathbb{Z}+1.\end{cases}$$

It is possible to use (2c), (4) and $2Q\subset\ker\Gamma$ to write $\Gamma(\alpha_i)\Gamma(\alpha)=\varepsilon(\alpha_i,\alpha)\Gamma(s_i.\alpha)$ if $(\alpha|\alpha_i)\in 2\mathbb{Z}+1$. With $r_i:=\hat{r}_i(\frac{\pi}{2})$, one then obtains the claimed relations

$$r_i\Gamma(\alpha)r_i^{-1}=\begin{cases}\Gamma(\alpha) & \text{if } (\alpha|\alpha_i)\in 2\mathbb{Z},\\ \varepsilon(\alpha_i,\alpha)\Gamma(s_i.\alpha) & \text{if } (\alpha|\alpha_i)\in 2\mathbb{Z}+1.\end{cases}\qquad\square$$

We had shown in Proposition 3.12 that $\rho(x_\alpha)=c(x_\alpha)\Gamma(\alpha)$ for all $x_\alpha\in\mathfrak{k}_\alpha$. We are now in the position to make a more precise statement on $c(x_\alpha)\in\mathbb{R}$.

Proposition 5.19. *Let A be simply laced and indecomposable and let (ρ,S) be a generalized spin representation of $\mathfrak{k}(A)$ as in Definition 3.1. If $x\in\mathfrak{k}_\alpha$ is nonzero, then*

$$\rho(x)=c\cdot\Gamma(\alpha)\ \textit{such that}\ c\neq 0\ \textit{if}\ \alpha\in\Delta^{\mathrm{re}}(A),$$
$$\rho(x)=0\ \textit{if}\ \alpha\ \textit{is an isotropic root.}$$

Furthermore, to $\alpha,\beta\in\Delta^{\mathrm{re}}$ such that $\alpha-\beta\in 2Q(A)$, $0\neq x_\alpha\in\mathfrak{k}_\alpha$, $0\neq x_\beta\in\mathfrak{k}_\beta$, there exists $c\in\mathbb{K}\setminus\{0\}$ such that $\rho(x_\alpha)=c\cdot\rho(x_\beta)$. If x_α and x_β are both conjugate to $\pm X_i$ for some $i=1,\dots,n$, then $c\in\{-1,+1\}$.

Remark 5.20. The close connection between generalized spin representations and generalized Γ-matrices has already been observed in [23] as well as the fact that elements associated to isotropic root spaces are represented trivially. The approach presented here differs in so far as we treat the Γ-matrices as abstract objects and not elements of a Clifford algebra, but more importantly, that we derive the parametrization result by the action of the Weyl group instead of successive commutators, which provides more control over the factor c, in particular if it is nonzero or not.

Proof. Let $w\in W(A)$ and let $\tilde{\omega}\in W^{\mathrm{ext}}(A)$ and $\hat{\omega}\in W^{\mathrm{spin}}(A)$ be the corresponding elements from Lemma 5.16. Together with Lemma 5.17, one has, for $x_\alpha\in\mathfrak{k}_\alpha$,

$$\rho(\mathrm{Ad}_{\tilde{\omega}}(x_\alpha))=\rho(\mathrm{Ad}_{\hat{\omega}}(x_\alpha))=\Omega(\hat{\omega})\rho(x_\alpha)\Omega(\hat{\omega})^{-1}.$$

Now $\Delta^{\mathrm{re}}=W(A)\cdot\{\alpha_1,\dots,\alpha_n\}$ and A indecomposable and simply laced imply that $\mathfrak{k}_\alpha$ for $\alpha\in\Delta^{\mathrm{re}}$ is conjugate to $\mathfrak{k}_{\alpha_i}=\mathbb{R}X_i$ for any $i=1,\dots,n$. As $\rho(X_i)\neq 0$ for all $i=1,\dots,n$, this shows $\rho(x)\neq 0$ for all $0\neq x\in\mathfrak{k}_\alpha$ for all $\alpha\in\Delta^{\mathrm{re}}$.

On $\mathfrak{k}(A)$, the invariant bilinear form is negative definite (a consequence of [18, Thm. 11.7]) and invariant under the action of $\mathfrak{k}(A)$ and hence invariant under the action of $K(A)$. Since $W^{\mathrm{ext}}(A)<K(A)$, this shows that the norm

(defined as the negative of the invariant bilinear form) of $\mathrm{Ad}_{\tilde{\omega}}(x_\alpha)$) is equal to that of x_α. Therefore, $c = \pm 1$ if the norm of x_α is that of an X_i.

