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On various notions of distance between subalgebras of operator algebras

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Abstract. Given any irreducible inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$, we show that the set of E-compatible intermediate C^*-subalgebras is finite, thereby generalizing a finiteness result of Ino and Watatani [11]. A finiteness result for a certain collection of intermediate C^*-subalgebras of a non-irreducible inclusion of simple unital C^*-algebras is also obtained, which provides a C^*-version of a finiteness result of Khoshkam and Mashood [18].

Apart from these finiteness results, comparisons between various notions of distance between subalgebras of operator algebras by Kadison–Kastler, Christensen and Mashood–Taylor are made. Further, these comparisons are used satisfactorily to provide some concrete calculations of distance between operator algebras associated to two distinct subgroups of a given discrete group.

1. Introduction

Watatani [25] (resp., Teruya and Watatani [23]) proved that the lattice of intermediate subfactors of an irreducible finite-index subfactor of type II_1 (resp., type III) is finite. This was then generalized to the C^*-context by Ino and Watatani [11], who proved that the set of intermediate C^*-subalgebras of an irreducible inclusion of simple unital C^*-algebras with a finite-index conditional expectation is finite [11, Cor. 3.9]. Further, Longo [19] obtained a bound for the cardinality of the lattice of intermediate subfactors of any finite-index irreducible inclusion of factors (of type II_1 or III). More recently, a similar bound was obtained for the cardinality of the lattice of intermediate C^*-algebras of a finite-index irreducible inclusion of simple unital C^*-algebras by Bakshi and the first named author in [3], which was achieved by introducing the notion of (interior) angle between intermediate C^*-subalgebras. This bound was further improved by Bakshi, Guin and Jana [2].

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One of the highlights of this paper is the following generalization of Ino–Watatani's finiteness result [11].

Corollary 4.3. *Let $\mathcal{B} \subset \mathcal{A}$ be an irreducible inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. Then the set* $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ *consisting of E-compatible intermediate C^*-subalgebras of $\mathcal{B} \subset \mathcal{A}$ is finite.*

Ino–Watatani's proof of finiteness was a clever compactness argument based on an appropriate estimate for $\|e_{\mathcal{C}} - e_{\mathcal{D}}\|$ for any two (E-compatible) intermediate C^*-subalgebras $\mathcal{C}$ and $\mathcal{D}$ (see [11, Lem. 3.3]), and a perturbation result established by them in [11, Thm. 3.8]. Corollary 4.3 is an immediate consequence of the following more general result, whose proof is an appropriate adaptation of the compactness argument of Ino–Watatani [11].

Theorem 4.2. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. If one (equivalently, any) of the algebras $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$, $\mathcal{Z}(\mathcal{B})$ and $\mathcal{Z}(\mathcal{A})$ is finite-dimensional, then the collection*

$$\mathcal{F}(\mathcal{B}, \mathcal{A}, E) := \{\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E) \mid \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{C}) \cup \mathcal{C}\}$$

is finite.

It is noteworthy that, in the above mentioned finiteness results (except Theorem 4.2), irreducibility of the initial inclusion was crucial. Somehow, not much is known about the finiteness of the lattice of intermediate subalgebras of non-irreducible inclusions. Interestingly, very recently, while looking for such finiteness results for the full lattice of intermediate von Neumann subalgebras of non-irreducible inclusions, the compactness argument of Ino and Watatani was also employed successfully by Bakshi and the first named author [3] to prove the following.

Theorem ([3, Thm. 6.4])**.** *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras with a faithful normal tracial state on $\mathcal{M}$ such that $\mathcal{Z}(\mathcal{N})$ is finite-dimensional and the trace preserving normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ has finite Watatani index. If the relative commutant $\mathcal{N}' \cap \mathcal{M}$ equals either $\mathcal{Z}(\mathcal{N})$ or $\mathcal{Z}(\mathcal{M})$, then the lattice consisting of intermediate von Neumann subalgebras of $\mathcal{N} \subset \mathcal{M}$ is finite.*

Prior to this, Khoshkam and Mashood [18, Thm. 1.3] had shown that, for any finite-index subfactor $N \subset M$ of type II_1, the subcollections

$$\mathcal{L}_1(N \subset M) := \{P \in \mathcal{L}(N \subset M) \mid N' \cap M \subset P\},$$
$$\mathcal{L}_2(N \subset M) := \{P \in \mathcal{L}(N \subset M) \mid N' \cap M = P' \cap M\}$$

are both finite, where $\mathcal{L}(N \subset M)$ denotes the lattice of intermediate subfactors of the inclusion $N \subset M$.

Note that Theorem 4.2 is a C^*-version of [18, Thm. 1.3] for non-irreducible inclusions of non-simple C^*-algebras. Moreover, in Section 4 itself, we also prove another variant of a C^*-version of [18, Thm. 1.3] for non-irreducible inclusions of simple unital C^*-algebras, which reads as follows.

Theorem 4.9. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^*-algebras with a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Then the sublattice $\mathcal{I}_1(\mathcal{B} \subset \mathcal{A}) := \{\mathcal{C} \in \mathcal{I}(\mathcal{B} \subset \mathcal{A}) \mid \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}\}$ is finite.*

A few words regarding the techniques employed to achieve the above mentioned finiteness results. Theorem 4.2 is achieved by directly employing the compactness argument of [11], wherein the main ingredients include a basic observation from [9] (that $e_{\mathcal{C}} \in \mathcal{A}_1$ for any $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$), an estimate for $\|e_{\mathcal{C}} - e_{\mathcal{D}}\|$ from [11] and a perturbation result from [8]. On the other hand, Theorem 4.9 is proved on similar lines by first showing that $e_{\mathcal{C}} \in \mathcal{A}_1$ for any intermediate C^*-subalgebra $\mathcal{C} \in \mathcal{I}_1$ (Proposition 4.7), then obtaining Ino–Watatani's estimate for $\|e_{\mathcal{C}} - e_{\mathcal{D}}\|$ (Lemma 4.8), where $\mathcal{C}$ and $\mathcal{D}$ are two intermediate C^*-subalgebras in $\mathcal{I}_1$, and finally, by exploiting a perturbation result by Dickson [8].

The above mentioned finiteness results are all achieved in Section 4. Prior to Section 4, we devote a short section (Section 2) on preliminaries and another short section (Section 3) recalling and proving some basic (yet interesting) observations related to the Kadison–Kastler distance. One interesting observation being that, for any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$, there exists an $\alpha > 0$ such that the set of unitaries

$$\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \mid \|u - \mathbf{1}\| < \alpha\} \subseteq \bigcap \{\mathcal{N}_{\mathcal{A}}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{F}(\mathcal{B}, \mathcal{A}, E)\};$$

see Corollary 3.7. A slightly stronger version holds for inclusions of simple unital C^*-algebras—see Corollary 3.9. These two observations, respectively, are immediate consequences of the above mentioned perturbation results by Ino–Watatani and Dickson, and a very simple minded inequality

$$d_{\mathrm{KK}}(\mathcal{B}, u\mathcal{B}u^*) \leq 2d(u, \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \leq 2\|u - 1\|,$$

obtained in Lemma 3.5, for every unitary u in $\mathcal{A}$.

Then, in Section 5, we first recall and make some basic observations related to the different notions of distance between subalgebras of C^*-algebras and tracial von Neumann algebras, introduced by Christensen [5, 6, 7] and Mashood and Taylor [20]. The essence of this section lies in making comparisons between them and the Kadison–Kastler distance, and a desirable (and useful) observation made in the following proposition.

Proposition 5.7. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state. Then*

$$d_{\mathrm{MT}}(P, Q) = d_{\mathrm{MT}}(P, \overline{Q}^{\mathrm{S.O.T.}}) = d_{\mathrm{MT}}(\overline{P}^{\mathrm{S.O.T.}}, \overline{Q}^{\mathrm{S.O.T.}})$$

for any two unital $*$-subalgebras P and Q of $\mathcal{M}$.

Finally, in Section 6, we make use of the comparisons made in Section 5 and provide some concrete calculations of various distances between subalgebras associated to distinct subgroups (via C^*-crossed products, Banach group algebras and group von Neumann algebras) of a given discrete group G. Interestingly, in all cases, they turn out to be distance 1 apart—see Corollary 6.7, Proposition 6.21 and Proposition 6.23.

2. Preliminaries

We briefly recall the notions of finite-index conditional expectations, Watatani's C^*-basic construction and compatible intermediate C^*-subalgebras.

2.1. **Watatani's C^*-basic construction.** For any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras with common unit and a faithful conditional expectation $E : \mathcal{A} \to \mathcal{B}$, $\mathcal{A}$ becomes a pre-Hilbert $\mathcal{B}$-module with respect to the $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}} : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ given by $\langle x, y \rangle_{\mathcal{B}} = E(x^*y)$ for $x, y \in \mathcal{A}$. We denote by $\mathcal{E}$ the Hilbert $\mathcal{B}$-module completion of $\mathcal{A}$.

In order to distinguish the elements of the C^*-algebra $\mathcal{A}$ and the pre-Hilbert $\mathcal{B}$-module $\mathcal{A}$, following [24], we consider the inclusion map $\eta : \mathcal{A} \to \mathcal{A} \subset \mathcal{E}$. Thus,

$$\|\eta(x)\| := \|E(x^*x)\|^{1/2} \leq \|x\|$$

for all $x \in \mathcal{A}$. Let $\mathcal{L}_{\mathcal{B}}(\mathcal{E})$ denote the unital C^*-algebra consisting of adjointable maps on $\mathcal{E}$. Every member of $\mathcal{L}_{\mathcal{B}}(\mathcal{E})$ is a $\mathcal{B}$-module map (see [24, §2.10]). Further, there is a natural C^*-embedding $\lambda : \mathcal{A} \to \mathcal{L}_{\mathcal{B}}(\mathcal{E})$ satisfying

$$\lambda(a)\eta(x) = \eta(ax) \quad \text{for all } a, x \in \mathcal{A}.$$

Thus, we can identify $\mathcal{A}$ with its image $\lambda(\mathcal{A})$ in $\mathcal{L}_{\mathcal{B}}(\mathcal{E})$. Further, there is a natural projection $e_1 \in \lambda(\mathcal{B})' \cap \mathcal{L}_{\mathcal{B}}(\mathcal{E})$ (called the Jones projection corresponding to E) satisfying, via the above identification, the relations

$$e_1(\eta(x)) = \eta(E(x)) \quad \text{and} \quad e_1 x e_1 = E(x)e_1$$

for all $x \in \mathcal{A}$. Watatani's C^*-basic construction for the inclusion $\mathcal{B} \subset \mathcal{A}$ with respect to the conditional expectation E is defined as the C^*-algebra

$$\mathcal{A}_1 = \overline{\operatorname{span}}\{x e_1 y \mid x, y \in \mathcal{A}\} \subseteq \mathcal{L}_{\mathcal{B}}(\mathcal{E}).$$

Like the "Jones" basic construction for a subfactor, Watatani's basic construction also satisfies a natural universal property—see [24, Prop. 2.2.11].

Some standard notation. Recall that, for an inclusion $\mathcal{B} \subset \mathcal{A}$ of C^*-algebras, the centralizer of $\mathcal{B}$ in $\mathcal{A}$ is defined by

$$\mathcal{C}_{\mathcal{A}}(\mathcal{B}) = \{x \in \mathcal{A} \mid bx = xb \text{ for all } b \in \mathcal{B}\},$$

and if $\mathcal{B} \subset \mathcal{A}$ are unital (with common unit), the normalizer of $\mathcal{B}$ in $\mathcal{A}$ is defined by

$$\mathcal{N}_{\mathcal{A}}(\mathcal{B}) = \{u \in \mathcal{U}(\mathcal{A}) \mid u\mathcal{B}u^* = \mathcal{B}\}.$$

The centralizer $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$ is a C^*-subalgebra of $\mathcal{A}$ and is also denoted by $\mathcal{B}' \cap \mathcal{A}$ and called the relative commutant of $\mathcal{B}$ in $\mathcal{A}$. The normalizer $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$ is a closed subgroup of $\mathcal{U}(\mathcal{A})$.

2.2. **Finite-index conditional expectations and index.** For any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras, a conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is said to have finite index if there exists a finite set $\{\lambda_i \mid 1 \leq i \leq n\} \subset \mathcal{A}$ such that $x = \sum_{i=1}^n E(x\lambda_i)\lambda_i^*$ for all $x \in \mathcal{A}$. Such a set $\{\lambda_i\}$ is called a quasi-basis for E and the Watatani index of E is defined as $\mathrm{Ind}_W(E) = \sum_{i=1}^n \lambda_i\lambda_i^*$, which is a positive invertible element in $\mathcal{Z}(\mathcal{A})$ and is independent of the quasi-basis $\{\lambda_i\}$. Every such E is faithful and preserves the unit, *i.e.*, $1_{\mathcal{B}} = E(1_{\mathcal{A}}) = 1_{\mathcal{A}}$.

The set of finite-index conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$ is denoted by $\mathcal{E}_0(A, B)$. If $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$ and $\mathcal{E}_0(A, B) \neq \varnothing$, then an $F \in \mathcal{E}_0(\mathcal{A}, \mathcal{B})$ is said to be minimal if

$$\mathrm{Ind}_W(F) = \inf\{\mathrm{Ind}_W(E) \mid E \in \mathcal{E}_0(\mathcal{A}, \mathcal{B})\}.$$

In some nice cases, a minimal conditional expectation exists and is unique as well.

Remark 2.3. Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras with

$$\mathcal{E}_0(\mathcal{A}, \mathcal{B}) \neq \varnothing \quad \text{and} \quad \mathcal{Z}(\mathcal{A}) = \mathbb{C} = \mathcal{Z}(\mathcal{B}).$$

(i) There exists a unique minimal conditional expectation (denoted usually by) $E_0 : \mathcal{B} \to \mathcal{A}$ (see [24, Thm. 2.12.3]).
(ii) The index of such an inclusion is defined as (see [24, Def. 2.12.4])

$$[\mathcal{A} : \mathcal{B}]_0 = \mathrm{Ind}_W(E_0) = \min\{\mathrm{Ind}_W(E) \mid E \in \mathcal{E}_0(\mathcal{A}, \mathcal{B})\}.$$

2.3.1. *Compatible intermediate C^*-subalgebras.* For any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras (with common unit), let $\mathcal{I}(\mathcal{B} \subset \mathcal{A})$ denote the collection of intermediate C^*-subalgebras of the inclusion $\mathcal{B} \subset \mathcal{A}$ and let $\mathcal{E}(\mathcal{A}, \mathcal{B})$ denote the set of conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$. Further, for any $E \in \mathcal{E}(\mathcal{A}, \mathcal{B})$, following [11] (also see [3, 9]), let

$$\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E) := \{\mathcal{C} \in \mathcal{I}(\mathcal{B} \subset \mathcal{A}) \mid \text{there exists an } F \in \mathcal{E}(\mathcal{A}, C) \\ \text{such that } E_{\restriction_C} \circ F = E\}.$$

If E has finite index, then for any $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$, a compatible conditional expectation from $\mathcal{A}$ onto $\mathcal{C}$ is unique and has finite index (see [11, p. 471]).

