Münster J. of Math. **17** (2024), 203–240
DOI 10.17879/51958617720
urn:nbn:de:hbz:6-51958707380

Münster Journal of Mathematics

The fundamental groupoid: Topology, Haar system and actions

Rohit Dilip Holkar, Md Amir Hossain, and Dheeraj Kulkarni

(Communicated by Siegfried Echterhoff)

Abstract. We equip the fundamental groupoid $\Pi_1(X)$ of a locally path connected and semilocally simply connected space X with the Whisker topology to make it a topological groupoid. Then the equivalence of the Whisker topology and the quotient of the compact-open topology on the path space of X is proved. This topological groupoid is *locally trivial* and not étale. We prove that the fundamental groupoid of a topological group, in particular a Lie group, is a *transformation groupoid*. To understand when $\Pi_1(X)$ is locally compact or Hausdorff, we study its point-set topology. We discuss the existence and give a description of a Haar system on $\Pi_1(X)$. For the last main result, let $\mathcal{E}$ be the category of $\Pi_1(X)$-spaces for which the momentum maps are local homeomorphisms. We show that this category is isomorphic to that of the covering spaces of X. Using this, we give different characterizations for the free or proper actions in $\mathcal{E}$.

1. Introduction

In this article, we study the fundamental groupoid as a topological groupoid. The project has two main goals: firstly, to describe when a fundamental groupoid is locally compact and can be equipped with a Haar system; secondly, to study the actions of this topological groupoid.

Three popular choices are available for the topology on the fundamental groupoid: the Whisker and Lasso topologies, and the quotient of the compact-open topology on the path space; we call the last topology the "CO′ topology". Brown [6, Lem. 2] seems to be the first to topologize the fundamental groupoid. He works with locally path connected and semilocally simply connected spaces, and he basically uses the Whisker topology. Recent articles of Pakdaman and Shahini show that the Lasso [27] (or the Whisker [28]) topology makes the fundamental groupoid of a *locally path connected* (respectively, *small loop transfer*) space a topological groupoid. In general, the fundamental

We thank our funding institutions: the first author was supported by SERB's grants MTR/2020/000198 and SRG/2020/001823 and the second one by CSIR grant 09/1020 (0159)/2019-EMR-I.

groupoid is not a topological groupoid with the CO′ topology. If the underlying space is locally path connected and semilocally simply connected, then the CO′ topology also makes the fundamental groupoid topological; this is noticed by Reinhart [29, Prop. 2.37 and the discussion preceding it] and Muhly [23, Ex. 5.33.4].

The restrictions of the above topologies on the universal covering space and the fundamental group are well-studied. It is customary to use the Whisker topology for constructing the universal covering space of a locally path connected and semilocally simply connected space, *e.g.* [32, 25, 18]. Brazas [3] proves the existence of an interesting topology—the finest topology that makes the fundamental group a topological group, and it also makes the quotient map from the path space continuous.

We study the Whisker topology on the fundamental groupoid of a locally path connected and semilocally simply connected space X (unlike [28] in which the spaces are slightly more general[1]). We show that the Whisker and CO′ topologies on the fundamental groupoid $\Pi_1(X)$ are equal. As X is semilocally simply connected, its universal covering space exists, and this allows us to describe the topological properties of $\Pi_1(X)$, such as Hausdorffness, local compactness (Section 5), in terms of those of X.

Once we have the locally compact fundamental groupoid, we can equip it with a Haar system. We show that the existence of a Haar system on the fundamental groupoid $\Pi_1(X)$ is equivalent to the existence of a *fully supported* measure on X or a $\pi_1(X,x)$-*invariant* measure (for the deck transformation action) on the universal covering space of X (Theorem 6.7). The point is that the fully supported measure on X can be *pulled back* to a $\pi_1(X,x)$-invariant measure on the covering space $\Pi_1(X)^x$ using the covering map; here $\Pi_1(X)^x = r^{-1}(x)$ for the range map $\Pi_1(X) \to X$. And such a measure, when *translated* to other cover copies, $\Pi_1(X)^z$ for $z \in X$, gives us the Haar system on the fundamental groupoid. We describe the concrete formulae for the Haar system using the measures on X and the universal covering space $\Pi_1(X)^x$ in Section 6.11.

It is natural to use the topological fundamental groupoid as an invariant for distinguishing spaces, *e.g.* [14, 4]. Although this is not our central goal, we give an example of two spaces whose algebraic fundamental groupoids as well as (topological) fundamental groups are isomorphic, but the topological fundamental groupoids are different (see Remark 3.14).

Example 2.1.4 in [20] states that the fundamental groupoid is étale. However, we note (Remark 3.13) that it is not étale but locally trivial in the sense of Ehresmann [12, p. 143, second paragraph]. An important observation of ours (Theorem 4.3) is that the fundamental groupoid of a path connected, locally path connected and semilocally simply connected group is a transformation groupoid—contrary to a common belief that fundamental groupoids are *far* away from being transformation groupoids.

[1]In [28], Pakdaman and Shahini assume that their spaces are "small loop transfer spaces" [5, Def. 4.7]; a locally path connected and semilocally simply connected space is such a space.

We observe that the covering spaces constitute an interesting category of actions of a fundamental groupoid. Theorem 7.9 characterizes the covering spaces as $\Pi_1(X)$-spaces. We describe free (Proposition 7.11) and proper (Theorem 8.6) actions in this category.

Organization of the article and chief results. Section 2 establishes notation, conventions and preliminaries for topological groupoids.

Section 3 starts by describing and then comparing the Whisker and CO′ topologies; Proposition 3.5 is the main result about this comparison. Theorem 3.9 proves that the fundamental groupoid is topological; its Corollary 3.10 shows that the topology on the fundamental groupoid is compatible with the universal covering space and the fundamental group.

Theorem 4.3 in Section 4 proves that the fundamental groupoid of a group is a transformation groupoid.

In Section 5, we first give an alternate description of the fundamental groupoid (Proposition 5.4). Then, in Section 5.5, we discuss the point-set topology of the fundamental groupoid in Propositions 5.7, 5.8, 5.9 and 5.10.

Section 6 discusses a Haar system on a locally compact, Hausdorff and locally trivial groupoid. Section 6.11 contains descriptions of the Haar system on $\Pi_1(X)$ in terms of certain fully supported measures on X or the universal cover of X.

In Section 7, we prove Theorems 7.9 and 7.12 which say that the category of covering spaces over a space can be identified with the category of $\Pi_1(X)$-spaces for which the momentum maps are local homeomorphisms.

In Section 8, we characterize the proper actions (Theorem 8.6) of the fundamental groupoid.

2. Notation, conventions and preliminaries

2.1. **Groupoids.** We work with topological groupoids. For locally compact spaces and groupoids, we follow Tu's conventions in [33]. Thus, by a *quasi-compact* space, we mean a space for which any open cover has a finite subcover. A quasi-compact and Hausdorff space is called *compact*. A topological space is called *locally compact* if every point in it has a compact neighborhood. Thus a locally compact space is locally Hausdorff and hence T_1. Note that a compact set K is also locally compact Hausdorff. Then the open set $K^0 \subseteq K$, the interior of K, is a locally compact Hausdorff space [25, Cor. 29.3]. A space is called *paracompact* if it is Hausdorff and every open cover of it has a locally finite open refinement.

For us, a *groupoid* is a *small* category in which every arrow is invertible. Thus a groupoid can be denoted as a quintuple $(G, G^{(0)}, r, s, \mathrm{inv})$, where G is the small category with $G^{(0)}$ as the set of objects, r and s are the range and source maps $G \rightrightarrows G^{(0)}$ and $\mathrm{inv} : G \to G$ is the inversion map. Thus, for an arrow $\gamma \in G$, $r(\gamma)$, $s(\gamma)$ and $\mathrm{inv}(\gamma)$ are, respectively, the range of γ, the source of γ and γ^{-1}. We shall not write the quintuple all the time but simply call G the groupoid.

For a groupoid G, we always consider $G^{(0)}$ as a subset of G by identifying it with the unit arrows on the objects. This convention implies that $s(\gamma)=\gamma^{-1}\gamma$ and $r(\gamma)=\gamma\gamma^{-1}$. Finally, we call the elements of $G^{(0)}$ the units instead of objects.

The elements of the fiber product

$$G \times_{s,G^{(0)},r} G := \{(\gamma,\eta) \in G \times G \mid s(\gamma)=r(\gamma)\}$$

are called *composable pairs* of G. The multiplication on G is the mapping

$$G \times_{s,G^{(0)},r} G \to G, \quad (\gamma,\eta) \mapsto \gamma\eta.$$

The groupoid G is called *topological* if G carries a topology such that the multiplication and inversion are continuous mappings; for the continuity of the multiplication, the fiber product $G \times_{s,G^{(0)},r} G$ is equipped with the subspace topology of $G \times G$. As a consequence, inv is a homeomorphism and $r,s : G \rightrightarrows G^{(0)}$ are continuous maps when $G^{(0)}$ is given the subspace topology.

A topological groupoid G is called *locally compact* if its topology is locally compact and its space of units is a Hausdorff subspace.

For $u \in G^{(0)}$, the symbols G^u, G_u and G^u_u have their standard meanings, namely,

$$G_u = s^{-1}(\{u\}), \quad G^u = r^{-1}(\{u\}) \quad \text{and} \quad G^u_u = G_u \cap G^u.$$

Note that G^u_u is the isotropy group at $u \in G^{(0)}$. In general, for sets $A,B \subseteq G^{(0)}$, we define

$$G^A = r^{-1}(A), \qquad G_B = s^{-1}(B) \qquad \text{and} \quad G^A_B = G^A \cap G_B.$$

The subspaces G_u or G^u of G are called *transversals* (see [24]).

Observation 2.2. Suppose that γ is an arrow in a groupoid G; write $x=s(\gamma)$ and $y=r(\gamma)$. Then the cardinality of the isotropy at x and G^y_x is the same; the function

$$\varphi : G^x_x \to G^y_x, \quad \eta \mapsto \gamma\eta \qquad \text{for } \eta \in G^x_x$$

is a bijection. The inverse of the function is given by

$$\varphi^{-1} : G^y_x \to G^x_x, \quad \eta' \mapsto \gamma^{-1}\eta' \quad \text{for } \eta' \in G^y_x.$$

Example 2.3. Given a topological space X, the product space $X \times X$ has the following groupoid structure: $(x,y),(z,w) \in X \times X$ are composable if and only if $y=z$ and the composition is $(x,y)(y,w)=(x,w)$; the inverse map is $(x,y)^{-1}=(y,x)$. The space of units of this groupoid is the diagonal in $X \times X$, which we often identify with X. This groupoid is called the *groupoid of the trivial equivalence relation*.

Example 2.4. Let G be a topological group acting continuously on the right on a space X. The *transformation groupoid* for this action, denoted by $X \rtimes G$, is defined as follows: the underlying space of this groupoid is the cartesian product $X \times G$; two elements (x,g) and (y,t) are composable if and only if $y = x \cdot g$ and the composition is given by $(x,g)(y,t)=(x,gt)$; the inverse map is given by $(x,g)^{-1} = (x \cdot g, g^{-1})$. Finally, if $e \in G$ is the unit, then the space of units of this groupoid is $X \times \{e\}$, which we identify with X.

For a left G-space X, the transformation groupoid is defined similarly and is denoted by $G \ltimes X$.

A map of spaces $f : X \to Y$ is called a *local homeomorphism* if any $x \in X$ has a neighborhood $U \subseteq X$ such that $f(U) \subseteq Y$ is open and $f|_U : U \to f(U)$ is a homeomorphism. A local homeomorphism is an open mapping. A covering map is a local homeomorphism.

Definition 2.5 (Locally trivial groupoid [12, p. 143])**.** A topological groupoid G is called *locally trivial* if, for each unit u, the restriction of the source map $G^u \to G^{(0)}$ is a local homeomorphism.

Definition 2.5 can be, equivalently, stated as $r|_{G_u} : G_u \to G^{(0)}$ is a local homeomorphism for every unit u.

Observation 2.6. Note that, since $s|_{G^u}$ is a local homeomorphism for a locally trivial groupoid G, the isotropy $G_u^u = s|_{G^u}^{-1}(\{u\})$ is discrete.

A groupoid G is called *transitive* if, given $x, y \in G^{(0)}$, there is an arrow $\gamma \in G$ with $x = r(\gamma)$ and $y = s(\gamma)$. Equivalently, G is transitive if $s|_{G^x} : G^x \to G^{(0)}$ is surjective for some $x \in G^{(0)}$. Therefore, a locally trivial groupoid is transitive if the local homeomorphism $s|_{G^x}$ is surjective for some $x \in G^{(0)}$.

Definition 2.7. A topological groupoid is called *étale* if its range (or, equivalently, source) map is a local homeomorphism.

The transformation groupoid in Example 2.4 is étale if and only if G is discrete.

2.8. **Actions of groupoids.** A *(left) action* of a topological groupoid G on a space X is a pair (r_X, σ), where $r_X : X \to G^{(0)}$ is an open surjection and $\sigma : G \times_{s,G^{(0)},r_X} X \to X$ is a function with the following properties:

(i) for any unit $u \in G^{(0)}$ and $x \in r_X^{-1}(u)$, $\sigma(u, x) = x$;

(ii) if $(\gamma, \eta) \in G \times G$ is a composable pair and $(\eta, x) \in G \times_{s,G^{(0)},r_X} X$, then $(\gamma, \sigma(\eta, x))$ is also in the fiber product $G \times_{s,G^{(0)},r_X} X$. Moreover, we have $\sigma(\gamma, \sigma(\eta, x)) = \sigma(\gamma\eta, x)$.

Here $G \times_{s,G^{(0)},r_X} X$ is the fiber product $\{(\gamma, x) \in G \times X \mid s(\gamma) = r_X(x)\}$. We call r_X the momentum map of the action. A *right action* is defined similarly. From now on, we shall simply write γx or $\gamma \cdot x$ instead of $\sigma(\gamma, x)$ for $(\gamma, x) \in G \times_{s,G^{(0)},r_X} X$.

A groupoid G acts on its space of units, from left, as follows: for $\gamma \in G$ and $x \in G$, the action is defined if $s(\gamma) = x$ and is given by $\gamma x = r(\gamma)$. The identity map on $G^{(0)}$ is the momentum map for this action. Similarly, the right action of G on $G^{(0)}$ is defined.

Given G-spaces X and Y, by an *equivariant map*, we mean a function $f : X \to Y$ such that $r_Y \circ f = r_X$ and $f(\gamma x) = \gamma f(x)$ for all composable pairs $(\gamma, x) \in G \times_{s,G^{(0)},r_X} X$.

Next is a characterization of spaces on which a transformation groupoid can act. We require this characterization in Example 7.10.

Lemma 2.9 ([13, Lem. 2.7]). *Let $G \ltimes X$ be a transformation groupoid for an action of a groupoid G on a space X. Then $G \ltimes X$ acts on a space Y with $\varrho : Y \to X$ as the momentum map if and only if ϱ is a G-equivariant map of spaces. Thus there is a one-to-one correspondence between G-equivariant maps $\varrho : Y \to X$ and $G \ltimes X$-spaces Y.*

Assume that X is a G-space for a groupoid G. While studying the groupoid actions, the map

$$a : G \times_{s,G^{(0)},r_X} X \to X \times X, \quad a : (\gamma, x) \mapsto (\gamma x, x) \tag{1}$$

turns out to be useful. Although, this map does not have a standard name, we call it *the kinetics* or *the map of the kinetics* of the action. Observation 3.11 shows that the map of kinetics for the action of $\Pi_1(X)$ on its space of units is a covering map, where X is a locally path connected and semilocally simply connected space.

Lemma and Definition 2.10. *Let G be a groupoid acting on a space X. Then the following statement are equivalent:*
(i) *the map of kinetics of the action is one-to-one;*
(ii) *for every $x \in X$, the isotropy $(G \ltimes X)_x^x$ is the trivial group.*
If any of the above conditions holds, we call the G-action on X free.