Now, by (4), one has for $\alpha, \beta \in \Delta^{\mathrm{re}}$ such that $2\gamma \coloneqq \alpha - \beta \in 2Q$ that

$$\Gamma(\alpha) = \Gamma(\beta + 2\gamma) = (-1)^{(\gamma|\gamma)}\Gamma(\beta) = \Gamma(\beta),$$

and as both $\rho(x_\alpha)$ and $\rho(x_\beta)$ are nonzero, they must be proportional.

Any isotropic root (roots $\alpha \in \Delta$ that satisfy $(\alpha|\alpha) = 0$) is $W(A)$-conjugate to the multiple of an affine null root, *i.e.*, a root whose support is an affine subdiagram of A (cp. [18, Prop. 5.7]). Given an affine null root δ, one has that $\mathfrak{k}_{m\delta}$ is spanned by all $[x_{\alpha_i}, x_{m\delta-\alpha_i}]$ with $i \in \operatorname{supp}\delta$. Since $(\alpha_i|\delta) = 0$ for all $i \in \operatorname{supp}\delta$, one has $(\alpha_i|m\delta - \alpha_i) = -2$ so that $[\Gamma(\alpha_i), \Gamma(m\delta - \alpha_i)] = 0$ shows $\mathfrak{k}_{m\delta} \subset \ker\rho$. One concludes with Lemma 5.16 that

$$\rho(\mathrm{Ad}_{\tilde{\omega}}(x_{m\delta})) = \Omega(\hat{\omega})\rho(x_{m\delta})\Omega(\hat{\omega})^{-1} = 0. \qquad \square$$

Proposition 5.21. *Let A be simply laced and indecomposable and let*

$$\sigma : \mathfrak{k}(A) \to \mathrm{End}(V) \otimes \mathrm{End}(S)$$

denote a higher spin representation from Theorem 3.18 or 3.22. Let $\alpha \in \Delta_+^{\mathrm{re}}$ and $0 \neq x_\alpha \in \mathfrak{k}_\alpha$; then there exists $c(x_\alpha) \neq 0$ such that

$$\sigma(x_\alpha) = c(x_\alpha) \cdot \tau(\alpha) \otimes \Gamma(\alpha).$$

If x_α has the same norm as X_j for $j = 1, \ldots, n$, then $c(x_\alpha) \in \{-1, +1\}$.

Remark 5.22. It was observed in [23] that the formula for the Berman generators' representation matrices can easily be used for any real root as well. It was shown further that if $\alpha = \beta + \gamma$ all real such that $(\beta|\gamma) = -1$ and the formula holds for $x \in \mathfrak{k}_\beta$ and $y \in \mathfrak{k}_\gamma$, then it holds for $[x, y] \in \mathfrak{k}_{\beta+\gamma} \oplus \mathfrak{k}_{\beta-\gamma}$. The formula therefore extends to all real roots that can be written as a successive sum of real roots with product equal to -1. However, it remains unclear if any real root can be decomposed this way in a simply laced root system. In a discussion, the first author of [23] expressed the idea of bypassing this problem by Weyl group conjugation, as any real root is $W(A)$-conjugate to a simple root. This proposition is the realization of this idea with a solid link to the mathematical literature and involved objects. In particular, we work in the required setting of spin covers and spin-extended Weyl groups developed in [11].

Proof. If σ is a representation as in Theorem 3.18, denote the lift to $\mathrm{Spin}(A)$ by Σ, and if it is a representation as in Theorem 3.22, denote the lift by $\tilde{\Sigma}$ so that the formulas of Propositions 5.11 and 5.12 apply verbatim. Also, denote the lifts to the fundamental rank-1 subgroups by Σ_i and $\tilde{\Sigma}_i$, respectively. By Propositions 5.11 and 5.12, one has (note that the formula for general argument ϕ differs)

$$\Sigma_i\Big(\pm\frac{\pi}{2}\Big) = \frac{1}{\sqrt{2}} \cdot \eta(s_i) \otimes (\mathrm{Id} \pm \Gamma(\alpha_i)),$$

$$\tilde{\Sigma}_i\Big(\pm\frac{\pi}{2}\Big) = \frac{1}{\sqrt{2}} \cdot \eta(s_i) \otimes (\mathrm{Id} \pm \Gamma(\alpha_i)).$$

