Münster J. of Math. **18** (2025), 1–26
DOI 10.17879/11958513543
urn:nbn:de:hbz:6-11958534157

Münster Journal of Mathematics

Universal properties of variations of the little cubes operads

Kensuke Arakawa

(Communicated by Michael Weiss)

Abstract. Given a map $B \to B\,\mathrm{Top}(n)$ of spaces, one can define a version $\mathbb{E}_B$ of the little cubes operad, whose construction is due to Lurie. We show that $\mathbb{E}_B$ enjoys the universal property that, for every ∞-operad $\mathcal{O}$, an operad map $\mathbb{E}_B \to \mathcal{O}$ is equivalent to a $\mathrm{Top}(n)$-equivariant map $B \times_{B\,\mathrm{Top}(n)} E\,\mathrm{Top}(n) \to \mathrm{Map}(\mathbb{E}_n, \mathcal{O})$. This gives us an explicit diagram exhibiting $\mathbb{E}_B$ as a colimit of $\mathbb{E}_n$ parametrized by B. It also shows that locally constant factorization algebras satisfy descent, reproving a recent theorem of Matsuoka.

1. Introduction

The operad of little n-cubes governs homotopy coherent multiplications by using rectilinear embeddings of cubes. They were first introduced in the works of Boardman–Vogt [4] and May [21] to study algebraic structures of iterated loop spaces, and since then, they have repeatedly appeared in many contexts, from mathematical physics to embedding calculus. (See [8] for a survey.)

As contexts shifted, various modifications of little n-cubes operads emerged. The framed little n-disks operad $f\mathcal{D}_n$ (see [9, 22]), in which we replace cubes with disks and allow rotations of disks, is one such example. To unify these variations, Markl and Wahl independently arrived at the notion of *semi-direct products* of operads [19, 22]: given a topological operad $\mathcal{O}$ equipped with a (left) action of a topological monoid M, the semi-direct product $M \ltimes \mathcal{O}$ is defined by setting $(M \ltimes \mathcal{O})(k) = M^k \times \mathcal{O}(k)$, with obvious structure maps. For example, the special orthogonal group $\mathrm{SO}(n)$ acts on the operad $\mathcal{D}_n$ of little n-cubes by moving the centers of the small disks (with their radii fixed) by using the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, and the resulting operad $\mathrm{SO}(n) \ltimes \mathcal{D}_n$ is nothing but the framed little n-disks operad. If we want to be more restrictive on the class of embeddings, we can just choose a subgroup $H \subset \mathrm{SO}(n)$ (or more generally, a group homomorphism $H \to \mathrm{SO}(n)$) and form the semi-direct product $H \ltimes \mathcal{D}_n$.

This work was partially supported by JSPS KAKENHI Grant Number 24KJ1443.

In [15], Lurie introduced an ∞-operadic analog of these variations, which we now recall.

Remark 1.1. The rest of this paper relies heavily on the theory of ∞-categories and ∞-operads. However, for the most part, one can get the idea of this section just by replacing these gadgets with topological categories and colored topological operads and by replacing various ∞-categorical constructions (limits and colimits, Kan extensions, sheaves, etc.) by the classical derived constructions (homotopy limits and homotopy colimits, homotopy Kan extensions, homotopy sheaves, etc.). Many ∞-operads have $\otimes$ (and sometimes $\amalg$) in their exponents, but this is just a notational convention. Remark 1.7 might also help to understand the main result.

Construction 1.2. Given a map $B \to B\,\mathrm{SO}(n)$ of spaces, we define an ∞-operad $\mathbb{E}_B^{\otimes}$ to be the pullback of the diagram

$$N(f\mathcal{D}_n)^{\otimes} \to B\,\mathrm{SO}(n)^{\amalg} \leftarrow B^{\amalg}$$

in the ∞-category $\mathcal{O}\mathsf{p}_\infty$ of ∞-operads, where

- $N(-)^{\otimes}$ denotes the operadic nerve functor, which converts a simplicial operad into an ∞-operad;
- $B^{\amalg}$ denotes the colimit of the constant diagram $\mathrm{Comm}^{\otimes} : B \to \mathcal{O}\mathsf{p}_\infty$ at the commutative ∞-operad.[1] In the case where $B = B\,\mathrm{SO}(n)$, a direct computation shows that $N(\mathrm{SO}(n) \ltimes \mathrm{Comm})^{\otimes} = B\,\mathrm{SO}(n)^{\amalg}$, and the left-hand map is induced by the map $f\mathcal{D}_n = \mathrm{SO}(n) \ltimes \mathcal{D}_n \to \mathrm{SO}(n) \ltimes \mathrm{Comm}$.

In fact, Lurie's definition is more general, as he defines the ∞-operad $\mathbb{E}_B^{\otimes}$ for any map $B \to B\,\mathrm{Top}(n)$ of spaces. Instead of spelling out the details here, we refer the readers to Section 2.1.

For example, given a homomorphism $H \to \mathrm{SO}(n)$ of topological groups, the ∞-operad $\mathbb{E}_{BH}^{\otimes}$ is equivalent to $N(H \ltimes \mathcal{D}_n)^{\otimes}$.

Notice that the above construction differs from the classical one in one crucial respect: Lurie's definition starts with the framed n-disks operad, instead of the action of $\mathrm{SO}(n)$ on the little n-disks operads. The reason for this is that the operadic nerve functor $N(-)^{\otimes}$ is not enriched (at least not in an obvious way), so it is difficult to construct a diagram $B\,\mathrm{SO}(n) \to \mathcal{O}\mathsf{p}_\infty$ encoding the action of $\mathrm{SO}(n)$ on $N(\mathcal{D}_n)^{\otimes}$.

Now the classical notion of semi-direct products of groups is a special case of the Grothendieck construction, which in turn is a special case of colimits. This suggests, as Lurie himself remarked informally in [15, Rem. 3.1.10], that the ∞-operad $\mathbb{E}_B^{\otimes}$ is the colimit of some diagram $B \to \mathcal{O}\mathsf{p}_\infty$ exhibiting an action of B on $\mathbb{E}_n^{\otimes}$. This is indeed true, although formal proofs had not appeared until quite recently (see, *e.g.*, [11, Prop. 2.2] and [7, Rem. 2.2]).

However, for the exact same reason as we had to start with the semi-direct product and not from the action, it seems difficult to pinpoint exactly *which*

[1] This description of $B^{\amalg}$ is not immediate from Lurie's definition, but we will see that it is correct (Lemma 2.21).

diagram's colimit $\mathbb{E}_B^\otimes$ is. Because of this, the description of $\mathbb{E}_B$-algebras has always been opaque, camouflaged with phrases like "$\mathbb{E}_n$-algebras with coherent actions of B." In this paper, we improve this situation by constructing an explicit diagram $B \to \mathcal{O}\mathsf{p}_\infty$ whose colimit is $\mathbb{E}_B^\otimes$, thereby making the universal property of $\mathbb{E}_B^\otimes$ lucid. We will achieve this by showing that the functor $\mathbb{E}_\bullet^\otimes$ (not just each $\mathbb{E}_B^\otimes$) enjoys a certain universal property.

As a motivation, consider the universal case where $B = B\,\mathrm{Top}(n)$. Our task is to find an ∞-operad which is equivalent to $\mathbb{E}_n^\otimes$ and admits a $\mathrm{Top}(n)$-action. This might be difficult if we are stuck with the picture of little disks, but there is in fact a canonical one: consider the classifying map $\mathbb{R}^n \to B\,\mathrm{Top}(n)$ of the tangent microbundle. Since the classifying maps of tangent microbundles are compatible with embeddings, the object $\mathbb{R}^n \in \mathcal{S}_{/B\,\mathrm{Top}(n)}$ has a $\mathrm{Top}(n)$-action. Since the construction of $\mathbb{E}_\bullet^\otimes$ is functorial, the ∞-operad $\mathbb{E}_{\mathbb{R}^n}^\otimes$ inherits a $\mathrm{Top}(n)$-action. On the other hand, since $\mathbb{R}^n$ is contractible, the ∞-operad $\mathbb{E}_{\mathbb{R}^n}^\otimes$ is equivalent to $\mathbb{E}_{\mathrm{pt}}^\otimes \simeq \mathbb{E}_n^\otimes$. Thus we obtain an ∞-operad equivalent to $\mathbb{E}_n^\otimes$, equipped with a $\mathrm{Top}(n)$-action. We will confirm that this is the action we were after.

Proposition 1.3. *For any map $B \to B\,\mathrm{Top}(n)$ of spaces, the ∞-operad $\mathbb{E}_B^\otimes$ is the colimit of the composite*

$$B \longrightarrow B\,\mathrm{Top}(n) \xrightarrow{\mathbb{R}^n} \mathcal{S}_{/B\,\mathrm{Top}(n)} \xrightarrow{\mathbb{E}_\bullet^\otimes} \mathcal{O}\mathsf{p}_\infty.$$

We can interpret Proposition 1.3 as describing the *local* universal property of $\mathbb{E}_B^\otimes$, in the sense that B is fixed. We will deduce this local property from the following *global* universal property of $\mathbb{E}_\bullet^\otimes$, which is the main result of this paper.

Theorem 1.4 (Theorem 2.20)**.** *The diagram*

$$\begin{array}{ccc} \mathrm{Fun}(B\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) & \xrightarrow{-\otimes_{\mathrm{Top}(n)}\mathbb{E}_{\mathbb{R}^n}^\otimes} & \mathcal{O}\mathsf{p}_\infty \\ & {\scriptstyle\simeq}\searrow \quad \nearrow{\scriptstyle\mathbb{E}_\bullet^\otimes} & \\ & \mathcal{S}_{/B\mathrm{Top}(n)} & \end{array}$$

commutes up to natural equivalence, where the left slanted arrow is the straightening-unstraightening equivalence and $-\otimes_{\mathrm{Top}(n)} \mathbb{E}_{\mathbb{R}^n}^\otimes$ is the left Kan extension of the functor $\mathbb{E}_{\mathbb{R}^n}^\otimes : B\,\mathrm{Top}(n) \to \mathcal{O}\mathsf{p}_\infty$ along the Yoneda embedding.

In other words, we have that, for any right $\mathrm{Top}(n)$-space X with classifying map $B \to B\,\mathrm{Top}(n)$, there is a natural equivalence of ∞-operads

$$X \otimes_{\mathrm{Top}(n)} \mathbb{E}_{\mathbb{R}^n}^\otimes \simeq \mathbb{E}_B^\otimes.$$

To explain why Theorem 1.4 implies Proposition 1.3, let X be a right $\mathrm{Top}(n)$-space with classifying map $B \to B\,\mathrm{Top}(n)$. By the colimit formula for Kan extensions, we have

$$X \otimes_{B\,\mathrm{Top}(n)} \mathbb{E}_{\mathbb{R}^n}^\otimes = \mathrm{colim}\big(B\,\mathrm{Top}(n)_{/X} \longrightarrow B\,\mathrm{Top}(n) \xrightarrow{\mathbb{E}_{\mathbb{R}^n}^\otimes} \mathcal{O}\mathsf{p}_\infty\big).$$

The Yoneda lemma implies that the right fibration $B\operatorname{Top}(n)_{/X} \to B\operatorname{Top}(n)$ classifies the right $\operatorname{Top}(n)$-space X, so B is equivalent to $B\operatorname{Top}(n)_{/X}$ as a space over $B\operatorname{Top}(n)$. Also, Theorem 1.4 says that $X \otimes_{B\operatorname{Top}(n)} \mathbb{E}^{\otimes}_{\mathbb{R}^n}$ is equivalent to $\mathbb{E}^{\otimes}_{B}$. Hence $\mathbb{E}^{\otimes}_{B}$ is the colimit of the diagram

$$B \longrightarrow B\operatorname{Top}(n) \xrightarrow{\mathbb{E}^{\otimes}_{\mathbb{R}^n}} \mathcal{O}\mathsf{p}_\infty,$$

as claimed.