As it is a standard fact, we leave the proof of the last lemma to the reader.
A map of locally compact, Hausdorff spaces $f : X \to Y$ is called *proper* if the inverse image of a compact set in Y under f is compact.

Lemma and Definition 2.11. *Let G be a locally compact Hausdorff groupoid acting on a locally compact Hausdorff space X. Then the following statements are equivalent:*
(i) *the map of the kinetics of the action is proper;*
(ii) *for any pair of compact subsets T, S of X, the set $\{\gamma \in G \mid \gamma T \cap S \neq \varnothing\}$ is compact in G.*
The action of G on X is called proper *if any of the above conditions holds. And then the transformation groupoid $G \ltimes X$ is called proper.*

The proof of the last lemma is also standard, *e.g.* see [35, Prop. 2.17]. A groupoid G is called proper if its action on the space of units is proper.

Observation 2.12. If a groupoid G acts properly on a space X, then the isotropy of the transformation groupoid at a point $x \in X$ is compact, and it is the inverse image of (x, x) under the map of kinetics of the action.

2.13. **Topology.** Our reference for algebraic topology is Hatcher's book [18]. We denote the unit closed interval $[0, 1] \subseteq \mathbb{R}$ by $\mathbb{I}$. For a topological space X, $\mathrm{P}X$ denotes the set of paths in X. That is, $\mathrm{P}X = \mathrm{C}(\mathbb{I}, X)$ is the set of all continuous functions from $\mathbb{I} \to X$. For $\gamma \in \mathrm{P}X$, we call $\gamma(0)$ and $\gamma(1)$, respectively, the starting and ending points of γ. For $x_0 \in X$, the constant function x_0 in $\mathrm{P}X$ is called the constant path at x_0, and is denoted by e_{x_0}.

Let $\gamma, \eta \in \mathrm{P}X$. If $\gamma(0) = \eta(1)$, we call the pair (γ, η) *composable* (or *concatenable*) and $\gamma \,\square\, \eta$ denotes the concatenated path. For a path γ, γ^- is its *opposite* path given by $\gamma^-(t) = \gamma(1-t)$.

For X as above, let $x \in X$. Then $\Pi_1(X)$ and $\pi_1(X, x)$ denote the *fundamental groupoid* of X and *fundamental group* of X based at x, respectively. Thus $\Pi_1(X)$ is the quotient of $\mathrm{P}X = \mathrm{C}(\mathbb{I}, X)$ by the *endpoint fixing* path homotopy equivalence relation. The (endpoint fixing) homotopy class of $\gamma \in \mathrm{P}X$ is denoted by $[\gamma]$. The quotient map from $\mathrm{P}X \to \Pi_1(X)$, corresponding to the equivalence relation mentioned above, need not be open in general [25, Ex. 1, §22].

For X as earlier, a set $V \subseteq X$ is called *relatively inessential* in X if the homomorphism $\pi_1(V, x) \to \pi_1(X, x)$ induced by the inclusion $(V, x) \hookrightarrow (X, x)$ is trivial for all $x \in V$. The space X is called *semilocally simply connected* if each $x \in X$ has a relatively inessential neighborhood. A covering map, covering space, locally path connected space, an evenly covered neighborhood and other standard terms regarding covering spaces and simply connected covers have their usual meanings as in Hatcher [18].

Let X and Y be spaces and let $\mathrm{C}(Y, X)$ denote the set of continuous functions from Y to X. For $U \subseteq Y$ and $V \subseteq X$, define the following subset of $\mathrm{C}(Y, X)$:

$$\mathcal{B}(U, V) = \{f \in \mathrm{C}(Y, X) \mid f(U) \subseteq V\}.$$

Recall from [10] that the sets of the form $\mathcal{B}(K, U)$, where $K \subseteq Y$ is compact and $U \subseteq X$ is open, constitute a subbasis for the compact-open topology on $\mathrm{C}(Y, X)$.

Lemma 2.14. *The following statements hold.*

(i) *Assume that $\mathcal{U}$ is a basis for the topology of a space X. Let $\mathcal{R}$ be the collection of subsets of $\mathrm{P}X$ of the form $\mathcal{B}(I, U)$, where $I \subseteq \mathbb{I}$ is a closed interval and $U \in \mathcal{U}$. Then $\mathcal{R}$ forms a subbasis for the compact-open topology on $\mathrm{P}X$.*

(ii) *In particular, if the space X above is locally path connected and semilocally simply connected, then the sets of the form $\mathcal{B}(I, U)$, with $I \subseteq \mathbb{I}$ a closed interval and $U \subseteq X$ a path connected and relatively inessential open neighborhood, form a subbasis for the compact-open topology on $\mathrm{P}X$.*

(iii) *In particular, if the space X above is locally path connected and semilocally simply connected, then the sets of the form $\mathcal{B}(I, U)$, with $I \subseteq \mathbb{I}$ a closed interval with* rational endpoints *and $U \subseteq X$ a path connected and relatively inessential open neighborhood, form a subbasis for the compact-open topology on $\mathrm{P}X$.*

Proof. (i): Every set in $\mathcal{R}$ is clearly open in $\mathrm{P}X$ in the compact-open topology. Let $\mathcal{B}(I, V)$ be a subbasic open set in $\mathrm{P}X$, where $I := [a, b] \subseteq \mathbb{I}$ is an interval and $V \subseteq X$ open. For given $\alpha \in \mathcal{B}(I, V)$, we construct finitely many sets $A_1, \dots, A_k \in \mathcal{R}$ with $\alpha \in \bigcap_{j=1}^k A_j \subseteq \mathcal{B}(I, V)$, which is the claim of the lemma.

Since $\mathcal{U}$ is a basis, we cover $V = \bigcup_{\vartheta \in J} U_\vartheta$ by open sets U_ϑ in $\mathcal{U}$, where J is some indexing set. Then $\{\alpha^{-1}(U_\vartheta)\}_{\vartheta \in J}$ is an open cover of $I = [a, b]$ in $\mathbb{I}$. Using a Lebesgue number for the open cover $\{\alpha^{-1}(U_\vartheta)\}_{\vartheta \in J}$, get a finite sequence $a_0 = a < a_1 < \cdots < a_l = b$ in $[a, b]$ such that, for each $0 \leq j \leq l$, there is an index $\vartheta_j \in J$ so that $[a_j, a_{j+1}]$ contained in $\alpha^{-1}(U_{\vartheta_j})$. Thus

$$\alpha \in \bigcap_{j=0}^{l} \mathcal{B}([a_j, a_{j+1}], U_{\vartheta_j}).$$

The third relation of [10, Chap. XII, Ex. 2] implies that

$$\bigcap_{j=0}^{l} \mathcal{B}([a_j, a_{j+1}], U_{\vartheta_j}) \subseteq \mathcal{B}\biggl(\bigcup_{j=0}^{l} [a_j, a_{j+1}], \bigcup_{j=0}^{l} U_{\vartheta_j}\biggr) \subseteq \mathcal{B}(I, V).$$

(ii) and (iii): These follow directly from (i) as the path connected relatively inessential neighborhoods form a basis for the topology on X. $\square$

Lemma 2.14 (iii) implies that $\mathrm{P}X$ is second countable if X is second countable, locally path connected and semilocally simply connected.

3. The fundamental groupoid as a topological groupoid

3.1. **The two topologies.** In this section, we describe the CO′ and the Whisker topologies on the fundamental groupoid, and show that they are equal for a locally path connected and semilocally simply connected space.

Definition 3.2 (CO′ topology)**.** Let X be a space. The *CO′ topology* on $\Pi_1(X)$ is the quotient topology induced by the compact-open topology on $\mathrm{P}X$ via the quotient map $q : \mathrm{P}X \to \Pi_1(X)$ that sends a path to its path homotopy class.

Let X be a topological space. For $\gamma \in \mathrm{P}X$ and neighborhoods $\gamma(1) \in U$ and $\gamma(0) \in V$ in X, define the following subsets of $\mathrm{P}X$ and $\Pi_1(X)$, respectively:

$$\begin{aligned}
\tilde{N}(\gamma, U, V) = \{\delta \,\Box\, (\gamma \,\Box\, \vartheta) \mid \delta \in \mathrm{P}U \text{ with } \delta(0) = \gamma(1) \text{ and}\\
\vartheta \in \mathrm{P}V \text{ with } \gamma(0) = \vartheta(1)\},\\
N([\gamma], U, V) = \{[\delta \,\Box\, \gamma \,\Box\, \vartheta] \mid \delta \in \mathrm{P}U \text{ with } \delta(0) = \gamma(1) \text{ and}\\
\vartheta \in \mathrm{P}V \text{ with } \gamma(0) = \vartheta(1)\}.
\end{aligned}$$

Thus, in short, $N([\gamma], U, V) = q(\tilde{N}(\gamma, U, V))$. For the sake of clarification, by $[\delta \,\Box\, \gamma \,\Box\, \vartheta]$, we mean $[\delta \,\Box\, (\gamma \,\Box\, \vartheta)] = [(\delta \,\Box\, \gamma) \,\Box\, \vartheta]$.

The sets of the form $N([\gamma], U, V)$, where $[\gamma] \in \Pi_1(X)$ and $U, V \subseteq X$ are neighborhoods of $\gamma(1)$ and $\gamma(0)$, constitute a basis for a topology on $\Pi_1(X)$. In particular, if the space X is locally path connected and semilocally simply connected, then the collection sets of the form $N([\gamma], U, V)$, where U and V are path connected and relatively inessential, form a basis for the topology on $\Pi_1(X)$ (see [6, Lem. 2]).

Definition 3.3 (The Whisker topology)**.** For a space X, the topology on $\Pi_1(X)$ generated by the basic open sets $N([\gamma], U, V)$, where $[\gamma] \in \Pi_1(X)$ and $U, V \subseteq X$ are neighborhoods of $\gamma(1)$ and $\gamma(0)$, respectively, is called the *Whisker topology*.

Assume that the space X in the previous definition is locally path connected and semilocally simply connected. Suppose $N([\alpha], U, V)$ is a neighborhood of $[\alpha] \in \Pi_1(X)$, where U and V are path connected and semilocally simply connected neighborhoods of $\alpha(1)$ and $\alpha(0)$, respectively. Then, for $[\beta] \in N([\alpha], U, V)$,

$$N([\beta], U, V) = N([\alpha], U, V).$$

The inclusion $N([\beta], U, V) \subseteq N([\alpha], U, V)$ is clear. Suppose $[\beta] = [\delta \,\square\, \alpha \,\square\, \sigma]$ for some paths δ and σ in U and V, respectively. Then $[\alpha] = [\delta^- \,\square\, \beta \,\square\, \sigma^-] \in N([\beta], U, V)$. Therefore, the reverse inclusion holds.

Our next goal is Proposition 3.5, which we state after the following two lemmas. We do not claim any originality for the proof of this proposition as it is basically the proof of [16, Lem. 2.1] written for *both ends* of a path.

Lemma 3.4. *For a topological space X, the CO′ topology is coarser than the Whisker topology. That is,* $\mathrm{CO}' \subseteq$ Whisker*.*

Proof. Assume that $\mathrm{P}X$ has the compact-open topology and $q : \mathrm{P}X \to \Pi_1(X)$ is the quotient map. Let a CO′-open set $M \subseteq \Pi_1(X)$ be given. That is, $q^{-1}(M)$ is open in $\mathrm{P}X$. Let $[\alpha] \in M$. Then we show that there is a Whisker-neighborhood $N([\alpha], V^1, V^0)$ of $[\alpha]$ that is contained in M.

Firstly, note that, since $q^{-1}(M) \subseteq \mathrm{P}X$ is open in the compact-open topology, α has a basic (compact-open) neighborhood $\bigcap_{i=1}^n \mathcal{B}(I_i, U_i) \subseteq q^{-1}(M)$, where $I_1, I_2, \ldots, I_n$ are closed intervals in $[0,1]$ and $U_1, U_2, \ldots, U_n$ are open subsets of X (cp. Lemma 2.14 (ii)).

Now we construct the required Whisker-neighborhood: for $i = 0, 1$, let S_i be the set of indices $k \in \{1, \ldots, n\}$ for which $i \in I_k$. Set $V^i = \bigcap_{k \in S_i} U_k$ for $i = 0, 1$. Then $\alpha(0) \in V^0$ and $\alpha(1) \in V^1$. The set $N([\alpha], V^1, V^0)$ is a neighborhood of $[\alpha]$ in the Whisker topology.

The neighborhood $N([\alpha], V^1, V^0)$ is contained in M: for that, assume that $[\beta'] \in N([\alpha], V^1, V^0)$ is given. Let $\beta = \vartheta \,\square\, (\alpha \,\square\, \gamma) \in [\beta']$ for some paths $\vartheta \in \mathrm{P}V^1$ and $\gamma \in \mathrm{P}V^0$ appropriately concatenable with α. Choose $\varepsilon \in (0,1)$ such that $\alpha([0,\varepsilon]) \subseteq V^0$ and $I_i \cap [0,\varepsilon] = \varnothing$ whenever $0 \notin I_i$. Similarly, let $\delta \in (0,1)$ be such that $\alpha([\delta,1]) \subseteq V^1$ and $I_i \cap [\delta,1] = \varnothing$ whenever $1 \notin I_i$. Due to this choice of ε and δ, the path $\beta = \vartheta \,\square\, (\alpha \,\square\, \gamma)$ looks as follows:

(i) $\beta([0,\varepsilon]) \subseteq V^0$;

(ii) β equals α on $[\varepsilon, \delta]$, that is, $\alpha(t) = \beta(t)$ for $t \in [\varepsilon, \delta]$.

(iii) $\beta([\delta,1]) \subseteq V^1$.

To show $\beta \in \bigcap_{i=1}^n \mathcal{B}(I_i, U_i)$, we note that, for a given index $k \in \{1, \ldots, n\}$, there are the following choices for I_k, namely,

(a) $0, 1 \notin I_k$,

(b) $0 \in I_k$ but $1 \notin I_k$,

(c) $0 \notin I_k$ but $1 \in I_k$,
(d) $0, 1 \in I_k$.
We note that, for all these choices, $\beta(I_k) \subseteq U_k$. Choice (a) is equivalent to saying that $k \notin S^i$ for $i = 0, 1$. Therefore, $I_k \subseteq [\varepsilon, \delta]$. Thus $\beta(I_k) = \alpha(I_k) \subseteq U_k$.

For choice (b), one can see that

$$\beta(t) = \begin{cases} \alpha(t) \in U_k & \text{for } \varepsilon \le t, \\ \beta(t) \in V^0 \subseteq U_k & \text{for } 0 \le t \le \varepsilon. \end{cases}$$

The first case in the above equation is because $\alpha \in \bigcap_{i=1}^n \mathcal{B}(I_i, U_i)$. A similar argument shows that $\beta(I_k) \subseteq U_k$ for choice (c).