The $\Sigma_i(\frac{\pi}{2})$ and $\tilde{\Sigma}_i(\frac{\pi}{2})$ generate the image of $W^{\text{spin}}(A)$ under Σ and $\tilde{\Sigma}$, respectively. Similar to the previous proof, we obtain the representation matrices by conjugation:

$$\sigma(\mathrm{Ad}_{\tilde{\omega}}(x_\alpha)) = \sigma(\mathrm{Ad}_{\hat{\omega}}(x_\alpha)) = \Sigma(\hat{\omega})\sigma(x_\alpha)\Sigma(\hat{\omega})^{-1},$$

where $\tilde{\omega} \in W^{\text{ext}}(A)$ is the projection of $\hat{\omega} \in W^{\text{spin}}(A)$. For both types of representations, one computes

$$\begin{aligned}
&\Sigma_i\Big(\frac{\pi}{2}\Big)\sigma(X_j)\Sigma_i\Big(-\frac{\pi}{2}\Big) \\
&\qquad = \frac{1}{2}\eta(s_i) \otimes (\mathrm{Id} + \Gamma(\alpha_i)) \cdot \tau(\alpha_j) \otimes \Gamma(\alpha_j) \cdot \eta(s_i) \otimes (\mathrm{Id} - \Gamma(\alpha_i)) \\
&\qquad = \frac{1}{2}(\eta(s_i)\tau(\alpha_j)\eta(s_i)) \otimes (\mathrm{Id} + \Gamma(\alpha_i))\Gamma(\alpha_j)(\mathrm{Id} - \Gamma(\alpha_i))
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{Id} + \Gamma(\alpha_i))\Gamma(\alpha_j)(\mathrm{Id} - \Gamma(\alpha_i)) \\
&\qquad = \Gamma(\alpha_j) + \Gamma(\alpha_i)\Gamma(\alpha_j) - \Gamma(\alpha_j)\Gamma(\alpha_i) - \Gamma(\alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_i) \\
&\qquad = \begin{cases} 2\Gamma(\alpha_j) & \text{if } (\alpha_i|\alpha_j) = 0 \\ 2\ \underbrace{\Gamma(\alpha_i)\Gamma(\alpha_j)}_{=\varepsilon(\alpha_i,\alpha_j)\Gamma(\alpha_i+\alpha_j)} & \text{if } (\alpha_i|\alpha_j) = -1. \end{cases}
\end{aligned}$$

For $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$, one has $\tau(\alpha_j) = \eta(s_j) - \frac{1}{2}\mathrm{Id}$ and computes further

$$\begin{aligned}
\eta(s_i)\tau(\alpha_j)\eta(s_i) &= \eta(s_i)\Big(\eta(s_j) - \frac{1}{2}\mathrm{Id}\Big)\eta(s_i) = \eta(s_is_js_i) - \frac{1}{2}\mathrm{Id} \\
&= \eta(s_{s_i.\alpha_j}) - \frac{1}{2}\mathrm{Id} = \tau(s_i.\alpha_j).
\end{aligned}$$

For $\mathcal{S}_{\frac{7}{2}}$, one has

$$\tau(\alpha_j) = \eta(s_j) - \frac{1}{2}\mathrm{Id} + f(\alpha_j),$$

and from Lemma 3.21, one derives that $\eta(s_i)f(\alpha_j)\eta(s_i) = f(s_i.\alpha_j)$ and therefore

$$\begin{aligned}
\eta(s_i)\tau(\alpha_j)\eta(s_i) &= \eta(s_i)\Big[\eta(s_j) - \frac{1}{2}\mathrm{Id} + f(\alpha_j)\Big]\eta(s_i) \\
&= \eta(s_{s_i.\alpha_j}) - \frac{1}{2}\mathrm{Id} + f(s_i.\alpha_j) = \tau(s_i.\alpha_j).
\end{aligned}$$

This yields for both cases (from now on, we denote the lift simply by Σ) that

$$\begin{aligned}
&\Sigma_i\Big(\frac{\pi}{2}\Big)\sigma(X_j)\Sigma_i\Big(-\frac{\pi}{2}\Big) \\
&\qquad = \begin{cases} \tau(\alpha_j) \otimes \Gamma(\alpha_j) & \text{if } (\alpha_i|\alpha_j) = 0, \\ \varepsilon(\alpha_i,\alpha_j)\tau(\alpha_i+\alpha_j) \otimes \Gamma(\alpha_i+\alpha_j) & \text{if } (\alpha_i|\alpha_j) = -1. \end{cases}
\end{aligned}$$