Remark 1.5. The above computation also explains why Theorem 1.4 ought to be true. Indeed, if $f : B\operatorname{Top}(n) \to \mathcal{O}\mathsf{p}_\infty$ is a diagram whose colimit is $\mathbb{E}^{\otimes}_{B\operatorname{Top}(n)}$, then $\mathbb{E}^{\otimes}_{B}$ should be the colimit of the composite

$$B \to B\operatorname{Top}(n) \xrightarrow{f} \mathcal{O}\mathsf{p}_\infty.$$

The above computation then shows that the composite

$$\operatorname{Fun}(B\operatorname{Top}(n)^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}_{/B\operatorname{Top}(n)} \xrightarrow{\mathbb{E}^{\otimes}_{\bullet}} \mathcal{O}\mathsf{p}_\infty$$

should be the left Kan extension of its restriction along the Yoneda embedding $y : B\operatorname{Top}(n) \to \operatorname{Fun}(B\operatorname{Top}(n)^{\mathrm{op}}, \mathcal{S})$. It is not hard to see that the composite

$$B\operatorname{Top}(n) \xrightarrow{y} \operatorname{Fun}(B\operatorname{Top}(n)^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}_{/B\operatorname{Top}(n)} \xrightarrow{\mathbb{E}^{\otimes}_{\bullet}} \mathcal{O}\mathsf{p}_\infty$$

is nothing but the functor $\mathbb{E}^{\otimes}_{\mathbb{R}^n} : B\operatorname{Top}(n) \to \mathcal{O}\mathsf{p}_\infty$. Thus we arrive at the statement of Theorem 1.4.

Remark 1.6. The action of $\operatorname{Top}(n)$ on $\mathbb{E}^{\otimes}_{\mathbb{R}^n}$ admits a nice geometric picture. Given an n-manifold M, regarded as a space over $B\operatorname{Top}(n)$ by the classifying map of its tangent microbundle, the ∞-operad $\mathbb{E}^{\otimes}_{M}$ can informally be described as follows.[2]

(i) The colors of $\mathbb{E}^{\otimes}_{M}$ are the embeddings $\mathbb{R}^n \to M$.
(ii) Given a collection $\iota_1, \dots, \iota_k,\ \iota : \mathbb{R}^n \to M$ of embeddings, a multiarrow $(\iota_1, \dots, \iota_k) \to \iota$ consists of an embedding $f : \coprod_{i=1}^{k} \mathbb{R}^n \to \mathbb{R}^n$ and a collection of isotopies $\{\iota_i \simeq \iota f\}_{1 \le i \le k}$.

In other words, the ∞-operad $\mathbb{E}^{\otimes}_{M}$ is something like the little n-cubes operad, but in which the little n-cubes are now embedded into the "ambient space" M. Every embedding $M \to N$ of n-manifolds induces a map $\mathbb{E}^{\otimes}_{M} \to \mathbb{E}^{\otimes}_{N}$ which embeds the little cubes into the larger ambient space N. The action of $\operatorname{Top}(n)$ on $\mathbb{E}^{\otimes}_{\mathbb{R}^n}$ arises when we take $M = N = \mathbb{R}^n$.

Remark 1.7. Although this is just a paraphrase of Theorem 2.20, the following reformulation of the universal property of $\mathbb{E}^{\otimes}_{B}$ is worth pointing out: for every ∞-operad $\mathcal{O}^{\otimes}$, a map $\mathbb{E}^{\otimes}_{B} \to \mathcal{O}^{\otimes}$ of ∞-operads is equivalent to a $\operatorname{Top}(n)$-equivariant map $B \times_{B\operatorname{Top}(n)} E\operatorname{Top}(n) \to \operatorname{Map}_{\mathcal{O}\mathsf{p}_\infty}(\mathbb{E}^{\otimes}_{\mathbb{R}_n}, \mathcal{O}^{\otimes})$. More precisely,

[2] Recall that, despite the name, ∞-operads are a generalization of colored operads.

there is a homotopy equivalence

$$\begin{aligned}&\mathrm{Map}_{\mathcal{O}\mathsf{p}_\infty}(\mathbb{E}_B^\otimes, \mathcal{O}^\otimes)\\ &\qquad \simeq \mathrm{Map}_{\mathrm{Fun}(B\,\mathrm{Top}(n)^{\mathrm{op}},\mathcal{S})}\big(B \times_{B\,\mathrm{Top}(n)} E\,\mathrm{Top}(n), \mathrm{Map}_{\mathcal{O}\mathsf{p}_\infty}(\mathbb{E}_{\mathbb{R}^n}^\otimes, \mathcal{O}^\otimes)\big)\end{aligned}$$

which is natural in $\mathcal{O}^\otimes \in \mathcal{O}\mathsf{p}_\infty$ and $B \in \mathcal{S}_{/B\,\mathrm{Top}(n)}$.

For instance, if M is an n-manifold, then a map $\mathbb{E}_M^\otimes \to \mathcal{O}^\otimes$ is equivalent to a $\mathrm{Top}(n)$-equivariant map from the topological frame bundle of M (*i.e.*, the principal $\mathrm{Top}(n)$-bundle associated with the tangent microbundle) to $\mathrm{Map}_{\mathcal{O}\mathsf{p}_\infty}(\mathbb{E}_{\mathbb{R}^n}^\otimes, \mathcal{O}^\otimes)$.

Related works. Algebras over the ∞-operad $\mathbb{E}_M^\otimes$, where M is an n-manifold, are called *locally constant factorization algebras*[3] on M and appear in classical and quantum field theory [5, 6]. Theorem 1.4 has an important antecedent from the theory of locally constant factorization algebras, as we now explain.

A basic problem with anything associated with manifolds is whether it obeys the local-to-global principle. Matsuoka showed that locally constant factorization algebras have this property.

Theorem 1.8 (Matsuoka [20, Thm. 1.3])**.** *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and let M be an n-manifold. The assignment $U \mapsto \mathrm{Alg}_{\mathbb{E}_U}(\mathcal{C})$ determines a sheaf of ∞-categories on M, where $\mathrm{Alg}_{\mathbb{E}_U}(\mathcal{C})$ denotes the ∞-category of $\mathbb{E}_U$-algebras in $\mathcal{C}$.*

Matsuoka's theorem is a special case of Theorem 1.4. To see this, recall that if $\mathcal{C}$ is a small ∞-category and $\mathcal{D}$ is an ∞-category with small colimits, then a functor $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ is a left Kan extension of its restriction along the Yoneda embedding if and only if it preserves small colimits. In particular, Theorem 1.4 implies that the functor $\mathbb{E}_\bullet^\otimes : \mathcal{S}_{/B\,\mathrm{Top}(n)} \to \mathcal{O}\mathsf{p}_\infty$ preserves small colimits. Since the tangent classifier functor $\tau : \mathrm{Open}(M) \to \mathcal{S}_{/B\,\mathrm{Top}(n)}$ is evidently a cosheaf,[4] this means that the composite

$$\mathrm{Open}(M) \xrightarrow{\tau} \mathcal{S}_{/B\,\mathrm{Top}(n)} \xrightarrow{\mathbb{E}_\bullet^\otimes} \mathcal{O}\mathsf{p}_\infty$$

is also a cosheaf. Matsuoka's theorem then follows from the observation that the functor $\mathrm{Alg}_\bullet(\mathcal{C}) : \mathcal{O}\mathsf{p}_\infty^{\mathrm{op}} \to \mathcal{C}\mathsf{at}_\infty$ preserves small limits. In this sense, Theorem 1.4 generalizes Matsuoka's theorem by allowing gluing of arbitrary spaces over $B\,\mathrm{Top}(n)$, not just manifolds.[5]

We should also remark that our technique is quite robust and applies equally well to general topological operads with actions of group-like topological monoids. We will work with the little n-cubes operads for concreteness, but readers interested in a version of Theorem 1.4 for other operads should have

[3] More precisely, $\mathbb{E}_M$-algebras are equivalent to locally constant factorization algebras; see [17, Thm. 5.4.5.9].

[4] For a formal proof, see the proof of Corollary 2.23.

[5] Another direction of generalization of Matsuoka's theorem is investigated in [14], where they consider the gluing properties of constructible factorization algebras on stratified manifolds.

no problem translating the arguments of this paper (see Section 2.19) to meet their needs.

Finally, algebras over $\mathbb{E}_B^{\otimes}$ play a central role in the theory of *topological chiral homology* [17, §5.5], alias *factorization homology* [1] (where they go under the name of $\mathcal{D}\mathsf{isk}_n^B$-algebras). We hope that this paper adds to the conceptual understanding of $\mathbb{E}_B^{\otimes}$-algebras.

Outline of the paper. In Section 2, we prove Theorem 1.4, assuming a lemma on some construction of ∞-operads. The lemma will then be proved in Section 3.

Notation and conventions.

- Following [16], we use the term ∞-category as a synonym for Joyal's quasi-category [12]. Unless stated otherwise, our notation and terminology follow Lurie's books [16, 17].
- If $\mathbf{A}$ is a simplicial model category, we let $\mathbf{A}^\circ \subset \mathbf{A}$ denote the full simplicial subcategory spanned by the fibrant-cofibrant objects.
- The simplicial category of simplicial sets, equipped with the Kan–Quillen model structure, will be denoted by sSet.
- The category of marked simplicial sets will be denoted by sSet^+.
- We will write Fin_* for the category whose objects are the finite pointed sets $\langle n\rangle = (\{*, 1, \dots, n\}, *)$ and whose morphisms are the maps of pointed sets.
- We let $\mathcal{O}\mathsf{p}_\infty^\Delta$ denote the simplicial category of ∞-operads, defined as in [17, Def. 2.1.4.1], and write $\mathcal{O}\mathsf{p}_\infty$ for its homotopy coherent nerve.
- If M and N are topological manifolds (without boundaries), we will write $\mathrm{Emb}(M, N)$ for the space of topological embeddings $M \to N$, topologized by the compact-open topology. We let $\mathrm{Top}(n) \subset \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ denote the subspace of self-homeomorphisms of $\mathbb{R}^n$.
- We will write $\mathcal{M}\mathsf{fld}_n^\Delta$ for the simplicial category whose objects are the topological n-manifolds (without boundary), and whose hom-simplicial sets are given by $\mathrm{Sing}\,\mathrm{Emb}(-,-)$. The homotopy coherent nerve of $\mathcal{M}\mathsf{fld}_n^\Delta$ will be denoted by $\mathcal{M}\mathsf{fld}_n$.
- We will write $B\,\mathrm{Top}(n)^\Delta \subset \mathcal{M}\mathsf{fld}_n^\Delta$ for the full simplicial subcategory spanned by $\mathbb{R}^n$, and let $B\,\mathrm{Top}(n)$ denote its homotopy coherent nerve. (This notation is justified by Kister–Mazur's theorem [13, Thm. 1], which says that the inclusion $\mathrm{Top}(n) \hookrightarrow \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ is a homotopy equivalence. We also remark that their theorem implies that $B\,\mathrm{Top}(n)$ is a Kan complex.)
- If K is a simplicial set, we will denote the cone point of the simplicial set $K^\triangleright$ by ∞.
- If $\mathcal{C}$ is an ∞-category, we will write $\mathcal{C}^\simeq \subset \mathcal{C}$ for the maximal sub-Kan complex of $\mathcal{C}$.

2. Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. We start by giving precise definitions of the functor $\mathbb{E}_\bullet^{\otimes}$ and related constructions in Section 2.1. We then prove some preliminary results in Section 2.13. After this, we prove the theorem in Section 2.19, assuming a lemma that we will prove in Section 3 (Lemma 2.21). In Section 2.22, we use the theorem to give an alternative proof of Matsuoka's gluing theorem on locally constant factorization algebras.

2.1. **Definitions.** In this subsection, we recall the definitions of the functor $\mathbb{E}_\bullet^\otimes : \mathcal{S}_{/B\,\mathrm{Top}(n)} \to \mathcal{O}\mathsf{p}_\infty$ and some related constructions.

Definition 2.2 ([17, Constr. 2.4.3.1])**.** We define a simplicial functor $(-)^{\amalg}$: $\mathsf{sSet}^\circ \to \mathcal{O}\mathsf{p}_\infty^\Delta$ as follows: Let $\iota : \mathsf{Fin}_* \to \mathsf{Set}$ denote the forgetful functor and let $\int \iota$ denote its category of elements. We let $\Gamma^* \subset \int \iota$ denote the full subcategory spanned by the objects $(\langle k\rangle, i)$ such that $i \in \langle k\rangle \setminus \{*\}$. We then define $(-)^{\amalg} : \mathsf{sSet} \to \mathsf{sSet}_{/N(\mathsf{Fin}_*)}$ as the right adjoint of the simplicial functor $X \mapsto X \times_{N(\mathsf{Fin}_*)} N(\Gamma^*)$. According to [17, Prop. 2.4.3.3], this simplicial functor lifts to a simplicial functor $(-)^{\amalg} : \mathsf{sSet}^\circ \to \mathcal{O}\mathsf{p}_\infty^\Delta$.