Finally, choice (d) forces $I_k = \mathbb{I}$ and then

$$\beta(t) = \begin{cases} \alpha(t) \in U_k & \text{when } \varepsilon \le t \le \delta, \\ \beta(t) \in V^0 \subseteq U_k & \text{when } 0 \le t \le \varepsilon, \\ \beta(t) \in V^1 \subseteq U_k & \text{when } \delta \le t \le 1. \end{cases}$$

Thus $\beta \in \bigcap_{i=1}^n \mathcal{B}(I_i, U_i)$ and hence $[\beta'] = [\beta] \in M$, which proves the lemma. □

Proposition 3.5. *For a locally path connected and semilocally simply connected space X, the CO′ and Whisker topologies on $\Pi_1(X)$ are equal.*

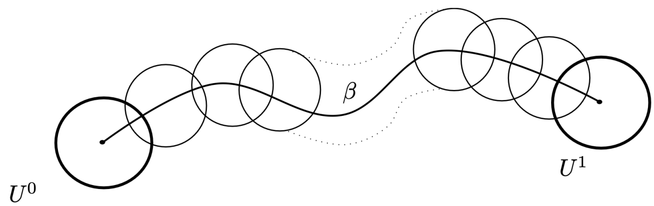

FIGURE 1

Proof. CO′ is coarser than Whisker by Lemma 3.4. For the converse, let $[\alpha] \in \Pi_1(X)$ and let $N([\alpha], U^1, U^0)$ be a basic Whisker-open neighborhood of it, and we may assume that U^0 and U^1 are path connected and relatively inessential; see Lemma 2.14 (ii). We need to show that $q^{-1}(N([\alpha], U^1, U^0)) \subseteq \mathrm{P}X$ is open in the compact-open topology. For this, let $\beta \in q^{-1}(N([\alpha], U^1, U^0))$ be given. We construct a compact-open neighborhood T of β that is contained in $q^{-1}(N([\alpha], U^1, U^0))$. Recall from the remark after Definition 3.3 that, in the current situation, $N([\alpha], U^1, U^0) = N([\beta], U^1, U^0)$. Firstly, using the continuity of β, choose $\varepsilon, \delta \in (0, 1)$ such that $\beta([0, \varepsilon]) \subseteq U^0$ and $\beta([\delta, 1]) \subseteq U^1$. Let $\mathcal{O}$ be an open cover of the compact set $\beta([\varepsilon, \delta])$ by path connected relatively inessential open sets; cp. Figure 1. Obtain a Lebesgue number $r > 0$ of the cover $\{\beta^{-1}(O) \mid O \in \mathcal{O}\}$ of the closed interval $[\varepsilon, \delta]$. Using this Lebesgue number r, we get a partition $\varepsilon = t_1 < t_2 < \cdots < t_n = \delta$ of $[\varepsilon, \delta]$ such that $\beta([t_i, t_{i+1}])$ lies in some path connected open relatively inessential neighborhood W_i of X for

$i = 1, 2, \dots, n-1$. We set $W_0 = U^0$ and $W_n = U^1$. And finally, let V_i be the path component of $W_i \cap W_{i+1}$ which contains $\beta(t_{i+1})$ for $i = 0, 1, 2, \dots, n-1$. Now we define our desired set $T \subseteq \mathrm{P}X$ as follows:

$$T = \mathcal{B}([0,\varepsilon], W_0) \cap \mathcal{B}(\{t_1\}, V_0) \cap \mathcal{B}([t_1,t_2], W_1) \cap \mathcal{B}(\{t_2\}, V_1) \cap \mathcal{B}([t_2,t_3], W_2) \cap \mathcal{B}(\{t_3\}, V_2) \cap \cdots \cap \mathcal{B}([t_{n-1},t_n], W_{n-1}) \cap \mathcal{B}(\{t_n\}, V_{n-1}) \cap \mathcal{B}([\delta,1], W_n). \tag{2}$$

By definition, T is open in the compact-open topology on $\mathrm{P}X$ and contains β. Lemma 3.6 shows that if $\xi \in T$, then $[\xi] \in N([\beta], U^1, U^0) = N([\alpha], U^1, U^0)$, that is, $T \subseteq q^{-1}(N([\alpha], U^1, U^0))$, which completes the proof. □

Lemma 3.6. *Let $T \subseteq \mathrm{P}X$ be as in equation* (2). *Assume the same conventions, hypotheses and notation as in the proof of Proposition 3.5. Then, for given $\xi \in T$, there are paths $\sigma_0 \in \mathrm{P}W_0$ and $\sigma_n \in \mathrm{P}W_n$ with $[\xi] = [\sigma_n \square \beta \square \sigma_0]$. In short, $[\xi] \in N([\alpha], U^1, U^0)$.*

Proof. Recall from the remark after Definition 3.3 that

$$N([\alpha], U^1, U^0) = N([\beta], U^1, U^0).$$

Using the path connectedness of W_0 and W_n, choose paths σ_0 from $\xi(0)$ to $\beta(0)$ and σ_n from $\beta(1)$ to $\xi(1)$. Since $\beta(t_i)$ and $\xi(t_i)$ are in the same path component V_i, choose a path η_i from $\beta(t_i)$ to $\xi(t_i)$ for $i = 1, 2, \dots, n$. Now we have

(a) $[\eta_1 \square \beta|_{[0,t_1]} \square \sigma_0] = [\xi|_{[0,t_1]}]$, because W_0 is relatively inessential;
(b) $[\eta_{i+1} \square \beta|_{[t_i,t_{i+1}]} \square \eta_i^-] = [\xi|_{[t_i,t_{i+1}]}]$, because W_i is relatively inessential for $1 \le i \le n-1$;
(c) $[\sigma_n \square \beta|_{[t_n,1]} \square \eta_n^-] = [\xi|_{[t_n,1]}]$, because W_n is relatively inessential.

Using (a), (b) and (c), we have

$$\begin{aligned}[\xi] &= [\sigma_n \square \beta|_{[t_n,1]} \square \eta_n^-][\eta_n \square \beta|_{[t_{n-1},t_n]} \square \eta_{n-1}^-] \cdots \\ &\qquad [\eta_2 \square \beta|_{[t_1,t_2]} \square \eta_1^-][\eta_1 \square \beta|_{[0,t_1]} \square \sigma_0] \\ &= [\sigma_n \square \beta|_{[t_n,1]} \square \beta|_{[t_{n-1},t_n]} \square \cdots \square \beta|_{[0,t_1]} \square \sigma_0] = [\sigma_n \square \beta \square \sigma_0];\end{aligned}$$

this is the claim of the lemma. □

Corollary 3.7 (Corollary of Proposition 3.5)**.** *Let X be locally path connected and semilocally simply connected space; equip $\mathrm{P}X$ with the compact-open topology and $\Pi_1(X)$ with the CO′ topology. Then the quotient map $q : \mathrm{P}X \to \Pi_1(X)$ is open.*

Proof. We use the fact that the CO′ and the Whisker topologies on $\Pi_1(X)$ are equal to prove this. Let $U = \bigcap_{j=1}^n \mathcal{B}(I_j, V_j) \subseteq \mathrm{P}X$ be a basic open set in the compact-open topology, where I_j are closed intervals and $V_j \subseteq X$ are neighborhoods. Let $[\alpha] \in q(U)$ be given. We construct a Whisker-neighborhood of $[\alpha]$ contained in $q(U)$. For that, we may assume that $\alpha(0) \in V_1$ and $\alpha(1) \in V_n$ (relabel the indices if required). Then note that $N([\alpha], V_n, V_1)$ is an open neighborhood of $[\alpha]$ contained in $q(U)$. □

3.8. **Topological fundamental groupoid.** Assume that X is a locally path connected and semilocally simply connected space. In this section, we prove that $\Pi_1(X)$ is a topological fundamental groupoid when equipped with CO′ or Whisker topology. Fabel [15] points out that if X is not semilocally simply connected, then the fundamental group equipped with the CO′ topology may fail to be topological. As a consequence, the fundamental groupoid fails to be topological.

For the fundamental groupoid $\Pi_1(X)$, the fiber product

$$\Pi_1(X) \times_{s,\Pi_1(X)^{(0)},r} \Pi_1(X),$$

which consists of composable arrows, shall be denoted by $\Pi_1(X)^{(2)}$.

Theorem 3.9. *Let X be a locally path connected and semilocally simply connected space. Then the fundamental groupoid $\Pi_1(X)$ is a topological groupoid when equipped with the Whisker or, equivalently, the CO′ topology.*

Proof. Continuity of multiplication: Let $([\alpha],[\beta]) \in \Pi_1(X)^{(2)}$. Consider an open set W of $\Pi_1(X)$ containing $[\alpha \square \beta]$. Take a basic Whisker-open neighborhood $N([\alpha \square \beta], U, V) \subseteq W$ of $[\alpha \square \beta]$. Choose a relatively inessential open neighborhood Z of $\alpha(0) = \beta(1)$. Consider the open set O in $\Pi_1(X)^{(2)}$ given by

$$O = \big(N([\alpha], U, Z) \times N([\beta], Z, V)\big) \cap \Pi_1(X)^{(2)}.$$

Then O is a neighborhood of $([\alpha],[\beta])$ in $\Pi_1(X)^{(2)}$.

We claim that the product $[\xi \square \eta]$ is in $N([\alpha \square \beta], U, V)$ for any $([\xi],[\eta]) \in O$. The validity of this claim proves the continuity of multiplication.

Proof of the claim: since $([\xi],[\eta]) \in O$, let

$$[\xi] = [\vartheta_1 \square \alpha \square \delta_1] \quad \text{and} \quad [\eta] = [\vartheta_2 \square \beta \square \delta_2],$$

where $\vartheta_1 \in \mathrm{P}U$, $\delta_1, \vartheta_2 \in \mathrm{P}Z$, $\delta_2 \in \mathrm{P}V$ and $\vartheta_2(1) = \delta_1(0)$. Now we have

$$\begin{aligned}[\xi \square \eta] &= [\vartheta_1 \square \alpha \square \delta_1 \square \vartheta_2 \square \beta \square \delta_2] = [\vartheta_1 \square \alpha][\delta_1 \square \vartheta_2][\beta \square \delta_2] \\ &= [\vartheta_1 \square \alpha][\beta \square \delta_2] = [\vartheta_1 \square \alpha \square \beta \square \delta_2] \in N([\alpha \square \beta], U, V).\end{aligned}$$

The third equality follows because Z is relatively inessential.

The continuity of the inverse map follows from the equation

$$\mathrm{inv}\big(N([\alpha^{-1}], V, U)\big) = N([\alpha], U, V). \qquad \square$$

Consider X as in the last theorem. Here we note some observations.

(a) Let $\Upsilon : X \to \Pi_1(X)^{(0)}$ be the correspondence that sends a point in X to the homotopy class of the corresponding constant path. Note that Υ is induced by the continuous mapping $X \xrightarrow{v} \mathrm{P}X$ that maps $x \in X$ to the constant path in x; here $\mathrm{P}X$ is equipped with the compact-open topology. The mapping v is continuous because the inverse image of a subbasic open set $\mathcal{B}(I, V) \subseteq \mathrm{P}X$ is V.

(b) For $[e_x] \in \Pi_1(X)^{(0)}$, where e_x is the constant path at $x \in X$, the fiber over its range is

$$\Pi_1(X)^{[e_x]} = \{[\gamma] \in \Pi_1(X) \mid \gamma(1) = x\}.$$

Then $\Pi_1(X)^{[e_x]}$ is in one-to-one correspondence with $\tilde{X}$. The covering map $p : \tilde{X} \to X$ is identified with the restriction $s|_{\Pi_1(X)^{[e_x]}}$ of the source map.

(c) Finally, for $x \in X$, the fundamental group $\pi_1(X, x)$ is $\Pi_1(X)^{[e_x]}_{[e_x]}$.

The next corollary shows that the above bijections of sets are actually homeomorphisms of spaces.

Corollary 3.10. *Let X be a path connected, locally path connected and semi-locally simply connected space.*

(i) *The mapping Υ in point* (a) *above is a homeomorphism of spaces.*

(ii) *In point* (b)*, the subspace topology on $\Pi_1(X)^{[e_x]}$ is the same as the covering space topology on it. By covering space topology, we mean the CO' or the Whisker topology on the universal covering space of X.*

(iii) *With the subspace topology of $\Pi_1(X)$, the fundamental group of X is a discrete group; cp. point* (c) *above.*

Proof. (i): The bijection Υ is continuous as it is the composite of continuous functions $X \xrightarrow{\upsilon} \mathrm{P}X \xrightarrow{q} \Pi_1(X)$. We show that Υ is also open. For that, let $U \subseteq X$ be a nonempty relatively inessential open set. Fix $z \in U$. Then we have $\Upsilon(U) = \Pi_1(X)^{(0)} \cap N([e_z], U, U)$. Thus $\Upsilon(U)$ is open in the subspace topology of $\Pi_1(X)^{(0)}$.

(ii): This is clear if one looks at the construction of the simply connected covering space of X. The covering map $\Pi_1(X)^{[e_x]} \to \Pi_1(X)^{(0)}$ is the restriction of the source map of the groupoid.

(iii): Due to (i) and (ii), we can identify the restriction of the source map $\Pi_1(X)^{[e_x]} \to \Pi_1(X)^{(0)}$ with the universal covering map $p : \tilde{X} \to X$, which is a local homeomorphism. As the fundamental group $\pi_1(X, x) = p^{-1}(\{x\})$, it is discrete. ☐

The reader may readily see that, in Corollary 3.10 (ii), $r^{-1}([e_x])$ can be replaced by $s^{-1}([e_x])$ with appropriate modifications.

Observation 3.11. For the space X in Corollary 3.10, consider the map

$$r \times s : \Pi_1(X) \to X \times X, \quad r \times s([\gamma]) = (\gamma(1), \gamma(0)) \quad \text{for } [\gamma] \in \Pi_1(X).$$

Consider a basic Whisker-open neighborhood $N([\alpha], U, V)$, wherein U and V are path connected, relatively inessential neighborhoods of $\alpha(1)$ and $\alpha(0)$, respectively. Then $(r \times s)(N([\alpha], U, V)) = U \times V$, where the latter set is a basic open set in $X \times X$. Since U and V are relatively inessential neighborhoods, the map $r \times s$ is injective on $N([\alpha], U, V)$. Thus the mapping $r \times s$ is a local homeomorphism. In fact, it is a covering map (see Proposition 8.4).

Remark 3.12. Let X be a locally path connected and simply connected space. Then the map $r \times s : \Pi_1(X) \to X \times X$ of Observation 3.11 is a bijection. It can be checked that $r \times s$ is an isomorphism of algebraic groupoids, where the latter is the groupoid of the trivial equivalence relation on X (see Example 2.3). Thus $r \times s$ is an isomorphism of topological groupoids as it is a local homeomorphism.

Remark 3.13. Corollary 3.10 (i) and (ii) clearly show that the fundamental groupoid is a locally trivial groupoid (Definition 2.5) but not étale (Definition 2.7) unless the path connected space is a point. Thus the claim in [20, Ex. 2.1.4] that a fundamental groupoid is étale is *not correct.*

Remark 3.14. Let X be the infinite-dimensional separable Hilbert space of square summable complex sequences. Let X^* be its dual space. Let X^*_{n} denote X^* with the norm topology and let X^*_{w} denote X^* with the weak*-topology; the weak*-topology is strictly coarser than the norm topology. Both topologies make X^* a locally convex topological vector space. Therefore, X^*_{n} and X^*_{w} are path connected, locally path connected and simply connected spaces. Now the identity map $\iota : X^*_{\mathrm{n}} \to X^*_{\mathrm{w}}$ is a continuous map which induces an isomorphism of the fundamental groups. In fact, ι also induces an isomorphism of algebraic fundamental groupoids—in both cases, the algebraic fundamental groupoid is the cartesian product $X^* \times X^*$, the groupoid of trivial equivalence on X^* due to Remark 3.12. Thus neither the fundamental group nor the algebraic fundamental groupoids can distinguish these spaces. However, the *topological* fundamental groupoids here are different: in the first case, it is the topological space $X^*_{\mathrm{n}} \times X^*_{\mathrm{n}}$, and in the second case, it is the space $X^*_{\mathrm{w}} \times X^*_{\mathrm{w}}$.

3.15. **Functoriality.**

Lemma 3.16. *Let $f : X \to Y$ be a continuous map between two locally path connected and semilocally simply connected topological spaces. Then*

$$f_* : \Pi_1(X) \to \Pi_1(Y) \quad \textit{given by} \quad [\alpha] \mapsto f_*([\alpha]) := [f \circ \alpha]$$

is a continuous groupoid homomorphism.