Spelled out differently, this yields

$$\Sigma_i\Big(\frac{\pi}{2}\Big)\sigma(X_j)\Sigma_i\Big(-\frac{\pi}{2}\Big) = \varepsilon(\alpha_i,\alpha_j) \cdot \tau(s_i.\alpha_j) \otimes \Gamma(s_i.\alpha_j),$$

since $\varepsilon(\alpha_i, \alpha_j) = 1$ if $(\alpha_i|\alpha_j) = 0$. In contrast to the formula before, this formula is also correct if α_j is replaced by an arbitrary real root $\beta \in \Delta_+^{\mathrm{re}}$: while the Γ-matrix only counts modulo $2Q$, one has $\tau(s_i.\beta) \neq \tau(\beta)$ if $(\alpha_i|\beta) \in 2\mathbb{Z}$. Therefore, it is important to include all actions of $W(A)$. By repeated action of $\Sigma_i(\frac{\pi}{2})$ for $i = 1, \ldots, n$, this yields, for a reduced expression $\hat{\omega} = r_{i_n} \cdots r_{i_1}$,

$$\Sigma(\hat{\omega})\sigma(X_j)\Sigma(\hat{\omega})^{-1} = c \cdot \tau(\omega.\alpha_j) \otimes \Gamma(\omega.\alpha_j), \tag{27a}$$

$$c = \prod_{k=1}^{n} \varepsilon(\alpha_{i_k}, s_{i_{k-1}} \cdots s_{i_1}.\alpha_j), \tag{27b}$$

where $\omega \in W$ is the projection of $\hat{\omega}$.

As before, $\mathfrak{k}_\alpha$ for $\alpha \in \Delta_+^{\mathrm{re}}$ is conjugate to $\mathfrak{k}_{\alpha_i} = \mathbb{R}X_i$ via $W^{\mathrm{ext}}(A)$ and hence also by $W^{\mathrm{spin}}(A)$. Thus, all nonzero $x_\alpha \in \mathfrak{k}_\alpha$ have a nontrivial image, and if the norm of x_α is equal to that of X_i, the constant in (27a) is ± 1. □

Remark 5.23. Note that (27b) suggests that the sign were fully under control. This is only the case if one fixes the basis of $\mathfrak{k}_\alpha$ to be $\mathrm{Ad}_{\tilde{\omega}}(X_j)$ for the appropriate $\tilde{\omega} \in W^{\mathrm{ext}}(A)$. In general, it is a hard problem to decide upon the sign if the basis of a real root space is already fixed (cp. [28, Ex. 4.25]). Thus, the difficulty of the sign is actually not due to the spin representation but due to the subtleties of the adjoint action of $W^{\mathrm{ext}}(A)$.

Acknowledgments. We gratefully acknowledge the support of the DFG under the grant KO 4323/13 and the Studienstiftung des Deutschen Volkes for partially funding the research presented in this paper. We would like to thank Axel Kleinschmidt and Hermann Nicolai for discussions and Paul Zellhofer for his comments on an earlier version of this manuscript. Last, we would like to thank an anonymous reviewer for helpful comments and suggestions that helped improve the quality of our manuscript.

References

[1] P. Abramenko and B. M. Mühlherr, Présentations de certaines BN-paires jumelées comme sommes amalgamées, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 7, 701–706. MR1483702

[2] S. Berman, On generators and relations for certain involutory subalgebras of Kac–Moody Lie algebras, Comm. Algebra **17** (1989), no. 12, 3165–3185. MR1030614

[3] N. Bourbaki, *Elements of mathematics 7: Lie groups and Lie algebras Pt I*, Hermann, Paris (1st edition), and Addison-Wesley, Reading MA, 1975.

[4] S. de Buyl, M. Henneaux, and L. Paulot, Extended E_8 invariance of 11-dimensional supergravity, J. High Energy Phys. **2006** (2006), no. 2, 056, 11 pp. MR2219449

[5] S. de Buyl, M. Henneaux, and L. Paulot, Hidden symmetries and Dirac fermions, Classical Quantum Gravity **22** (2005), no. 17, 3595–3622. MR2168546

[6] T. Damour, A. Kleinschmidt, and H. Nicolai, Hidden symmetries and the fermionic sector of eleven-dimensional supergravity, Phys. Lett. B **634** (2006), no. 2–3, 319–324. MR2202945

[7] T. Damour, A. Kleinschmidt, and H. Nicolai, $K(E_{10})$, supergravity and fermions, J. High Energy Phys. **2006** (2006), no. 8, 046, 40 pp. MR2249924

[8] W. Fulton and J. D. Harris, *Representation theory*, Grad. Texts in Math. 129, Springer, New York, 1991. MR1153249

[9] O. Gabber and V. G. Kac, On defining relations of certain infinite-dimensional Lie algebras, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 2, 185–189. MR0621889

[10] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Math. Soc. Monogr. (N.S.) 21, The Clarendon Press, Oxford University Press, New York, 2000. MR1778802