2.3.2. *Dual conditional expectation and iterated tower of basic constructions.*

Remark 2.4. Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras and let $E : \mathcal{A} \to \mathcal{B}$ be a finite-index conditional expectation. Let $\mathcal{A}_1$ denote the C^*- basic construction of $\mathcal{B} \subset \mathcal{A}$ with respect to the conditional expectation E and let e_1 denote the corresponding Jones projection. The following facts are noteworthy and shall be needed ahead.
(i) $\sum_i \lambda_i e_1 \lambda_i^* = 1$ for any quasi-basis $\{\lambda_i\}$ of E.
(ii) $\mathcal{A}_1 = \mathrm{span}\{xe_1y \mid x, y \in \mathcal{A}\} = C^*(\mathcal{A}, e_1) = \mathcal{L}_{\mathcal{B}}(\mathcal{A})$ (see [24, Prop. 1.3.3]).
(iii) There exists a unique finite-index conditional expectation $\tilde{E} : \mathcal{A}_1 \to \mathcal{A}$ satisfying $\tilde{E}(xe_1y) = (\mathrm{Ind}_W(E))^{-1}xy$ for all $x, y \in \mathcal{A}$ (see [24, Prop. 1.6.1]). ($\tilde{E}$ is called the dual conditional expectation of E and is often denoted by E_1.)

(iv) By iteration, one obtains a tower of unital C^*-algebras (see [13, §3.1], [15, Prop. 3.18] and [3, §2.3])

$$\mathcal{A}_{-1} := \mathcal{B} \subset \mathcal{A}_0 := \mathcal{A} \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_k \subset \cdots$$

with finite-index conditional expectations $E_k : \mathcal{A}_k \to \mathcal{A}_{k-1}$ and Jones projections $e_k \in \mathcal{A}_k$, $k \geq 1$, with $\mathcal{A}_k = C^*(\mathcal{A}_{k-1}, e_k)$ for all $k \geq 1$. Also, E_{k+1} is the dual of E_k for all $k \geq 0$, with $E_0 := E$.

(v) If $\mathrm{Ind}_W(E) \in B$, then $\mathrm{Ind}_W(\tilde{E}) = \mathrm{Ind}_W(E)$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are both simple and E is minimal, then $\mathcal{A}_1$ is simple and $\tilde{E}$ is minimal ([24, Cor. 2.2.14, Prop. 2.3.4] and [15, Cor. 3.4]).

(vi) (Pushdown Lemma.) For each $x_1 \in \mathcal{A}_1$, there exists a unique $x_0 \in \mathcal{A}$ such that $x_1 e_1 = x_0 e_1$ and x_0 is given by $x_0 = \mathrm{Ind}_W(E)\tilde{E}(x_1 e_1)$ (see [13, Lem. 3.7]).

In [3, Prop. 3.2], it was shown (using the so-called "Fourier transforms") that if $\mathcal{B} \subset \mathcal{A}$ is an inclusion of simple unital C^*-algebras with $\mathcal{E}_0(\mathcal{A}, \mathcal{B}) \neq \varnothing$, then $\mathcal{B}' \cap \mathcal{A}_k \cong \mathcal{A}' \cap \mathcal{A}_{k+1}$ (as vector spaces) for all $k \geq 0$. In particular, if $\mathcal{B} \subset \mathcal{A}$ is irreducible, then so is $\mathcal{A} \subset \mathcal{A}_1$. It turns out that the last inference is true for more general inclusions and will be needed ahead.

Lemma 2.5. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. Then*

(i) $E_k(\mathcal{A}'_{k-2} \cap \mathcal{A}_k) = \mathcal{A}'_{k-2} \cap \mathcal{A}_{k-1}$ *for every $k \geq 1$, and*

(ii) *if, in addition, the inclusion $\mathcal{B} \subset \mathcal{A}$ is irreducible, then the inclusions $\mathcal{A}_{k-1} \subset \mathcal{A}_k$, $k \geq 1$, are all irreducible.*

Proof. It is enough to prove for $k = 1$.

(i) Clearly, for each $x_1 \in \mathcal{B}' \cap \mathcal{A}_1$, $\tilde{E}(x_1)b = \tilde{E}(x_1 b) = \tilde{E}(b x_1) = b\tilde{E}(x_1)$ for all $b \in \mathcal{B}$. Hence, $\tilde{E}(\mathcal{B}' \cap \mathcal{A}_1) \subseteq \mathcal{B}' \cap \mathcal{A}$. For the reverse inclusion, note that, for each $a \in \mathcal{B}' \cap \mathcal{A}$, $x_1 := \mathrm{Ind}_W(E) a e_1 \in \mathcal{B}' \cap \mathcal{A}_1$ and $\tilde{E}(x_1) = a$ by Remark 2.4. Hence, $\mathcal{B}' \cap \mathcal{A} \subseteq \tilde{E}(\mathcal{B}' \cap \mathcal{A}_1)$.

(ii) Now, suppose that $\mathcal{B} \subset \mathcal{A}$ is irreducible and that $x_1 \in \mathcal{A}' \cap \mathcal{A}_1$. Let $\{\lambda_i \mid 1 \leq i \leq n\} \subset \mathcal{A}$ be a quasi basis for E. Then $x_0 := \mathrm{Ind}_W(E)\tilde{E}(x_1 e_1) \in \mathcal{A}$ and $x_1 e_1 = x_0 e_1$ by Remark 2.4 (vi). Further, for every $b \in B$, we have

$$\begin{aligned} x_0 b &= \mathrm{Ind}_W(E)\tilde{E}(x_1 e_1) b = \mathrm{Ind}_W(E)\tilde{E}(x_1 e_1 b) \\ &= \mathrm{Ind}_W(E)\tilde{E}(b x_1 e_1) = b\mathrm{Ind}_W(E)\tilde{E}(x_1 e_1) = b x_0. \end{aligned}$$

Hence, $x_0 \in \mathcal{B}' \cap \mathcal{A} = \mathbb{C}$ so that $x_0 = \beta\mathbf{1}$ for some $\beta \in \mathbb{C}$, which then shows that

$$x_1 = x_1 \sum_{i=1}^{n} \lambda_i e_1 \lambda_i^* = \sum_{i=1}^{n} \lambda_i x_1 e_1 \lambda_i^* = \beta \sum_{i=1}^{n} \lambda_i e_1 \lambda_i^* = \beta\mathbf{1},$$

where the first equality holds because of Remark 2.4 (i). This implies that $\mathcal{A}' \cap \mathcal{A}_1 = \mathbb{C}\mathbf{1}$, and we are done. □

See [24, 11] for more on Watatani index, C^*-basic construction and compatible intermediate C^*-subalgebras.

3. Kadison–Kastler distance

For any normed space X, as is standard, its closed unit ball will be denoted by $B_1(X)$, and for any subset K of X and an element $x \in X$, the distance between x and K is defined as

$$d(x, K) = \inf\{\|x - y\| \mid y \in K\}.$$

Definition 3.1 ([14])**.** The Kadison–Kastler distance between any two subalgebras $\mathcal{C}$ and $\mathcal{D}$ of a normed algebra $\mathcal{A}$ (which we denote by $d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D})$) is defined as the Hausdorff distance between the closed unit balls of $\mathcal{C}$ and $\mathcal{D}$, *i.e.*,

$$d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) = \max\Big\{ \sup_{x \in B_1(\mathcal{C})} d(x, B_1(\mathcal{D})), \sup_{z \in B_1(\mathcal{D})} d(z, B_1(\mathcal{C}))\Big\}.$$

The Kadison–Kastler distance makes sense even for two subspaces of a normed space, but we shall work mainly with the distance between subalgebras of a normed algebra.

We must remark that the notation d_{KK} is not standard. We have used it for the Kadison–Kastler distance in order to keep a distinction between it and two other notions of distance introduced by Christensen and a distance introduced by Mashood and Taylor, which shall be discussed in Section 5.

Notation. For a normed algebra $\mathcal{A}$, let

$$\mathrm{Sub}_{\mathcal{A}} := \{\text{subalgebras of } \mathcal{A}\},$$
$$C\text{-}\mathrm{Sub}_{\mathcal{A}} := \{\text{closed subalgebras of } \mathcal{A}\},$$

and, if $\mathcal{A}$ is a C^*-algebra, then let

$$C^*\text{-}\mathrm{Sub}_{\mathcal{A}} := \{C^*\text{-subalgebras of } \mathcal{A}\}.$$

Here are some well-known elementary observations related to the Kadison–Kastler distance.

Remark 3.2. Let $\mathcal{A}$ be a normed algebra.
(i) $d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) \leq 1$ for all $\mathcal{C}, \mathcal{D} \in \mathrm{Sub}_{\mathcal{A}}$.
(ii) If $\mathcal{A}$ is a Banach algebra, then d_{KK} is a metric on C-$\mathrm{Sub}_{\mathcal{A}}$.
(iii) If $\mathcal{C}, \mathcal{D} \in C$-$\mathrm{Sub}_{\mathcal{A}}$ and $\mathcal{C}$ is a proper subalgebra of $\mathcal{D}$, then $d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) = 1$ (see [11, Lem. 2.1]).

The following elementary observation is obvious and will be used to calculate the distance between certain C^*-subalgebras in Section 6.

Lemma 3.3. *Let $\mathcal{A}$ be a normed algebra. Then*

$$d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) = d_{\mathrm{KK}}(\mathcal{C}, \overline{\mathcal{D}}) = d_{\mathrm{KK}}(\overline{\mathcal{C}}, \mathcal{D}) = d_{\mathrm{KK}}(\overline{\mathcal{C}}, \overline{\mathcal{D}})$$

for all $\mathcal{C}, \mathcal{D} \in \mathrm{Sub}_{\mathcal{A}}$.

It is natural to ask whether, for a given subalgebra $\mathcal{B}$ of a C^*-algebra $\mathcal{A}$, one can always find a subalgebra as close as one desires. In [10, Ex. 2.2.2], it was shown that $d_{\mathrm{KK}}(\mathcal{B}, u\mathcal{B}u^*) \leq \|u - \mathbf{1}\|$ for all $u \in \mathcal{U}(\mathcal{A})$. Thus, because of the following lemma, one can get a conjugate subalgebra as close as one wishes. (Its proof follows from an elementary continuous functional calculus argument and we leave it to the reader.)

Lemma 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra. Then, for each $\epsilon > 0$, there exists a unitary u in $\mathcal{A}$ such that $0 < \|u - \mathbf{1}\| < \epsilon$.*

Further, for any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras, it is also quite natural to ask whether, for a unitary $u \in \mathcal{U}(A)$, there exists any relationship between its distance from $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$ and the Kadison–Kastler distance between $\mathcal{B}$ and its conjugate $u\mathcal{B}u^*$.

Interestingly, motivated by an inequality given by Popa–Sinclair–Smith [22, Lem. 6.3], we obtain the following pleasing relationship without much effort, which then has a nice consequence that if $E \in \mathcal{E}_0(\mathcal{A}, \mathcal{B})$, then a unitary which normalizes $\mathcal{B}$ and is sufficiently close to $\mathcal{N}_{\mathcal{A}}(\mathcal{C})$ must belong to $\mathcal{N}_{\mathcal{A}}(\mathcal{C})$ for any $\mathcal{C} \in \mathcal{F}(\mathcal{B}, \mathcal{A}, E)$ (see (1))—see Proposition 3.6.

Lemma 3.5. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ a unital C^*-subalgebra of $\mathcal{A}$. Then*

$$d_{\mathrm{KK}}(\mathcal{B}, u\mathcal{B}u^*) \leq 2d(u, \mathcal{N}_{\mathcal{A}}(\mathcal{B})) \leq 2\|u - \mathbf{1}\|$$

for all $u \in \mathcal{U}(\mathcal{A})$.

Proof. Let $u \in \mathcal{U}(\mathcal{A})$. Clearly, $d(u, \mathcal{N}_{\mathcal{A}}(\mathcal{B})) \leq \|u - \mathbf{1}\|$. Next, let $v \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$ and $w := uv^*$. Then $w\mathcal{B}w^* = (uv^*)\mathcal{B}(uv^*)^* = u(v^*\mathcal{B}v)u^* = u\mathcal{B}u^*$. So, for any $x \in B_1(\mathcal{B})$,

$$\|x - wxw^*\| = \|xw - wx\| \leq \|xw - x\| + \|x - wx\| \leq 2\|w - \mathbf{1}\|.$$

This implies that $d(x, B_1(w\mathcal{B}w^*)) \leq 2\|w - \mathbf{1}\|$. Hence,

$$\sup_{x \in B_1(\mathcal{B})} d(x, B_1(w\mathcal{B}w^*)) \leq 2\|w - \mathbf{1}\|.$$

Likewise,

$$\sup_{y \in B_1(w\mathcal{B}w^*)} d(y, B_1(\mathcal{B})) \leq 2\|w - \mathbf{1}\|.$$

So

$$d_{\mathrm{KK}}(\mathcal{B}, w\mathcal{B}w^*) \leq 2\|w - \mathbf{1}\|.$$

Thus, $d_{\mathrm{KK}}(\mathcal{B}, w\mathcal{B}w^*) \leq 2\|u - v\|$ for all $v \in N_{\mathcal{A}}(\mathcal{B})$. Hence,

$$d_{\mathrm{KK}}(\mathcal{B}, u\mathcal{B}u^*) = d_{\mathrm{KK}}(\mathcal{B}, w\mathcal{B}w^*) \leq 2d(u, N_{\mathcal{A}}(\mathcal{B})). \qquad \square$$

In the reverse direction, Kadison and Kastler [14] had conjectured that sufficiently close subalgebras must be conjugates of each other. This was answered in the affirmative for various cases in a series of some fundamental papers by Christensen, Phillips, Raeburn and others in the 1970s. Since then, there have been several other such so-called "perturbation results". People have also employed such perturbation results to answer other important questions. One such result with a nice application is due to Ino and Watatani [11]. In fact, some of the results in this article are direct applications of the perturbation result of Ino and Watatani.

Consider the following question, which arises naturally from Lemma 3.5.

Question. Given a unital C^*-algebra $\mathcal{A}$ and a C^*-subalgebra $\mathcal{B}$, is every unitary sufficiently close to $\mathbf{1}$ (or to $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$) in the normalizer of the subalgebra $\mathcal{B}$?

It is easily seen—see, for instance, Example 6.18—that it has a negative answer. However, interestingly, Lemma 3.5 along with a perturbation result of Ino and Watatani [11, Prop. 3.6] immediately yields a somewhat positive answer for compatible intermediate C^*-subalgebras—see Proposition 3.6 and Corollary 3.7 below.

Notation. Given an inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$, let

$$\mathcal{F}(\mathcal{B}, \mathcal{A}, E) := \{\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E) \mid \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{C}) \cup \mathcal{C}\}. \tag{1}$$

Note that $\mathcal{F}(\mathcal{B}, \mathcal{A}, E) = \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ if $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$.

Proposition 3.6. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. Then there exists a constant $\alpha > 0$ such that $\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \mid d(u, \mathcal{N}_{\mathcal{A}}(\mathcal{C})) < \alpha\} \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{C})$ for every $\mathcal{C} \in \mathcal{F}(\mathcal{B}, \mathcal{A}, E)$.*

In particular, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$, then $\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \mid d(u, \mathcal{N}_{\mathcal{A}}(\mathcal{C})) < \alpha\} \subseteq \mathcal{N}_{\mathcal{A}}(\mathcal{C})$ for every $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$.