Proof. Let $N([f \circ \alpha], V, U)$ be a Whisker-open neighborhood of $[f \circ \alpha] \in \Pi_1(Y)$, where V and U are open neighborhoods of $f(\alpha(1))$ and $f(\alpha(0))$, respectively. By continuity of f, there are open neighborhoods V' of $\alpha(1)$ and U' of $\alpha(0)$ satisfying $f(V') \subseteq V$ and $f(U') \subseteq U$. Now it can be readily seen that

$$f_*\big(N([\alpha], U', U')\big) \subseteq N([f \circ \alpha], U, V).$$

The rest is a standard fact. □

Let $\mathcal{T}$ be the category of locally path connected and semilocally simply connected topological spaces, $\mathcal{G}^{\mathrm{A}}$ the category of algebraic groupoids with algebraic homomorphisms as morphisms and let $\mathcal{G}^{\mathrm{T}}$ be obtained by enriching $\mathcal{G}^{\mathrm{A}}$ with topology. Thus $\mathcal{G}^{\mathrm{T}}$ is the category of topological groupoids with continuous groupoid homomorphism as morphisms. It is a standard fact that $X \mapsto \Pi_1(X)$ is a functor $\mathcal{T} \to \mathcal{G}^{\mathrm{A}}$. Lemma 3.16 combined with this standard fact gives us the next result.

Theorem 3.17. *The assignment $\Pi_1 : \mathcal{T} \to \mathcal{G}^{\mathrm{T}}$ that sends a space $X \mapsto \Pi_1(X)$ and a map of spaces $X \xrightarrow{f} Y$ to $\Pi_1(X) \xrightarrow{f_*} \Pi_1(Y)$ as in Lemma 3.16 is a covariant functor.*

4. The fundamental groupoid of a group

Let $p : H \to G$ be a homomorphism of topological groups. Then H acts on the left of G continuously through p as $\eta \cdot \gamma = p(\eta)\gamma$ for $\gamma \in G$ and $\eta \in H$. We denote the corresponding transformation groupoid by $H \ltimes_p G$.

For a path connected group G, we follow the standard custom of considering $\pi_1(G)$ as $\pi_1(G, 1_G)$, where $1_G \in G$ is the identity. Recall that, for a path connected, locally path connected and semilocally simply connected group G, its simply connected cover H is a *group*, and the covering map $p : H \to G$ is a group homomorphism. In this section, we fix to G, H and p as in the last sentence. Our goal is to prove that the fundamental groupoid of G is isomorphic to the transformation groupoid $H \ltimes_p G$.

Throughout this section, we shall consider the universal cover H as the fiber $\Pi_1(G)_{1_G}$. In this case, the covering map $p : H \to G$ is given by $p([\gamma]) = \gamma(1)$ for $[\gamma] \in H$. For $[\eta], [\gamma] \in H$, the group multiplication $\otimes$ in H is given by $[\gamma] \otimes [\eta] = [\gamma \otimes' \eta]$, where $\gamma \otimes' \eta$ is the path $t \mapsto \gamma(t)\eta(t)$. We shall abuse the notation and write $\otimes$ instead of $\otimes'$ from now on.

For a path γ in G and $g \in G$, define the translated path $\gamma \cdot g$ as $t \mapsto \gamma(t)g$, where $t \in \mathbb{I}$. The concatenation, product and translates of paths have the following properties: for paths $\gamma, \eta, \delta, \vartheta \in \mathrm{P}G$ with $\delta(0) = \vartheta(1)$, and $g \in G$,

$$(\eta \otimes \gamma) \cdot g = \eta \otimes (\gamma \cdot g),$$
$$(\delta \,\Box\, \vartheta) \cdot g = (\delta \cdot g) \,\Box\, (\vartheta \cdot g).$$

The multiplication by g above is a homeomorphism of $\mathrm{P}G$ which respects (endpoint preserving) path homotopy, that is, $[\eta \cdot g] = [\eta] \cdot g$ for all $\eta \in \mathrm{P}G$. Therefore, for the paths as above, we have

$$[\eta \otimes \gamma] \cdot g = [\eta] \otimes [\gamma \cdot g], \qquad (3)$$
$$[\delta \,\Box\, \vartheta] \cdot g = [\delta \cdot g] \,\Box\, [\vartheta \cdot g].$$

We realize the space $\Pi_1(G)$ as the fiber product

$$\Pi_1(G) \times_{s, \Pi_1(G)^{(0)}, \mathrm{id}_G} G := \{([\gamma], g) \mid g \in G \text{ and } \gamma(0) = g\};$$

this identification is useful to prove the main result of this section. Note that the basic open set in $\Pi_1(G) \times_{s, \Pi_1(G)^{(0)}, \mathrm{id}_G} G$ corresponding to the basic open set $N([\alpha], U, V) \subseteq \Pi_1(G)$ is $N([\alpha], U, V) \times_{s, \Pi_1(G)^{(0)}, \mathrm{id}_G} V$.

Lemma 4.1. *The mapping* $J : H \times G \to \Pi_1(G)$ *given by* $([\gamma], g) \mapsto ([\gamma \cdot g], g)$ *is a homeomorphism.*

Proof. Let $N([\alpha], U, V) \times_{s, \Pi_1(G)^{(0)}, \mathrm{id}_G} V$ be a basic open set in $\Pi_1(G)$, where $U, V \subseteq G$ are path connected and relatively inessential open sets. Its inverse image under J is the set $N([\alpha], U) \times V$, which is a basic open set in the product topology. Conversely, if $N([\alpha], U) \times V \subseteq H \times G$ is a basic open set, where U and V are path connected and relatively inessential open sets in G (such sets form a basis of G), then $J(N([\alpha], U) \times V) = N([\alpha], U, V) \times_{s, \Pi_1(G)^{(0)}, \mathrm{id}_G} V$. Therefore, J is open. One can easily check that J is invertible with its inverse given by $J^{-1}([\gamma], g) = ([\gamma \cdot g^{-1}], g)$. □

The basic open subset $N([\alpha], U)$ of H in the last proof is given by

$$N([\alpha], U) = \{[\delta \mathbin{\square} \alpha] \mid \delta \in \mathrm{P}U \text{ with } \delta(0) = \alpha(1)\}.$$

Let η, γ be paths in G starting at the identity 1_G. Then the path $\eta \cdot \gamma(1)$ is concatenable with γ as $(\eta \cdot \gamma(1))(0) = \eta(0)\gamma(1) = \gamma(1)$. Therefore, the statement of the next lemma makes sense.

Lemma 4.2. *Let η, γ be paths in G starting at the identity 1_G. Then, for any $g \in G$, $[(\eta \cdot \gamma(1) \mathbin{\square} \gamma) \cdot g] = [(\eta \otimes \gamma) \cdot g]$.*

Proof. Recall that, for two continuous functions, their minimum and maximum functions are continuous. Then, for $0 \le t \le 1$,

$$\begin{aligned}
\big((\eta \cdot \gamma(1) \mathbin{\square} \gamma) \cdot g\big)(t) &= \big((\eta \cdot \gamma(1) \mathbin{\square} \gamma)(t)\big) \cdot g \\
&= \begin{cases} \gamma(2t)g & \text{if } 0 \le t \le \frac{1}{2}, \\ \eta(2t-1)\gamma(1)g & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \\
&= \eta(\max\{2t-1, 0\}) \cdot \gamma(\min\{2t, 1\}) \cdot g.
\end{aligned}$$

For $s, t \in [0, 1]$, consider the function

$$H(s,t) = \eta\big(\max\{t + (t-1)s, 0\}\big) \cdot \gamma(\min\{t + ts, 1\}) \cdot g.$$

Then H implements a path homotopy between

$$H(0, *) = (\eta \otimes \gamma) \cdot g \quad \text{and} \quad H(1, *) = (\eta \cdot \gamma(1) \mathbin{\square} \gamma) \cdot g.$$

This is because H is continuous, and

$$\begin{aligned}
&H(s, 0) = g = \eta(0)\gamma(0)g = \big((\eta \cdot \gamma(1) \mathbin{\square} \gamma) \cdot g\big)(0), \\
&H(s, 1) = \eta(1)\gamma(1)g = \big((\eta \otimes \gamma) \cdot g\big)(1) = \big((\eta\gamma(1) \mathbin{\square} \gamma)g\big)(1)
\end{aligned}$$

for $0 \le s \le 1$. □

If $g = 1_G$, then Lemma 4.2 allows us to express the product in the universal covering group H in terms of homotopies of paths. We need this observation to prove the next theorem.

Theorem 4.3. *Let G be a locally path connected, path connected, semilocally simply connected topological group, let $p\colon H \to G$ be its universal covering space and H be given its canonical group structure. Then the topological fundamental groupoid $\Pi_1(G)$ is isomorphic to the transformation groupoid $H \ltimes_p G$.*

Proof. Define

$$J : H \ltimes_p G \to \Pi_1(G) \quad \text{by} \quad J(([\gamma], g)) = ([\gamma \cdot g], g)$$

for $([\gamma], g) \in H \times G$. Lemma 4.1 shows that J is a homeomorphism. It remains to prove that J is an (algebraic) homomorphism of groupoids. For that, take two composable elements $([\eta], h), ([\gamma], g) \in H \ltimes_p G$. Then

- $h = p([\gamma])g := \gamma(1)g$, and
- $([\eta], h)([\gamma], g) = ([\eta] \otimes [\gamma], g) = ([\eta \otimes \gamma], g)$.

As a consequence,

$$J\big(([\eta], h)([\gamma], g)\big) = J([\eta \otimes \gamma], g) = \big([(\eta \otimes \gamma) \cdot g], g\big), \tag{4}$$

the images of these elements $J([\eta],h) = ([\eta \cdot h],h)$ and $J([\gamma],g) = ([\gamma \cdot g],g)$ are composable in $\Pi_1(G)$ as $(\gamma \cdot g)(1) = \gamma(1)g = h = (\eta \cdot h)(1)$. Furthermore, the concatenation of these images in $\Pi_1(G)$ is

$$J([\eta],h)J([\gamma],g) = ([\eta \cdot h],h)([\gamma \cdot g],g) = \big([(\eta \cdot h) \square (\gamma \cdot g)],g\big).$$

By using property (3), take the g out in the path homotopy $[(\eta \cdot h) \square (\gamma \cdot g)]$ above; then the last term in the computation above equals

$$\big([(\eta \cdot hg^{-1}) \square \gamma] \cdot g, g\big).$$

Now apply Lemma 4.2 to the first coordinate above, so the last term equals

$$\big([(\eta \otimes \gamma) \cdot g], g\big) = J\big(([\eta],h)([\gamma],g)\big)$$

as we wanted; cp. equation (4) above. □

Since the groupoids $\Pi_1(G)$ and $H \ltimes_p G$ in Theorem 4.3 are isomorphic, the isotropies of the points are also isomorphic. The isotropy of the unit $1_G \in G$ in the transformation groupoid $H \ltimes_p G$ is $\{\eta \in H \mid p(\eta) = e\} = \ker(p)$. Therefore, $\pi_1(G,e)$ is isomorphic to the kernel of p.

We end this section with the following remark. Theorem 4.3 shows that, for the fundamental groupoid not to have the form of a transformation groupoid, it is necessary that the underlying space is not a topological group. The following are some path connected, locally path connected and semilocally simply connected spaces which are known to carry no group structure.

(i) The sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ for $n \neq 0, 1, 3$.
(ii) The figure of eight in the plane. The fundamental group of a topological group is abelian [25, Ex. 7 (d), §52]. The fundamental group of the figure of eight is not abelian.
(iii) The wedge sum of n-circles $\bigvee_n \mathbb{S}^1$ for $n > 1$, for the same reason as in (ii).

5. THE POINT-SET TOPOLOGY OF THE FUNDAMENTAL GROUPOID

The goal of this section is to describe the point-set topology of the fundamental groupoid $\Pi_1(X)$. We relate this topology with that of X and the simply connected covering space of X. For this purpose, we need an alternate construction of the fundamental groupoid involving equivalence of groupoids.

5.1. **One more description of the fundamental groupoid.** Recall actions of groupoids on topological spaces from Section 2.8.

Definition 5.2 (Equivalence of groupoids [24])**.** Let G and H be groupoids. We call a space X a *G-H-equivalence* if X is a left G- and right H-space, and the following properties are satisfied:

(i) the actions commute, that is, $\gamma(x\eta) = (\gamma x)\eta$ for all appropriate $\gamma \in G, x \in X$ and $\eta \in H$;
(ii) both the actions are free and proper;
(iii) the momentum maps induce homeomorphisms

$$G \setminus X \to H^{(0)} \quad \text{and} \quad X/H \to G^{(0)}.$$

The next is a well-known example of equivalence of groupoids from [24].

Example 5.3 ([35, Prop. 2.47])**.** Let H be a group and X a right H-space. Assume that the action of H is free and proper. Then $G := (X \times X)/H$ is a topological groupoid; here the action of H on the product space is given by $(x, y)\eta = (x\eta, y\eta)$, where $x, y \in X$ and $\eta \in H$. The groupoid G acts freely and properly on X on the left. Moreover, this X implements an equivalence between G and H.

For a Hausdorff, path connected, locally path connected and semilocally simply connected space X, the deck transformation action of $\pi_1(X)$ on $\tilde{X}$ is free and proper (the properness is a consequence of [22, Lem. 21.11]). Therefore, using Example 5.3, we notice that the quotient space $(\tilde{X} \times \tilde{X})/\pi_1(X)$ is a groupoid equivalent to $\pi_1(X)$ where the equivalence is implemented by the simply connected covering space $\tilde{X}$. But [24, Ex. 2.3] already shows that $\pi_1(X)$ is equivalent to $\Pi_1(X)$ via the simply connected covering space. Then [35, Lem. 2.44] implies that $(\tilde{X} \times \tilde{X})/\pi_1(X)$ and $\Pi_1(X)$ are isomorphic as topological groupoids. This result is summarized as follows.

Proposition 5.4. *Let X be a Hausdorff, path connected, locally path connected and semilocally simply connected space, and let $\tilde{X}$ be its simply connected cover. Let G be the topological groupoid $(\tilde{X} \times \tilde{X})/\pi_1(X)$ associated with the free and proper action of $\pi_1(X)$ on $\tilde{X}$ as in Example 5.3. Then G is isomorphic to the topological fundamental groupoid of X.*

5.5. **The point-set topology.** We study the relation between the point-set topologies of X, $\tilde{X}$, $\mathrm{P}X$ and the fundamental groupoid $\Pi_1(X)$. Our main goal is to investigate when the groupoid is Hausdorff or locally compact or second countable. Before we start proving the main propositions, we need a lemma.

Lemma 5.6. *Assume X is a topological space with an equivalence relation $\sim$ having open quotient map $q : X \to X/{\sim}$. Then $X/{\sim}$ is Hausdorff if and only if $\sim \subseteq X \times X$ is a closed subset.*

Proof. Write Y for $X/{\sim}$. Let $q \times q : X \times X \to Y \times Y$ be the continuous mapping $(x, y) \mapsto (q(x), q(y))$, where $x, y \in X$. If the diagonal $\mathrm{dia}(Y) \subseteq Y \times Y$ is closed, then $\sim = (q \times q)^{-1}(\mathrm{dia}(Y))$ is a closed subspace of $X \times X$.

For the converse, note that $q \times q$ is an open map since q is open. Now assume that $\sim$ is a closed subspace of $X \times X$. Then note that

$$(q \times q)((X \times X) - \sim) = (Y \times Y) - \mathrm{dia}(Y).$$

The left-hand side, and hence the right side, of this equation is an open set. Thus the diagonal $\mathrm{dia}(Y) \subseteq Y \times Y$ is closed. □

Proposition 5.7. *Let X be a locally path connected and semilocally simply connected space. Let $\sim$ be the equivalence relation of path homotopy on $\mathrm{P}X$. Consider the following assertions.*

(i) *X is Hausdorff.*
(ii) *$\tilde{X}$ is Hausdorff.*
(iii) *$\mathrm{P}X$ is Hausdorff.*

(iv) $\Pi_1(X)$ *is Hausdorff.*
(v) $\sim \subseteq \mathrm{P}X \times \mathrm{P}X$ *is a closed subspace.*
Then (i)*,* (iii)*,* (iv) *and* (v) *are equivalent, and* (i) $\Rightarrow$ (ii)*.*

Proof. (i) $\Leftrightarrow$ (iii) follows from [10, Chap. XII, 1.3].