Remark 2.3. Let $\mathcal{C}$ be a simplicial category. The simplicial set $N(\mathcal{C})^{\amalg}$ is isomorphic to the homotopy coherent nerve of the simplicial category $\mathcal{C}^{\amalg}$ whose objects are the (possibly empty) sequences $(C_1, \ldots, C_k)$ of objects of $\mathcal{C}$, and whose hom-simplicial sets are given by

$$\mathcal{C}^{\amalg}\big((C_1, \ldots, C_k), (D_1, \ldots, D_l)\big) = \coprod_{\alpha:\langle k\rangle \to \langle l\rangle} \prod_{j=1}^{l} \prod_{i \in \alpha^{-1}(j)} \mathcal{C}(C_i, D_j),$$

where the coproduct ranges over the morphisms $\langle k\rangle \to \langle l\rangle$ in Fin_*.

Remark 2.4. Using [17, Rem. B.3.9], we can check that the functor $N(\Gamma^*) \to N(\mathsf{Fin}_*)$ is a flat inner fibration. Therefore, the functor

$$- \times_{N(\mathsf{Fin}_*)} N(\Gamma^*) : \mathsf{sSet}_{/N(\mathsf{Fin}_*)} \to \mathsf{sSet}$$

is left Quillen with respect to the Joyal model structures [17, Cor. B.3.15]. It follows that every Kan fibration $X \to Y$ of Kan complexes induces a fibration of ∞-operads $X^{\amalg} \to Y^{\amalg}$.

Definition 2.5 ([17, Def. 5.4.2.10])**.** We define a simplicial functor

$$\mathbb{E}_\bullet^\otimes : (\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ \to \mathcal{O}\mathsf{p}_\infty^\Delta$$

as follows. Let $\mathbb{E}_{B\,\mathrm{Top}(n)}^\otimes$ denote the operadic nerve of the topological operad whose space of operations of arity k is $\mathrm{Emb}(\mathbb{R}^n \times \{1, \ldots, k\}, \mathbb{R}^n)$. The restriction map

$$\mathrm{Emb}(\mathbb{R}^n \times \{1, \ldots, k\}, \mathbb{R}^n) \to \prod_{i=1}^{k} \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$$

determines a functor $\mathbb{E}_{B\,\mathrm{Top}(n)}^\otimes \to B\,\mathrm{Top}(n)^{\amalg}$. Given a Kan fibration $B \to B\,\mathrm{Top}(n)$ of Kan complexes, we set

$$\mathbb{E}_B^\otimes = \mathbb{E}_{B\,\mathrm{Top}(n)}^\otimes \times_{B\,\mathrm{Top}(n)^{\amalg}} B^{\amalg}.$$

Note that $\mathbb{E}_B^\otimes$ is an ∞-operad by Remark 2.4.

Definition 2.6 ([1, §2.1])**.** We define the *tangent classifier functor*

$$\tau : \mathcal{M}\mathsf{fld}_n^\Delta \to (\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ$$

to be the composite simplicial functor

$$\begin{aligned}\mathcal{M}\mathsf{fld}_n^\Delta &\xrightarrow{\text{yoneda}} \mathrm{Fun}^s((\mathcal{M}\mathsf{fld}_n^\Delta)^{\mathrm{op}}, \mathsf{sSet}^\circ)\\ &\xrightarrow{\text{restriction}} \mathrm{Fun}^s((B\,\mathrm{Top}(n)^\Delta)^{\mathrm{op}}, \mathsf{sSet}^\circ)\\ &\xrightarrow{\mathrm{Un}_\varepsilon} (\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ,\end{aligned}$$

where $\mathrm{Fun}^s(-,-)$ denotes the simplicial category of simplicial functors, and Un_ε denotes the unstraightening functor [16, §2.2.1] with respect to the counit map $\varepsilon : \mathfrak{C}[B\,\mathrm{Top}(n)] \to B\,\mathrm{Top}(n)^\Delta$.

Remark 2.7. The name "tangent classifier" comes from the observation that if M is an n-manifold, then $\tau(M)$ can be regarded as a model of the classifying map of the tangent microbundle of M. (In particular, the total space of $\tau(M)$ has the homotopy type of $\mathrm{Sing}\,M$.) More precisely, let $\xi : TM \to M$ denote the fiber bundle with fiber $\mathbb{R}^n$ associated to the tangent microbundle of M, and let $\pi : \mathrm{Fr}(M) \to M$ denote the associated $\mathrm{Top}_0(n)$-bundle, where $\mathrm{Top}_0(n) \subset \mathrm{Top}(n)$ denotes the subgroup of homeomorphisms that fixes the origin. (Thus, for each point $p \in M$, the space $T_pM = \xi^{-1}(p)$ is an open subset of $\{p\} \times M$, and $\mathrm{Fr}_p(M) = \pi^{-1}(p)$ is the subspace $\mathrm{Homeo}_0(\mathbb{R}^n, T_pM) \subset \mathrm{Emb}(\mathbb{R}^n, T_pM)$ of homeomorphisms $\mathbb{R}^n \to T_pM$ carrying the origin to $(p,p) \in T_pM$.) We will see in the next paragraph that the map $\theta : \mathrm{Fr}(M) \to \mathrm{Emb}(\mathbb{R}^n, M)$ is a weak homotopy equivalence. Since $\tau(M)$ is the classifying map of $\mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n, M) \in \mathrm{Fun}(B\,\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S})$ by definition, and since the inclusion $\mathrm{Top}_0(n) \hookrightarrow \mathrm{Top}(n)$ is a homotopy equivalence, this proves that $\tau(M)$ can be identified with the classifying map of the tangent microbundle of M.

To see that θ is a weak homotopy equivalence, observe that $\mathrm{Fr}(M)$ and $\mathrm{Emb}(\mathbb{R}^n, M)$ project to M via Serre fibrations. (For $\mathrm{Fr}(M)$, this is clear because π is a fiber bundle. For $\mathrm{Emb}(\mathbb{R}^n, M)$, this is proved in [2, Prop. 2.7].) It will therefore suffice to show that, for each $p \in M$, the inclusion

$$\theta_p : \mathrm{Fr}_p(M) = \mathrm{Homeo}_0(\mathbb{R}^n, T_pM) \hookrightarrow \mathrm{Emb}(\mathbb{R}^n, M) \times_M \{p\}$$

is a weak homotopy equivalence. We can factor θ_p as

$$\mathrm{Homeo}_0(\mathbb{R}^n, T_pM) \xrightarrow{\phi} \mathrm{Emb}(\mathbb{R}^n, T_pM) \times_{T_pM} \{(p,p)\} \xrightarrow{\psi} \mathrm{Emb}(\mathbb{R}^n, M) \times_M \{p\}.$$

The map ϕ is a homotopy equivalence by Kister–Mazur's theorem [13, Thm. 1]. The map ψ is a weak homotopy equivalence by [2, Prop. 2.9], and we are done.

Definition 2.8. We define a simplicial functor $\mathbb{E}_\bullet^\otimes : \mathcal{M}\mathsf{fld}_n^\Delta \to \mathcal{O}\mathsf{p}_\infty^\Delta$ to be the composite

$$\mathcal{M}\mathsf{fld}_n^\Delta \xrightarrow{\tau} (\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ \xrightarrow{\mathbb{E}_\bullet^\otimes} \mathcal{O}\mathsf{p}_\infty^\Delta.$$

The restriction $\mathbb{E}_\bullet^\otimes | B\,\mathrm{Top}(n)^\Delta$ will be denoted by $\mathbb{E}_{\mathbb{R}^n}^\otimes : B\,\mathrm{Top}(n)^\Delta \to \mathcal{O}\mathsf{p}_\infty^\Delta$.

Remark 2.9. By direct computation, we can check that if M is an n-manifold, then the object $\tau(M) \in \mathsf{sSet}_{/B\,\mathrm{Top}(n)}$ is (isomorphic to) the projection

$$B\,\mathrm{Top}(n)_{/M} = B\,\mathrm{Top}(n) \times_{\mathcal{M}\mathsf{fld}_n} \mathcal{M}\mathsf{fld}_{n/M} \to B\,\mathrm{Top}(n).$$

(Readers unfamiliar with the explicit description of the unstraightening functor should consult [3, §4].) In particular, our definition of the ∞-operad $\mathbb{E}_M^{\otimes} = \mathbb{E}_{\tau(M)}^{\otimes}$ coincides with Lurie's definition [17, Def. 5.4.5.1].

Definition 2.10. We will write $(-)^{\amalg} : \mathcal{S} \to \mathcal{O}\mathsf{p}_\infty$ for the homotopy coherent nerve of the simplicial functor $(-)^{\amalg} : \mathsf{sSet}^\circ \to \mathcal{O}\mathsf{p}_\infty^\Delta$. We define functors $\mathbb{E}_\bullet^\otimes : N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ) \to \mathcal{O}\mathsf{p}_\infty$, $\tau : \mathcal{M}\mathsf{fld}_n \to N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ)$, and $\mathbb{E}_\bullet^\otimes : \mathcal{M}\mathsf{fld}_n \to \mathcal{O}\mathsf{p}_\infty$ similarly.

By slightly abusing notation, we will use the symbol $\mathbb{E}_\bullet^\otimes$ for any functor $F : \mathcal{S}_{/B\,\mathrm{Top}(n)} \to \mathcal{O}\mathsf{p}_\infty$ rendering the diagram

$$\begin{array}{ccc} N((\mathsf{sSet}_{/B\mathrm{Top}(n)})^\circ) & \xrightarrow{N(\mathbb{E}_\bullet^\otimes)} & \mathcal{O}\mathsf{p}_\infty \\ & \searrow^{\simeq} \quad \nearrow_{F} & \\ & \mathcal{S}_{/B\mathrm{Top}(n)} & \end{array}$$

commutative up to natural equivalence, where the slanted arrow on the left is the categorical equivalence of [18, Tag 01ZT].

Definition 2.11. Let $\mathcal{C}^\Delta$ be a small simplicial category whose hom-simplicial sets are Kan complexes, and let $\mathcal{C} = N(\mathcal{C}^\Delta)$ be its homotopy coherent nerve. By the *unstraightening functor* (or the *unstraightening equivalence*), we mean any functor $\mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to N((\mathsf{sSet}_{/\mathcal{C}})^\circ_{\mathrm{contra}})$ rendering the diagram

$$\begin{array}{ccc} N(\mathrm{Fun}^s(\mathcal{C}_\Delta^{\mathrm{op}}, \mathsf{sSet})^\circ) & \underset{\simeq}{\xrightarrow{N(\mathrm{Un}_\varepsilon)}} & N((\mathsf{sSet}_{/\mathcal{C}})^\circ_{\mathrm{contra}}) \\ \downarrow{\scriptstyle\simeq} & \nearrow & \\ \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) & & \end{array}$$

commutative, where $(\mathsf{sSet}_{/\mathcal{C}})_{\mathrm{contra}}$ denotes the contravariant model structure [16, §2.1.4] and Un_ε denotes the unstraightening functor [16, §2.2.1] with respect to the counit map $\varepsilon : \mathfrak{C}[\mathcal{C}] \to \mathcal{C}^\Delta$, and the left vertical arrow is the categorical equivalence of [16, Prop. 4.2.4.4].

Suppose now that $\mathcal{C}$ is a Kan complex. Then the contravariant model structure on $\mathsf{sSet}_{/\mathcal{C}}$ coincides with the Kan–Quillen model structure, so there is a categorical equivalence $N((\mathsf{sSet}_{/\mathcal{C}})^\circ_{\mathrm{contra}}) \xrightarrow{\simeq} \mathcal{S}_{/\mathcal{C}}$ (see [18, Tag 01ZT]). We will refer to the composite

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \longrightarrow N((\mathsf{sSet}_{/\mathcal{C}})^\circ_{\mathrm{contra}}) \xrightarrow{\simeq} \mathcal{S}_{/\mathcal{C}}$$

also as the *unstraightening functor*.

Remark 2.12. The functor $\tau : \mathcal{M}\mathsf{fld}_n \to N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ)$ is naturally equivalent to the composite

$$\mathcal{M}\mathsf{fld}_n \xrightarrow{y} \mathrm{Fun}(\mathcal{M}\mathsf{fld}_n^{\mathrm{op}}, \mathcal{S}) \xrightarrow{i^*} \mathrm{Fun}(B\,\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\simeq} N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ),$$

where y denotes the Yoneda embedding, the second functor is the restriction along the inclusion $i : B\,\mathrm{Top}(n) \hookrightarrow \mathcal{M}\mathsf{fld}_n$, and the last equivalence is the unstraightening equivalence. Indeed, replacing $\mathcal{M}\mathsf{fld}_n$ by its full subcategory spanned by the manifolds embedded in some Euclidean space, we may assume that $\mathcal{M}\mathsf{fld}_n$ is small. The naturality of unstraightening [10, App. A] implies that the diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{M}\mathsf{fld}_n^{\mathrm{op}}, \mathcal{S}) & \xrightarrow{\simeq} & N((\mathsf{sSet}_{/\mathcal{M}\mathsf{fld}_n})^\circ_{\mathrm{contra}}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(B\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) & \xrightarrow[\simeq]{} & N((\mathsf{sSet}_{/B\mathrm{Top}(n)})^\circ) \end{array}$$

commutes up to natural equivalence, where the horizontal arrows are the unstraightening equivalences. Thus it suffices to show that τ is naturally equivalent to the composite

$$\mathcal{M}\mathsf{fld}_n \xrightarrow{y} \mathrm{Fun}(\mathcal{M}\mathsf{fld}_n^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\simeq} N((\mathsf{sSet}_{/\mathcal{M}\mathsf{fld}_n})^\circ_{\mathrm{contra}}) \longrightarrow N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ),$$

which follows from the definitions.