Proof. Let N be the number of elements (which is not unique) in a quasi-basis for E and let

$$\alpha = \frac{0.5}{(10N)^4}.$$

First, note that, for any $\mathcal{C} \in \mathrm{IMS}(B, A, E)$, with respect to the compatible finite-index conditional expectation $F : \mathcal{A} \to \mathcal{C}$, and any $u \in \mathcal{U}(\mathcal{A})$, there exists a faithful conditional expectation $F_u : \mathcal{A} \to u\mathcal{C}u^*$ given by $F_u = \mathrm{Ad}_u \circ F \circ \mathrm{Ad}_{u^*}$.

Now, let $\mathcal{C} \in \mathcal{F}(B, A, E)$ and $u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$ with $d(u, \mathcal{N}_{\mathcal{A}}(\mathcal{B})) < \alpha$. Then

$$\mathcal{B} \subseteq u\mathcal{C}u^* \subseteq \mathcal{A} \quad \text{and} \quad d_{\mathrm{KK}}(\mathcal{C}, u\mathcal{C}u^*) < \frac{1}{(10N)^4},$$

by Lemma 3.5. Thus, by [11, Prop. 3.6], there exists a $v \in \mathcal{U}(\mathcal{B}' \cap \mathcal{A})$ such that $v\mathcal{C}v^* = u\mathcal{C}u^*$. Since $\mathcal{B}' \cap \mathcal{A} \subseteq (\mathcal{C}' \cap \mathcal{A}) \cup \mathcal{C}$, it follows that $v\mathcal{C}v^* = \mathcal{C}$. Hence, $u \in \mathcal{N}_{\mathcal{A}}(\mathcal{C})$, and we are done. ☐

Corollary 3.7. *Let $\mathcal{A}$, $\mathcal{B}$, E and α be as in Proposition 3.6. Then*

$$\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \mid \|u - \mathbf{1}\| < \alpha\} \subseteq \bigcap \{\mathcal{N}_{\mathcal{A}}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{F}(\mathcal{B}, \mathcal{A}, E)\}.$$

In particular, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$, then

$$\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \mid \|u - \mathbf{1}\| < \alpha\} \subseteq \bigcap \{\mathcal{N}_{\mathcal{A}}(\mathcal{C}) \mid \mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)\}.$$

As an application of a perturbation result by Dickson [8] and a fundamental result regarding finiteness of the index of a conditional expectation by Izumi [12], we have a slightly more general variant of Proposition 3.6 for inclusions of simple unital C^*-algebras.

Notation. For any inclusion $\mathcal{B}\subset\mathcal{A}$ of unital C^*-algebras (with common unit), let
$$\mathcal{I}_0(\mathcal{B}\subset\mathcal{A}) := \{\mathcal{C}\in\mathcal{I}(\mathcal{B}\subset\mathcal{A}) \mid \mathcal{C}_{\mathcal{A}}(\mathcal{B})\subseteq\mathcal{C}_{\mathcal{A}}(\mathcal{C})\cup\mathcal{C}\}.$$
Clearly, $\mathcal{F}(\mathcal{B},\mathcal{A},E)\subseteq\mathcal{I}_0(\mathcal{B}\subset\mathcal{A})$ for every $E\in\mathcal{E}_0(\mathcal{A},\mathcal{B})$, and if $\mathcal{C}_{\mathcal{A}}(\mathcal{B})\subseteq\mathcal{B}$, then $\mathcal{I}_0(\mathcal{B}\subset\mathcal{A})=\mathcal{I}(\mathcal{B}\subset\mathcal{A})$.

Proposition 3.8. *Let $\mathcal{B}\subset\mathcal{A}$ be an inclusion of simple unital C^*-algebras with a finite-index conditional expectation $E:\mathcal{A}\to\mathcal{B}$. Then, for any $0<\gamma<10^{-6}$, $\{u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})\mid d(u,\mathcal{N}_{\mathcal{A}}(\mathcal{C}))<\frac{\gamma}{2}\}\subseteq\mathcal{N}_{\mathcal{A}}(\mathcal{C})$ for every $\mathcal{C}\in\mathcal{I}_0(\mathcal{B}\subset\mathcal{A})$.*

In particular, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B})\subseteq\mathcal{B}$, then $\{u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})\mid d(u,\mathcal{N}_{\mathcal{A}}(\mathcal{C}))<\frac{\gamma}{2}\}\subseteq\mathcal{N}_{\mathcal{A}}(\mathcal{C})$ for every $\mathcal{C}\in\mathcal{I}(\mathcal{B}\subset\mathcal{A})$.

Proof. First, note that, for any $\mathcal{C}\in\mathcal{I}(\mathcal{B}\subset\mathcal{A})$, by [12, Prop. 6.1], there exists a conditional expectation $F:\mathcal{A}\to\mathcal{C}$ of finite index. Hence, it is faithful.

Now, let $\mathcal{C}\in\mathcal{I}_0(\mathcal{B}\subset\mathcal{A})$, $0<\gamma<10^{-6}$ and $u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})$ with $d(u,\mathcal{N}_{\mathcal{A}}(\mathcal{C}))<\frac{\gamma}{2}$. Then $\mathcal{B}\subseteq u\mathcal{C}u^*\subseteq\mathcal{A}$ and $d_{\mathrm{KK}}(\mathcal{C},u\mathcal{C}u^*)<\gamma<10^{-6}$ by Lemma 3.5. Note that, as E is of finite Watatani index, it satisfies the Pimsner–Popa inequality; in particular, so does the restriction $E_{\restriction u\mathcal{C}u^*}:u\mathcal{C}u^*\to\mathcal{B}$. Since $\mathcal{B}$ is simple, it follows from [12, Cor. 3.4] that $E_{\restriction u\mathcal{C}u^*}$ also has a finite quasi-basis. Thus, by [8, Thm. 3.7], there exists a $v\in\mathcal{U}(\mathcal{B}'\cap\mathcal{A})$ such that $v\mathcal{C}v^*=u\mathcal{C}u^*$. Since $\mathcal{B}'\cap\mathcal{A}\subseteq(\mathcal{C}'\cap\mathcal{A})\cup\mathcal{C}$, it follows that $v\mathcal{C}v^*=\mathcal{C}$. Hence, $u\in\mathcal{N}_{\mathcal{A}}(\mathcal{C})$, and we are done. □

Corollary 3.9. *Let $\mathcal{A}$, $\mathcal{B}$ and E be as in Proposition 3.8. Then, for any $0<\gamma<10^{-6}$,*
$$\Big\{u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})\;\Big|\;\|u-\mathbf{1}\|<\frac{\gamma}{2}\Big\}\subseteq\bigcap\{\mathcal{N}_{\mathcal{A}}(\mathcal{C})\mid\mathcal{C}\in\mathcal{I}_0(\mathcal{B}\subset\mathcal{A})\}.$$
In particular, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B})\subseteq\mathcal{B}$, then
$$\Big\{u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})\;\Big|\;\|u-\mathbf{1}\|<\frac{\gamma}{2}\Big\}\subseteq\bigcap\{\mathcal{N}_{\mathcal{A}}(\mathcal{C})\mid\mathcal{C}\in\mathcal{I}(\mathcal{B}\subset\mathcal{A})\}.$$

For any group G, let Sub_G denote the collection of subgroups of G. Then G admits a canonical action on Sub_G via conjugation, *i.e.*,
$$G\times\mathrm{Sub}_G\ni(g,H)\mapsto gHg^{-1}\in\mathrm{Sub}_G.$$

Corollary 3.10. *Let $\mathcal{A}$, $\mathcal{B}$, E and α be as in Proposition 3.6 and let $G:=\mathcal{N}_{\mathcal{A}}(\mathcal{B})$. Then, with respect to the natural conjugation action of G on Sub_G, the following hold.*

(i) *The open ball*
$$\begin{aligned}&\{u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})\mid\|u-\mathbf{1}\|<\alpha\}\\&\qquad\subseteq\bigcap\big\{\mathrm{Stab}_G\big(\mathcal{N}_{\mathcal{A}}(\mathcal{B})\cap\mathcal{N}_{\mathcal{A}}(\mathcal{C})\big)\;\big|\;\mathcal{C}\in\mathcal{F}(\mathcal{B},\mathcal{A},E)\big\},\end{aligned}$$
and in particular, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B})\subseteq\mathcal{B}$, then
$$\begin{aligned}&\{u\in\mathcal{N}_{\mathcal{A}}(\mathcal{B})\mid\|u-\mathbf{1}\|<\alpha\}\\&\qquad\subseteq\bigcap\big\{\mathrm{Stab}_G\big(\mathcal{N}_{\mathcal{A}}(\mathcal{B})\cap\mathcal{N}_{\mathcal{A}}(\mathcal{C})\big)\;\big|\;\mathcal{C}\in\mathrm{IMS}(\mathcal{B},\mathcal{A},E)\big\}.\end{aligned}$$

(ii) *In addition, if $\mathcal{A}$ and $\mathcal{B}$ are both simple, then for any $0 < \gamma < 10^{-6}$, the open ball*

$$\Big\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \;\Big|\; \|u - \mathbf{1}\| < \frac{\gamma}{2}\Big\} \subseteq \bigcap \big\{\mathrm{Stab}_G\big(\mathcal{N}_{\mathcal{A}}(\mathcal{B}) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C})\big) \;\big|\; \mathcal{C} \in \mathcal{I}_0(\mathcal{B} \subset \mathcal{A})\big\},$$

and in particular, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$, then

$$\Big\{u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \;\Big|\; \|u - \mathbf{1}\| < \frac{\gamma}{2}\Big\} \subseteq \bigcap \big\{\mathrm{Stab}_G\big(\mathcal{N}_{\mathcal{A}}(\mathcal{B}) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C})\big) \;\big|\; \mathcal{C} \in \mathcal{I}(\mathcal{B} \subset \mathcal{A})\big\}.$$

4. Some finiteness results

This section is devoted to some more applications of certain perturbation results from [11, 8] to generalize some finiteness results by Ino–Watatani [11] and Khoshkam–Mashood [18].

4.1. Finiteness of certain compatible intermediate C^*-subalgebras.

Theorem 4.2. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. If one (equivalently, any) of the algebras $\mathcal{C}_{\mathcal{A}}(\mathcal{B}), \mathcal{Z}(\mathcal{B}), \mathcal{Z}(\mathcal{A})$ is finite-dimensional, then the collection $\mathcal{F}(\mathcal{B}, \mathcal{A}, E)$ (as in* (1)*) is finite.*

Proof. First, note that, since there exists a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$, it follows from [24, Prop. 2.7.3] that the C^*-subalgebras $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$, $\mathcal{Z}(\mathcal{B})$ and $\mathcal{Z}(\mathcal{A})$ are either all finite-dimensional or none of them is finite-dimensional.

Consider Watatani's C^*-basic construction $\mathcal{A}_1 := C^*(\mathcal{A}, e_1)$ of the inclusion $\mathcal{B} \subset \mathcal{A}$ with respect to the conditional expectation E and Jones projection e_1. In view of the preceding paragraph and the given hypothesis, $\mathcal{Z}(\mathcal{B})$ is finite-dimensional, so by [24, Prop. 2.7.3] again, the relative commutant $\mathcal{B}' \cap \mathcal{A}_1$ is finite-dimensional. Therefore, the set

$$\mathcal{P} := \{p \in \mathcal{B}' \cap \mathcal{A}_1 \mid p \text{ is a projection}\}$$

is a compact Hausdorff space with respect to the operator norm.

Further, note that if $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the compatible conditional expectation $F : \mathcal{A} \to \mathcal{C}$, then F has finite index and $\mathcal{C}_1 \subseteq \mathcal{A}_1$ (see [9, Prop. 2.7 (2)], so the corresponding Jones projection $e_{\mathcal{C}}$ belongs to

$$\mathcal{C}' \cap \mathcal{C}_1 \subseteq \mathcal{B}' \cap \mathcal{A}_1,$$

where $\mathcal{C}_1$ denotes the C^*-basic construction of the inclusion $\mathcal{C} \subset \mathcal{A}$ with respect to the finite-index conditional expectation F and Jones projection $e_{\mathcal{C}}$.

Fix a $0 < \gamma < 10^{-6}$ and let

$$\varepsilon = \frac{\gamma}{2\|\mathrm{Ind}_W(E)\|}.$$

By the compactness of $\mathcal{P}$, there exists a finite cover of $\mathcal{P}$ consisting of open balls of radius ε. So it suffices to show that each such ε-ball contains only finitely many Jones projections corresponding to the members of $\mathcal{F}(\mathcal{B},\mathcal{A},E)$.

Note that, for any two $\mathcal{C},\mathcal{D}\in \mathrm{IMS}(\mathcal{B},\mathcal{A},E)$, their corresponding Jones projections $e_{\mathcal{C}}$ and $e_{\mathcal{D}}$ are in $\mathcal{B}'\cap\mathcal{A}_1$ and satisfy

$$d_{\mathrm{KK}}(\mathcal{C},\mathcal{D})\leq \|\mathrm{Ind}_W(E)\|\|e_{\mathcal{C}}-e_{\mathcal{D}}\|,$$

by [11, Lem. 3.3]. In particular, if $e_{\mathcal{C}}$ and $e_{\mathcal{D}}$ are in one of the ε-open balls, then $d_{\mathrm{KK}}(\mathcal{C},\mathcal{D})\leq\frac{\gamma}{2}<\gamma$. Thus, by [8, Thm. 3.7], there exists a unitary u in $\mathcal{B}'\cap\mathcal{A}$ such that $\mathcal{C}=u\mathcal{D}u^*$ (and $\|u-1\|\leq 16\sqrt{110}\gamma^{\frac{1}{2}}+880\gamma$).

Let $\mathcal{C},\mathcal{D}\in\mathcal{F}(\mathcal{B},\mathcal{A},E)$ and let they be in one ε-ball as above. As $u\in\mathcal{B}'\cap\mathcal{A}$ and $\mathcal{B}'\cap\mathcal{A}\subseteq\mathcal{C}_{\mathcal{A}}(\mathcal{D})\cup\mathcal{D}$, it follows that $\mathcal{C}=u\mathcal{D}u^*=\mathcal{D}$. Thus, each ε-open ball of the cover contains at most one Jones projection for some member of $\mathcal{F}(\mathcal{B},\mathcal{A},E)$, as was desired. □

We can immediately deduce the following generalization of [11, Cor. 3.9]. (The second part follows from Lemma 2.5.)

Corollary 4.3. *Let $\mathcal{B}\subset\mathcal{A}$ be an irreducible inclusion of unital C^*-algebras with a finite-index conditional expectation $E:\mathcal{A}\to\mathcal{B}$. Then* $\mathrm{IMS}(\mathcal{B},\mathcal{A},E)$ *is finite. In particular,* $\mathrm{IMS}(\mathcal{A}_k,\mathcal{A}_{k+1},E_{k+1})$ *is finite for every $k\geq 0$.*

For any unital inclusion of von Neumann algebras $\mathcal{N}\subset\mathcal{M}$, let $\mathcal{L}(\mathcal{N}\subset\mathcal{M})$ denote the lattice of intermediate von Neumann subalgebras. For any such inclusion, if there exists a faithful normal tracial state tr on $\mathcal{M}$ and the unique tr-preserving conditional expectation $E_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ has finite Watatani index, then it was shown in [3] that if $\mathcal{Z}(\mathcal{N})$ is finite-dimensional and the relative commutant $\mathcal{N}'\cap\mathcal{M}$ equals either $\mathcal{Z}(\mathcal{N})$ or $\mathcal{Z}(\mathcal{M})$, then $\mathcal{L}(\mathcal{N}\subset\mathcal{M})$ is finite. Analogous to [18, Thm. 1.3] and Theorem 4.2, we now have the following.