(iv) $\Leftrightarrow$ (v): Since the quotient map $\mathrm{P}X \to \Pi_1(X)$ is open, this follows from Lemma 5.6.

(iv) $\Rightarrow$ (i), as $X, \tilde{X} \subseteq \Pi_1(X)$ are subspaces by Corollary 3.10.

(i) $\Rightarrow$ (iv): Suppose X is Hausdorff and two distinct points $[\alpha], [\beta] \in \Pi_1(X)$ are given. Then there are two cases: one of the endpoints of α and β differ, or both these paths have the same initial points and the same terminal points. In the first case, first assume that $\alpha(0) \neq \beta(0)$. Choose neighborhoods $\alpha(0) \in W$ and $\beta(0) \in W'$ in X with $W \cap W' = \varnothing$. And choose any neighborhoods $\alpha(1) \in V$ and $\beta(1) \in V'$. Then, clearly, $N([\alpha], V, W) \cap N([\beta], V', W') = \varnothing$. A similar argument works if $\alpha(1) \neq \beta(1)$.

Consider the second case, namely, $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Since X is semilocally simply connected, choose relatively inessential neighborhoods U of $\alpha(0)$ and V of $\alpha(1)$. Then $N([\alpha], V, U) \cap N([\beta], V, U) = \varnothing$. If this intersection is not empty, we get a contradiction as follows: assume that $[\gamma] \in N([\alpha], V, U) \cap N([\beta], V, U)$. Then $[\gamma] = [\eta \,\Box\, \alpha \,\Box\, \vartheta] = [\eta' \,\Box\, \beta \,\Box\, \vartheta']$ for some $\eta, \eta' \in \mathrm{P}V$ and $\vartheta, \vartheta' \in \mathrm{P}U$ with $\gamma(0) = \vartheta(0) = \vartheta'(0)$, $\gamma(1) = \eta(1) = \eta'(1)$. Moreover, since α and β have the same endpoints, we also have that $\vartheta(1) = \vartheta'(1)$ and $\eta(0) = \eta'(0)$. Therefore,

$$[\alpha] = [\eta^{-1} \,\Box\, \eta' \,\Box\, \beta \,\Box\, \vartheta' \,\Box\, \vartheta^{-1}] = [\eta^{-1} \,\Box\, \eta'][\beta][\vartheta' \,\Box\, \vartheta^{-1}] = [\beta]. \tag{5}$$

The last equality holds because U and V are relatively inessential neighborhoods. Equation (5) contradicts that $[\alpha]$ and $[\beta]$ are distinct.

Finally, for the proof of (i) $\Rightarrow$ (ii), consider $\tilde{X}$ as a subspace of $\Pi_1(X)$. Then the same proof as in the first case above works. □

The proof of (i) $\Rightarrow$ (iv) in Proposition 5.7 uses that X is semilocally simply connected.

Recall from Section 2.1 that, for us, locally compact spaces are not necessarily Hausdorff.

Proposition 5.8. *Consider the following statements about a path connected, locally path connected and semilocally simply connected space X.*
(i) X *is locally compact.*
(ii) $\tilde{X}$ *is locally compact.*
(iii) $\Pi_1(X)$ *is locally compact.*
Then (i) $\Leftrightarrow$ (ii)*, and* (iii) $\Rightarrow$ (ii), (i)*. If X is also Hausdorff, all are equivalent.*

Proof. (i) $\Leftrightarrow$ (ii): This follows because X and $\tilde{X}$ are locally homeomorphic via the covering map.

(iii) $\Rightarrow$ (ii): A locally compact space in our sense is T_1. Therefore, the covering space $\tilde{X} \subseteq \Pi_1(X)$ is a closed subspace (remark after Corollary 3.10). Being a closed subspace of a locally compact space [33, Prop. 1.2], $\tilde{X}$ is closed.

(i) $\Rightarrow$ (iii): Assume that X is Hausdorff. Then $\tilde{X}$ is Hausdorff due to Proposition 5.7. Therefore, $\tilde{X} \times \tilde{X}$ is a locally compact Hausdorff space. Now Proposition 5.4 implies that $\Pi_1(X)$ is locally compact Hausdorff, being the quotient space of a proper action. □

Recall from our convention, Section 2.1, that paracompact and second countable spaces are Hausdorff.

Proposition 5.9. *Let X be a locally path connected and semilocally simply connected space. Then the following are equivalent.*
(i) *X is second countable.*
(ii) *$\tilde{X}$ is second countable.*
(iii) *$\mathrm{P}X$ is second countable.*
(iv) *$\Pi_1(X)$ is second countable.*

Proof. (iii) $\Rightarrow$ (iv): Assume $\mathrm{P}X$ is second countable. Then the fundamental groupoid is second countable as it is the image of $\mathrm{P}X$ under the quotient map, which is open due to Corollary 3.7.

(iv) $\Rightarrow$ (i) and (ii): Since X and $\tilde{X}$ are subspaces of $\Pi_1(X)$, they are second countable [25, Thm. 30.2].

(i) $\Leftrightarrow$ (iii): Due to [11, Thm. 3.7].

(ii) $\Rightarrow$ (iv): This is a consequence of Proposition 5.4. □

The proof of the last proposition, except (ii) $\Rightarrow$ (iv), holds even if second countable spaces are not assumed to be Hausdorff.

Before we proceed, note that the space of units of a Hausdorff groupoid G is a closed subspace of G: consider the map $G \to G \times G$ that sends $\gamma \in G$ to (γ, γ^{-1}). The diagonal in $G \times G$ is closed, and $G^{(0)}$ is the inverse image of the diagonal under the above continuous map.

Proposition 5.10. *Let X be a path connected, locally path connected and semilocally simply connected space. Consider the following statements.*
(i) *X is paracompact.*
(ii) *$\tilde{X}$ is paracompact.*
(iii) *$\Pi_1(X)$ is paracompact.*
Then (iii) $\Rightarrow$ (ii), (i). *Additionally, if we assume $\pi_1(X) < \infty$, then* (i) $\Rightarrow$ (ii).

Proof. (iii) $\Rightarrow$ (ii), (i): This follows because $X, \tilde{X} \subseteq \Pi_1(X)$ are closed subspaces—X is closed as it is the space of units, and $\tilde{X}$ is closed for being the inverse image of a point under the range map.

(i) $\Rightarrow$ (ii): The hypothesis implies that the covering map $p : \tilde{X} \to X$ is a perfect map. Therefore, $\tilde{X}$ is paracompact if X is so; see [25, Ex. 8, p. 260]. □

If Y and X are metric spaces with Y separable, then $\mathrm{C}(Y, X)$ is paracompact in the compact-open topology [26]. As a consequence, the path space of a metric space is paracompact. Zabrodsky discusses the metrizability of $\tilde{X}$ in [36].

6. A Haar system on a fundamental groupoid

Theorem 6.7 in this section gives conditions for the existence of the Haar system on the fundamental groupoid. More concretely, in Section 6.7, we describe formulae for this Haar system, provided that a measure on the universal covering space of the space X as in Theorem 6.7 is given. All the computations required for this purpose directly generalize to locally trivial groupoids. Therefore, in this section, we prove the results for locally trivial groupoids.

6.1. **Families of measures.** By a measure on a locally compact space, we mean a positive Radon measure. Such measures are exactly the positive linear functional $\mathrm{C_c}(X) \to \mathbb{C}$ due to the Riesz theorem [31], where X is a locally compact Hausdorff space. Therefore, if μ is such a measure on a locally compact space X, then we also write $\mu(f)$ for $\int_X f\,\mathrm{d}\mu$. The support of μ is the smallest closed set $S \subseteq X$ with $\mu(X \setminus S) = \varnothing$. We shall work with fully supported measures, that is, the measures whose support is the whole space. Here $\mathrm{C_c}(X)$ denotes the normed involutive algebra of compactly supported continuous complex-valued functions on X equipped with the uniform norm.

Consider a homeomorphism $\varphi\colon X \to Y$ of locally compact spaces. A measure μ on Y induces the *pullback* measure $\varphi^*\mu$ on X which is defined by $\varphi^*\mu(f) = \mu(f \circ \varphi^{-1})$ for $f \in \mathrm{C_c}(X)$.

Let X and Y be locally compact, Hausdorff spaces and $\pi : X \to Y$ a continuous open surjection. By a *continuous family of measures along* π, we mean a family of measures $\mu := \{\mu_y\}_{y\in Y}$ such that, for each $y \in Y$, μ_y is a measure on X supported in $\pi^{-1}(y)$, and for $f \in \mathrm{C_c}(X)$, the complex-valued function on Y given by $y \mapsto \mu_y(f)$, for $y \in Y$, is continuous on Y. We denote the above function $y \mapsto \mu_y(f)$ by $\mu(f)$. In fact, the function $\mu(f)$ is in $\mathrm{C_c}(Y)$ as it vanishes outside $\pi(\mathrm{supp}(f))$. Thus $\mu : \mathrm{C_c}(X) \to \mathrm{C_c}(Y)$ is a map of complex vector spaces that is continuous in the inductive limit topologies. Lemme 1.1 in [30] says that μ is surjective.

If ν is a measure on Y, then $\nu \circ \mu$ is a measure on X; for $f \in \mathrm{C_c}(X)$, $\nu \circ \mu(f) = \nu(\mu(f))$.

We frequently call a continuous family of measures as a family of measures.

Example 6.2. Let $\pi : X \to Y$ be a surjective local homeomorphism. For $y \in Y$, define μ_y to be the counting measure on $\pi^{-1}(y)$. Then $\mu := \{\mu_y\}_{y\in Y}$ is a continuous family of measures along π. For $f \in \mathrm{C_c}(X)$ and $y \in Y$,

$$\mu_y(f) = \sum_{x\in\pi^{-1}(y)} f(x).$$

In particular, if f is supported in an open set U such that $\pi|_U$ is a homeomorphism, then

$$\mu_y(f) = \begin{cases} f((\pi|_U)^{-1}(y)) & \text{if } y \in \pi(U), \\ 0 & \text{otherwise.} \end{cases}$$

Assume that X and Y are G-spaces for a groupoid G and $\pi : X \to Y$ is a G-equivariant open mapping. Suppose a continuous family of measures

$\mu := \{\mu_y\}_{y\in Y}$ along π is given. We say that μ is *G-equivariant* if, for each composable pair $(\gamma, y) \in G \times_{s,G^{(0)},r_Y} Y$,

$$\int_{\pi^{-1}(\gamma y)} f(x)\,\mathrm{d}\mu_{\gamma y}(x) = \int_{\pi^{-1}(y)} f(\gamma^{-1}x)\,\mathrm{d}\mu_y(x)$$

for $f \in \mathrm{C_c}(X)$.

Note that if G is a group, then a G-equivariant family for the constant map $X \to \{*\}$ is essentially a G-invariant measure.

A *Haar system* on a locally compact groupoid G is a continuous family of fully supported measures $\alpha := \{\alpha^u\}_{u\in G^{(0)}}$ along the range map $r : G \to G^{(0)}$ that is also G-equivariant for the left multiplication action of G on itself and the standard action of G on $G^{(0)}$. Thus the measure α^u is supported on G^u, and for $f \in \mathrm{C_c}(G)$ and $\gamma \in G$,

$$\int_{G^{s(\gamma)}} f(\eta)\,\mathrm{d}\alpha^{s(\gamma)}(\eta) = \int_{G^{r(\gamma)}} f(\gamma\eta)\,\mathrm{d}\alpha^{r(\gamma)}(\eta).$$

It is well-known that, unlike groups, a locally compact groupoid may not have a Haar system.

Example 6.3. (i) The Haar measure on a locally compact group is a Haar system.

(ii) The transformation groupoid of group action, $X \rtimes G$, in Example 2.4 carries a Haar system. Let λ be the Haar measure on G, and for $x \in X$, let δ_x be the point mass at x in X. For each $x \in X$, define $\alpha_x := \delta_x \times \lambda$. Then $\{\alpha_x\}_{x\in X}$ is a Haar system on $X \rtimes G$.

(iii) Let X be a locally compact space and $X \times X$ the groupoid of trivial equivalence on X (see Example 2.3). Assume that λ is a fully supported measure on X. Then $\{\delta_x \times \lambda\}_{x\in X}$ is a Haar system on $X \times X$.

Lemma and Definition 6.4 ([2, Lem. 1, App. I])**.** *Let X be a locally compact Hausdorff space, R an open equivalence relation on X, such that the quotient space X/R is paracompact. Let $q : X \to X/R$ be the canonical quotient map. Then there is a real-valued continuous function $e \geq 0$ on X such that*
(i) *e is not identically zero on any equivalence for R;*
(ii) *for every compact subset K of X/R, $q^{-1}(K) \cap \mathrm{supp}(e)$ is a compact set.*
We call the function e a cutoff function for q.

In particular, if $p : \tilde{X} \to X$ is a covering map over a locally path connected, semilocally simply connected and paracompact space X, then p has an associated cutoff function.

6.5. Haar system on locally trivial groupoids.

6.5.1. *Second countable case.* If G is a locally compact, Hausdorff, second countable, locally trivial and transitive groupoid, then G has a Haar system. To see this, let $x \in G^{(0)}$. Since G is second countable, the isotropy G^x_x is a countable discrete group; the counting measure is the Haar measure on G^x_x. As G^x establishes an equivalence between G^x_x and G, the required claim follows from Williams' [34, Thm. 2.1]. As a consequence, if X is a path connected, locally path connected, semilocally simply connected, locally compact, Hausdorff and second countable space, then the fundamental groupoid $\Pi_1(X)$ has a Haar system.

6.5.2. *Non-second countable case.*

Theorem 6.6. *Let X be a path connected, locally path connected, semilocally simply connected and locally compact topological group. Then its fundamental groupoid $\Pi_1(X)$ has a Haar system.*

Proof. Theorem 4.3 says that $\Pi_1(X)$ is a transformation groupoid $H \ltimes X$; here H is the universal covering group of X. Therefore, the Haar measure on H induces a Haar system on $\Pi_1(X)$ (see Example 6.3 (ii)). □

Theorem 6.7. *Let G be a locally compact, locally trivial, transitive groupoid, and let $x \in G^{(0)}$. Consider the following statements:*
(i) *G has a Haar system;*
(ii) *the transversal G^x has a G^x_x-invariant fully supported invariant Radon measure for the left multiplication action of G^x_x on G^x;*
(iii) *$G^{(0)}$ has a fully supported Radon measure.*
Then (i) $\Leftrightarrow$ (ii) *and* (iii) $\Rightarrow$ (ii)*. If $G^{(0)}$ is paracompact, then* (ii) $\Rightarrow$ (iii)*.*

Note that a G^x_x-invariant measure λ on G^x is fully supported if and only if λ is nonzero.

Proof of Theorem 6.7. (i) $\Leftrightarrow$ (ii): The transversal G^x implements equivalence between G^x_x and G. Therefore, the claim follows from [21, Prop. 5.2].

(iii) $\Rightarrow$ (ii): Assume that ν is a fully supported measure on $G^{(0)}$. Let $\mu := \{\mu_x\}_{x\in G^{(0)}}$ be the family of counting measures along the local homeomorphism $s|_{G^x} : G^x \to G^{(0)}$. Then the measure $\nu \circ \mu$ is a G^x_x-invariant fully supported measure on G^x. Since μ and ν have full support, so does $\nu \circ \mu$. Also, note that μ is invariant for the left multiplication action of G^x_x on G^x. Now G^x_x-invariance of $\nu \circ \mu$ follows from a standard computation; for example, see [19, Prop. 3.1 (i)].

(ii) $\Rightarrow$ (iii): Assume that $G^{(0)}$ is paracompact, and let λ be a G^x_x-invariant measure on G^x having full support.