2.13. **Universal colimit diagrams.** Let $\mathcal{C}$ be an ordinary category with pullbacks and small colimits. In ordinary category theory, we say that *colimits in $\mathcal{C}$ are universal* if, for each morphism $f : X \to Y$ in $\mathcal{C}$, the pullback functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ preserves small colimits. In some cases, colimits in $\mathcal{C}$ may not be universal, but some particular colimit diagram in $\mathcal{C}_{/Y}$ is preserved by the pullback functor f^*. The goal of this subsection is to record some basic facts on such colimit diagrams in the setting of ∞-categories.

Definition 2.14. Let $\mathcal{C}$ be an ∞-category with pullbacks, let K be a simplicial set, and let $p : K^\triangleright \to \mathcal{C}$ be a diagram. We say that p is a *universal colimit diagram* if, for each cartesian natural transformation $\alpha : K^\triangleright \times \Delta^1 \to \mathcal{C}$ such that $\alpha | K^\triangleright \times \{1\} = p$, the diagram $\alpha | K^\triangleright \times \{0\}$ is a colimit diagram.

Definition 2.15. Let $\mathcal{C}$ be an ∞-category with pullbacks. Given a morphism $f : X \to Y$ in $\mathcal{C}$, we let $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ denote the right adjoint of the composite

$$\mathcal{C}_{/X} \underset{\phi}{\xrightarrow{\simeq}} \mathcal{C}_{/f} \longrightarrow \mathcal{C}_{/Y},$$

where the functor ϕ is any section of the trivial fibration $\mathcal{C}_{/f} \xrightarrow{\simeq} \mathcal{C}_{/X}$.

Remark 2.16 ([17, Lem. 6.1.3.3])**.** Let $\mathcal{C}$ be an ∞-category with pullbacks, and let $p : K^\triangleright \to \mathcal{C}$ be a diagram. The following conditions are equivalent.

(i) The diagram p is a universal colimit diagram.

(ii) For every morphism $f : X \to Y$ in $\mathcal{C}$ and for every diagram $p' : K^{\triangleright} \to \mathcal{C}_{/Y}$ lifting p, the composite

$$K^{\triangleright} \xrightarrow{p'} \mathcal{C}_{/Y} \xrightarrow{f^*} \mathcal{C}_{/X}$$

is a colimit diagram.

Definition 2.17. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory. Suppose that $\mathcal{D}$ has pullbacks. We say that f is a *universal left Kan extension* of $f|\mathcal{C}'$ if, for each object $X \in \mathcal{C}$, the diagram

$$(\mathcal{C}'_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{f} \mathcal{D}$$

is a universal colimit diagram.

Proposition 2.18. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory. If f is a universal left Kan extension of $f|\mathcal{C}'$, then for every object $X \in \mathcal{C}$, the composite*

$$f_X : \mathcal{C}_{/X} \to \mathcal{C} \xrightarrow{f} \mathcal{D}$$

is a universal left Kan extension of $f_X|\mathcal{C}'_{/X}$.

Proof. It suffices to show that, for each object $p : C \to X$ in $\mathcal{C}_{/X}$, the functor

$$\mathcal{C}'_{/X} \times_{\mathcal{C}_{/X}} \mathcal{C}_{/p} \to \mathcal{C}' \times_{\mathcal{C}} \mathcal{C}_{/C}$$

is final. But this is a trivial fibration, being a pullback of the trivial fibration $\mathcal{C}_{/p} \to \mathcal{C}_{/C}$. □

2.19. **Main result.** We now prove the main theorem of this paper (Theorem 1.4). Let us recall the statement of the theorem once again.

Theorem 2.20. *The diagram*

$$\begin{array}{ccc} \mathrm{Fun}(B\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) & \xrightarrow{-\otimes_{\mathrm{Top}(n)} \mathbb{E}^{\otimes}_{\mathbb{R}^n}} & \mathcal{O}\mathsf{p}_{\infty} \\ & {\scriptstyle \simeq} \searrow \qquad \nearrow {\scriptstyle \mathbb{E}^{\otimes}_{\bullet}} & \\ & \mathcal{S}_{/B\mathrm{Top}(n)} & \end{array}$$

of ∞-categories commutes up to natural equivalences, where the left slanted arrow is the unstraightening functor (Definition 2.11).

The proof relies on the following lemma, which we will prove in Section 3.

Lemma 2.21. *Let $\mathcal{D} \subset \mathcal{S}$ denote the full subcategory spanned by the contractible Kan complexes. The functor*

$$(-)^{\amalg} : \mathcal{S} \to \mathcal{O}\mathsf{p}_{\infty}$$

is a universal left Kan extension of $(-)^{\amalg}|\mathcal{D}$.

Proof of Theorem 2.20, assuming Lemma 2.21. By construction, the diagram of ∞-categories

$$\begin{array}{ccccc} B\mathrm{Top}(n) & & & \xrightarrow{\mathbb{E}^{\otimes}_{\mathbb{R}^n}} & \\ \downarrow{\scriptstyle y} & & & & \\ \mathrm{Fun}(B\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) & \xrightarrow[\simeq]{} & \mathcal{S}_{/B\mathrm{Top}(n)} & \xrightarrow[\mathbb{E}^{\otimes}_{\bullet}]{} & \mathcal{O}\mathsf{p}_\infty \end{array}$$

commutes up to natural equivalence, where y denotes the Yoneda embedding. Therefore, it will suffice to show that the functor $\mathbb{E}^{\otimes}_{\bullet} : \mathcal{S}_{/B\,\mathrm{Top}(n)} \to \mathcal{O}\mathsf{p}_\infty$ is a left Kan extension of its restriction to $B\,\mathrm{Top}(n)$.

Let $\mathcal{D} \subset \mathcal{S}$ denote the full subcategory spanned by the contractible Kan complexes. Then the essential image of the composite

$$B\,\mathrm{Top}(n) \xrightarrow{y} \mathrm{Fun}(B\,\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\simeq} \mathcal{S}_{/B\,\mathrm{Top}(n)}$$

is equal to $\mathcal{D}_{/B\,\mathrm{Top}(n)}$. Since the Yoneda embedding is fully faithful, this implies that the functor $B\,\mathrm{Top}(n) \to \mathcal{S}_{/B\,\mathrm{Top}(n)}$ restricts to a categorical equivalence $B\,\mathrm{Top}(n) \xrightarrow{\simeq} \mathcal{D}_{/B\,\mathrm{Top}(n)}$. Therefore, we are reduced to showing that the functor $\mathbb{E}^{\otimes}_{\bullet} : \mathcal{S}_{/B\,\mathrm{Top}(n)} \to \mathcal{O}\mathsf{p}_\infty$ is a left Kan extension of $\mathbb{E}^{\otimes}_{\bullet}|\mathcal{D}_{/B\,\mathrm{Top}(n)}$. Let

$$\iota : \mathbb{E}^{\otimes}_{B\,\mathrm{Top}(n)} \to B\,\mathrm{Top}(n)^{\amalg}$$

denote the inclusion. By Remark 2.4, the functor $\mathbb{E}^{\otimes}_{\bullet}$ is naturally equivalent to the composite

$$\mathcal{S}_{/B\,\mathrm{Top}(n)} \xrightarrow{(-)^{\amalg}} (\mathcal{O}\mathsf{p}_\infty)_{/B\,\mathrm{Top}(n)^{\amalg}} \xrightarrow{\iota^*} (\mathcal{O}\mathsf{p}_\infty)_{/\mathbb{E}^{\otimes}_{B\,\mathrm{Top}(n)}} \xrightarrow{U} \mathcal{O}\mathsf{p}_\infty,$$

where U denotes the forgetful functor. According to Proposition 2.18 and Lemma 2.21, the composite

$$\mathcal{S}_{/B\,\mathrm{Top}(n)} \xrightarrow{(-)^{\amalg}} (\mathcal{O}\mathsf{p}_\infty)_{/B\,\mathrm{Top}(n)} \longrightarrow \mathcal{O}\mathsf{p}_\infty$$

is a universal left Kan extension of $\mathcal{D}_{/B\,\mathrm{Top}(n)}$, where $\mathcal{D} \subset \mathcal{S}$ denotes the full subcategory spanned by the contractible Kan complexes. It follows that the functor

$$\iota^* \circ (-)^{\amalg} : \mathcal{S}_{/B\,\mathrm{Top}(n)} \to (\mathcal{O}\mathsf{p}_\infty)_{/\mathbb{E}^{\otimes}_{B\,\mathrm{Top}(n)}}$$

is a left Kan extension of its restriction to $\mathcal{D}_{/B\,\mathrm{Top}(n)}$. Since U preserves small colimits, we are done. □

2.22. **Matsuoka's gluing theorem.** As an application of Theorem 1.4, we give an alternative proof of Matsuoka's gluing theorem on locally constant factorization algebras. We say that a functor $F : \mathcal{M}\mathsf{fld}_n \to \mathcal{C}$ of ∞-categories is a *cosheaf* if, for each n-manifold M and each open cover $\mathcal{U}$ of M which is downward-closed (*i.e.*, if $U \in \mathcal{U}$, then every open set of U belongs to $\mathcal{U}$), the map $\mathrm{colim}_{U \in N(\mathcal{U})} FU \to FM$ is an equivalence.[6]

---

[6]Cosheaves on $\mathcal{M}\mathsf{fld}_n$ with values in ∞-categories with small colimits admit various characterizations, such as being left Kan extended from their restriction to $B\,\mathrm{Top}(n)$, or being (1-)excisive and exhaustive. For a proof, see [2, Thm. 5.3].

Corollary 2.23 ([20]). *The functor*

$$\mathbb{E}_\bullet^\otimes : \mathcal{M}\mathsf{fld}_n \to \mathcal{O}\mathsf{p}_\infty$$

(of Definition 2.8) is a cosheaf.

Proof. By Theorem 1.4 and [16, Thm. 5.1.5.6], the functor

$$\mathbb{E}_\bullet^\otimes : N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ) \to \mathcal{O}\mathsf{p}_\infty$$

preserves small colimits. Therefore, it suffices to show that the tangent classifier functor $\tau : \mathcal{M}\mathsf{fld}_n \to N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ)$ is a cosheaf. As observed in Remark 2.12, the composite

$$\mathcal{M}\mathsf{fld}_n \xrightarrow{\tau} N((\mathsf{sSet}_{/B\,\mathrm{Top}(n)})^\circ) \xrightarrow{\simeq} \mathcal{S}_{/B\,\mathrm{Top}(n)}$$

is naturally equivalent to the composite

$$\mathcal{M}\mathsf{fld}_n \xrightarrow{y} \mathrm{Fun}(\mathcal{M}\mathsf{fld}_n^{\mathrm{op}}, \mathcal{S}) \xrightarrow{i^*} \mathrm{Fun}(B\,\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}_{/B\,\mathrm{Top}(n)},$$

where y denotes the Yoneda embedding, the second functor is the restriction along the inclusion $i : B\,\mathrm{Top}(n) \hookrightarrow \mathcal{M}\mathsf{fld}_n$, and the last equivalence is the unstraightening equivalence. It will therefore suffice to show that the composite $i^* \circ y$ is a cosheaf (Remark 2.12). Since colimits in functor categories can be computed pointwise [18, Tag 02XK], it suffices to show that the composite

$$\mathcal{M}\mathsf{fld}_n \xrightarrow{\ \ y\ \ } \mathrm{Fun}(\mathcal{M}\mathsf{fld}_n^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\ \ i^*\ \ } \mathrm{Fun}(B\,\mathrm{Top}(n)^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{ev}_{\mathbb{R}^n}} \mathcal{S}$$

is a cosheaf. In other words, we are reduced to showing that the functor

$$\mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n, -) : \mathcal{M}\mathsf{fld}_n \to \mathcal{S}$$

is a cosheaf.