Proposition 4.4. *Let $\mathcal{N}\subset\mathcal{M}$ be a unital inclusion of finite von Neumann algebras with a (fixed) faithful normal tracial state* tr *on $\mathcal{M}$ such that the unique* tr*-preserving conditional expectation $E_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ has finite Watatani index. If one (equivalently, any) of the algebras $\mathcal{C}_{\mathcal{M}}(\mathcal{N})$, $\mathcal{Z}(\mathcal{N})$ and $\mathcal{Z}(\mathcal{M})$ is finite-dimensional, then the subcollection*

$$\mathcal{L}_0(\mathcal{N}\subset\mathcal{M}):=\{\mathcal{P}\in\mathcal{L}(\mathcal{N}\subset\mathcal{M}):\mathcal{N}'\cap\mathcal{M}\subseteq\mathcal{P}\cup(\mathcal{P}'\cap\mathcal{M})\}$$

is finite.

Proof. Clearly, $\mathcal{L}(\mathcal{N}\subset\mathcal{M})\subseteq\mathrm{IMS}(\mathcal{N},\mathcal{M},E_{\mathcal{N}})$. The rest follows from Theorem 4.2. □

Note that Proposition 4.4 also generalizes Watatani's finiteness result [25, Thm. 2.2] to a non-irreducible setting.

4.5. **Finiteness results for non-irreducible inclusions of simple C^*-algebras.** In general, if $\mathcal{B}\subset\mathcal{A}$ are simple unital C^*-algebras and the inclusion is not irreducible, then the lattice $\mathcal{I}(\mathcal{B}\subset\mathcal{A})$ need not be finite. For instance, consider the following easy example.

Example 4.6. Let $\mathcal{A} = M_2(\mathbb{C})$, $\mathcal{B} = \mathbb{C}I_2$, $\Delta = \{\mathrm{diag}(\lambda, \mu) \mid \lambda, \mu \in \mathbb{C}\}$, let $E : \mathcal{A} \to \mathcal{B}$ denote the canonical (tracial) conditional expectation given by

$$E([a_{ij}]) = \frac{(a_{11} + a_{22})}{2} I_2, \quad [a_{ij}] \in \mathcal{A},$$

and let $F : \mathcal{A} \to \Delta$ denote another canonical conditional expectation given by $F([a_{ij}]) = \mathrm{diag}(a_{11}, a_{22})$, $[a_{ij}] \in \mathcal{A}$.

Note that $u\Delta u^* \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ for all $u \in U(2)$ (by [9, Lem. 2.8]). Also, the set $\{u\Delta u^* \mid u \in U(2) \setminus \mathcal{N}_{\mathcal{A}}(\Delta)\}$ is infinite because the set of left cosets of $\mathcal{N}_{\mathcal{A}}(\Delta)$ in $U(2)$ is infinite. Hence, $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ and, therefore, $\mathcal{I}(\mathcal{B} \subset \mathcal{A})$ are infinite sets.

Here is an indirect way of seeing why $U(2)/\mathcal{N}_{\mathcal{A}}(\Delta)$ is infinite. Suppose, on the contrary, that $\{[u] := u\mathcal{N}_{\mathcal{A}}(\Delta) \mid u \in U(2)\}$ is finite. Then, for any element $w \in u\mathcal{N}_{\mathcal{A}}(\Delta)$, $w = uv$ for some $v \in \mathcal{N}_{\mathcal{A}}(\Delta)$, so

$$\alpha(\Delta, w\Delta w^*) = \alpha(\Delta, uv\Delta(uv)^*) = \alpha(\Delta, u\Delta u^*),$$

where α is the interior angle (see [9]). This implies $\alpha(\Delta, w\Delta w^*) = \alpha(\Delta, u\Delta u^*)$ for every $w \in [u]$. Hence, the set

$$\{\alpha(\Delta, u\Delta u^*) \mid u \in U(2)\} = \left\{\alpha(\Delta, u\Delta u^*) \;\middle|\; [u] \in \frac{U(2)}{\mathcal{N}_{\mathcal{A}}(\Delta)}\right\}$$

is finite. This contradicts the fact that $\{\alpha(\Delta, u\Delta u^*) \mid u \in U(2)\} = [0, \frac{\pi}{2}]$ (see [9, Cor. 4.6]). Thus, the set of left cosets of $\mathcal{N}_{\mathcal{A}}(\Delta)$ in $U(2)$ must be infinite.

However, for a non-irreducible inclusion $\mathcal{B} \subset A$ of simple unital C^*-algebras with a finite-index conditional expectation, we shall show in this section that the sublattice consisting of intermediate C^*-subalgebras of $\mathcal{B} \subset \mathcal{A}$ which contain the centralizer algebra $\mathcal{C}_{\mathcal{A}}(\mathcal{B})$ is finite.

The following useful observation (whose first part comes from [12] and the second part is comparable with [9, Prop. 2.7 (1), (2)]) will be needed ahead.

Proposition 4.7. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. Then, for any $\mathcal{C} \in \mathcal{I}(\mathcal{B} \subset \mathcal{A})$, there exists a finite-index conditional expectation $F : \mathcal{A} \to \mathcal{C}$ and, moreover, if $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}$, then $\mathcal{L}_{\mathcal{C}}(\mathcal{A}) \subseteq \mathcal{L}_{\mathcal{B}}(\mathcal{A})$, where $\mathcal{L}_{\mathcal{C}}(\mathcal{A})$ is defined with respect to F.*

In particular, $\mathcal{C}_1 \subset \mathcal{A}_1$, where $\mathcal{A}_1$ (resp., $\mathcal{C}_1$) denotes the C^*-basic construction of the inclusion $\mathcal{B} \subset \mathcal{A}$ (resp., $\mathcal{C} \subset \mathcal{A}$) with respect to E (resp., F).

Proof. Let $\mathcal{C} \in \mathcal{I}(\mathcal{B} \subset \mathcal{A})$. Then, by [12, Prop. 6.1], there exists a conditional expectation $F : \mathcal{A} \to \mathcal{C}$ of finite index. Next, suppose that $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) = \mathcal{B}' \cap \mathcal{A} \subseteq \mathcal{C}$.

Note that, as E is of finite Watatani index, it satisfies the Pimsner–Popa inequality [24]; in particular, so does the restriction $E_{\restriction C} : \mathcal{C} \to \mathcal{B}$. Since $\mathcal{B}$ is simple, it then follows from [12, Cor. 3.4] that $E_{\restriction \mathcal{C}}$ also has a finite quasi-basis. Thus, $G := E_{\restriction \mathcal{C}} \circ F$ is of finite Watatani index by [24, Prop. 1.7.1]. Since E and G are two finite-index conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$, by [24, Prop. 2.10.9], there exists a $q \in \mathcal{B}' \cap \mathcal{A}$ such that $E(x) = G(q^* x q)$ for all $x \in \mathcal{A}$.

Then, for any $T \in \mathcal{L}_C(\mathcal{A})$, we observe that

$$\begin{aligned}\langle T(x), y\rangle_{\mathcal{B}} &= E(T(x)^*y) = G(q^*T(x)^*yq) = E_{\restriction_C}(F((T(x)q)^*yq))\\ &= E_{\restriction_C}(\langle T(x)q, yq\rangle_C) = E_{\restriction C}(\langle xq, T^*(yq)\rangle_C)\\ &= E_{\restriction C}\big(F(q^*x^*T^*(yq))\big) = E_{\restriction C}\big(F(q^*x^*T^*(y)q)\big)\\ &= G(q^*x^*T^*(y)q) = E(x^*T^*(y)) = \langle x, T^*(y)\rangle_{\mathcal{B}}\end{aligned}$$

for all $x, y \in \mathcal{A}$. Hence, $T \in \mathcal{L}_{\mathcal{B}}(\mathcal{A})$. □

Notation. For any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras, let

$$\mathcal{I}_1(\mathcal{B} \subset \mathcal{A}) := \{\mathcal{C} \in \mathcal{I}(\mathcal{B} \subset \mathcal{A}) \mid \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}\}.$$

Clearly, $\mathcal{I}_1$ a sublattice of $\mathcal{I}(\mathcal{B} \subset \mathcal{A})$, $\mathcal{I}_1(\mathcal{B} \subset \mathcal{A}) \subseteq \mathcal{I}_0(\mathcal{B} \subset \mathcal{A})$ and $\mathcal{I}_1(\mathcal{B} \subset \mathcal{A}) = \mathcal{I}(\mathcal{B} \subset \mathcal{A})$ if $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$.

We shall need the following adaptation of [11, Lem. 3.3].

Lemma 4.8. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. Then*

$$d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) \leq \mathrm{Ind}_W(E)\|e_{\mathcal{C}} - e_{\mathcal{D}}\|$$

for all $\mathcal{C}, \mathcal{D} \in \mathcal{I}_1(\mathcal{B} \subset \mathcal{A})$.

Proof. We shall simply write $\mathcal{I}_1$ for the set $\mathcal{I}_1(\mathcal{B} \subset \mathcal{A})$. Since E has finite index, it satisfies the Pimsner–Popa inequality (by [24, Prop. 2.6.2]), *i.e.*, $E(x^*x) \geq (\mathrm{Ind}_W(E))^{-1}x^*x$ for all $x \in \mathcal{A}$, and since $\mathrm{Ind}_W(E) \geq 1$ (by [24, Lem. 2.3.1]), it follows that

$$E(x^*x) \geq (\mathrm{Ind}_W(E))^{-2}x^*x$$

for all $x \in \mathcal{A}$. In particular, $\|\eta(x)\| \geq (\mathrm{Ind}_W(E))^{-1}\|x\|$ for all $x \in \mathcal{A}$. Moreover, $\|\eta(a)\| \leq \|a\| \leq 1$ for every $a \in B_1(\mathcal{C})$.

Since $\mathcal{B} \subset \mathcal{A}$ are both simple, it follows from Proposition 4.7 that $\mathcal{L}_{\mathcal{C}}(\mathcal{A}) \cup \mathcal{L}_{\mathcal{D}}(\mathcal{A}) \subseteq \mathcal{L}_{\mathcal{B}}(\mathcal{A})$ for any two $\mathcal{C}, \mathcal{D} \in \mathcal{I}_1$. Thus, $e_{\mathcal{C}}, e_{\mathcal{D}} \in \mathcal{L}_{\mathcal{B}}(\mathcal{A})$, and

$$\begin{aligned}\|e_{\mathcal{C}} - e_{\mathcal{D}}\| &\geq \frac{\|\eta(E_{\mathcal{C}}(a) - E_{\mathcal{D}}(a))\|}{\|\eta(a)\|} \geq \|\eta(a - E_{\mathcal{D}}(a))\|\\ &\geq \frac{1}{\mathrm{Ind}_W(E)}\|a - E_{\mathcal{D}}(a)\|\end{aligned}$$

for all $a \in B_1(\mathcal{C}) \setminus \{0\}$. Therefore, for each $a \in B_1(\mathcal{C})$, there exists an element $b := E_{\mathcal{D}}(a) \in B_1(\mathcal{D})$ such that $\|a - b\| \leq \mathrm{Ind}_W(E)\|e_{\mathcal{C}} - e_{\mathcal{D}}\|$. By a symmetric argument, we deduce that $d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) \leq \mathrm{Ind}_W(E)\|e_{\mathcal{C}} - e_{\mathcal{D}}\|$. □

The following finiteness result is an adaptation (as well as a mild generalization) of [11, Cor. 3.9] (also see [18, Thm. 1.3]). Its proof is an imitation of that of Theorem 4.2. However, we provide all steps for the sake of completeness.

Theorem 4.9. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of simple unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$. Then the sublattice $\mathcal{I}_1(\mathcal{B} \subset \mathcal{A})$ is finite.*

Proof. Consider Watatani's C^*-basic construction $\mathcal{A}_1 := C^*(\mathcal{A}, e_1)$ of the inclusion $\mathcal{B} \subset \mathcal{A}$ with respect to the finite-index conditional expectation E and Jones projection e_1. Let $\tilde{E} : \mathcal{A}_1 \to \mathcal{A}$ be the finite-index dual conditional expectation of E. Then $\tilde{E} \circ E : \mathcal{A}_1 \to \mathcal{B}$ is also a conditional expectation of finite index. Since $\mathcal{A}$ is simple, it follows from [24, Prop. 2.7.3] that the relative commutant $\mathcal{B}' \cap \mathcal{A}_1$ is finite-dimensional. Therefore, the set

$$\mathcal{P} := \{p \in \mathcal{B}' \cap \mathcal{A}_1 \mid p \text{ is a projection}\}$$

is a compact Hausdorff space with respect to the operator norm.

Since $\mathcal{A}$ is simple, $\mathrm{Ind}_W(E)$ is a positive scalar. Fix a $0 < \gamma < 10^{-6}$ and let

$$\varepsilon = \frac{\gamma}{2\mathrm{Ind}_W(E)}.$$

By the compactness of $\mathcal{P}$, there exists a finite cover of $\mathcal{P}$ consisting of open balls of radius ϵ. Now, it suffices to show that each such ϵ-ball contains only finitely many Jones projections corresponding to members of $\mathcal{I}_1$.

Note that, for any two $\mathcal{C}, \mathcal{D} \in \mathcal{I}_1$, their corresponding Jones projections $e_{\mathcal{C}}$ and $e_{\mathcal{D}}$ are in $\mathcal{B}' \cap \mathcal{L}_{\mathcal{B}}(\mathcal{A}) = \mathcal{B}' \cap \mathcal{A}_1$ by Proposition 4.7, and by Lemma 4.8, they also satisfy the inequality

$$d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) \leq \mathrm{Ind}_W(E)\|e_{\mathcal{C}} - e_{\mathcal{D}}\|.$$

So if $e_{\mathcal{C}}$ and $e_{\mathcal{D}}$ are in one of the ε-open balls, then $d_{\mathrm{KK}}(\mathcal{C}, \mathcal{D}) \leq \frac{\gamma}{2} < \gamma$. Thus, by [8, Thm. 3.7], there exists a unitary u in $\mathcal{B}' \cap \mathcal{A}$ such that $\mathcal{D} = u\mathcal{C}u^*$ (and $\|u - 1_{\mathcal{A}}\| \leq 16\sqrt{110}\gamma^{\frac{1}{2}} + 880\gamma$). Since $\mathcal{B}' \cap \mathcal{A} = \mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{C}$, this implies that $u \in \mathcal{C}$. Hence, $\mathcal{D} = u\mathcal{C}u^* = \mathcal{C}$. This shows that each ε-open ball of the cover contains at most one Jones projection for some intermediate C^*-subalgebra in $\mathcal{I}_1$, and we are done. □

Since countably decomposable type III factors are simple, we obtain the following generalization of [23, Thm. 2.5] (and a partial generalization of [18, Thm. 1.3]).