Let e be the cutoff function for the local homeomorphism $s|_{G^x} : G^x \to G^{(0)}$. Getting the measure ν on $G^{(0)}$ from λ is a standard process; for example, see [19, Prop. 3.2 (ii)]. We describe this construction next, which is basically [19, Prop. 3.1 (i)].

Note that $s|_{G^x}^{-1}(z) \cap \operatorname{supp}(e)$ is a finite set for $z \in G^{(0)}$. Define $h : G^x \to [0, \infty)$ as

$$h(\tilde{x}) = \frac{e(\tilde{x})}{\sum_{\tilde{y}\in(s|_{G^x})^{-1}(s|_{G^x}(\tilde{x}))} e(\tilde{y})}$$

for $\tilde{x} \in G^x$. Then h is also a cutoff function. Additionally, for $\tilde{z} \in G^x$ and $x \in G^{(0)}$,

$$0 \le h(\tilde{z}) \le 1 \quad \text{and} \quad \sum_{\tilde{x}\in s|_{G^x}^{-1}(x)} h(\tilde{x}) = 1.$$

Now, for $f \in \mathrm{C_c}(G^{(0)})$, define the positive Radon measure ν as follows:

$$\nu(f) := \int_{G^x} f \circ s|_{G^x}(\tilde{x}) \cdot h(\tilde{x})\, \mathrm{d}\lambda(\tilde{x}).$$

The fact that ν is a G^x_x-invariant measure on G^x is a special case of [19, Prop. 3.2 (ii)]. The measure ν has full support because λ has full support and the cutoff function h does not identically vanish on any fiber $s|_{G^x}^{-1}(z)$ for $z \in G^{(0)}$.

In fact, [19, Prop. 3.2 (iii)] says that, for given λ, this ν is the unique measure with the property that $\lambda = \nu \circ \mu$. □

Corollary 6.8 (Corollary of Theorem 6.7)**.** *Consider the following statements for a locally compact, path connected, locally path connected and semilocally simply connected topological space X:*

(i) *the fundamental groupoid $\Pi_1(X)$ has a Haar system;*
(ii) *the universal covering space $\tilde{X}$ has a $\pi_1(X)$-invariant fully supported Radon measure;*
(iii) *X has a fully supported Radon measure.*

Then (i) $\Leftrightarrow$ (ii) *and* (iii) $\Rightarrow$ (ii)*. If X is paracompact, then* (ii) $\Rightarrow$ (iii)*.*

We derive the following standard fact from the last corollary.

Corollary 6.9. *Let X be a locally compact, path connected, locally path connected and semilocally simply connected topological group. Then the Haar measure on its universal cover $\tilde{X}$ is $\pi_1(X)$-invariant.*

Proof. This is a consequence of Theorem 6.6 and Theorem 6.7 (i)–(ii). □

Example 6.10. Consider a locally compact, path connected, locally path connected and semilocally simply connected space X. Theorem 6.7 implies that, in the following cases, $\Pi_1(X)$ has a Haar system.

(i) If X is a homogeneous G-space for some locally compact group G, then it is well-known that X carries a quasi-invariant measure (see [17]). Therefore, the measure has full support.
(ii) Let X be a compact Hausdorff space and σ a homeomorphism of it. Then $\mathbb{Z}$ acts on X via this homeomorphism which gives the classical dynamical system (X, σ). Assume that this action is transitive. Then X has a σ-invariant measure μ (see [7, Thm. VII.3.1]). Due to the transitivity of the action, this is a measure on X with full support.

Two examples of compact Hausdorff spaces that cannot be supports of any Radon measure are given in [8].

6.11. **Formulae for the Haar system.**

6.11.1. *Haar system in terms of measure on the transversal.* Fix the locally compact, locally trivial, transitive groupoid G as in Theorem 6.7. Let $x \in G^{(0)}$, and suppose a G^x_x-invariant measure λ on the transversal G^x is given.

Let $\eta \in G$. Define the left translation $\ell_\eta g$ of a function $g \in \mathrm{C_c}(G^{s(\eta)})$ as the function $\ell_\eta g : z \mapsto g(\eta^{-1}z)$ on $G^{r(\eta)}$. Then $\ell_\eta g \in \mathrm{C_c}(G^{r(\eta)})$.

Let $\eta \in G^x_w$. Define the measure $\ell_{\eta^{-1}}\lambda$ on G^w by

$$\int_{G^w} f(\xi)\,\mathrm{d}(\ell_{\eta^{-1}}\lambda)(\xi) := \int_{G^x} \ell_\eta f(\gamma)\,\mathrm{d}\lambda(\gamma),$$

where $f \in \mathrm{C_c}(G^w)$.

Lemma 6.12. *For $\eta, \xi \in G^x_w$, the measures $\lambda_{\eta^{-1}} = \lambda_{\xi^{-1}}$.*

Proof. For $f \in \mathrm{C_c}(G^w)$,

$$\int_{G^x} f(\eta^{-1}\gamma)\,\mathrm{d}\lambda(\gamma) = \int_{G^x} f(\xi^{-1}(\xi\eta^{-1}\gamma))\,\mathrm{d}\lambda(\gamma) = \int_{G^x} f(\xi^{-1}\gamma)\,\mathrm{d}\lambda(\gamma)$$

because λ is G^x_x-invariant. □

Choose $C \subseteq G$ formed by collecting *exactly* one arrow ${}_w\eta \in G^x_w$ for the given $x \in G^{(0)}$ and w varying over $G^{(0)}$. For $w \in G^{(0)}$, define $\tilde\lambda^w = \ell_{{}_w\eta^{-1}}\lambda$, which is well-defined due to the last lemma. Since G is a transitive groupoid, $\tilde\lambda := \{\tilde\lambda^w\}_{w\in G^{(0)}}$ is a family of measures along $r : G \to G^{(0)}$.

Proposition 6.13. *Let G be a locally compact, locally trivial, transitive groupoid. Let $x \in G^{(0)}$. Assume that λ is a G^x_x-invariant measure on the transversal G^x for the left multiplication of G^x_x. Then $\tilde\lambda := \{\tilde\lambda^w\}_{w\in G^{(0)}}$ defined above is a Haar system on G.*

Proof. Support condition: The multiplication by ${}_w\eta^{-1}$ is a homeomorphism $G^x \to G^w$ and $\ell_{{}_w\eta^{-1}}\lambda$ is the push forward of the measure λ along this homeomorphism. As λ has full support, $\ell_{{}_w\eta^{-1}}\lambda$ has full support.

Translation invariance: This is a straight-forward computation: let $g \in \mathrm{C_c}(G)$ and $\xi \in G$. Let $w = s(\xi)$. Then

$$\int_G g(\xi\gamma)\,\mathrm{d}\tilde\lambda^{s(\xi)}(\gamma) = \int_G \ell_{\xi^{-1}} g(\gamma)\,\mathrm{d}\tilde\lambda^{s(\xi)}(\gamma) := \int_{G^x} \ell_{{}_w\eta}(\ell_{\xi^{-1}} g)(\gamma)\,\mathrm{d}\lambda(\gamma).$$

But $\ell_{{}_w\eta}\ell_{\xi^{-1}} = \ell_{({}_w\eta)\xi^{-1}}$ and ${}_w\eta \cdot \xi^{-1} \in G^x_{r(\xi)}$. Therefore, the last term above equals

$$\int_{G^x} \ell_{\eta\xi^{-1}} g(\gamma)\,\mathrm{d}\lambda(\gamma) := \int_G g(\gamma)\,\mathrm{d}\tilde\lambda^{r(\xi)}(\gamma)$$

due to Lemma 6.12.

Continuity: Let $f \in \mathrm{C_c}(G)$. We want to show that $\tilde\lambda(f)$ is continuous on $G^{(0)}$. Let $w \in G^{(0)}$ and consider a net $(w_i)_{i\in I}$ with $w_i \to w$. We know that $s|_{G^x} : G^x \to G^{(0)}$ is a local homeomorphism. Choose an open set $P \subseteq G^x$ such that $s(P)$ is an open neighborhood of w and $s|_P$ is a homeomorphism. Choose a cocompact neighborhood U of w such that $\overline{U} \subseteq s(P)$. We may assume that $w_i \in U$ for all i. Let η_i be preimage of w_i and η that of w in P under $s|_P$. Then $\eta_i \to \eta$. And $\ell_{\eta_i} f \to \ell_\eta f$ pointwise on the compact set $(\overline{U}^{-1} \cdot \mathrm{supp}(f)) \cap G^x$. Therefore, $\ell_{\eta_i} f \to \ell_\eta f$ uniformly on the compact set $(\overline{U}^{-1} \cdot \mathrm{supp}(f)) \cap G^x$. The dominated convergence theorem implies that $\tilde\lambda(f)(w_i) \to \tilde\lambda(f)(w)$. □

For a space X, the simply connected covering spaces $\Pi_1(X)^x$ are homeomorphic for all $x \in X$. The last discussion shows that the Haar system on $\Pi_1(X)$ depends on the covering space $\Pi_1(X)^x$ and $\pi_1(X,x)$-invariant measure on it. Compare this with Takesaki's remark in the second last paragraph in [24, p. 19].

6.13.1. *Haar system in terms of the measure on the space of units.* Let G be a locally compact, locally trivial, transitive groupoid and let ν be a fully supported measure on $G^{(0)}$. Fix $x \in G^{(0)}$. Recall from the proof of (iii) $\Rightarrow$ (ii) in Theorem 6.7 that the G_x^x-invariant measure λ that ν induces on G^x is $\nu \circ \mu$, where μ is the continuous family of measures along the local homeomorphism $s|_{G^x} : G^x \to G^{(0)}$, μ consisting of atomic measures. Thus, for $g \in \mathrm{C_c}(G^x)$,

$$\int_{G^x} g(\xi)\,\mathrm{d}\lambda(\xi) := \int_{G^{(0)}} \Big(\sum_{\gamma \in s|_{G^x}^{-1}(w)} g(\gamma) \Big)\,\mathrm{d}\nu(w).$$

For this measure λ, the Haar system $\tilde\lambda := \{\tilde\lambda^w\}_{w \in G^{(0)}}$ on G in Proposition 6.13 is

$$\int_G f(\xi)\,\mathrm{d}\tilde\lambda^w(\xi) = \int_{G^x} f({}_w\eta^{-1}\xi)\,\mathrm{d}\lambda(\xi) = \int_{G^{(0)}} \Big(\sum_{\gamma \in s|_{G^x}^{-1}(w)} f({}_w\eta^{-1}\gamma) \Big)\,\mathrm{d}\nu(w).$$

The existence of a Haar system on a fundamental groupoid allows us to talk about its C*-algebra. Following are some observations regarding this.

Remark 6.14. Let X be a path connected, locally path connected, semilocally simply connected, locally compact, Hausdorff and second countable space. Then $\Pi_1(X)$ carries a Haar system, and as consequence of the equivalence of groupoids $\pi_1(X,x)$ and $\Pi_1(X)$, we have $\mathrm{C}^*(\Pi_1(X)) \simeq \mathrm{C}^*(\pi_1(X,x)) \otimes \mathbb{K}$, where $\mathbb{K}$ is the C*-algebra of compact operators on a separable Hilbert space. Compare this remark with [24, Thm. 3.1].

Remark 6.15. Let X be a locally compact, path connected, locally path connected and semilocally simply connected group. Let $p : H \to X$ be its universal cover. Then Theorems 4.3 and 6.6 together imply that $\mathrm{C}^*(\Pi_1(X))$ is isomorphic to the crossed product $H \ltimes_p \mathrm{C}_0(X)$.

The last two remarks together imply the following: assume $p : H \to X$ as in Remark 6.15, plus that X is second countable. Let K be the kernel of p. Then $K \simeq \pi_1(X, e)$, where $e \in X$ is the unit; see the remark after Theorem 4.3. Then the crossed product $H \ltimes_p \mathrm{C}_0(X) \simeq \mathrm{C}^*(K) \otimes \mathbb{K}$.

Remark 6.16. Suppose G is a locally compact, locally trivial groupoid that is not necessarily transitive. For $x \in G^{(0)}$, the orbit of x, for the obvious right G action on $G^{(0)}$, is the set $s(G^x)$. As G is locally trivial, $s(G^x)$ is an open subset of $G^{(0)}$. Since each orbit $s(G^x)$ is open in $G^{(0)}$, it is also closed in $G^{(0)}$. Thus

$$G^{(0)} = \bigsqcup_{s(G^x) \in G^{(0)}/G} s(G^x),$$

where $s(G^x) \subseteq G^{(0)}$ is closed as well as open. Let $G|_{s(G^x)}^{s(G^x)}$ be the closed and open subgroupoid of G consisting of those arrows in G whose ranges and sources lie in $s(G^x)$. Now

$$G = \bigsqcup_{s(G^x) \in G^{(0)}/G} G|_{s(G^x)}^{s(G^x)},$$

where each subgroupoid $G|_{s(G^x)}^{s(G^x)}$ is open as well as closed in G; plus it is locally compact, transitive and locally trivial. Thus each subgroupoid has a Haar system. The disjoint union of these Haar systems is a Haar system on G.

7. An action category of a fundamental groupoid and free actions

Definition 7.1. Let X and Y be spaces, and $f : Y \to X$ a continuous surjection. We say that f has

(i) the *path lifting property* (or *the unique path lifting property*) if, for any path $\gamma : \mathbb{I} \to X$ and a point $y \in f^{-1}(\gamma(0))$, there is a path (respectively, a unique path) $\tilde{f} : \mathbb{I} \to Y$ starting at y and $f \circ \tilde{f} = \gamma$;

(ii) *the homotopy lifting property* (or *the unique homotopy lifting property*) if f has path lifting property (respectively, the unique path lifting property) and for two given paths $\gamma, \alpha : \mathbb{I} \to X$ with $\gamma(0) = \alpha(0)$ and $\gamma(1) = \alpha(1)$, an endpoint fixing homotopy $\Gamma : \mathbb{I} \times \mathbb{I} \to X$ of γ with α and a point $y \in f^{-1}(\gamma(0))$, there is a function (respectively, a unique function) $\tilde{\Gamma} : \mathbb{I} \times \mathbb{I} \to Y$ with the properties that $\tilde{\Gamma}|_{\{0\}\times\mathbb{I}}$ is a lift (respectively, the unique lift) $\tilde{\gamma}$ of γ starting at $y \in Y$; $\tilde{\Gamma}|_{\{1\}\times\mathbb{I}}$ is a lift (respectively, the unique lift) $\tilde{\alpha}$ of α starting at $y \in Y$, and $f \circ \tilde{\Gamma} = \Gamma$.

It is a standard fact that covering maps have the unique path lifting and unique homotopy lifting properties. Proposition 3 of [9, Chap. 5-6A] says that a local homeomorphism having the unique path lifting property also has unique homotopy lifting property.

Remark 7.2 (Functoriality of the unique path lifting property)**.** Suppose $Y_1 \xrightarrow{p_1} X \xleftarrow{p_2} Y_2$ are two mappings which have unique path lifting properties. Assume that $f : Y_1 \to Y_2$ is a continuous map such that $p_1 = p_2 \circ f$. For a given path γ in X, choose $y_1 \in Y_1$ with $p_1(y_1) = \gamma(0)$. Let $y_2 = f(y_1)$. If $\tilde{\gamma}_{y_1}$ is the unique lift of γ in Y_1 starting at y_1, then $f \circ \tilde{\gamma}_{y_1}$ is the unique lift of γ in Y_2 starting at y_2 as $p_1 = p_2 \circ f$.