Let M be an n-manifold and let $\mathcal{U}$ be an open cover of M which is downward-closed. We wish to show that the map

$$\mathrm{colim}_{U \in N(\mathcal{U})} \mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n, U) \to \mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n, M)$$

is an equivalence. According to [2, Prop. 2.19], for each $U \in \mathcal{U}$, the evaluation at the origin determines a pullback square

$$\begin{array}{ccc} \mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n, U) & \longrightarrow & \mathrm{Sing}\,\mathrm{Emb}(\mathbb{R}^n, M) \\ \downarrow & & \downarrow \\ \mathrm{Sing}\,U & \longrightarrow & \mathrm{Sing}\,M \end{array}$$

in $\mathcal{S}$. Since colimits in $\mathcal{S}$ are universal [16, Lem. 6.1.3.14], we are reduced to showing that the map

$$\mathrm{colim}_{U \in N(\mathcal{U})} \mathrm{Sing}\,U \to \mathrm{Sing}\,M$$

is an equivalence. This follows from [17, Thm. A.3.1]. □

3. Proof of Lemma 2.21

The goal of this section is to prove Lemma 2.21, which asserts that the functor $(-)^{\mathrm{II}} : \mathcal{S} \to \mathcal{O}\mathsf{p}_\infty$ is a universal left Kan extension of its restriction to the full subcategory of contractible Kan complexes.

Warning 3.1. (Casual readers may safely ignore this warning.) There are two conflicting conventions of the homotopy coherent nerve functor.
(i) The homotopy coherent nerve functor N_{I}, defined as in [16].
(ii) The homotopy coherent nerve functor N_{II}, defined as follows: if $\mathcal{C}$ is a simplicial category, we set $N_{\mathrm{II}}(\mathcal{C}) = N_{\mathrm{I}}(\mathcal{C}^c)$, where $\mathcal{C}^c$ denotes the simplicial category obtained from $\mathcal{C}$ by replacing each hom-simplicial sets by its opposites. (This is the convention adopted in [18].)

As we remarked in Section 1, this paper generally follows [16, 17] in its terminology and notation. This means that, so far, we have adopted convention (i). However, in [3], which we will frequently refer to below, the author used convention (ii) (as it seemed more natural to do so[7]). Because of this, we will henceforth switch to convention (ii). Thus, from now on, the ∞-categories such as $\mathcal{S}$, $\mathcal{C}\mathsf{at}_\infty$, $\mathcal{O}\mathsf{p}_\infty$ will be defined by applying the functor N_{II} to the simplicial categories sSet°, $(\mathsf{sSet}^+)^\circ$, and $\mathcal{O}\mathsf{p}_\infty^\Delta$. Note that this is allowed as far as Lemma 2.21 is concerned, for the validity of the lemma does not depend on the choice of the convention.

3.2. **Recollections.** In this subsection, we review some results on categorical patterns [17, App. B] that are proved in [3].

Definition 3.3 ([17, Def. B.0.19])**.** Let S be a simplicial set. A *categorical pattern* $\mathfrak{P} = (M_S, T, \{p_\alpha\}_{\alpha\in A})$ on S consists of a set M_S of edges of S containing all degenerate edges, a set T of 2-simplices of S containing all degenerate 2-simplices, and a set $\{p_\alpha : K_\alpha^\triangleleft \to S\}_{\alpha\in A}$ of diagrams of S such that p_α carries all edges and 2-simplices of $K_\alpha^\triangleleft$ into M_S and T, respectively.

In the case where T contains all 2-simplices of S, we will omit T from the notation and write $\mathfrak{P} = (M_S, \{p_\alpha\}_{\alpha\in A})$. (All categorical patterns we consider in this paper are of this form.) If further S is an ∞-category and M_S contains all equivalences of S, we say that $\mathfrak{P}$ is a *commutative categorical pattern* [3, Def. 2.14].

Definition 3.4 ([17, Def. B.0.19], [3, Rem. 2.5])**.** Let $\mathfrak{P} = (M_S, \{p_\alpha\}_{\alpha\in A})$ be a categorical pattern on a simplicial set S. A map $p : X \to S$ of simplicial sets is said to be $\mathfrak{P}$-*fibered* if it satisfies the following conditions.
(i) The map p is an inner fibration.
(ii) For every edge $s \to s'$ in M_S and every vertex $x \in X$ lying over s, there is a p-cocartesian edge $x \to x'$ lying over M_S.
(iii) Each p_α lifts to a map $\tilde{p}_\alpha : K_\alpha^\triangleleft \to X$ which carries all edges to p-cocartesian edges. Moreover, any such lift is a p-limit diagram.

[7] A rule of thumb is that, when we want to consider straightening-unstraightening of cartesian fibrations, we should use N_{I}, while when we consider that of cocartesian fibrations, we should use N_{II}.

If p satisfies these conditions, we will write $X_\natural$ for the marked simplicial set obtained from X by marking the p-cocartesian morphisms whose images in S belong to M_S.

The totality of $\mathfrak{P}$-fibered objects can be organized into an ∞-category, because of the following theorem.

Theorem 3.5 ([17, Thm. B.0.20]). *Let $\mathfrak{P} = (M_S, \{p_\alpha\}_{\alpha\in A})$ be a categorical pattern on a simplicial set S. There is a combinatorial simplicial model structure on $\mathsf{sSet}^+_{/(S,M_S)}$, denoted by $\mathsf{sSet}^+_{/\mathfrak{P}}$, whose cofibrations are the monomorphisms and whose fibrant objects are the objects of the form $X_\natural$, where $X \to S$ is $\mathfrak{P}$-fibered.*

Definition 3.6. Let $\mathfrak{P} = (M_S, \{p_\alpha\}_{\alpha\in A})$ be a categorical pattern on a simplicial set S. We write $\mathfrak{P}\mathcal{F}\mathsf{ib}$ for the homotopy coherent nerve of the full simplicial subcategory of $\mathsf{sSet}^+_{/\mathfrak{P}}$ spanned by the fibrant-cofibrant objects.

Example 3.7. For each $n \geq 0$, let $\rho_n : (\{1, \dots, n\})^\triangleleft \to N(\mathsf{Fin}_*)$ denote the functor which classifies the n inert morphisms $\langle n\rangle \to \langle 1\rangle$. (When $n = 0$, the diagram ρ_n classifies the object $\langle 0\rangle \in N(\mathsf{Fin}_*)$.) A functor $\mathcal{E} \to N(\mathsf{Fin}_*)$ is fibered over the categorical pattern $\mathfrak{Op} = (\{\text{inert maps}\}, \{\rho_n\}_{n\geq 0})$ if and only if it is an ∞-operad. By definition, we have $\mathfrak{Op}\mathcal{F}\mathsf{ib} = \mathcal{O}\mathsf{p}_\infty$.

Just like the ordinary straightening-unstraightening, functors with values in $\mathfrak{P}\mathcal{F}\mathsf{ib}$ can equivalently be specified by a fibrational structure, which we now review.

Definition 3.8 ([3, Def. 3.1]). Let $\mathfrak{P} = (M_\mathcal{D}, \{p_\alpha\}_{\alpha\in A})$ be a commutative categorical pattern on an ∞-category $\mathcal{D}$, and let S be a simplicial set. A $\mathfrak{P}$-*bundle* (*over* S) is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & S \times \mathcal{D} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & S & \end{array}$$

of simplicial sets which satisfies the following conditions.

(a) The map $q : X \to S$ is a cocartesian fibration.
(b) The map p lifts to a fibration of fibrant objects of $\mathsf{sSet}^+_{/S}$ with respect to the cocartesian model structure.
(c) For each vertex $v \in S$, the map $X_v = X \times_S \{v\} \to \mathcal{D}$ is $\mathfrak{P}$-fibered.
(d) For each edge $f : v \to v'$ in S, the induced functor $f_! : X_v \to X_{v'}$ is a morphism of $\mathfrak{P}$-fibered objects.

We will often say that the map p (or X) is a $\mathfrak{P}$-bundle over S. Given a $\mathfrak{P}$-bundle $p : X \to S \times \mathcal{D}$, we will write $X_\natural$ for the marked simplicial set obtained from X by marking the p-cocartesian edges whose images in $\mathcal{D}$ belong to $M_\mathcal{D}$. This does not conflict with the notation in Definition 3.4, because of the following reason. Let $S \times \mathfrak{P}$ denote the categorical pattern on $S \times \mathcal{D}$ given by

$$S \times \mathfrak{P} = (S_1 \times M_\mathcal{D}, \{\{v\} \times p_\alpha\}_{v\in S_0,\, \alpha\in A}).$$

We can show that the fibrant-cofibrant objects of $\mathsf{sSet}^+_{/S\times\mathfrak{P}}$ are precisely the objects of the form $(X, M_X) \to S^\sharp \times (\mathcal{D}, M_D)$, where X is a $\mathfrak{P}$-bundle and M_X is the set of p-cocartesian edges whose images in $\mathcal{D}$ belong to $M_{\mathcal{D}}$ (see [3, Prop. 3.5]).

We will write $\mathfrak{P}\mathcal{B}\mathsf{und}(S)$ for the homotopy coherent nerve of the full simplicial subcategory of $\mathsf{sSet}^+_{/S\times\mathfrak{P}}$ spanned by the fibrant–cofibrant objects.

The following is the main result of [3].

Theorem 3.9 ([3, Cor. 5.10])**.** *Let $\mathfrak{P}$ be a commutative categorical pattern, and let S be a small simplicial set. There is a categorical equivalence*

$$\mathfrak{P}\mathcal{B}\mathsf{und}(S) \simeq \mathrm{Fun}(S, \mathfrak{P}\mathcal{F}\mathsf{ib})$$

which lifts the ordinary straightening-unstraightening equivalence.

Remark 3.10. Let CCP denote the category whose objects are the pairs $(\mathcal{D}, \mathfrak{P})$, where $\mathcal{D}$ is an ∞-category and $\mathfrak{P}$ is a commutative categorical pattern on $\mathcal{D}$, and whose morphisms $(\mathcal{D}, \mathfrak{P}) \to (\mathcal{D}', \mathfrak{P}')$ are functors $\mathcal{D} \to \mathcal{D}'$ that carry each edge and diagram in $\mathfrak{P}$ into those of $\mathfrak{P}$. Then the assignments $(S, (\mathcal{D}, \mathfrak{P})) \mapsto \mathfrak{P}\mathcal{B}\mathsf{und}(S)$ and $(S, (\mathcal{D}, \mathfrak{P})) \mapsto \mathrm{Fun}(S, \mathfrak{P}\mathcal{F}\mathsf{ib})$ determine a functor $N(\mathsf{sSet}^{\mathrm{op}} \times \mathsf{CCP}^{\mathrm{op}}) \to \widehat{\mathcal{C}\mathsf{at}}_\infty$, and the equivalence of Theorem 3.9 can be promoted to a natural equivalence of these functors. This follows from the proof of [3, Cor. 5.10] (and arguing as in the proof of the naturality of the ordinary straightening-unstraightening [10, App. A]).

In the situation of Theorem 3.9, we say that a $\mathfrak{P}$-bundle $p : X \to S \times \mathcal{D}$ is *classified* by a functor $f : S \to \mathfrak{P}\mathcal{F}\mathsf{ib}$ if the equivalence of Theorem 3.9 carries p to an object equivalent to f. The naturality property discussed in the previous paragraph implies that classifying maps are compatible with pullback in the following two senses.

- If $S' \to S$ is a map of simplicial sets, then the $\mathfrak{P}$-bundle $X \times_S S' \to S' \times \mathcal{D}$ is classified by the composite
$$S' \to S \xrightarrow{f} \mathfrak{P}\mathcal{F}\mathsf{ib}.$$
- If $\mathfrak{P}'$ is a commutative categorical pattern on an ∞-category $\mathcal{D}'$ and $\mathcal{D}' \to \mathcal{D}$ is a functor which carries each edge and each diagram of $\mathfrak{P}'$ into those of $\mathfrak{P}$, then the p-bundle $X \times_{\mathcal{D}} \mathcal{D}' \to S \times \mathcal{D}'$ is classified by the composite
$$S \xrightarrow{f} \mathfrak{P}\mathcal{F}\mathsf{ib} \to \mathfrak{P}'\mathcal{F}\mathsf{ib}.$$

Example 3.11 ([3, Prop. 5.12])**.** Let $\mathfrak{P} = (M_{\mathcal{D}}, \{p_\alpha\}_\alpha)$ be a commutative categorical pattern on an ∞-category $\mathcal{D}$, let $\mathcal{C}$ be an ordinary category, and let $F : \mathcal{C} \to \mathsf{sSet}^+_{/\mathfrak{P}}$ be a functor which carries every object to a fibrant object. Then the nerve of F classifies the $\mathfrak{P}$-bundle

$$\int_{\mathcal{C}} F_\flat \to \int_{\mathcal{C}} \delta(\mathcal{D}) \cong N(\mathcal{C}) \times \mathcal{D},$$

where $F_\flat$ denotes the composite

$$\mathcal{C} \longrightarrow \mathsf{sSet}^+_{/\mathfrak{P}} \xrightarrow{\text{forget}} \mathsf{sSet},$$

$\int_{\mathcal{C}} F_\flat$ denotes the relative nerve of $F_\flat$ (see [16, §3.2.5]), and $\delta(\mathcal{D}) : \mathcal{C} \to \mathsf{sSet}$ denotes the constant functor at $\mathcal{D}$.