Corollary 4.10. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of countably decomposable type III factors with a finite-index conditional expectation. Then the set $\mathcal{L}_1(\mathcal{N} \subset \mathcal{M}) := \{\mathcal{Q} \in \mathcal{L}(\mathcal{N} \subset \mathcal{M}) \mid \mathcal{N}' \cap \mathcal{M} \subseteq \mathcal{Q}\}$ is finite.*

4.11. **Cardinality of $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$.** Longo [19] showed that the cardinality of the lattice of intermediate subfactors of any finite-index irreducible inclusion $N \subset M$ of factors (of type II_1 or III) is bounded by $([M:N]^2)^{[M:N]^2}$. More recently, a similar bound was obtained for the lattice of intermediate C^*-algebras of a finite-index inclusion of simple unital C^*-algebras by Bakshi and the first named author in [3], which was achieved by introducing the notion of (interior) angle between two intermediate C^*-subalgebras. Further, Bakshi, Guin and Jana [2] improved the bound by proving that the cardinality of the lattice of intermediate C^*-subalgebras of an irreducible inclusion $\mathcal{B} \subset \mathcal{A}$ of simple unital C^*-algebras with finite index is bounded by $9^{[\mathcal{A}:\mathcal{B}]_0}$.

In this subsection, for any irreducible inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras with a finite index conditional expectation $E : \mathcal{A} \to \mathcal{B}$, through some elementary group action technique, we provide an expression for the cardinality of $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ in terms of the indices of the unitary normalizers of its members in $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$; see Proposition 4.13. On face, the expression looks a bit dry, but it might prove useful while performing some concrete calculations.

In [9, Lem. 2.8 (2)], it was shown that, for any inclusion $\mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras with a tracial finite-index conditional expectation $E : \mathcal{A} \to B$, $u\mathcal{C}u^* \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ for every $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ and $u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$. We now have its following (more obvious) variant (which does not require E to be tracial), which gives an abundance of compatible intermediate C^*-algebras and will also allow us to get an expression for the cardinality of $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ when the inclusion $\mathcal{B} \subset \mathcal{A}$ is irreducible.

Lemma 4.12. *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$. Suppose that $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the compatible finite-index conditional expectation $F : \mathcal{A} \to \mathcal{C}$. Then, for every $u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$, $u\mathcal{C}u^* \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the conditional expectation $F_u : \mathcal{A} \to u\mathcal{C}u^*$ given by $F_u = \mathrm{Ad}_u \circ F \circ \mathrm{Ad}_{u^*}$.*

Proof. As $\mathcal{C}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{B}$ and $\mathcal{E}_0(\mathcal{A}, \mathcal{B}) \neq \varnothing$, it follows from [24, Cor. 1.4.3] that E is the unique conditional from $\mathcal{A}$ onto $\mathcal{B}$. Also, from [9, Lem. 2.8 (1)], we know that F_u is of finite index. So $E_{\restriction_{u\mathcal{C}u^*}} \circ F_u$ is again a finite-index conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. Thus, by uniqueness of E, we get $E_{\restriction_{u\mathcal{C}u^*}} \circ F_u = E$. Hence, $u\mathcal{C}u^* \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the compatible conditional expectation F_u. □

With notation as in Lemma 4.12, we thus observe that the group $G := \mathcal{N}_{\mathcal{A}}(\mathcal{B})$ admits an action on the set $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ via the map

$$G \times \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E) \ni (u, \mathcal{C}) \mapsto u\mathcal{C}u^* \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E).$$

For any $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$, its stabilizer

$$\mathrm{Stab}_G(\mathcal{C}) = \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C}).$$

Let $\widehat{\mathrm{IMS}(\mathcal{B}, \mathcal{A}}, E)$ denote a set of representatives of the orbits of $\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to this action. Then, in view of Corollary 4.3, from some basic theory of group actions, we immediately conclude the following.

Proposition 4.13. *Let $\mathcal{B} \subset \mathcal{A}$ be an irreducible inclusion of unital C^*-algebras with a finite-index conditional expectation $E : \mathcal{A} \to \mathcal{B}$ and $G := \mathcal{N}_{\mathcal{A}}(\mathcal{B})$. Then*
(i) $\big[\mathcal{N}_{\mathcal{A}}(\mathcal{B}) : \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C})\big] < \infty$ *for every* $C \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$, *and*
(ii) $|\mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)| = \sum_{\mathcal{C} \in \widehat{\mathrm{IMS}(\mathcal{B}, \mathcal{A}}, E)} \big[\mathcal{N}_{\mathcal{A}}(\mathcal{B}) : \mathcal{N}_{\mathcal{A}}(\mathcal{B}) \cap \mathcal{N}_{\mathcal{A}}(\mathcal{C})\big]$.

Lemma 4.14. *Let $\mathcal{A}, \mathcal{B}, E, G$ be as in Proposition 4.13 and $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$. Then*

$$[\mathcal{A} : \mathcal{D}]_0 = [\mathcal{A} : \mathcal{C}]_0 \quad \textit{and} \quad [\mathcal{D} : \mathcal{B}]_0 = [\mathcal{C} : \mathcal{B}]_0$$

for every $\mathcal{D} \in \mathrm{Orb}_G(\mathcal{C})$.

Proof. Since $\mathcal{B} \subset \mathcal{A}$ is an irreducible inclusion, the conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is unique by [24, Cor. 1.4.3]. Hence, it is minimal.

Suppose that $\mathcal{C} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the compatible finite-index conditional expectation $F : \mathcal{A} \to C$. Note that $\mathrm{Orb}_G(\mathcal{C}) = \{u\mathcal{C}u^* \mid u \in G\}$.

Let $\mathcal{D} \in \mathrm{Orb}_G(\mathcal{C})$. Then $\mathcal{D} = u\mathcal{C}u^*$ for some $u \in \mathcal{N}_{\mathcal{A}}(\mathcal{B})$ and $\mathcal{D} \in \mathrm{IMS}(\mathcal{B}, \mathcal{A}, E)$ with respect to the conditional expectation $F_u := \mathrm{Ad}_u \circ F \circ \mathrm{Ad}_{u^*}$. Also,

$$\mathcal{Z}(\mathcal{D}) = \mathcal{Z}(u\mathcal{C}u^*) = u\mathcal{Z}(\mathcal{C})u^* = \mathbb{C} = \mathcal{Z}(\mathcal{A}),$$

$$\mathcal{D}' \cap \mathcal{A} \subseteq \mathcal{B}' \cap \mathcal{A} = \mathbb{C} \quad \text{and} \quad \mathcal{B}' \cap \mathcal{D} \subseteq \mathcal{B}' \cap \mathcal{A} = \mathbb{C},$$

so, by [24, Cor. 1.4.3] again, F_u and $E_{\restriction_{\mathcal{D}}}$ are both unique and hence minimal. Thus, by [16, Thm. 3], we get $[\mathcal{A} : \mathcal{B}]_0 = [\mathcal{A} : \mathcal{D}]_0 [\mathcal{D} : \mathcal{B}]_0$ for every $\mathcal{D} \in \mathrm{Orb}_G(\mathcal{C})$. Note that $[\mathcal{A} : \mathcal{D}]_0 = [\mathcal{A} : u\mathcal{C}u^*]_0 = [\mathcal{A} : \mathcal{C}]_0$ by [9, Lem. 2.8], which then shows that $[\mathcal{D} : \mathcal{B}]_0 = [\mathcal{C} : \mathcal{B}]_0$ as well. □

5. Comparisons between various notions of distance

In this section, we discuss the various notions of distance between subalgebras of a given operator algebra by Christensen and Mashood–Taylor and make comparisons between them and the Kadison–Kastler distance.

5.1. **Christensen's distances.** Christensen has given two notions of distances and used them effectively in proving some significant perturbation results. The first one is defined between subalgebras of any tracial von Neumann algebra [5] and, more generally, the other one is defined between subalgebras of any C^*-algebra [6]. We briefly recall both notions in reversed order and state some facts related to them.

5.1.1. *Christensen's distance between subalgebras of a normed algebra.* Though Christensen gave the notion of distance between subalgebras of a C^*-algebra, the same can be used in the normed algebra context as well.

Let A be a normed algebra. Recall from [6] that, for $C, D \in \mathrm{Sub}_A$ and a scalar $\gamma > 0$, $C \subseteq_\gamma D$ if, for each $x \in B_1(C)$, there exists a $y \in D$ such that $\|x - y\| \le \gamma$, and the Christensen distance between C and D is defined by

$$d_0(C, D) = \inf\{\gamma > 0 \mid C \subseteq_\gamma D \text{ and } D \subseteq_\gamma C\}.$$

Here are some elementary observations regarding the Christensen distance, some of which will be used ahead.

Remark 5.2. Let A be a normed algebra.
(i) $d_0(C, D) \le 1$ for all $C, D \in \mathrm{Sub}_A$.
(ii) $d_0(C, D) = d_0(\overline{C}, D) = d_0(\overline{C}, \overline{D})$ for all $C, D \in \mathrm{Sub}_A$.
(iii) d_0 is not a metric on Sub_A (as it does not satisfy the triangle inequality). However, d_0 and d_{KK} are "equivalent" in the sense that

$$d_0(C, D) \le d_{\mathrm{KK}}(C, D) \le 2d_0(C, D)$$

for all $C, D \in \mathrm{Sub}_A$ (see [7, Rem. 2.3]).

(iv) If $C, D \in \mathrm{Sub}_A$ and C is a norm closed proper subalgebra of D, then $d_0(C, D) = 1$. (This follows from [7, Prop. 2.4]—also compare with Remark 3.2 (iii).)

5.2.1. *Christensen's distance between subalgebras of a tracial von Neumann algebra.* For a von Neumann algebra $\mathcal{M}$, let

$$*\text{-}\mathrm{Sub}^u_{\mathcal{M}} := \{{}^*\text{-subalgebras of } \mathcal{M} \text{ containing } 1_{\mathcal{M}}\}$$

and

$$W^*\text{-}\mathrm{Sub}_{\mathcal{M}} := \{\text{von Neumann subalgebras of } \mathcal{M} \\ \text{(possibly with different unit)}\}.$$

Let $\mathcal{M}$ be a von Neumann algebra with a (fixed) faithful normal tracial state τ. Then $\mathcal{M}$ inherits a natural inner-product structure with respect to the inner product $\langle x, y\rangle_\tau = \tau(y^*x)$, $x, y \in \mathcal{M}$. The corresponding norm on $\mathcal{M}$ will be denoted by $\|x\|_\tau$, *i.e.*, $\|x\|_\tau = \tau(x^*x)^{1/2}$, $x \in \mathcal{M}$, and the Hilbert space completion of $\mathcal{M}$ is denoted by $L^2(\mathcal{M}, \tau)$. There is a natural embedding of $\mathcal{M}$ into $B(L^2(\mathcal{M}, \tau))$ via left multiplication and this allows us to consider $\mathcal{M}$ as a von Neumann algebra on $L^2(\mathcal{M}, \tau)$.

Further, in order to distinguish the elements of the inner-product structure $\mathcal{M}$ from that of the von Neumann algebra $\mathcal{M}$, as is standard, we shall use the notation $\hat{x}$ for the elements of the inner-product space $\mathcal{M}$. Thus, we have $\langle \hat{x}, \hat{y}\rangle_\tau = \tau(y^*x)$ and $x(\hat{1}) = \hat{x}$ for all $x, y \in \mathcal{M}$.

Recall from [5] that, for $P, Q \in \mathrm{Sub}_{\mathcal{M}}$ and a scalar $\gamma > 0$, $P \subset_\gamma Q$ if, for each $x \in B_1(P)$ (with respect to $\|\cdot\|$), there exists a $y \in Q$ such that $\|\hat{x} - \hat{y}\|_\tau \leq \gamma$, and the Christensen distance between P and Q is defined by

$$d_{\mathrm{C}}(P, Q) = \inf\{\gamma > 0 \mid P \subset_\gamma Q \text{ and } Q \subset_\gamma P\}.$$

Here are some facts related to the Christensen distance d_{C}.

Remark 5.3. With running notation, the following hold.
(i) $d_{\mathrm{C}}(P, Q) \leq 1$ for all $P, Q \in \mathrm{Sub}_{\mathcal{M}}$.
(ii) d_{C} is a complete metric on $W^*\text{-}\mathrm{Sub}_{\mathcal{M}}$ (see [5, Thm. 5.1]).
(iii) From Proposition 5.11 and Proposition 5.7 ahead, it follows that

$$d_{\mathrm{C}}(P, Q) = d_{\mathrm{C}}(\overline{P}^{\mathrm{S.O.T.}}, \overline{Q}^{\mathrm{S.O.T.}}) = d_{\mathrm{C}}(P'', Q'')$$

for all $P, Q \in *\text{-}\mathrm{Sub}^u_{\mathcal{M}}$.

Here is a well-known observation which will be essential for our discussion—see, for instance, [1, Prop. 2.6.4].

Proposition 5.4. *Let M be a unital C^*-algebra equipped with a faithful tracial state τ. Then M is a von Neumann algebra on $L^2(M, \tau)$ if and only if $\widehat{B_1(M)}$ is complete with respect to $\|\cdot\|_\tau$.*

In particular, M is a W^*-algebra if and only if $\widehat{B_1(M)}$ is complete with respect to $\|\cdot\|_\tau$.

5.5. **Mashood–Taylor distance between subalgebras of a tracial von Neumann algebra.** Given a II_1-factor M with a unique faithful normal tracial state τ and any two subfactors P and Q of M, Mashood and Taylor [20] consider the Hausdorff distance (with respect to $\|\cdot\|_\tau$) between $B_1(P)$ and $B_1(Q)$ and prove some nice results related to continuity of “Jones” index. They also mention [20, p. 56] that this distance gives the same topology on the set of subfactors of M as the one given by the Christensen distance d_{C}. We shall show in this section that $d_{\mathrm{C}} = d_{\mathrm{MT}}$.

Mashood–Taylor distance can, in fact, be defined in a slightly more general set-up as follows. For a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ, for any pair $P, Q \in \mathrm{Sub}_{\mathcal{M}}$, the Mashood–Taylor distance between P and Q is defined as

$$d_{\mathrm{MT}}(P,Q) = d_{H,\|\cdot\|_\tau}(\widehat{B_1(P)}, \widehat{B_1(Q)}),$$

where $d_{H,\|\cdot\|_\tau}$ denotes the Hausdorff distance with respect to the metric induced by the norm $\|\cdot\|_\tau$ and $\hat{S} \subseteq L^2(\mathcal{M},\tau)$ for $S \subseteq \mathcal{M}$ (as in Section 5.2.1).

Lemma 5.6. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ. Then the following hold.*
(i) d_{MT} *is a semi-metric on* $\mathrm{Sub}_{\mathcal{M}}$.
(ii) d_{MT} *is a metric on* W^*-$\mathrm{Sub}_{\mathcal{M}}$.
(iii) $d_{\mathrm{C}}(P,Q) \leq d_{\mathrm{MT}}(P,Q)$ *for all* $P, Q \in \mathrm{Sub}_{\mathcal{M}}$.

Proof. (i) follows readily from the definition.