[11] D. Ghatei, M. Horn, R. Köhl, and S. Weiss, Spin covers of maximal compact subgroups of Kac–Moody groups and spin-extended Weyl groups, J. Group Theory **20** (2017), no. 3, 401–504. MR3641685

[12] H. Glöckner, R. Köhl, and T. Hartnick, Final group topologies, Kac–Moody groups and Pontryagin duality, Israel J. Math. **177** (2010), 49–101. MR2684413

[13] G. Hainke, R. Köhl, and P. D. Levy, Generalized spin representations, Münster J. Math. **8** (2015), no. 1, 181–210. MR3549525

[14] T. Hartnick, R. Köhl, and A. Mars, On topological twin buildings and topological split Kac–Moody groups, Innov. Incidence Geom. **13** (2013), 1–71. MR3173010

[15] A. A. Ivanov and S. V. Shpectorov, *Geometry of sporadic groups. II*, Encyclopedia Math. Appl. 91, Cambridge Univ. Press, Cambridge, 2002. MR1894594

[16] V. G. Kac and D. H. Peterson, Regular functions on certain infinite-dimensional groups, in *Arithmetic and geometry, Vol. II*, 141–166, Progr. Math. 36, Birkhäuser Boston, Boston, MA, 1983. MR0717610

[17] V. G. Kac and D. H. Peterson, Defining relations of certain infinite-dimensional groups, Astérisque **1985** (1985), Numéro Hors Série, 165–208. MR0837201

[18] V. G. Kac, *Infinite-dimensional Lie algebras*, third edition, Cambridge Univ. Press, Cambridge, 1990. MR1104219

[19] V. G. Kac, *Vertex algebras for beginners*, second edition, Univ. Lect. Ser. 10, Amer. Math. Soc., Providence, RI, 1998. MR1651389

[20] A. Keurentjes, E_{11}: sign of the times, Nuclear Phys. B **697** (2004), no. 1–2, 302–318. MR2091162

[21] A. Kleinschmidt, H. Nicolai, and A. Viganò, On spinorial representations of involutory subalgebras of Kac–Moody algebras, in *Partition functions and automorphic forms*, 179–215, Mosc. Lect. 5, Springer, Cham, 2020. MR4359195

[22] A. Kleinschmidt, R. Köhl, R. Lautenbacher, and H. Nicolai, Representations of involutory subalgebras of affine Kac–Moody algebras, Comm. Math. Phys. **392** (2022), no. 1, 89–123. MR4410058

[23] A. Kleinschmidt and H. Nicolai, On higher spin realizations of $K(E_{10})$, J. High Energy Phys. **2013** (2013), no. 8, 041, front matter+27 pp. MR3106397

[24] A. Kleinschmidt and H. Nicolai, Higher spin representations of $K(E_{10})$, in *Higher spin gauge theories*, 25–38, World Scientific, London, 2017.

[25] A. Kleinschmidt and H. Nicolai, Generalised holonomies and $K(E_9)$, J. High Energy Phys. **2021** (2021), no. 9, Paper No. 054, 22 pp. MR4327262

[26] R. Lautenbacher, *Higher spin representations in Kac–Moody theory*, PhD thesis, Justus-Liebig-Universität Gießen (2022).

[27] R. Lautenbacher and R. Köhl, Extending generalized spin representations, J. Lie Theory **28** (2018), no. 4, 915–940. MR3818651

[28] T. Marquis, *An introduction to Kac–Moody groups over fields*, EMS Textb. Math., Eur. Math. Soc., Zürich, 2018. MR3838421

[29] P. Slodowy, *Singularitäten, Kac–Moody-Liealgebren, assoziierte Gruppen und Verallgemeinerungen*, Habiliationsschrift, Univ. Bonn (1984).

[30] J. Tits, Uniqueness and presentation of Kac–Moody groups over fields, J. Algebra **105** (1987), no. 2, 542–573. MR0873684

[31] P. C. West, E_{11}, SL(32) and central charges, Phys. Lett. B **575** (2003), no. 3–4, 333–342; MR2044710

Received September 16, 2024; accepted February 12, 2025

Robin Lautenbacher
Christian-Albrechts-Universität zu Kiel, Mathematisches Seminar
Heinrich-Hecht-Platz 6, 24118 Kiel, Germany
E-mail: `lautenbacher@math.uni-kiel.de`

Ralf Köhl
Christian-Albrechts-Universität zu Kiel, Mathematisches Seminar
Heinrich-Hecht-Platz 6, 24118 Kiel, Germany
E-mail: `koehl@math.uni-kiel.de`