As in ordinary straightening-unstraightening, we can use Theorem 3.9 to give a criterion for a diagram in $\mathfrak{P}\mathcal{F}\mathsf{ib}$ to be a colimit diagram. The criterion relies on the following preliminary construction.

Definition 3.12 ([3, Def. 6.7])**.** Let $\mathfrak{P} = (M_{\mathcal{D}}, \{p_\alpha\}_{\alpha\in A})$ be a commutative categorical pattern on an ∞-category $\mathcal{D}$, let K be a simplicial set, and let $p' : X' \to K^{\triangleright} \times \mathcal{D}$ be a $\mathfrak{P}$-bundle over $K^{\triangleright}$. Set $X = X' \times_{K^{\triangleright}} K$. Regarding X' and X as $\mathfrak{P}$-bundles over $K^{\triangleright}$ and K, respectively, we define objects $X'_\natural \in \mathsf{sSet}^+_{/K^{\triangleright}\times\mathfrak{P}}$ and $X_\natural \in \mathsf{sSet}^+_{/K\times\mathfrak{P}}$ as in Definition 3.8. A map $\mathrm{Rf} : X_\natural \to X'_\natural \times_{(K^{\triangleright})^\sharp} \{\infty\}^\sharp$ of $\mathsf{sSet}^+_{/\mathfrak{P}}$ is called a *refraction map* if there is a morphism $H : (\Delta^1)^\sharp \times X_\natural \to X'_\natural$ in $\mathsf{sSet}^+_{/K^{\triangleright}\times\mathfrak{P}}$ satisfying the following conditions.

(i) The diagram

$$\begin{array}{ccccc} \{0\}^\sharp \times X_\natural & \longrightarrow & & \rightsquigarrow & X'_\natural \\ \big\downarrow & & \overset{H}{\nearrow} & & \big\downarrow p \\ (\Delta^1)^\sharp \times X_\natural & \xrightarrow[\mathrm{id}\times p]{} & (\Delta^1)^\sharp \times (K)^\sharp \times \overline{\mathcal{D}} & \xrightarrow[h\times\mathrm{id}]{} & (K^{\triangleright})^\sharp \times \overline{\mathcal{D}} \end{array}$$

is commutative, where $h : \Delta^1 \times K \to K$ is the map determined by the inclusion $K \times \{0\} \hookrightarrow K^{\triangleright}$ and the projection $K \times \{1\} \to \{\infty\}$ and $\overline{\mathcal{D}} = (\mathcal{D}, M_{\mathcal{D}})$.

(ii) The restriction $H|\{1\}^\sharp \times X_\natural$ is equal to Rf.

Note that refraction maps exist and are unique up to homotopy in the model category $\mathsf{sSet}^+_{/\mathfrak{P}}$.

Here is the colimit criterion.

Proposition 3.13 ([3, Prop. 6.8])**.** *Let $\mathfrak{P}$ be a commutative categorical pattern on an ∞-category $\mathcal{D}$, let K be a small simplicial set, let $f : K^{\triangleright} \to \mathfrak{P}\mathcal{F}\mathsf{ib}$ be a diagram which classifies a $\mathfrak{P}$-bundle $X' \to K^{\triangleright} \times \mathcal{D}$. Set $X = X' \times_{K^{\triangleright}} K$. The following conditions are equivalent.*

(i) *The diagram f is a colimit diagram.*

(ii) *The refraction map $X_\natural \to X'_\natural \times_{(K^{\triangleright})^\sharp} \{\infty\}$ is a $\mathfrak{P}$-equivalence.*

We conclude this section with a remark comparing bundles of ∞-operads with Lurie's *families of ∞-operads.*

Remark 3.14. For the categorical pattern $\mathfrak{Op}$ for ∞-operads (Example 3.7), the notion of $\mathfrak{Op}$-bundles is closely related to that of families of ∞-operads [17, Def. 2.3.2.10]. More precisely, if $\mathcal{C}$ is an ∞-category, then every $\mathfrak{Op}$-bundle over $\mathcal{C}$ is a family of ∞-operads. This is immediate from the definitions and [16, Cor. 4.3.1.15].

We can also prove this by using model categories. To see this, it will be convenient to introduce some notation. Given a marked simplicial set $\overline{S} = (S, M_S)$ and a commutative categorical pattern $\mathfrak{P} = (M_{\mathcal{D}}, \{p_\alpha\}_{\alpha\in A})$ on an ∞-category $\mathcal{D}$, let $\overline{S} \times \mathfrak{P}$ denote the pair

$$(M_S \times M_{\mathcal{D}}, \{\{v\} \times p_\alpha : \{v\} \times K_\alpha^{\triangleleft} \to S \times \mathcal{D}\}_{v\in S_0,\, \alpha\in A}).$$

Unwinding the definitions, the fibrant objects of $\mathsf{sSet}^+_{/\mathcal{C}^\natural \times \mathfrak{Op}}$ are precisely the $\mathcal{C}$-families of ∞-operads whose inert morphisms are marked. Since the pullback functor

$$\mathsf{sSet}^+_{/\mathcal{C}\times\mathfrak{Op}} = \mathsf{sSet}^+_{/\mathcal{C}^\sharp\times\mathfrak{Op}} \to \mathsf{sSet}^+_{/\mathcal{C}^\natural\times\mathfrak{Op}}$$

is right Quillen [17, Prop. B.2.9], every $\mathfrak{Op}$-bundle over $\mathcal{C}$ is a $\mathcal{C}$-family of ∞-operads.

Note that the above argument shows that if $\mathcal{C}$ is a Kan complex (so that $\mathcal{C}^\natural = \mathcal{C}^\sharp$), then $\mathcal{C}$-families of ∞-operads and bundles of ∞-operads over $\mathcal{C}$ are exactly the same things.

3.15. **Universal weak equivalences.** Recall that our goal of this section is to show that a certain diagram in $\mathcal{O}\mathsf{p}_\infty$ is a universal colimit diagram. For this, we will need a version of Proposition 3.13 for universal colimit diagrams (Proposition 3.18), which is the subject of this subsection.

To state the main result of this subsection, we need a model-categorical notion associated with universal colimit diagrams, called universal weak equivalences.

Definition 3.16. Let $\mathbf{A}$ be a model category. A morphism $A \to B$ of $\mathbf{A}$ is called a *universal weak equivalence* if, for each fibration $X \to B$ in $\mathbf{A}$, the map $A \times_B X \to X$ is a weak equivalence.

If $\mathfrak{P}$ is a categorical pattern, we will refer to universal weak equivalences of $\mathsf{sSet}^+_{/\mathfrak{P}}$ as *universal* $\mathfrak{P}$*-equivalences*.

Example 3.17. Let $\mathbf{A}$ be a model category.

(i) Every weak equivalence of fibrant objects of $\mathbf{A}$ is a universal weak equivalence. This follows from [16, Lem. A.2.4.3].

(ii) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms of $\mathbf{A}$. Suppose that f is a universal weak equivalence. Then g is a universal weak equivalence if and only if gf is a universal weak equivalence.

(iii) Suppose that $\mathbf{A}$ is a simplicial model category in which every object is cofibrant. We say that a morphism $i : A \to B$ is a *right deformation retract* if there is a retraction $r : B \to A$ of i and a map $h : \Delta^1 \otimes B \to B$ such that $h|\{0\} \otimes B = \mathrm{id}_B$ and $h|\{1\} \otimes B = ir$. Every right deformation retract of $\mathbf{A}$ is a universal weak equivalence. This follows from [3, Prop. 6.15].

(iv) Suppose that $\mathbf{A}$ is a simplicial model category in which every object is cofibrant. Part (iii) (and its dual) implies that, for each $A \in \mathbf{A}$ and $i \in \{0,1\}$, the map $\{i\} \otimes A \to \Delta^1 \otimes A$ is a universal weak equivalence. Hence, by part (ii), universal weak equivalences of $\mathbf{A}$ are stable under left homotopy.

Here is the main result of this subsection.

Proposition 3.18. *Let $\mathfrak{P}$ be a commutative categorical pattern on an ∞-category $\mathcal{D}$, let K be a small simplicial set, and let $f : K^\triangleright \to \mathfrak{P}\mathcal{F}\mathsf{ib}$ be a diagram which classifies a $\mathfrak{P}$-bundle $X' \to K^\triangleright \times \mathcal{D}$. Set $X_\natural = X'_\natural \times_{(K^\triangleright)^\sharp} K^\sharp$ and $(X'_\infty)_\natural = X'_\natural \times_{(K^\triangleright)^\sharp} \{\infty\}^\sharp$. The following conditions are equivalent.*

(i) *The diagram f is a universal colimit diagram.*

(ii) *The refraction map $X_\natural \to (X'_\infty)_\natural$ is a universal $\mathfrak{P}$-equivalence of $\mathsf{sSet}^+_{/\mathfrak{P}}$.*

Remark 3.19. In the situation of Proposition 3.18, if some refraction map is a universal $\mathfrak{P}$-equivalence, so is any other map (by point (iv) of Example 3.17). This justifies the usage of the definite article ("the") in condition (ii).

The proof of Proposition 3.18 relies on a few preliminaries.

Definition 3.20. Let K be a weakly contractible simplicial set, and let $\mathcal{C}$ be an ∞-category. We say that a diagram $f : K \to \mathcal{C}$ is *essentially constant* if f carries each morphism to an equivalence in $\mathcal{C}$.

Remark 3.21. Let K be a weakly contractible simplicial set, and let $\mathcal{C}$ be an ∞-category. Then
(i) the diagonal functor $\delta : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ is fully faithful;
(ii) the essential image of δ consists of the essentially constant diagrams.
For part (i), we may assume that $\mathcal{C}$ has colimits of shape K (by embedding $\mathcal{C}$ into a larger ∞-category if necessary). In this case, δ is a fully faithful right adjoint by [16, Cor. 4.4.4.10]. Part (ii) follows from the observation that every diagram $K \to \mathcal{C}^{\simeq}$ is equivalent to a constant diagram, because K is weakly contractible.

Lemma 3.22. *Let $\mathcal{C}$ be an ∞-category with pullbacks, let K be a simplicial set, and let $p : K^{\triangleright} \to \mathcal{C}$ be a diagram. The following conditions are equivalent.*
(i) *The diagram p is a universal colimit diagram.*
(ii) *For every pullback diagram*

$$\begin{array}{ccc} p' & \longrightarrow & p \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ q' & \longrightarrow & q \end{array}$$

in $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$, if q and q' are essentially constant, then the diagram p' is a colimit diagram.
(iii) *There exists a morphism $\alpha : p \to q$ in $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ satisfying the following conditions.*
(a) *The map $\alpha_\infty : p(\infty) \to q(\infty)$ is an equivalence.*
(b) *The diagram q is essentially constant.*
(c) *For every pullback diagram*

$$\begin{array}{ccc} p' & \longrightarrow & p \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ q' & \longrightarrow & q \end{array}$$

in $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$, if q' is essentially constant, then p' is a colimit diagram.

Proof. We first prove that (i) $\Rightarrow$ (ii). Suppose that condition (i) is satisfied. In the situation of (ii), iterated applications of the pasting law of pullbacks [16, Lem. 4.4.2.1] shows that the natural transformation $p' \to p$ is cartesian. Hence p' is a colimit diagram, proving (i) $\Rightarrow$ (ii).

Next, we prove (ii) $\Rightarrow$ (i). Suppose that condition (ii) is satisfied. Let $\alpha : p' \to p$ be a cartesian natural transformation of diagrams $K^{\triangleright} \to \mathcal{C}$. We wish to show that p' is a colimit diagram. Pulling back α along the natural transformation $K^{\triangleright} \times \Delta^1 \to K^{\triangleright}$ from the identity map to the constant map at the cone point, we obtain a cartesian square

$$\begin{array}{ccc} p' & \longrightarrow & p \\ \downarrow & & \downarrow \\ \delta(p'(\infty)) & \longrightarrow & \delta(p(\infty)) \end{array}$$

in $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$, where $\delta : \mathcal{C} \to \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ denotes the diagonal functor. Condition (ii) then tells us that p' is a colimit diagram. Hence (ii) $\Rightarrow$ (i).