(ii) Note that $\widehat{B_1(\mathcal{Q})}$ is closed and bounded in $L^2(\mathcal{M},\tau)$ for every $\mathcal{Q} \in W^*$-$\mathrm{Sub}_{\mathcal{M}}$ by Proposition 5.4. And it is well-known that the Hausdorff distance is a metric on the collection of closed and bounded subsets of a metric space.

(iii) Let $P, Q \in \mathrm{Sub}_{\mathcal{M}}$ and $\epsilon > 0$. By the definition of $d_{\mathrm{MT}}(P,Q)$, for each $x \in B_1(P)$, there exists a $z \in B_1(Q)$ such that $\|\hat{x} - \hat{z}\|_\tau \leq d_{\mathrm{MT}}(P,Q) + \epsilon$. This implies that $P \subset_{d_{\mathrm{MT}}(P,Q)+\epsilon} Q$. Similarly, we have $Q \subset_{d_{\mathrm{MT}}(P,Q)+\epsilon} P$. Thus, $d_{\mathrm{C}}(P,Q) \leq d_{\mathrm{MT}}(P,Q) + \epsilon$ for all $\epsilon > 0$, and we are done. $\square$

The following observation is a nice tool to calculate the distance between two von Neumann subalgebras and will be used ahead on more than one occasion.

Proposition 5.7. *Let $(\mathcal{M},\tau)$ be as in Lemma 5.6. Then*

$$d_{\mathrm{MT}}(P,Q) = d_{\mathrm{MT}}(P, \overline{Q}^{\mathrm{S.O.T.}}) = d_{\mathrm{MT}}(\overline{P}^{\mathrm{S.O.T.}}, \overline{Q}^{\mathrm{S.O.T.}}) = d_{\mathrm{MT}}(P'', Q'')$$

for all $P, Q \in *$-$\mathrm{Sub}^u_{\mathcal{M}}$.

Proof. In order to avoid any possible confusion, for every $x \in \mathcal{M}$ and $S \subseteq \mathcal{M}$, let $d_\tau(\hat{x}, \hat{S}) := \inf\{\|\hat{x} - \hat{s}\|_\tau \mid s \in S\}$. Now, let $P, Q \in *$-$\mathrm{Sub}^u_{\mathcal{M}}$. We first assert that

$$d_\tau(\hat{a}, \widehat{B_1(Q)}) = d_\tau(\hat{a}, B_1(\widehat{\overline{Q}^{\mathrm{S.O.T.}}}))$$

for all $a \in B_1(P)$. Clearly,

$$d_\tau(\hat{a}, B_1(\widehat{\overline{Q}^{\mathrm{S.O.T.}}})) \leq d_\tau(\hat{a}, \widehat{B_1(Q)})$$

for all $a \in B_1(P)$. For the reverse inequality, fix an $a \in B_1(P)$ and an $\epsilon > 0$. Then there exists a $b_0 \in B_1(\overline{Q}^{\text{S.O.T.}})$ such that

$$d_\tau(\hat{a}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) \leq \|\hat{a} - \widehat{b_0}\|_\tau < d_\tau(\hat{a}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) + \epsilon.$$

As

$$B_1(\overline{Q}^{\text{S.O.T.}}) = \overline{B_1(Q)}^{\text{S.O.T.}}$$

(by Kaplansky density theorem), there exists a net

$$\{b_\alpha\} \subseteq B_1(Q) \quad \text{such that} \quad b_\alpha \xrightarrow{\text{S.O.T.}} b_0.$$

Hence, there exists an α_0 such that

$$\|\hat{a} - \hat{b}_{\alpha_0}\|_\tau < d_\tau(\hat{a}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) + \epsilon.$$

This implies that

$$d_\tau(\hat{a}, \widehat{B_1(Q)}) \leq \|\hat{a} - \hat{b}_{\alpha_0}\|_\tau < d_\tau(\hat{a}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, we get

$$d_\tau(\hat{a}, \widehat{B_1(Q)}) \leq d_\tau(\hat{a}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}).$$

This proves our assertion.

Then we get

$$\begin{aligned}\beta := \sup_{a \in B_1(P)} d_\tau(\hat{a}, \widehat{B_1(Q)}) &= \sup_{a \in B_1(P)} d_\tau(\hat{a}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) \\ &\leq \sup_{z \in B_1(\overline{P}^{\text{S.O.T.}})} d_\tau(\hat{z}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) =: \alpha.\end{aligned}$$

Further, for any $\eta > 0$, there exists a $z_0 \in B_1(\overline{P}^{\text{S.O.T.}})$ such that

$$\alpha - \eta < d_\tau(\widehat{z_0}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) \leq \alpha.$$

Then, by Kaplansky density theorem again, there exists a net

$$\{z_\lambda\} \subset B_1(P) \quad \text{such that} \quad z_\lambda \xrightarrow{\text{S.O.T.}} z_0.$$

In particular, $\widehat{z_\lambda} \to \widehat{z_0}$ with respect to $\|\cdot\|_\tau$, and since d_τ is continuous with respect to $\|\cdot\|_\tau$, it follows that

$$d_\tau(\widehat{z_\lambda}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) \to d_\tau(\widehat{z_0}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}).$$

Thus, there exists a λ_0 such that

$$d_\tau(\hat{z}_{\lambda_0}, B_1\widehat{(\overline{Q}^{\text{S.O.T.}})}) > \alpha - \eta,$$

so that $\beta \geq \alpha - \eta$. Since $\eta > 0$ was arbitrary, we have $\beta \geq \alpha$. Thus, $\alpha = \beta$, *i.e.*,

$$d_{\text{MT}}(P, Q) = d_{\text{MT}}(P, \overline{Q}^{\text{S.O.T.}}).$$

By a similar argument, we conclude that

$$d_{\text{MT}}(P, Q) = d_{\text{MT}}(\overline{P}^{\text{S.O.T.}}, \overline{Q}^{\text{S.O.T.}}).$$

The rest follows from von Neumann's Double Commutant Theorem. $\square$

5.8. **Comparisons between d_{KK}, d_{C} and d_{MT}.** We first present the following comparison between d_{MT} and d_{KK}.

Lemma 5.9. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ. Then $d_{\mathrm{MT}}(P,Q) \leq d_{\mathrm{KK}}(P,Q)$ for all $P,Q \in \mathrm{Sub}_{\mathcal{M}}$.*

Proof. Let $P,Q \in \mathrm{Sub}_{\mathcal{M}}$ and $p \in B_1(P)$. Then

$$d_\tau(\hat{p}, \widehat{B_1(Q)}) \leq \|\hat{p} - \hat{q}\|_\tau \leq \|p-q\|$$

for all $q \in B_1(Q)$. Thus,

$$d_\tau(\hat{p}, \widehat{B_1(Q)}) \leq d(p, B_1(Q))$$

for all $p \in B_1(P)$. In particular,

$$\sup_{p\in B_1(P)} d_\tau(\hat{p}, \widehat{B_1(Q)}) \leq \sup_{p\in B_1(P)} d(p, B_1(Q)).$$

Similarly,

$$\sup_{q\in B_1(Q)} d_\tau(\hat{q}, \widehat{B_1(P)}) \leq \sup_{q\in B_1(Q)} d(q, B_1(P)).$$

Hence, $d_{\mathrm{MT}}(P,Q) \leq d_{\mathrm{KK}}(P,Q)$. $\square$

The following useful observation is well-known to the experts (see, for instance, [5, Eqn. (6)]).

Lemma 5.10. *Let $(\mathcal{M},\tau)$ be as in Lemma 5.9 and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ (with common unit). Then*

$$d_\tau(\hat{x}, \widehat{B_1(\mathcal{N})}) = \|\hat{x} - \widehat{E_{\mathcal{N}}(x)}\|_\tau = d_\tau(\hat{x}, L^2(\mathcal{N}))$$

for all $x \in B_1(\mathcal{M})$, where $E_{\mathcal{N}}$ denotes the unique τ-preserving normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$.

We are now all set to show that the Mashood–Taylor distance agrees with the Christensen distance on $*\text{-}\mathrm{Sub}^u_M$.

Proposition 5.11. *Let $(\mathcal{M},\tau)$ be as in Lemma 5.10. Then*

$$d_{\mathrm{C}}(P,Q) = d_{\mathrm{MT}}(P,Q)$$

for all $P,Q \in *\text{-}\mathrm{Sub}^u_{\mathcal{M}}$.

Proof. Let $P,Q \in *\text{-}\mathrm{Sub}^u_{\mathcal{M}}$ and consider

$$\tilde{P} := \overline{P}^{\mathrm{S.O.T.}} \quad \text{and} \quad \tilde{Q} := \overline{Q}^{\mathrm{S.O.T.}}.$$

In view of Lemma 5.6 and Proposition 5.7, it suffices to show that

$$d_{\mathrm{C}}(P,Q) \geq d_{\mathrm{MT}}(\tilde{P},\tilde{Q}).$$

We first assert that

$$d_\tau(\hat{p}, (\widehat{B_1(\tilde{Q})})) = d_\tau(\hat{p}, L^2(\tilde{Q})) = d_\tau(\hat{p}, \hat{Q})$$

for all $p \in B_1(P)$.

Let $p \in B_1(P)$. Then, by Lemma 5.10,

$$d_\tau(\hat{p}, \widehat{B_1(\tilde{Q})}) = \|\hat{p} - \widehat{E_{\tilde{Q}}(p)}\|_\tau = d_\tau(\hat{p}, L^2(\tilde{Q})).$$

Thus, as $\widehat{B_1(Q)} \subset \hat{Q} \subset L^2(\tilde{Q})$, we see that

$$d_\tau(\hat{p}, \widehat{B_1(\tilde{Q})}) = d_\tau(\hat{p}, L^2(\tilde{Q})) \leq d_\tau(\hat{p}, \hat{Q}) \leq d_\tau\big(\hat{p}, \widehat{B_1(Q)}\big) = d_\tau(\hat{p}, \widehat{B_1(\tilde{Q})})$$

for all $p \in B_1(P)$. Hence, our first assertion is true.

Now, suppose that $P \subset_\gamma Q$ and $Q \subset_\gamma P$ for some $\gamma > 0$. We then assert that

$$d_\tau(\hat{p}, \widehat{B_1(\tilde{Q})}) \leq \gamma \quad \text{for all } p \in B_1(\tilde{P}).$$

So let $p \in B_1(\tilde{P})$ and, by Kaplansky density theorem, fix a net

$$\{p_\alpha\} \subset B_1(P) \quad \text{such that} \quad p_\alpha \xrightarrow{\text{S.O.T.}} p \quad \text{in } B(L^2(\mathcal{M}, \tau)).$$

Since $P \subset_\gamma Q$, for each α, there exists a $q_\alpha \in Q$ such that $\|\hat{p}_\alpha - \hat{q}_\alpha\|_\tau \leq \gamma$. So

$$\gamma \geq \|\hat{p}_\alpha - \hat{q}_\alpha\|_\tau \geq d_\tau(\hat{p}_\alpha, \hat{Q}) = d_\tau(\hat{p}_\alpha, \widehat{B_1(\tilde{Q})}) = \|\hat{p}_\alpha - \widehat{E_{\tilde{Q}}(p_\alpha)}\|_\tau$$

for all α, where $E_{\tilde{Q}}$ is the unique τ-preserving normal conditional expectation from $\mathcal{M}$ onto $\tilde{Q}$. Thus,

$$d_\tau(\hat{p}, \widehat{B_1(\tilde{Q})}) \leq \inf_\alpha \|\hat{p} - \widehat{E_{\tilde{Q}}(p_\alpha)}\|_\tau \leq \inf_\alpha \|\hat{p} - \hat{p}_\alpha\|_\tau + \inf_\alpha \|\hat{p}_\alpha - \widehat{E_{\tilde{Q}}(p_\alpha)}\|_\tau \leq \gamma,$$

where the last inequality holds because $\|\hat{p}_\alpha - \hat{p}\|_\tau \to 0$ (as $p_\alpha \to p$ in S.O.T.) and

$$\|\widehat{p_\alpha} - \widehat{E_{\tilde{Q}}(p_\alpha)}\|_\tau \leq \gamma$$

for all α. Thus, our second assertion is also true.

As a consequence,

$$\sup_{p \in B_1(\tilde{P})} d_\tau(\hat{p}, \widehat{B_1(\tilde{Q})}) \leq \gamma.$$

Likewise,

$$\sup_{q \in B_1(\tilde{Q})} d_\tau(\hat{q}, \widehat{B_1(\tilde{P})}) \leq \gamma$$

as well. Hence, $d_{\mathrm{MT}}(\tilde{P}, \tilde{Q}) \leq \gamma$, which then implies that

$$d_{\mathrm{MT}}(\tilde{P}, \tilde{Q}) \leq d_{\mathrm{C}}(P, Q),$$

and we are done. □

In view of Lemma 5.9, we also have the following.

Corollary 5.12. *Let $(\mathcal{M}, \tau)$ be as in Lemma 5.10. Then*

$$d_{\mathrm{C}}(P, Q) \leq d_{\mathrm{KK}}(P, Q)$$

for all $P, Q \in \mathrm{Sub}_{\mathcal{M}}$.

6. Kadison–Kastler and Christensen distance between subalgebras corresponding to subgroups

6.1. **Distances between crossed-product subalgebras associated to subgroups.** Let G be a discrete group acting on a C^*-algebra $\mathcal{A}$ via the map $\alpha : G \to \mathrm{Aut}(\mathcal{A})$. Consider the space $C_c(G, \mathcal{A})$ consisting of compactly supported $\mathcal{A}$-valued functions on G, which can be identified naturally with the vector space $\{\sum_{\text{finite}} a_g g \mid a_g \in \mathcal{A},\, g \in G\}$ consisting of formal finite sums. Further, $C_c(G, \mathcal{A})$ is a $*$-algebra with respect to the (so-called twisted) multiplication given by the convolution operation

$$\Big(\sum_{s\in I} a_s s\Big)\Big(\sum_{t\in J} b_t t\Big) = \sum_{s\in I,\, t\in J} a_s \alpha_s(b_t) st$$

and involution given by

$$\Big(\sum_{s\in I} a_s s\Big)^* = \sum_{s\in I} \alpha_{s^{-1}}(a_s^*) s^{-1}$$

for any two finite sets I and J in G. The reduced crossed product $\mathcal{A} \rtimes_\alpha^r G$ and the universal crossed product $\mathcal{A} \rtimes_\alpha^u G$ are defined, respectively, as the completions of $C_c(G, \mathcal{A})$ with respect to the reduced norm and the universal norm on $C_c(G, \mathcal{A})$, as described below.

The reduced norm is given by

$$\Big\|\sum_{\text{finite}} a_g g\Big\|_r := \Big\|\sum_{\text{finite}} \pi(a_g)(1 \otimes \lambda_g)\Big\|_{B(H\otimes l^2(G))},$$

where $\mathcal{A} \subset B(H)$ is an (equivalently, any) fixed faithful representation of $\mathcal{A}$, $\lambda : G \to B(l^2(G))$ is the left regular representation and $\pi : \mathcal{A} \to B(H \otimes l^2(G))$ is the representation satisfying $\pi(a)(\xi \otimes \delta_g) = \alpha_{g^{-1}}(a)(\xi) \otimes \delta_g$ for all $\xi \in H$ and $g \in G$.