7.3. **Covering spaces as $\Pi_1(X)$-spaces.** Let X be a locally path connected and semilocally simply connected space, and $\mathrm{c} : Y \to X$ a covering map. For a path $\gamma \in \mathrm{P}X$ and $y \in \mathrm{c}^{-1}(\gamma(0))$, by $\tilde{\gamma}_y$, we shall denote the unique lift of γ starting at y. As c has the unique homotopy lifting property, each pair $([\gamma], y)$, where γ is a path in X and $y \in \mathrm{c}^{-1}(\gamma(0))$, determines the unique element $[\tilde{\gamma}_y] \in \Pi_1(Y)$. Using this observation, we define a (left) action of $\Pi_1(X)$ on Y as follows.

(i) c is the momentum map for the action.

(ii) For each pair $([\gamma], y)$ in the fiber product $\Pi_1(X) \times_{s,X,\mathrm{c}} Y$, the action $[\gamma]y := \tilde{\gamma}_y(1)$.

For the sake of clarity, the fiber product is

$$\Pi_1(X) \times_{s,X,\mathrm{c}} Y = \{([\gamma], y) \in \Pi_1(X) \times Y \mid \gamma(0) = \mathrm{c}(y)\}.$$

We shall refer to this action of the fundamental groupoid $\Pi_1(X)$ on Y as *the* (left) action of the groupoid on the covering space. A right action can be defined similarly.

Notice that the last action can be defined, in general, for any mapping having the unique path and homotopy lifting properties.

For a covering map $\mathrm{c} : Y \to X$, an *evenly covered neighborhood* of a point has the standard meaning as in Hatcher [18]. Consider an evenly covered open set $U \subseteq X$; write $\mathrm{c}^{-1}(U) = \bigsqcup_\alpha \tilde{U}_\alpha$, where each $\tilde{U}_\alpha$ is homeomorphic to U via c. We call each $\tilde{U}_\alpha$ a *slice* over U. Since evenly covered neighborhoods form a basis for the topology of X, the slices also form a basis for the topology of Y.

Proposition 7.4. *Let X be a locally path connected and semilocally simply connected space and $\mathrm{c} : Y \to X$ a covering map.*

(i) *The action of $\Pi_1(X)$ on Y is continuous.*

(ii) *The stabilizer of $y \in Y$ is the subgroup $\mathrm{c}_*(\pi_1(Y, y)) \simeq \pi_1(Y, y)$ of the fundamental group $\pi_1(X, \mathrm{c}(y))$; here c_* is the homomorphism of fundamental group(oid)s that* c *induces.*

(iii) *Assume Y is also path connected. Then the action of $\Pi_1(X)$ on Y is free if and only if Y is simply connected.*

Proof. (i): The momentum map c is continuous. So we only need to show that the map

$$\sigma : \Pi_1(X) \times_{s,X,\mathrm{c}} Y \to Y, \quad \sigma : ([\gamma], y) \mapsto [\gamma]y := \tilde{\gamma}_y(1)$$

is continuous. Let $W \subseteq Y$ be a given open set. To prove the continuity of σ, for a given point $([\gamma], y) \in \sigma^{-1}(W)$, we construct a basic open neighborhood Q of $([\gamma], y)$ such that $Q \subseteq \sigma^{-1}(W)$.

Let y' denote $\tilde{\gamma}_y(1)$. Thus $\tilde{\gamma}_y$ starts at y and ends at $y' \in W$. Choose path connected relatively inessential slices V and U over some evenly covered neighborhoods of $\mathrm{c}(y')$ and $\mathrm{c}(y)$, respectively. Additionally, as slices form a basis for the topology of Y, we can choose $V \subseteq W$. Then

$$Q := \big(N([\gamma], \mathrm{c}(V), \mathrm{c}(U)) \times U\big) \cap \big(\Pi_1(X) \times_{s,X,\mathrm{c}} Y\big)$$

is a nonempty basic open neighborhood of $([\gamma], y)$ in $\Pi_1(X) \times_{s,X,\mathrm{c}} Y$, and clearly, $\sigma(Q) = V \subseteq W$.

(ii): Here we basically want to describe the homotopy classes of paths in X starting at $\mathrm{c}(y)$ which lift to homotopy classes of loops at y. It is a standard result [18, Prop. 1.31] that such homotopy classes of paths starting at $\mathrm{c}(y)$ are exactly the subgroup $\mathrm{c}_*(\pi_1(Y, y)) \subseteq \pi_1(X, \mathrm{c}(y))$.

(iii): The action is free if and only if the stabilizer at each point of Y is trivial. Due to (ii) above, this means the action is free if and only if $\mathrm{c}_*(\pi_1(Y, y))$ is the trivial subgroup of $\pi_1(X, \mathrm{c}(y))$ for each $y \in Y$. This happens if and only if Y is the universal covering space. □

Remark 7.5. In Proposition 7.4, let $\alpha : \Pi_1(X) \times_{s,X,\mathrm{c}} Y \to Y$ denote the action of $\Pi_1(X)$ on Y. For $x \in X$, let α_x be the restriction of this action to $\pi_1(X, x) \subseteq \Pi_1(X)$. Then note that the proof of (ii) in the proposition also implies that the isotropy of α and α_x is the same.

Proposition 7.4 describes covering spaces as $\Pi_1(X)$-spaces (in which the covering maps are serving as the momentum maps for the actions). But not every $\Pi_1(X)$-space can be a covering space as we can easily construct $\Pi_1(X)$-spaces in which the momentum maps are not local homeomorphisms; see Example 7.10. However, adding the extra hypothesis that the momentum map of the action is a local homeomorphism produces the converse of Proposition 7.4, which is our next main result, Theorem 7.9. By an étale map, we mean a local homeomorphism. Next we discuss some lemmas required to prove Theorem 7.9.

Lemma 7.6. *Let X be a locally path connected and semilocally simply connected space. Let $p : Y \to X$ be an open surjection; assume that p has the unique homotopy lifting property. Then p is a covering map.*

Proof. Let $x \in X$, and let U be a path connected and relatively inessential neighborhood of x. Let $\mathrm{P}U_x$ be the set of all paths in U starting at x. For given $\tilde{x} \in p^{-1}(x)$, define the set $\tilde{U}_{\tilde{x}} := \{\tilde{\gamma}_{\tilde{x}}(1) \mid \gamma \in \mathrm{P}U_x\}$. Since p has the unique path lifting property, a standard argument shows that, for two preimages $\tilde{x} \neq \tilde{y}$ of x, $\tilde{U}_{\tilde{x}} \cap \tilde{U}_{\tilde{y}} = \varnothing$.

We show that

$$p^{-1}(U) = \bigsqcup_{\tilde{x} \in p^{-1}(x)} \tilde{U}_{\tilde{x}}.$$

By definition of $\tilde{U}_{\tilde{x}}$, it is clear that $\tilde{U}_{\tilde{x}} \subseteq p^{-1}(U)$ for each $\tilde{x} \in p^{-1}(x)$. Therefore,

$$\bigsqcup_{\tilde{x} \in p^{-1}(x)} \tilde{U}_{\tilde{x}} \subseteq p^{-1}(U).$$

Conversely, suppose that $y \in p^{-1}(U)$. Let γ be a path in U connecting x to $p(y)$; let γ^- be the path obtained by traversing γ in the opposite direction. Let $\tilde{\gamma}^-_y$ be the unique lift of γ^- starting at y. Put $\tilde{x} = \tilde{\gamma}^-_y(1)$. Then $\tilde{x} \in p^{-1}(x)$ and $y \in \tilde{U}_{\tilde{x}}$.

We now show that, for any $\tilde{x} \in p^{-1}(x)$, $p|_{\tilde{U}_{\tilde{x}}} : \tilde{U}_{\tilde{x}} \to U$ is a homeomorphism. Firstly, note that the restricted map $p|_{\tilde{U}_{\tilde{x}}}$ is surjective as U is path connected. The map is injective because U is relatively inessential and p has the unique homotopy lifting property. Finally, we prove that $p|_{\tilde{U}_{\tilde{x}}}$ is open. As p was an open map, to prove that $p|_{\tilde{U}_{\tilde{x}}}$ is open, it is sufficient to show $\tilde{U}_{\tilde{x}}$ is an open set. This can be proved as follows: let $\tilde{\gamma}_{\tilde{x}}(1) \in \tilde{U}_{\tilde{x}}$ be a point. Using the continuity of p, choose an open set $V \subseteq Y$ containing $\tilde{\gamma}_{\tilde{x}}(1)$ and with $p(V) \subseteq U$. Then V is, in fact, contained in $\tilde{U}_{\tilde{x}}$. To see this, let $v \in V$, and choose a path $\xi \in \mathrm{P}U$ from x to $p(v)$. By the uniqueness of the path lifting property, we have $v = \tilde{\xi}_{\tilde{x}}(1) \in \tilde{U}_{\tilde{x}}$. □

Lemma 7.7. *Let $p : Y \to X$ be a local homeomorphism having the path lifting property. Then*
(i) *p has unique path lifting property;*
(ii) *p has unique homotopy lifting property.*

Proof. (i): Let $\gamma : \mathbb{I} \to X$ be a path with two lifts $\tilde{\gamma}, \eta$ starting at $y \in p^{-1}(\gamma(0))$. Let $A = \{s \in \mathbb{I} \mid \tilde{\gamma}(t) = \eta(t) \text{ for all } t \leq s\}$. Then A is a nonempty closed subset of the unit interval: $A \neq \varnothing$ for $0 \in A$, and the closedness of A follows from the continuity of the maps $\tilde{\gamma}$ and η and Hausdorffness of Y.

Now our claim is that $\sup(A) := s_0 = 1$. On the contrary, suppose that $s_0 < 1$. Since $A \subseteq \mathbb{I}$ is closed, $s_0 \in A$, that is, $\tilde{\gamma}(s_0) = \eta(s_0) = y_0$. Choose a neighborhood U of y_0 such that $p|_U$ is homeomorphism onto its image and $p(U) \subseteq X$ is open. Note that $\gamma = p \circ \tilde{\gamma} = p \circ \eta$. The continuity of γ at s_0 gives us $\varepsilon > 0$ such that $\gamma((s_0 - \varepsilon, s_0 + \varepsilon)) \subseteq p(U)$. As $p|_U : U \to p(U)$ is a homeomorphism, $\tilde{\gamma}(s_0 + \varepsilon/2) = \eta(s_0 + \varepsilon/2)$, which contradicts $s_0 = \sup(A)$.

(ii): p is a local homeomorphism with unique path lifting property (see (i)). The current claim follows from [9, Chap. 5-6A, Prop. 3]. □

Lemma 7.8. *Let Y be a left $\Pi_1(X)$-space, where X is locally path connected and semilocally simply connected. Suppose the momentum map $r_Y : Y \to X$ is a local homeomorphism. Then the momentum map r_Y has unique path lifting property and unique homotopy lifting property.*

Proof. Given a path γ in X and $y \in r_Y^{-1}(\gamma(0))$, define the path $\tilde{\gamma}$ in Y starting at y as follows:

$$\tilde{\gamma}(t) = [\gamma|_{[0,t]}]y \quad \text{for } 0 \leq t \leq 1.$$

The continuity of $\tilde{\gamma}$ follows from the continuity of the action. Furthermore,

$$r_Y \circ \tilde{\gamma}(t) = r_Y([\gamma|_{[0,t]}]y) = r([\gamma|_{[0,t]}]) = \gamma(t),$$

where r is the range map of $\Pi_1(X)$. Thus $\tilde{\gamma}$ is a lift of γ at y in Y. This shows that r_Y has path lifting property. Now Lemma 7.7 implies the required claim. □

Theorem 7.9. *Let X be a locally path connected and semilocally simply connected space. Suppose $p : Y \to X$ is a (surjective) local homeomorphism. Then the following statements are equivalent:*
(i) *Y is a $\Pi_1(X)$-space;*
(ii) *p has unique path lifting property;*
(iii) *p is a covering map;*
(iv) *p has unique homotopy lifting property.*

Proof. (i) ⇒ (ii) or (iv): This follows from Lemma 7.8.

(ii) ⇒ (iii): Since p is a local homeomorphism with unique path lifting property, Lemma 7.8 (ii) says that p has unique homotopy lifting property. Now Lemma 7.6 shows that p is a covering map.

(iii) ⇒ (i): This follows from the first part of Proposition 7.4.

Finally, (iv) ⇒ (ii) is clear. □

The next example describes a $\Pi_1(X)$-space in which the momentum map is not a local homeomorphism.

Example 7.10. Consider the map $p : \mathbb{R} \times \mathbb{R} \to S^1$ given by $p(x, y) = \mathrm{e}^{2\pi i x}$; this map is not a local homeomorphism. Equip $\mathbb{R} \times \mathbb{R}$ with the translation (in both variables) action of $\mathbb{R}$; equip the unit circle S^1 with another $\mathbb{R}$-action: $t \cdot \mathrm{e}^{2\pi i x} = \mathrm{e}^{2\pi i (t+x)}$, where $t, x \in \mathbb{R}$. Then p is an $\mathbb{R}$-equivariant map. Now Lemma 2.9 implies that $\mathbb{R} \times \mathbb{R}$ carries an action of $\mathbb{R} \ltimes S^1$ with p as the momentum map. We identify $\Pi_1(S^1) \simeq \mathbb{R} \ltimes S^1$ using Theorem 4.3. Thus $\mathbb{R} \times \mathbb{R}$ is a $\Pi_1(S^1)$-space, but p is not a covering map.

From now on, we shall restrict our study to the category of $\Pi_1(X)$-spaces having the momentum maps local homeomorphisms. Our next quests are to characterize free and proper $\Pi_1(X)$-actions on such spaces. The next result gives us a necessary and sufficient condition for freeness of such actions.

Proposition 7.11. *Suppose X is a locally path connected and semilocally simply connected space and Y a path connected $\Pi_1(X)$-space. Suppose the momentum map $r_Y : Y \to X$ is a local homeomorphism. Then the action of $\Pi_1(X)$ on Y is free if and only if Y is simply connected.*

Proof. Recall from Theorem 7.9 that r_Y is a covering map. The given action is free if and only if the stabilizer of any point $y \in Y$ is trivial. Recall from Proposition 7.4 (ii) that the stabilizer of y is the subgroup $r_{Y*}(\pi_1(Y, y)) \subseteq \Pi_1(X)$. This subgroup is trivial if and only if $r_Y : Y \to X$ is the universal covering space. □

Consider a path connected, locally path connected and semilocally simply connected space X. Let $\mathcal{A}_{\Pi_1(X)}$ denote the category of $\Pi_1(X)$-spaces—the objects of this category are $\Pi_1(X)$-spaces, and $\Pi_1(X)$-equivariant maps are arrows between objects. Consider the subcategory $\mathcal{E}_{\Pi_1(X)}$ of $\mathcal{A}_{\Pi_1(X)}$ consisting of path connected $\Pi_1(X)$-spaces whose momentum maps are local homeomorphisms. On the other hand, let $\mathcal{COV}_X$ denote the category of covering *maps*[2] of X that Spanier defines [32, Chap. 2, §5]—the objects in this category are covering maps and arrows are the continuous maps of covering spaces which preserve those covering maps. Then Theorem 7.9 establishes an isomorphism of categories $\mathcal{COV}_X \simeq \mathcal{E}_{\Pi_1(X)}$: (i) and (iii) of this theorem clearly establish the isomorphism of objects. To show that arrows are also well behaved, firstly, take two covering spaces $Y_1 \xrightarrow{p_1} X \xleftarrow{p_2} Y_2$ and consider a morphism $f : Y_1 \to Y_2$. Then, as a consequence of Remark 7.2, f is $\Pi_1(X)$-equivariant map. Conversely, given a $\Pi_1(X)$-equivariant map $g : Y_1 \to Y_2$, by definition of equivariant map, $r_{Y_2} \circ g = r_{Y_1}$. That means g is a morphism of covering spaces $(Y_1, p_1) \to (Y_2, p_2)$. This discussion is summarized in the next theorem.

Theorem 7.12. *Let X be a path connected, locally path connected and semilocally simply connected space. Then the categories $\mathcal{E}_{\Pi_1(X)}$ and $\mathcal{COV}_X$ are isomorphic.*

[2]We assume that the corresponding covering spaces are path connected.