Finally, we prove (ii) $\Leftrightarrow$ (iii). It is clear that (ii) $\Rightarrow$ (iii). For the converse, it suffices to show that, for every morphism $\beta : p \to r$ in $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ such that r is essentially constant, there is a diagram $\Delta^2 \to \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ whose boundary is depicted as

$$\begin{array}{ccccc} & & p & & \\ & \swarrow^{\alpha} & & \searrow^{\beta} & \\ q & & \xrightarrow[\gamma]{} & & r. \end{array}$$

According to [16, Prop. 4.3.2.17], for any pair of diagrams $f, g : K^{\triangleright} \to \mathcal{C}$ such that g is essentially constant, the map

$$\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright}, \mathcal{C})}(f, g) \to \mathrm{Hom}_{\mathcal{C}}(f(\infty), g(\infty))$$

is a homotopy equivalence. Therefore, it suffices to show that there is a diagram $\Delta^2 \to \mathcal{C}$ of the form

$$\begin{array}{ccccc} & & p(\infty) & & \\ & \swarrow^{\alpha_\infty} & & \searrow^{\beta_\infty} & \\ q(\infty) & & \longrightarrow & & r(\infty), \end{array}$$

which is clear because α_∞ is an equivalence. $\square$

Lemma 3.23. *Let $\mathfrak{P}$ be a commutative categorical pattern on an ∞-category $\mathcal{D}$, let K be a small simplicial set, and let $X' \to K^{\triangleright} \times \mathcal{D}$ be a $\mathfrak{P}$-bundle over $K^{\triangleright}$. There is a map $r : X'_{\natural} \to (X'_{\infty})_{\natural}$ rendering the diagram*

(1)
$$\begin{array}{ccc} (X'_\infty)_\natural & \xrightarrow{\mathrm{id}} & (X'_\infty)_\natural \\ \downarrow & \overset{r}{\nearrow} & \downarrow \\ X'_\natural & \longrightarrow & \overline{\mathcal{D}} \end{array}$$

commutative. Moreover, such an r is unique up to homotopy, and the restriction $r|X_\natural$ is a refraction map for p.

Proof. For the existence of the map r, it suffices to show that the inclusion $(X'_\infty)_\natural \subset X'_\natural$ induces a trivial fibration

$$\theta : \mathrm{Map}_{\mathsf{sSet}^+_{/K^\triangleright \times \mathfrak{P}}}(X'_\natural, (K^\triangleright)^\sharp \times (X'_\infty)_\natural) \xrightarrow{\simeq} \mathrm{Map}_{\mathsf{sSet}^+_{/\mathfrak{P}}}((X'_\infty)_\natural, (X'_\infty)_\natural)$$

of Kan complexes. The map θ is a Kan fibration (because $\mathsf{sSet}^+_{/K^\triangleright \times \mathfrak{P}}$ is a simplicial model category and the inclusion $(X'_\infty)_\natural \subset X'_\natural$ is its cofibration), so it suffices to show that θ is a homotopy equivalence. Using Theorem 3.9, we are reduced to showing that, for every pair of diagrams $f, g : K^\triangleright \to \mathfrak{P}\mathcal{F}\mathsf{ib}$ with g essentially constant, the map

$$\mathrm{Hom}_{\mathrm{Fun}(K^\triangleright, \mathfrak{P}\mathcal{F}\mathsf{ib})}(f, g) \to \mathrm{Hom}_{\mathfrak{P}\mathcal{F}\mathsf{ib}}(f(\infty), g(\infty))$$

is a homotopy equivalence, which is the content of [16, Prop. 4.3.2.17].

To complete the proof, we must show that there is some filler of diagram (1) that restricts to a refraction map of p. For this, let $h' : \Delta^1 \times K^\triangleright \to K^\triangleright$ denote the natural transformation from the identity map to the constant map at the base point. Since $\mathsf{sSet}^+_{/K^\triangleright \times \mathfrak{P}}$ is a simplicial model category, the left vertical arrow of the diagram

$$\begin{array}{ccc}
(\{0\}^\sharp \times X'_\natural) \amalg_{\{0\}^\sharp \times (X'_\infty)_\natural} ((\Delta^1)^\sharp \times (X'_\infty)_\natural) & \longrightarrow & X'_\natural \\
\big\downarrow & \overset{H'}{\nearrow} & \big\downarrow p' \\
(\Delta^1)^\sharp \times X'_\natural \xrightarrow{\mathrm{id} \times p'} (\Delta^1)^\sharp \times (K^\triangleright)^\sharp \times \overline{\mathcal{D}} & \xrightarrow{h' \times \mathrm{id}} & (K^\triangleright)^\sharp \times \overline{\mathcal{D}}
\end{array}$$

is a trivial cofibration. (Here the top arrow is the amalgamation of the identity map of $X'_\natural$ and the projection $(\Delta^1)^\sharp \times (X'_\infty)_\natural \to (X'_\infty)_\natural$.) Then there is a dashed filler H' as indicated in the diagram. The restriction $r = H'|\{1\}^\sharp \times (X'_\infty)_\natural$ is a filler of (1) which restricts to a refraction map, and the proof is complete. $\square$

We now arrive at the proof of Proposition 3.18.

Proof of Proposition 3.18. Choose a retraction $r : X'_\natural \to (X'_\infty)_\natural$ as in Lemma 3.23. Given a fibration $\pi : Z_\natural \to (X'_\infty)_\natural$ of $\mathsf{sSet}^+_{/\mathfrak{P}}$, form pullback squares as in the diagram

$$\begin{array}{ccc}
Y_\natural & \longrightarrow & X_\natural \\
\big\downarrow & & \big\downarrow \\
Y'_\natural & \longrightarrow & X'_\natural \\
\zeta \big\downarrow & & \big\downarrow r \\
Z_\natural & \xrightarrow[\pi]{} & (X'_\infty)_\natural.
\end{array}$$

The map $(Y'_\infty)_\natural \to Z_\natural$ is an isomorphism, and under this isomorphism, we can identify $\zeta|Y_\natural$ with the refraction map of $Y'_\natural$ (by Lemma 3.23). Therefore, by Proposition 3.13, we can rephrase condition (ii) as follows.

(ii′) For every fibration $Z_\natural \to (X'_\infty)_\natural$ in $\mathsf{sSet}^+_{/\mathfrak{P}}$, the $\mathfrak{P}$-bundle $Z_\natural \times_{X'_\infty} X'_\natural$ over $K^\triangleright$ is classified by a colimit diagram $K^\triangleright \to \mathfrak{P}\mathcal{F}\mathsf{ib}$.

Since $K^\triangleright$ is weakly contractible, the diagonal functor $\mathfrak{P}\mathcal{F}\mathsf{ib} \to \mathrm{Fun}(K^\triangleright, \mathfrak{P}\mathcal{F}\mathsf{ib})$ is fully faithful, with essential image consisting of essentially constant functors (Remark 3.21). It follows from Theorem 3.9 that the functor $K^\triangleright \times - : \mathfrak{P}\mathcal{F}\mathsf{ib} \to \mathfrak{P}\mathcal{B}\mathsf{und}(K^\triangleright)$ is also fully faithful. Combining this observation with [16, Prop. 4.2.4.1], we can further rephrase (ii′) as follows:

(ii″) For every pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & X'_\natural \\ \downarrow & & \downarrow \\ B & \longrightarrow & (K^\triangleright)^\sharp \times (X'_\infty)_\natural \end{array}$$

in $\mathfrak{P}\mathcal{B}\mathsf{und}(K^\triangleright)$ (where the right-hand map is induced by r), if B is classified by an essentially constant diagram $K^\triangleright \to \mathfrak{P}\mathcal{F}\mathsf{ib}$, then A is a colimit diagram.

Lemma 3.22 and Theorem 3.9 now show that (ii″) is equivalent to (i), and the proof is complete. □

3.24. **Universal weak equivalence of $\mathsf{sSet}^+_{/\mathfrak{Op}}$.** Let K be a Kan complex. As we saw in Remark 3.14, if $\mathcal{E} \to K \times N(\mathsf{Fin}_*)$ is a K-bundle of ∞-operads, then it is a K-family of ∞-operads (in particular, a generalized ∞-operad [17, Prop. 2.3.2.11]) and the marked edges of $\mathcal{E}_\natural$ are precisely the inert morphisms. This, together with the universal colimit criterion we established in Section 3.15 (Proposition 3.18), motivates the following question: Let $\mathcal{A}^\otimes \to \mathcal{B}^\otimes$ be a morphism of generalized ∞-operads, where $\mathcal{B}^\otimes$ is an ∞-operad. Let $\mathcal{A}^{\otimes,\natural}, \mathcal{B}^{\otimes,\natural}$ denote the marked simplicial sets obtained from $\mathcal{A}^\otimes, \mathcal{B}^\otimes$ by marking the inert maps. When is the map $\mathcal{A}^{\otimes,\natural} \to \mathcal{B}^{\otimes,\natural}$ a universal weak equivalence of $\mathsf{sSet}^+_{/\mathfrak{Op}}$?

A more general question has been posed by Lurie in [17, §2.3.3] for (non-universal) weak equivalences of $\mathsf{sSet}^+_{/\mathfrak{Op}}$. While this was not explicitly stated by him, we will see that his answer in fact accommodates universal weak equivalences of $\mathsf{sSet}^+_{/\mathfrak{Op}}$ (Proposition 3.27).

We start by recalling the following theorem, which is a special case of [17, Thm. 2.3.3.23].

Theorem 3.25. *Let $f : \mathcal{A}^\otimes \to \mathcal{B}^\otimes$ be a morphism of generalized ∞-operads, where $\mathcal{B}^\otimes$ is an ∞-operad. Suppose that f satisfies the following conditions.*

(i) *The functor $\mathcal{A} \to \mathcal{B}$ is a categorical equivalence.*

(ii) *For each object $B \in \mathcal{B}^\otimes$, the inclusion $(\mathcal{A}^\otimes_{B/})' \subset \mathcal{A}^\otimes_{B/}$ is initial, where $(\mathcal{A}^\otimes_{B/})' \subset \mathcal{A}^\otimes_{B/}$ denotes the full subcategory spanned by the objects $(A, \alpha : B \to f(A))$ such that α is inert.*

Then the map $\mathcal{A}^{\otimes,\natural} \to \mathcal{B}^{\otimes,\natural}$ is a weak equivalence of $\mathsf{sSet}^+_{/\mathfrak{Op}}$.

Condition (ii) of Theorem 3.25 admits the following reformulation, which follows from the argument of [17, 2.3.3.11].

Proposition 3.26. *Let $f : \mathcal{A}^\otimes \to \mathcal{B}^\otimes$ be a map of generalized ∞-operads. The following conditions are equivalent.*

(i) *For each object $B \in \mathcal{B}^\otimes$, the inclusion $(\mathcal{A}^\otimes_{B/})' \subset \mathcal{A}^\otimes_{B/}$ is initial, where $(\mathcal{A}^\otimes_{B/})'$ is defined as in Theorem 3.25.*
(ii) *For each object $A \in \mathcal{A}^\otimes$, the homotopy fibers (in the Joyal model structure) of the functor*

$$(\mathcal{A}^\otimes_{\mathrm{act}})_{/A} \to (\mathcal{B}^\otimes_{\mathrm{act}})_{/f(A)}$$

are weakly contractible.[8]

Combining Theorem 3.25 and Proposition 3.26, we obtain the following criterion for universal weak equivalences in $\mathsf{sSet}^+_{/\mathfrak{Op}}$.

Proposition 3.27. *Let $f : \mathcal{A}^\otimes \to \mathcal{B}^\otimes$ be a morphism of generalized ∞-operads, where $\mathcal{B}^\otimes$ is an ∞-operad. Suppose that the functor $\mathcal{A} \to \mathcal{B}$ is a categorical equivalence and that, for each object $A \in \mathcal{A}^\otimes$, the homotopy fibers (in the Joyal model structure) of the functor*

$$(\mathcal{A}^\otimes_{\mathrm{act}})_{/A} \to (\mathcal{B}^\otimes_{\mathrm{act}})_{/f(A)}$$

are weakly contractible. Then the map $f : \mathcal{A}^{\otimes,\natural} \to \mathcal{B}^{\otimes,\natural}$ is a universal weak equivalence of $\mathsf{sSet}^+_{/\mathfrak{Op}}$.