The universal norm is given by

$$\|x\|_u := \sup_\pi \|\pi(x)\|_{B(H_\pi)} \quad \text{for } x \in C_c(G, \mathcal{A}),$$

where the supremum runs over all (cyclic) $*$-homomorphisms $\pi : C_c(G, \mathcal{A}) \to B(H_\pi)$. Note that $\|x\|_r \le \|x\|_u$ for all $x \in C_c(G, \mathcal{A})$.

We refer the reader to [4] for more on crossed products.

6.1.1. *Distances between subalgebras of reduced crossed product.* First, we gather some relevant facts related to the reduced crossed product construction.

Remark 6.2. Let G and $\mathcal{A}$ be as above and H be a subgroup of G.

(i) The canonical injective $*$-homomorphism

$$C_c(H, \mathcal{A}) \ni \sum_{\text{finite}} a_h h \mapsto \sum_{\text{finite}} a_h h \in C_c(G, \mathcal{A})$$

extends to an injective $*$-homomorphism from $\mathcal{A} \rtimes_\alpha^r H$ into $\mathcal{A} \rtimes_\alpha^r G$. So we can consider $\mathcal{A} \rtimes_\alpha^r H$ as C^*-subalgebra of $\mathcal{A} \rtimes_\alpha^r G$ (see [17, Rem. 3.2]).

(ii) There exists a faithful conditional expectation $E_H : \mathcal{A} \rtimes_\alpha^r G \to \mathcal{A} \rtimes_\alpha^r H$ satisfying

$$E_H\Big(\sum_{g\in I} a_g g\Big) = \sum_{h\in I\cap H} a_h h \quad \text{for all } \sum_{g\in I} a_g g \in C_c(G,\mathcal{A})$$

(see [17, Rem. 3.2]), and if $[G:H]<\infty$, then E_H has finite (Watatani) index with a quasi-basis given by any set of left coset representatives $\{g_i \mid 1\le i\le [G:H]\}$ of H in G.

(iii) Thus, as in Section 2, the vector space $\mathcal{A} \rtimes_\alpha^r G$ is a natural left $(\mathcal{A} \rtimes_\alpha^r H)$-module via multiplication from left. Hence, following [24], we consider $\mathcal{A} \rtimes_\alpha^r G$ as a pre-Hilbert (left) $(\mathcal{A} \rtimes_\alpha^r H)$-module with respect to the $(\mathcal{A} \rtimes_\alpha^r H)$-valued inner product given by

$$\langle x,y\rangle = E_H(x^*y) \quad \text{for } x,y\in \mathcal{A} \rtimes_\alpha^r G.$$

Further, in order to distinguish the elements of the C^*-algebra $\mathcal{A} \rtimes_\alpha^r G$ and the pre-Hilbert $(\mathcal{A} \rtimes_\alpha^r H)$-module $\mathcal{A} \rtimes_\alpha^r G$, following [24], we consider the identity map $\eta_H : \mathcal{A} \rtimes_\alpha^r G \to \mathcal{A} \rtimes_\alpha^r G$, where we consider the codomain as the pre-Hilbert $(\mathcal{A} \rtimes_\alpha^r H)$-module. Since E_H is a contraction, we have

$$\|\eta_H(x)\| = \|E_H(x^*x)\|_r^{1/2} \le \|x\|_r \tag{2}$$

for all $x\in \mathcal{A} \rtimes_\alpha^r G$.

(iv) Note that, for two subgroups $H \subsetneq K$ of a discrete group G, $\|\eta_H(x)\|$ need not be equal to $\|\eta_K(x)\|$ for every $x\in \mathcal{A} \rtimes_\alpha^r G$.

For instance, consider the trivial action of the finite permutation group $G=S_3$ on the C^*-algebra $\mathbb{C}$. Let $H=\{e\}$, $K=A_3$ and

$$x = a_1(123)+a_2(132) \in C_r^*(G) = \mathbb{C}[G] \quad \text{with } a_1,a_2\neq 0.$$

Then $x^*x = (|a_1|^2+|a_2|^2)e + \bar{a_1}a_2(123) + \bar{a_2}a_1(132)$ and

$$\begin{aligned}
\|\eta_K(x)\|^2 &= \|E_K(x^*x)\|_r \\
&= \|(|a_1|^2+|a_2|^2)e + (\bar{a_1}a_2)(123) + (\bar{a_2}a_1)(132)\|_r \\
&= \|(|a_1|^2+|a_2|^2)\lambda_e + (\bar{a_1}a_2)\lambda_{(123)} + (\bar{a_2}a_1)\lambda_{(132)}\|_{B(\ell^2(S_3))} \\
&\ge \big\|\big((|a_1|^2+|a_2|^2)\lambda_e + (\bar{a_1}a_2)\lambda_{(123)} + (\bar{a_2}a_1)\lambda_{(132)}\big)(\delta_e)\big\|_{\ell^2(S_3)} \\
&= \|(|a_1|^2+|a_2|^2)\delta_e + (\bar{a_1}a_2)\delta_{(123)} + (\bar{a_2}a_1)\delta_{(132)}\|_{\ell^2(S_3)} \\
&= \big((|a_1|^2+|a_2|^2)^2 + |\bar{a_1}a_2|^2 + |\bar{a_2}a_1|^2\big)^{\frac{1}{2}} \\
&> \big((|a_1|^2+|a_2|^2)^2\big)^{\frac{1}{2}} = (|a_1|^2+|a_2|^2) = \|\eta_H(x)\|^2.
\end{aligned}$$

(v) We shall denote $E_{\{e\}}$ (resp., $\eta_{\{e\}}$) simply by E (resp., η).

The following is an elementary and useful observation. A more general version of it was proved by Phillips (see [21, Prop. 9.16 (3)]).

Lemma 6.3. *Let G be a discrete group acting on a C^*-algebra $\mathcal{A}$. If*

$$a = \sum_{g\in I} a_g g \in C_c(G,\mathcal{A}),$$

then

$$\|a_g\|^2 \leq \|E(a^*a)\|_r = \|\eta(a)\|^2 \quad \textit{for all } g \in I.$$

Proposition 6.4. *Let $G, \mathcal{A}$ be as in Lemma 6.3 and let H and K be two distinct subgroups of G. Then*

$$d_{\mathrm{KK}}(C_c(H,\mathcal{A}), C_c(K,\mathcal{A})) = 1 = d_0(C_c(H,\mathcal{A}), C_c(K,\mathcal{A}))$$

in $\mathcal{A} \rtimes^r_\alpha G$.

Proof. Note that $d_{\mathrm{KK}}(C_c(H,\mathcal{A}), C_c(K,\mathcal{A})) \leq 1$ by definition, and

$$d_0(C_c(H,\mathcal{A}), C_c(K,\mathcal{A})) \leq d_{\mathrm{KK}}(C_c(H,\mathcal{A}), C_c(K,\mathcal{A}))$$

by Remark 5.2. So it just remains to show that

$$d_0(C_c(H,\mathcal{A}), C_c(K,\mathcal{A})) \geq 1.$$

Since H and K are distinct, either $H \neq H \cap K$ or $K \neq H \cap K$. Without loss of generality, we can assume that $H \neq H \cap K$. Then, in view of (2) and Lemma 6.3, we observe that

$$\|h - x\|_r \geq \|\eta(h - x)\| \geq 1 > \gamma$$

for all $h \in H \setminus H \cap K$, $x \in B_1(C_c(K,\mathcal{A}))$ and for every $0 < \gamma < 1$. Thus, if $C_c(H,\mathcal{A}) \subseteq_\gamma C_c(K,\mathcal{A})$ for some $\gamma > 0$, then $\gamma \geq 1$. So, by the definition of d_0, we must have

$$d_0(C_c(H,\mathcal{A}), C_c(K,\mathcal{A})) \geq 1. \qquad \square$$

6.4.1. *Distances between subalgebras of universal crossed product.*

Remark 6.5. Let G, $\mathcal{A}$ and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be as in the preceding subsection and let H be a subgroup of G. Then the canonical injective $*$-homomorphism

$$C_c(H,\mathcal{A}) \ni \sum_{\text{finite}} a_h h \mapsto \sum_{\text{finite}} a_h h \in C_c(G,\mathcal{A})$$

extends to an injective $*$-homomorphism from $\mathcal{A} \rtimes^u_\alpha H$ into $\mathcal{A} \rtimes^u_\alpha G$. Hence, we can consider $\mathcal{A} \rtimes^u_\alpha H$ as C^*-subalgebra of $\mathcal{A} \rtimes^u_\alpha G$ (see [17, Prop. 3.1]).

Since $\|x\|_u \geq \|x\|_r$ on $C_c(G,\mathcal{A})$, the following is immediate from the proof of Proposition 6.4.

Proposition 6.6. *Let G, $\mathcal{A}$ be as in Lemma 6.3 and let H and K be two distinct subgroups of G. Then*

$$d_{\mathrm{KK}}(C_c(H,\mathcal{A}), C_c(K,\mathcal{A})) = 1 = d_0(C_c(H,\mathcal{A}), C_c(K,\mathcal{A}))$$

in $\mathcal{A} \rtimes^u_\alpha G$.

In view of Lemma 3.3, its analog in Remark 5.2 and the preceding two propositions, we obtain the following corollary.

Corollary 6.7. *Let G, H, K, $\mathcal{A}$ and α be as in Proposition 6.4. Then*
(i) $d_{\mathrm{KK}}(A \rtimes^r_\alpha H, A \rtimes^r_\alpha K) = 1 = d_0(A \rtimes^r_\alpha H, A \rtimes^r_\alpha K)$ *in $\mathcal{A} \rtimes^r G$, and*
(ii) $d_{\mathrm{KK}}(A \rtimes^u_\alpha H, A \rtimes^u_\alpha K) = 1 = d_0(A \rtimes^u_\alpha H, A \rtimes^u_\alpha K)$ *in $A \rtimes^u_\alpha G$.*

6.7.1. *Some observations related to the C^*-algebras associated to groups.* Recall that, for any discrete group G, if $\mathcal{A} = \mathbb{C}$ and $\alpha : G \to \mathrm{Aut}(G)$ is the trivial action, then $\mathcal{A} \rtimes_\alpha^r G$ (resp., $\mathcal{A} \rtimes_\alpha^u G$) is just the reduced group C^*-algebra $C_r^*(G)$ (resp., the (universal) group C^*-algebra $C_u^*(G)$). Thus, we readily obtain the following.

Corollary 6.8. *Let G be a discrete group and let H and K be two distinct subgroups of G. Then*

$$\begin{aligned}
d_{\mathrm{KK}}(\mathbb{C}[H], \mathbb{C}[K]) &= d_{\mathrm{KK}}(C_r^*(H), C_r^*(K)) = 1\\
&= d_0(\mathbb{C}[H], \mathbb{C}[K]) = d_0(C_r^*(H), C_r^*(K)) \quad \textit{in } C_r^*(G),\\
d_{\mathrm{KK}}(\mathbb{C}[H], \mathbb{C}[K]) &= d_{\mathrm{KK}}(C_u^*(H), C_u^*(K)) = 1\\
&= d_0(\mathbb{C}[H], \mathbb{C}[K]) = d_0(C_u^*(H), C_u^*(K)) \quad \textit{in } C_u^*(G).
\end{aligned}$$

As in Remark 6.2, for any subgroup H of G, we consider the identity map $\eta_H : C_r^*(G) \to C_r^*(G)$ with the pre-Hilbert $C_r^*(H)$-norm

$$\|\eta_H(x)\| := \|E_H(x^*x)\|_r^{1/2}$$

for all $x \in \mathbb{C}[G]$. Also, we simply write η for η_e.

Lemma 6.9. *With running notation,*

$$\Big\|\eta\Big(\sum_{g\in G} \alpha_g g\Big)\Big\|^2 \leq \sum_{g\in G} |\alpha_g| \Big(\sum_{x\in gH} |\alpha_x|\Big)$$

for all $\sum_g \alpha_g g \in \mathbb{C}[G]$.

Proof. Note that

$$E\Big(\Big(\sum_{g\in G} \alpha_g g\Big)^* \Big(\sum_{g\in G} \alpha_g g\Big)\Big) = \sum_{g\in G}\sum_{x\in gH} \overline{\alpha_g}\alpha_x g^{-1}x = \sum_g \overline{\alpha_g} g^{-1}\Big(\sum_{x\in gH} \alpha_x x\Big)$$

for all $\sum_g \alpha_g g \in \mathbb{C}[G]$. □

Further, the reduced group C^*-algebra always admits a canonical faithful tracial state $\tau : C_r^*(G) \to \mathbb{C}$ which satisfies $\tau(x) = \langle \lambda(x)\delta_e, \delta_e\rangle$ for all $x \in \mathbb{C}[G]$, where $\lambda : \mathbb{C}[G] \to B(\ell^2(G))$ is the faithful $*$-representation induced by the (left) regular representation of G.

Remark 6.10. Let G be a discrete group.

(i) If H is the trivial subgroup of G, then the conditional expectation E from $C_r^*(G)$ onto $\mathbb{C}$ ($\equiv C_r^*(H)$) is just the tracial state τ on $C_r^*(G)$, which satisfies

$$E\Big(\sum_{g\in G} \alpha_g g\Big) = \alpha_e \quad \text{for all } \sum_g \alpha_g g \in \mathbb{C}[G],$$

the $\mathbb{C}$-valued inner product induced by E is just the usual inner product on $C_r^*(G)$ induced by τ, and

$$\Big\|\eta\Big(\sum_{g\in G} \alpha_g g\Big)\Big\| = \sqrt{\sum_{g\in G} |\alpha_g|^2}$$

for all $\sum_g \alpha_g g \in \mathbb{C}[G]$ because

$$E\Big(\Big(\sum_{g\in G}\alpha_g g\Big)^*\Big(\sum_{g\in G}\alpha_g g\Big)\Big) = \sum_{g\in G}|\alpha_g|^2.$$

(ii) We thus deduce that

$$\Big\|\sum_g \alpha_g g\Big\|_r \geq \sqrt{\Big(\sum_g |\alpha_g|^2\Big)} \quad \text{for all } \sum_g \alpha_g g \in \mathbb{C}[G].$$

In particular, $\sqrt{2} \leq \|g-h\|_r \leq 2$ for any two distinct elements g, h of G. Thus, $\{g \mid g \in G\}$ is a discrete linearly independent subset of the unit sphere of $C_r^*(G)$.

(iii) If G is finite, then $C_r^*(G) = \mathbb{C}[G]$ and

$$\|\eta(x)\| \leq \|x\|_r \leq |G|\|\eta(x)\| \tag{3}$$

for all $x \in \mathbb{C}[G]$, because

$$\|\eta(x)\| = \sqrt{\sum_{g\in G}|\alpha_g|^2} \quad \text{for every } x = \sum_g \alpha_g g \in \mathbb{C}[G]$$

(and the last inequality in (3) follows from Hölder's inequality).

Moreover, $E : \mathbb{C}[G] \to \mathbb{C}$ has finite index with a quasi-basis $\{g \mid g \in G\}$ and $\mathrm{Ind}_W(E) = |G|$.

Remark 6.11. If $\mathcal{B}$ is a $*$-subalgebra of a unital C^*-algebra $\mathcal{A}$, then $\mathcal{N}_{\mathcal{A}}(\mathcal{B})$ is a subgroup of $\mathcal{N}_{\mathcal{A}}(\overline{\mathcal{B}})$. In particular, for any subgroup H of a discrete group G,

$$\mathcal{N}_G(H) \leq \mathcal{N}_{C_r^*(G)}(\mathbb{C}[H]) \leq \mathcal{N}_{C_r^*(G)}(C_r^*(H)).$$

Given a subgroup H of a discrete group G and a unitary u in $\mathcal{U}(C_r^*(G)) \setminus \mathcal{N}_{\mathbb{C}[G]}(\mathbb{C}[H])$, it is natural to ask whether

$$u\mathbb{C}[H]u^* = \mathbb{C}[K] \quad \text{or} \quad uC_r^*(H)u^* = C_r^*(K)$$

for some subgroup K of G or not. Obviously, for every $g \in G$,

$$g\mathbb{C}[H]g^* = \mathbb{C}[gHg^{-1}], \quad gC_r^*(H)g^* = C_r^*(gHg^{-1}) \quad \text{and} \quad \|g-\mathbf{1}\|_r \geq \sqrt{2}.$$

However, when $u \notin G$, then it is not clear when $u\mathbb{C}[H]u^*$ equals $\mathbb{C}[K]$ for some subgroup K of G. Corollary 6.8 allows us to deduce the following partial answer to this question.

Proposition 6.12. *Let H be a proper subgroup of a discrete group G and let u be a unitary in $\mathbb{C}[G]$ (resp., $C_r^*(G)$) such that $\|u-\mathbf{1}\|_r < 1/2$. If $u\mathbb{C}[H]u^* = \mathbb{C}[K]$ (resp., $uC_r^*(H)u^* = C_r^*(K)$) for some subgroup K of G, then $K = H$ and, in particular, $u \in \mathcal{N}_{\mathbb{C}[G]}(\mathbb{C}[H])$ (resp., $u \in \mathcal{N}_{C_r^*(G))}(C_r^*(H))$),*

Proof. For the group algebra case, let $u \in \mathbb{C}[G]$ with $\|u-\mathbf{1}\|_r < 1/2$. Suppose that $u\mathbb{C}[H]u^* = \mathbb{C}[K]$ for some subgroup K of G. Then, by Lemma 3.5, we observe that

$$d_{\mathrm{KK}}(\mathbb{C}[H],\mathbb{C}[K]) = d_{\mathrm{KK}}(\mathbb{C}[H],u\mathbb{C}[H]u^*) \leq 2\|u-\mathbf{1}\|_r < 1.$$

Thus, $K = H$ by Corollary 6.8.

Even in the reduced group C^*-algebra case, when u is a unitary in $C^*_r(G)$ satisfying the inequality $\|u-\mathbf{1}\|_r < 1/2$ and that $uC^*_r(H)u^* = C^*_r(K)$ for some subgroup K of G, the same argument shows that $d_{\mathrm{KK}}(C^*_r(H), C^*_r(K)) < 1$, so that $K = H$ (by Corollary 6.8 again). □

Here are two obvious reformulations of the preceding corollary.

Remark 6.13. Let H, G be as in Proposition 6.12 and let u be a unitary in $\mathbb{C}[G]$ (resp., $C^*_r(G)$).

(i) If $u\mathbb{C}[H]u^* = \mathbb{C}[K]$ (resp., $uC^*_r(H)u^* = C^*_r(K)$) for some subgroup K other than H, then $\|u-\mathbf{1}\|_r \geq 1/2$.

(ii) If $\|u-\mathbf{1}\|_r < 1/2$, the conjugate $*$-subalgebra $u\mathbb{C}[H]u^*$ (resp., $uC^*_r(H)u^*$) is not equal to $\mathbb{C}[K]$ (resp., $C^*_r(K)$) for any subgroup K other than H.

Remark 6.14. One could ask whether every unitary u in $\mathcal{N}_{\mathbb{C}[G]}(\mathbb{C}[H])$ (resp., in $\mathcal{N}_{C^*_r(G)}(C^*_r(H))$) satisfies the inequality $\|u-\mathbf{1}\|_r < 1/2$. This is trivially seen to be false.

Indeed, if H is a subgroup of G with a nontrivial normalizer, then $g\mathbb{C}[H]g^* = \mathbb{C}[gHg^{-1}] = \mathbb{C}[H]$ for any $e \neq g \in \mathcal{N}_G(H)$, whereas, as noted in Remark 6.10 (ii), we have

$$\|g-\mathbf{1}\|_r = \|g-e\|_r \geq \sqrt{2} > 1/2.$$

By the same argument, one also concludes that if $u \in \mathcal{N}_{C^*_r(G)}(C^*_r(H))$, then the inequality $\|u-\mathbf{1}\|_r < 1/2$ need not be true.

Remark 6.15. Note that, by Lemma 3.4, we can always find a unitary u in $C^*_r(G)$ such that $0 < \|\mathbf{1}-u\|_r < 1/2$. Hence, as is well-known, not every C^*-subalgebra of $C^*_r(G)$ is a reduced subgroup C^*-algebra.

The following recipe provides a concrete way of obtaining unitaries arbitrarily close to $\mathbf{1}$.

Lemma 6.16. *Let G be a discrete group. If there exists an element $g \in G$ such that $g = g^{-1}$, then $u_\theta := \cos(\theta)e + i\sin(\theta)g$ is a unitary in $\mathbb{C}[G]$ for every $\theta \in \mathbb{R}$. Also, for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\|u_\theta - \mathbf{1}\|_r < \epsilon$ whenever $|\theta| < \delta$.*

Proof. Let $\theta \in \mathbb{R}$. Then

$$\begin{aligned} u_\theta u_\theta^* &= \big(\cos(\theta)e + i\sin(\theta)g\big)\big(\cos(\theta)e - i\sin(\theta)g\big) \\ &= \big(\cos^2(\theta)+\sin^2(\theta)\big)e + \big(i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta)\big)g = e. \end{aligned}$$

Similarly, $u_\theta^* u_\theta = e$. Hence, u_θ is a unitary in $\mathbb{C}[G]$ for every $\theta \in \mathbb{R}$.

Note that, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\cos(\theta)-1|$, $|\sin(\theta)| < \epsilon/2$ whenever $|\theta| < \delta$. So

$$\|u_\theta - \mathbf{1}\|_r = \|(\cos(\theta)-1)e + i\sin(\theta)g\|_r \leq |\cos(\theta)-1| + |\sin(\theta)| < \epsilon$$

whenever $|\theta| < \delta$. □

We can now find unitaries arbitrarily close to $\mathbf{1}$ yet not in the normalizer of a subalgebra. Compare with Corollary 3.7.

Lemma 6.17. *Let H be a proper subgroup of a discrete group G. If there exists an element $g \in G \setminus H$ such that $g = g^{-1}$ and $g \notin \mathcal{C}_G(H)$ (centralizer of H in G), then $u_\theta \notin \mathcal{N}_{C_r^*(G)}(C_r^*(H))$ for every $\theta \in \mathbb{R} \setminus \{\frac{n\pi}{2} \mid n \in \mathbb{Z}\}$.*

Proof. Since $g \notin \mathcal{C}_G(H)$, there exists an $h \in H$ such that $gh \neq hg$; also, we have $gh, hg \notin H$. Then, for this $h \in \mathbb{C}[H]$ and $\theta \in \mathbb{R} \setminus \{\frac{n\pi}{2} \mid n \in \mathbb{Z}\}$,

$$u_\theta h u_\theta^* = \cos^2(\theta)h + i\sin(\theta)\cos(\theta)gh - i\sin(\theta)\cos(\theta)hg + \sin^2(\theta)ghg.$$

Hence, $u_\theta \notin \mathcal{N}_{C_r^*(G)}(C_r^*(H))$ for every $\theta \in \mathbb{R} \setminus \{\frac{n\pi}{2} \mid n \in \mathbb{Z}\}$. ☐

Example 6.18. Let $G = S_3$ and $H = A_3$. Then $u_\theta := \cos(\theta)e + i\sin(\theta)(12)$ is a unitary in $\mathbb{C}[S_3]$ for every $\theta \in \mathbb{R}$. Clearly, by the preceding lemma, we have $u_\theta \notin \mathcal{N}_{\mathbb{C}[S_3]}(\mathbb{C}[A_3])$ for every $\theta \in \mathbb{R} \setminus \{\frac{n\pi}{2} \mid n \in \mathbb{Z}\}$, because $(12) \notin \mathcal{C}_{S_3}(A_3)$. Also, we can choose $0 < \theta < \pi/2$ small enough so that $\|u_\theta - \mathbf{1}\|_r$ is as small as we wish. Thus,

(i) for each $\epsilon > 0$, there exists a unitary u in $\mathbb{C}[S_3]$ such that

$$0 < d_{\mathrm{KK}}(\mathbb{C}[A_3], u\mathbb{C}[A_3]u^*) < \epsilon,$$

(ii) $\mathbf{1}$ is not an interior point of $\mathcal{N}_{\mathbb{C}[S_3]}(\mathbb{C}[A_3])$ in $\mathcal{U}(\mathbb{C}[S_3])$.

6.19. **Kadison–Kastler distance between (Banach) subgroup algebras.** Recall that, for a discrete group G, the Banach space

$$\ell^1(G) := \Big\{ f : G \to \mathbb{C} \;\Big|\; \sum_{g\in G} |f(g)| < \infty \Big\}$$

is a unital Banach $*$-algebra with multiplication given by convolution, *i.e.*, for $a, b \in \ell^1(G)$,

$$(ab)(g) := \sum_{h \in G} a(h)b(h^{-1}g),$$

and involution given by

$$a^*(g) := \overline{a(g^{-1})}$$

for $a \in \ell^1(G)$, $g \in G$.

Further, for each $g \in G$, we define $u_g \in \ell^1(G)$ by $u_g(h) = \delta_{g,h}$. Then u_e is the multiplicative identity for $\ell^1(G)$.

Remark 6.20. With running notation,

(i) there exists an injective unital *-homomorphism $i : \mathbb{C}[G] \to \ell^1(G)$ such that $i(g) = u_g$ for all $g \in G$ and the image of i is dense in $\ell^1(G)$. In particular, $\mathbb{C}[G]$ can be considered as a dense subspace of $\ell^1(G)$.

(ii) For any subgroup H of G, the natural embedding of $\mathbb{C}[H]$ into $\mathbb{C}[G]$ extends to an isometric unital $*$-homomorphism from $\ell^1(H)$ into $\ell^1(G)$. We can thus identify $\ell^1(H)$ with a unital Banach $*$-subalgebra of $\ell^1(G)$.

Proposition 6.21. *Let H and K be two distinct subgroups of a discrete group G. Then*

$$d_{\mathrm{KK}}(\mathbb{C}[H], \mathbb{C}[K]) = d_{\mathrm{KK}}(\ell^1(H), \ell^1(K)) = 1$$

in $\ell^1(G)$.

Proof. Since $\mathbb{C}[G]$ is dense in $\ell^1(G)$, by Lemma 3.3, it is enough to show that $d_{\mathrm{KK}}(\mathbb{C}[H]), \mathbb{C}[K]) = 1$ in $\ell^1(G)$. In view of Remark 3.2 (iii), we can assume that $H \neq H \cap K \neq K$.

For convenience, let $C := \mathbb{C}[H]$ and $D := \mathbb{C}[K]$ and let $h \in H \setminus H \cap K$. Then

$$\|h - \sum_{k \in K} \alpha_k k\|_1 \geq 1 \quad \text{for all} \sum_{k \in K} \alpha_k k \in B_1(D).$$

This implies that $d(h, B_1(D)) \geq 1$. Thus, by definition, we get

$$d_{\mathrm{KK}}(\mathbb{C}[H], \mathbb{C}[K]) \geq 1,$$

and we are done. □

6.22. **Kadison–Kastler and Christensen distances between subgroup von Neumann subalgebras.** Recall that, for any discrete group G, the group von Neumann algebra associated to G is the von Neumann algebra given by

$$L(G) = \{\lambda(G)\}'' \subseteq B(\ell^2(G)),$$

where $\lambda : G \to B(\ell^2(G))$ is the left regular (unitary) representation of G. Also, there is a natural $*$-isomorphism between $\mathbb{C}[G]$ and $*$-alg$(\lambda(G))$, and for any subgroup H of G, there is a natural $*$-isomorphism from $L(H)$ onto $\lambda(H)'' \subset L(G)$; thus, $L(H)$ can be considered as a von Neumann subalgebra of $L(G)$. Further, $L(G)$ always admits a faithful normal tracial state τ given by $\tau(x) = \langle x(\delta_e), \delta_e \rangle$ for $x \in L(G)$. Thus, $L(G)$ admits an inner-product structure via τ which induces a norm $\|\cdot\|_\tau$ on $L(G)$ given by $\|x\|_\tau = \tau(x^*x)^{1/2}$, $x \in L(G)$.

Proposition 6.23. *Let G be a discrete group and let H and K be two distinct nontrivial subgroups of G. Then, in $L(G)$, the distances*

$$d_{\mathrm{C}}(\mathbb{C}[H], \mathbb{C}[K]),\ d_{\mathrm{MT}}(\mathbb{C}[H], \mathbb{C}[K]),\ d_{\mathrm{KK}}(\mathbb{C}[H], \mathbb{C}[K]),$$
$$d_{\mathrm{C}}(L(H), L(K)),\ d_{\mathrm{MT}}(L(H), L(K)) \text{ and } d_{\mathrm{KK}}(L(H), L(K))$$

are all equal to 1.

Proof. By Lemma 5.9, $d_{\mathrm{MT}}(\mathbb{C}[H], \mathbb{C}[K]) \leq d_{\mathrm{KK}}(\mathbb{C}[H], \mathbb{C}[K]) \leq 1$. So, in view of Proposition 5.7 and Proposition 5.11, it is enough to show that

$$d_{\mathrm{MT}}(\mathbb{C}[H], \mathbb{C}[K]) \geq 1.$$

We prove this by considering the following two possibilities separately.

Case 1. Suppose that $H \subsetneq K$ or $K \subsetneq H$. Without loss of generality, assume that $H \subsetneq K$. Then $\mathbb{C}[H] \subsetneq \mathbb{C}[K]$ and

$$d_\tau(\hat{a}, \widehat{B_1(\mathbb{C}[K])}) = 0$$

for all $a \in B_1(\mathbb{C}[H])$. Further, for any $k \in K \setminus H \cap K$, we have $\|\hat{k} - \hat{x}\|_\tau \geq 1$ for all $x \in B_1(\mathbb{C}[H])$. Thus,

$$\sup_{z \in B_1(\mathbb{C}[K]))} d_\tau(\hat{z}, \widehat{B_1(\mathbb{C}[H])}) \geq 1,$$

and hence $d_{\mathrm{MT}}(\mathbb{C}[H], \mathbb{C}[K]) \geq 1$.

Case 2. Suppose that $H \neq H \cap K \neq K$. Then, again for any $h \in H \setminus H \cap K$, $\|\hat{h} - \hat{x}\|_\tau \geq 1$ for all $x \in B_1(\mathbb{C}[K])$. Thus, as above, $d_{\mathrm{MT}}(\mathbb{C}[H], \mathbb{C}[K]) \geq 1$, and we are done. $\square$

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