Proof. Let $\mathcal{D}^\otimes \to \mathcal{B}^\otimes$ be a fibration of ∞-operads and set $\mathcal{C}^\otimes = \mathcal{A}^\otimes \times_{\mathcal{B}^\otimes} \mathcal{D}^\otimes$. We must show that the map $g : \mathcal{C}^{\otimes,\natural} \to \mathcal{D}^{\otimes,\natural}$ is a weak equivalence of $\mathsf{sSet}^+_{/\mathfrak{Op}}$. By Theorem 3.25 and Proposition 3.26, it suffices to show that, for each object $C \in \mathcal{C}^\otimes$, the functor

$$\theta : (\mathcal{C}^\otimes_{\mathrm{act}})_{/C} \to (\mathcal{D}^\otimes_{\mathrm{act}})_{/g(C)}$$

has weakly contractible homotopy fibers. If $A \in \mathcal{A}^\otimes$ denotes the image of C, then the map θ is a pullback of the functor $(\mathcal{A}^\otimes_{\mathrm{act}})_{/A} \to (\mathcal{B}^\otimes_{\mathrm{act}})_{/f(A)}$. Since the latter has weakly contractible homotopy fibers, we are done. □

3.28. **Proof of Lemma 2.21.** In this subsection, we will prove Lemma 2.21, the main result of this section, by using results in Sections 3.15 and 3.24.

We begin with a lemma.

Lemma 3.29. *Let K be a Kan complex, and let $f : \mathcal{O}^\otimes \to K^{\amalg}$ be a morphism of generalized ∞-operads. Suppose that f satisfies the following conditions.*
(i) *The ∞-category $\mathcal{O}$ is a Kan complex, and the map $\mathcal{O} \to K$ is a homotopy equivalence of Kan complexes.*
(ii) *The functor $\mathcal{O}^\otimes \to N(\mathsf{Fin}_*)$ is conservative.*
(iii) *For each $n \geq 0$ and each object $X \in \mathcal{O}^\otimes_{\langle n\rangle}$, the map*

$$((\mathcal{O}^\otimes_{\mathrm{act}})_{/X})^{\simeq} \to ((N(\mathsf{Fin}_*)_{\mathrm{act}})_{/\langle n\rangle})^{\simeq}$$

is a homotopy equivalence.
Then f is a universal weak equivalence of $\mathsf{sSet}^+_{/\mathfrak{Op}}$.

[8] Here $(\mathcal{A}^\otimes_{\mathrm{act}})_{/A}$ denotes the slice of $\mathcal{A}^\otimes_{\mathrm{act}}$ with respect to A, not the fiber product $\mathcal{A}^\otimes_{\mathrm{act}} \times_{\mathcal{A}^\otimes} \mathcal{A}^\otimes_{/A}$.

Proof. According to Proposition 3.27, it will suffice to show that, for each object $X \in \mathcal{O}^{\otimes}$, the functor

$$(\mathcal{O}^{\otimes}_{\mathrm{act}})_{/X} \to (K^{\amalg}_{\mathrm{act}})_{/f(X)}$$

has weakly contractible homotopy fibers. By condition (ii), this functor is conservative. Therefore, it suffices to show that the map

$$((\mathcal{O}^{\otimes}_{\mathrm{act}})_{/X})^{\simeq} \to ((K^{\amalg}_{\mathrm{act}})_{/f(X)})^{\simeq}$$

is a homotopy equivalence of Kan complexes. Condition (iii) now shows that this is equivalent to the condition that the map

$$\pi : ((K^{\amalg}_{\mathrm{act}})_{/f(X)})^{\simeq} \to ((N(\mathsf{Fin}_*)_{\mathrm{act}})_{/\langle n\rangle})^{\simeq}$$

be a homotopy equivalence, where $\langle n\rangle \in \mathsf{Fin}_*$ denotes the image of X. Since π is a Kan fibration, it suffices to show that its fibers are contractible. But the fibers of π are products of simplicial sets of the form $K_{/v}$, where v is some vertex of K. In particular, the fibers of π are contractible. The proof is now complete. □

Proof of Lemma 2.21. Set $K' = \mathcal{D}_{/K}$ and let $\mathcal{O}^{\otimes}_{\natural} \to (K'^{\triangleright})^{\sharp} \times N(\mathsf{Fin}_*)_{\natural}$ be an $\mathfrak{Op}$-bundle classified by the composite

$$K'^{\triangleright} = (\mathcal{D}_{/K})^{\triangleright} \longrightarrow \mathcal{S} \xrightarrow{(-)^{\amalg}} \mathcal{O}\mathsf{p}_{\infty}.$$

Let $\chi : \mathcal{O}^{\otimes}_{\natural} \times_{(K'^{\triangleright})^{\sharp}} (K')^{\sharp} \to \mathcal{O}^{\otimes}_{\natural} \times_{(K'^{\triangleright})^{\sharp}} \{\infty\}^{\sharp}$ denote the refraction map. By Proposition 3.18, it will suffice to show that the map χ is a universal weak equivalence of $\mathsf{sSet}^{+}_{/\mathfrak{Op}}$. According to Lemma 3.29, it will suffice to prove the following.

(i) The simplicial set $\mathcal{O} \times_{(K'^{\triangleright})^{\triangleright}} K'$ is a Kan complex, and the refraction map χ restricts to a homotopy equivalence

$$\alpha : \mathcal{O} \times_{(K'^{\triangleright})^{\triangleright}} K' \xrightarrow{\simeq} \mathcal{O} \times_{(K'^{\triangleright})} \{\infty\}$$

of Kan complexes.

(ii) The functor $\mathcal{O}^{\otimes} \times_{K'^{\triangleright}} K' \to N(\mathsf{Fin}_*)$ is conservative.

(iii) For each $n \geq 0$ and each object $X \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, the map

$$\gamma : ((\mathcal{O}^{\otimes} \times_{K'^{\triangleright}} K')_{/X})^{\simeq} \to ((N(\mathsf{Fin}_*))_{/\langle n\rangle})^{\simeq}$$

is a trivial fibration.

We begin with (i). By the definition of bundles, the functor $\mathcal{O} \to K'^{\triangleright}$ is a cocartesian fibration. Moreover, the fibers of the map $\mathcal{O} \to K'^{\triangleright}$ are Kan complexes (Remark 3.10), so it is a left fibration. Since K' is a Kan complex, it follows that $\mathcal{O} \times_{K'^{\triangleright}} K'$ is a Kan complex. By Remark 3.10, we can identify α with the refraction map of the $K'^{\triangleright}$-bundle classified by the composite

$$K'^{\triangleright} = (\mathcal{D}_{/K})^{\triangleright} \longrightarrow \mathcal{S} \xrightarrow{(-)^{\amalg}} \mathcal{O}\mathsf{p}_{\infty} \xrightarrow{(-)_{\langle 1\rangle}} \mathcal{C}\mathsf{at}_{\infty},$$

where $(-)_{\langle 1\rangle}$ denotes the functor which assigns to each ∞-operad $\mathcal{O}^{\otimes}$ the fiber $\mathcal{O}^{\otimes}_{\langle 1\rangle} = \mathcal{O}^{\otimes} \times_{N(\mathsf{Fin}_*)} \{\langle 1\rangle\}$. But the composite $(-)_{\langle 1\rangle} \circ (-)^{\amalg} : \mathcal{S} \to \mathcal{C}\mathsf{at}_{\infty}$ is just the inclusion, so we are reduced to showing that the inclusion $\mathcal{S} \to \mathcal{C}\mathsf{at}_{\infty}$ is a left Kan extension of its restriction to $\mathcal{D}$. This is clear, because the functor $\mathcal{S} \to \mathcal{C}\mathsf{at}_\infty$ is a left adjoint and the identity functor of $\mathcal{S}$ is a left Kan extension of its restriction to $\mathcal{D}$ (see [16, Lem. 5.1.5.3]).

Next, we prove (ii). Since the K'-bundle $\mathcal{O}^\otimes \times_{K'^\triangleright} K'$ is equivalent to the terminal K'-bundle $K' \times N(\mathsf{Fin}_*)$ (because the constant diagram at $N(\mathsf{Fin}_*)$ is a terminal object of $\mathrm{Fun}(K', \mathcal{O}\mathsf{p}_\infty)$), it suffices to show that the functor $K' \times N(\mathsf{Fin}_*) \to N(\mathsf{Fin}_*)$ is conservative, which is obvious.

Finally, we prove (iii). As in (ii), it will suffice to show that, for each object $(v, \langle n \rangle) \in K' \times N(\mathsf{Fin}_*)$, the map

$$((K' \times N(\mathsf{Fin}_*))_{/(v,\langle n \rangle)})^\simeq = K'_{/v} \times (N(\mathsf{Fin}_*)_{/\langle n \rangle})^\simeq \to (N(\mathsf{Fin}_*)_{/\langle n \rangle})^\simeq$$

is a trivial fibration. This is clear, since $K'_{/v}$ is a contractible Kan complex. The proof is now complete. □

Acknowledgments. I am grateful to an anonymous referee for carefully reading a manuscript of this paper, offering valuable feedback, and suggesting a simple proof of Proposition 3.18. I also thank Michael Weiss for his comment, which improved the exposition of this paper. Finally, I appreciate Daisuke Kishimoto and Mitsunobu Tsutaya for their constant support and encouragement.

References

[1] D. Ayala and J. N. K. Francis, Factorization homology of topological manifolds, J. Topol. **8** (2015), no. 4, 1045–1084. MR3431668

[2] K. Arakawa, A context for manifold calculus, arXiv:2403.03321v2 [math.AT] (2024)

[3] K. Arakawa, The Grothendieck construction for ∞-categories fibered over categorical patterns, 2024.

[4] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347, Springer, Berlin, 1973. MR0420609

[5] K. J. Costello and O. Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, New Math. Monogr. 31, Cambridge Univ. Press, Cambridge, 2017. MR3586504

[6] K. J. Costello and O. Gwilliam, *Factorization algebras in quantum field theory. Vol. 2*, New Math. Monogr. 41, Cambridge Univ. Press, Cambridge, 2021. MR4300181

[7] S. Devalapurkar, J. Hahn, T. Lawson, A. Senger, and D. Wilson, Examples of disk algebras, arXiv:2302.11702v2 [math.AT] (2025)

[8] B. Fresse, Little discs operads, graph complexes and Grothendieck-Teichmüller groups, in *Handbook of homotopy theory*, 405–441, CRC Press/Chapman Hall Handb. Math. Ser, CRC, Boca Raton, FL, 2020. MR4197991

[9] E. Getzler, Batalin–Vilkovisky algebras and two-dimensional topological field theories, Comm. Math. Phys. **159** (1994), no. 2, 265–285. MR1256989

[10] D. Gepner, R. Haugseng, and T. Nikolaus, Lax colimits and free fibrations in ∞-categories, Doc. Math. **22** (2017), 1225–1266. MR3690268

[11] G. Horel, M. Krannich, and A. Kupers, Two remarks on spaces of maps between operads of little cubes, arXiv:2211.00908v2 [math.AT] (2022)

[12] A. Joyal, Quasi-categories and Kan complexes, J. Pure Appl. Algebra **175** (2002), no. 1–3, 207–222. MR1935979

[13] J. M. Kister, Microbundles are fibre bundles, Ann. of Math. (2) **80** (1964), 190–199. MR0180986

[14] E. Karlsson, C. I. Scheimbauer, and T. Walde, Assembly of constructible factorization algebras, arXiv:2403.19472v3 [math.AT] (2025)

[15] J. Lurie, Derived algebraic geometry VI: $\mathbb{E}[k]$-algebras, arXiv:0911.0018 [math.AT] (2009)

[16] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, 170, Princeton Univ. Press, Princeton, NJ, 2009. MR2522659

[17] J. Lurie, *Higher algebra*, `https://www.math.ias.edu/~lurie/papers/HA.pdf`, 2017.

[18] J. Lurie, *Kerodon*, `https://kerodon.net`, 2024.

[19] M. Markl, A compactification of the real configuration space as an operadic completion, J. Algebra **215** (1999), no. 1, 185–204. MR1684178

[20] T. Matsuoka, Descent properties of topological chiral homology, Münster J. Math. **10** (2017), no. 1, 83–118. MR3624103

[21] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 271, Springer, Berlin, 1972. MR0420610

[22] N. Wahl, *Ribbon braids and related operads*, Ph.D. thesis, University of Oxford, 2001, `https://web.math.ku.dk/~wahl/wahlthesis.pdf`.

Received June 17, 2024; accepted January 14, 2025

Kensuke Arakawa
Department of Mathematics, Kyoto University
Kyoto, 606-8502, Japan
E-mail: `arakawa.kensuke.22c@st.kyoto-u.ac.jp`