Furthermore, Proposition 7.11 identifies the *universal* covering space with a free $\Pi_1(X)$-space in $\mathcal{E}_{\Pi_1(X)}$. Therefore, up to equivariant homeomorphism, there is a *unique path connected free* $\Pi_1(X)$*-space* having the momentum map a local homeomorphism. In fact, other $\Pi_1(X)$-spaces are quotients of this free space; using this observation, proposing a universal property for the free $\Pi_1(X)$-space which makes the universal covering space as the *universal* $\Pi_1(X)$*-space* should be a good exercise.

Since other $\Pi_1(X)$ spaces are quotients of the universal covering spaces, the universal covering space *seems* the initial object of $\mathcal{E}_{\Pi_1(X)}$, unlike the classifying space of proper G-actions in [1, Def. 1.6] which is the *terminal* object in appropriate sense.

Rephrasing the standard results about covering spaces using the identification $\mathcal{E}_{\Pi_1(X)} \simeq \mathcal{COV}_X$ can be an interesting exercise; next are two instances of it.

Proposition 7.13 (Consequence of [25, Lem. 80.2] and Theorem 7.12)**.** *Let Y and Z be $\Pi_1(X)$-spaces and $\omega : Y \to Z$ a map of spaces. The next two statements are equivalent.*
(i) *ω is a $\Pi_1(X)$-equivariant map.*
(ii) *$\omega \circ r_Z = r_Y$.*
Moreover, if any one of the above holds, then the following hold.
(iii) *ω is a covering map.*
(iv) *Y is a $\Pi_1(Z)$-space with ω as the momentum map (and the action is given by evaluation of lifted path homotopies at 1).*

Proposition 7.14 ([25, Thm. 80.1] stated using Theorem 7.12)**.** *Let Y be a $\Pi_1(X)$-space and $y_0 \in Y$. Let $H \subseteq \Pi_1(X)$ be the stabilizer at y_0. Then the group of $\Pi_1(X)$-equivariant homeomorphisms of Y is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, p(y_0))$.*

The last proposition can be proved using Proposition 7.4 (ii).

8. Proper actions

8.1. **The kinetics of action.** Let $A \xrightarrow{f} X \xleftarrow{g} B$ be maps of spaces. For $P \subseteq A$ and $Q \subseteq B$, we denote the subset $(P \times Q) \cap (A \times_{f,X,g} B)$ of the fiber product $A \times_{f,X,g} B$ by $P \times_{f,X,g} Q$. The set $P \times_{f,X,g} Q$ can be empty.

Recall the definition of kinetics of a groupoid action from Section 2.8. Before moving to the proper actions of $\Pi_1(X)$, we discuss a technical property of the kinetics of the $\Pi_1(X)$-action, namely, Lemma 8.2 (iii) and Proposition 8.4. Fix a path connected covering space $p : Y \to X$, equivalently, a path connected $\Pi_1(X)$-space in which the momentum map is a local homeomorphism. Recall from equation (1) that the map of kinetics of the action is given by

$$a : \Pi_1(X) \times_{s,X,p} Y \to Y \times Y, \quad a([\gamma], y) = (\widetilde{\gamma}_y(1), y),$$

where $([\gamma], y) \in \Pi_1(X) \times_{s,X,p} Y$, and $\widetilde{\gamma}_y$ is the unique lift of γ in Y starting at y. In other words, $a([\gamma], y) = (\widetilde{\gamma}_y(1), \widetilde{\gamma}_y(0))$.

Since the action of $\Pi_1(X)$ on Y is continuous, a is continuous. As Y is path connected, a is surjective. Lemma 2.10 and Proposition 7.11 imply that the kinetics a is one-to-one if and only if Y is the simply connected covering space of X. In what follows, we show that a is a covering map, and we shall describe slices of a in Proposition 8.4. This technical observation shall be useful to describe proper actions of $\Pi_1(X)$.

Consider the following collection $\mathcal{L}$ of open subsets of $\Pi_1(X) \times_{s,X,p} Y$: an element in $\mathcal{L}$ is of the form $N([\gamma], p(U), p(V)) \times_{s,X,p} V$, where

- $[\gamma] \in \Pi_1(X)$;
- $U, V \subseteq Y$ are path connected and relatively inessential slices of p such that $\gamma(1) \in p(U)$ and $\gamma(0) \in p(V)$.

Since path connected and relatively inessential slices of p form a basis of Y, $\mathcal{L}$ is a basis of $\Pi_1(X) \times_{s,X,p} Y$. Moreover, if U and V are nonempty, then so is $N([\gamma], p(U), p(V)) \times_{s,X,p} V$.

Lemma 8.2. *Let $U, V \subseteq Y$ be nonempty path connected relatively inessential open slices.*

(i) *The map of kinetics a maps the basic open set $N([\gamma], p(U), p(V)) \times_{s,X,p} V$ bijectively onto the basic open set $U \times V$ of $Y \times Y$.*

(ii) *a is an open map.*

(iii) *a is a local homeomorphism. In particular, restriction of a to the basic open set $N([\gamma], p(U), p(V)) \times_{s,X,p} V$ is a homeomorphism onto the basic open set $U \times V$.*

Proof. (i): Write $B := N([\gamma], p(U), p(V)) \times_{s,X,p} V$. Since U and V are path connected and relatively inessential slices, $a|_B : B \to U \times V$ is a continuous bijection.

(ii): The last argument shows that a maps a basic open set in $\Pi_1(X) \times_{s,X,p} Y$ to a basic open set in $Y \times Y$. Therefore, a is an open mapping.

(iii): This follows from (i) and (ii) above. □

Fix two path connected relatively inessential slices $U, V \subseteq Y$. We next want to describe $a^{-1}(U \times V)$. For that, fix points $y \in U$ and $z \in V$. Denote the transformation groupoid $\Pi_1(X) \ltimes Y$ by A. Consider the set A_z^y of arrows in A which *take* the unit z in A to y for the obvious action of A on $A^{(0)} \approx Y$. To be precise, $A_z^y := \{([\gamma], y) \in A \mid \widetilde{\gamma}_z(1) = y\}$. In other words, A_z^y consists of arrows in $\Pi_1(X)$ which take $z \in Y$ to y under the action of the fundamental groupoid. Therefore, we may also write

$$A_z^y = \{[\gamma] \in \Pi_1(X) \mid \widetilde{\gamma}_z(1) = y\}; \tag{6}$$

this identification is more comfortable to use than the earlier one.

Lemma 8.3. *Let $[\gamma_1], [\gamma_2] \in A_z^y$; define $B_i = N([\gamma_i], p(U), p(V)) \times_{s,X,p} V$ for $i = 1, 2$. Then*

$$B_1 \cap B_2 = \begin{cases} \varnothing & \text{if } [\gamma_1] \neq [\gamma_2], \\ B_1 & \text{if } [\gamma_1] = [\gamma_2]. \end{cases}$$

Proof. Assume $B_1 \cap B_2 \neq \varnothing$, and let $[\eta]$ be in the intersection. Then we have $[\eta] = [\delta_2 \square \gamma_2 \square \delta_1] = [\varepsilon_2 \square \gamma_1 \square \varepsilon_1]$ for paths δ_2, ε_2 laying in $p(U)$ starting at $p(y)$, and paths δ_1, ε_1 laying in $p(V)$ ending at $p(z)$. Therefore,

$$[\gamma_2] = [\delta_2^- \square \varepsilon_2] \square [\gamma_1] \square [\varepsilon_1 \square \delta_1^-],$$

where δ_i^- is the reverse of δ_i for $i = 1, 2$. Now, since $p(U)$ is relatively inessential, the *loop* $\delta_2^- \square \varepsilon_2$ at y is null homotopic. So is $\varepsilon_1 \square \delta_1^-$. Therefore, the last equation implies that $[\gamma_1] = [\gamma_2]$. Thus $[\gamma_1] \neq [\gamma_2]$ gives $B_1 \cap B_2 = \varnothing$. The other case is clear. □

As a consequence of the last lemma, using equation (6), we can see that

$$a^{-1}(U \times V) = \bigsqcup_{[\gamma] \in A_z^y} \big(N([\gamma], p(U), p(V)) \times_{s,X,p} V\big).$$

Moreover, Lemma 8.2 shows that restriction of the kinetics to each

$$N\big([\gamma], p(U), p(V)\big) \times_{s,X,p} V$$

above is a homeomorphism onto $U \times V$. Thus we have proved the following results.

Proposition 8.4. *Let X be path connected, locally path connected and semi-locally simply connected space. Let Y be a path connected $\Pi_1(X)$-space having étale momentum map r_Y. Then the map $a : \Pi_1(X) \times_{s,X,r_Y} Y \to Y \times Y$ of the kinetics of the action is a covering map. In fact, for path connected relatively inessential slices $U, V \subseteq Y$,*

$$a^{-1}(U \times V) = \bigsqcup_{[\gamma] \in A_z^y} \big(N([\gamma], p(U), p(V)) \times_{s,X,p} V\big),$$

where each $N([\gamma], p(U), p(V)) \times_{s,X,p} V$ is a slice over $U \times V$ under a.

The reader may compare Proposition 8.4 and Observation 3.11; in the latter one, one should consider the action of $\Pi_1(X)$ on X. The last proposition generalizes [29, Prop. 2.37].

Using Proposition 8.4, we can describe *small* compact sets in the transformation groupoid A as follows. Assume Y is locally compact, and consider the same open sets U and V as in last proposition. Let $U', V' \subseteq Y$ be path connected relatively inessential slices whose closures are compact and $\overline{U'} \subseteq U$ and $\overline{V'} \subseteq V$. Then, for $[\gamma] \in A_z^y$, the closure

$$\overline{N([\gamma], p(U'), p(V')) \times_{s,X,p} V'}$$

is homeomorphic to the compact subset $\overline{U'} \times \overline{V'} = \overline{U' \times V'} \subseteq U \times V$. Then the proposition implies that

$$a^{-1}(\overline{U' \times V'}) = \bigsqcup_{[\gamma] \in A_z^y} \big(\overline{N([\gamma], p(U'), p(V')) \times_{s,X,p} V'}\big). \tag{7}$$

As a closing remark, we note that the collection $\mathcal{L}'$ consisting of path connected, relatively inessential open sets U' such that the closure of U' is compact and the closure is contained in a path connected, relatively inessential slice forms a basis for the topology on Y when Y is locally compact, Hausdorff, path connected, locally path connected and semilocally simply connected. Moreover, $\mathcal{L}'$ is a refinement of $\mathcal{L}$.

8.5. **Proper actions of the fundamental groupoid.** In this section, we study proper actions of the locally compact groupoid $\Pi_1(X)$. We prove Theorem 8.6 which characterizes proper actions using isotropy at a point. Indeed, we focus on the $\Pi_1(X)$-spaces in the category $\mathcal{E}_X$.

Suppose a path connected, locally path connected and semilocally simply connected space X is given. Let $p : Y \to X$ be a path connected covering space. If X is locally compact and Hausdorff, then the transformation groupoid $A := \Pi_1(X) \ltimes Y$ is locally compact and Hausdorff. This can be seen as follows: Section 5.5 implies that the simply connected covering space $\tilde{X}$ is Hausdorff and locally compact. In this case, the deck transformation action on $\tilde{X}$ is proper; see [22, Chap. 21, § Covering manifold]. As a consequence, Y, which is quotient of X by a subgroup of the deck transformation, is also locally compact and Hausdorff. Next, $\Pi_1(X)$ is locally compact and Hausdorff if X is so. Therefore, the transformation groupoid $\Pi_1(X) \ltimes Y$ is locally compact and Hausdorff.

Theorem 8.6. *Let X be a locally compact, Hausdorff, path connected, locally path connected and semilocally simply connected space. Suppose Y is a path connected $\Pi_1(X)$-space with momentum map $p : Y \to X$ a surjective local homeomorphism. Then the following are equivalent.*
(i) *The action of $\Pi_1(X)$ on Y is proper.*
(ii) *The fundamental group of Y is finite.*
(iii) *$p_*(\pi_1(Y))$ is a finite subgroup of $\pi_1(X)$.*
(iv) *The restricted action of the fundamental group $\pi_1(X)$ on Y is proper.*

Proof. (ii) $\Leftrightarrow$ (iii): Theorem 7.9 implies that Y is a covering space of X with p the covering map. Therefore, (ii) and (iii) are clearly equivalent as the group homomorphism $p_* : \pi_1(Y) \to \pi_1(X)$ is injective.

(i) $\Rightarrow$ (ii): Observation 2.12 says that, for proper action, the isotropy is a compact set. Proposition 7.4 (ii) says that the isotropy in the current case is the fundamental group $\pi_1(Y)$ which is compact if and only if it is finite.

(iii) $\Rightarrow$ (i): This is the longest part of the proof and will be done in the end.

(i) $\Rightarrow$ (iv): For any $x \in X$, $\pi_1(X, x) \subseteq \Pi_1(X)$ is a closed subgroup. Therefore, if the action of $\Pi_1(X)$ is proper, the restriction of the action to $\pi_1(X, x)$ is also proper.

(iv) $\Rightarrow$ (ii): Remark 7.5 identifies the isotropy of the restricted action $\pi_1(X)$ with $\pi_1(Y) \simeq p_*(\pi_1(Y))$. Therefore, the restricted action being proper implies the isotropy is finite.

Finally, we prove only the unjustified claim (iii) $\Rightarrow$ (i), which shall complete the proof. Let $K \subseteq Y \times Y$ be compact. We need to show that

$$a^{-1}(K) \subseteq \Pi_1(X) \times_{s,X,p} Y$$

is compact, where a is the kinetics of the action. The current claim is proved in three steps:
(i) firstly, when $K \subseteq Y \times Y$ is singleton,
(ii) then, when K is a small compact set as in equation (7),
(iii) finally, for a general compact set K.
Before we start, let A denote the transformation groupoid $\Pi_1(X) \ltimes Y$. And note that the isotropy at $z \in Y$ for the action of $\Pi_1(X)$ is the stabilizer subgroup $A_z^z \subseteq A$ which is assumed to be finite. Recall equation (6), and write an enumeration of $A_z^z = \{[\gamma_1], \ldots, [\gamma_n]\}$ for some $n \in \mathbb{N}$.

Then, in the first case, when $K = \{(y,z)\}$, $a^{-1}(\{(y,z)\}) = A_z^y$. Observation 2.2 implies that A_z^y is in bijection with A_z^z; hence it is compact.

Now consider the basis $\mathcal{L}'$ for the topology of X discussed just after equation (7). This basis consists of relatively compact open sets U' whose closures are contained in a path connected relatively inessential slice over $p: Y \to X$. Then the sets of the form $U' \times V'$, where $U', V' \in \mathcal{L}'$, form a basis of relatively compact sets for the topology of $Y \times Y$. For $U', V' \in \mathcal{L}'$, equation (7) and case (i) above imply that

$$a^{-1}(\overline{U' \times V'}) = \bigsqcup_{i=1}^{n} \big(\overline{N([\gamma_i], p(U'), p(V')) \times_{s,X,p} V'}\big),$$

where each

$$\overline{N([\gamma_i], p(U'), p(V')) \times_{s,X,p} V'}$$

is a homeomorphic copy of $\overline{U' \times V'}$ (see Proposition 8.4). Thus $a^{-1}(\overline{U' \times V'})$ is a compact set, being a union of finitely many compact sets.

Finally, let $K \subseteq Y \times Y$ be any compact set. Cover K by finitely many relatively compact sets $U_1 \times V_1, \ldots, U_m \times V_m$, where $U_j, V_j \in \mathcal{L}'$. Then

$$a^{-1}(K) \subseteq \bigcup_{i=1}^{m} a^{-1}(\overline{U_j \times V_j}),$$

where each $a^{-1}(\overline{U_j \times V_j})$ is compact by the last argument. Thus $a^{-1}(K) \subseteq A$ is compact. □

Following are some immediate consequences of Theorem 8.6 (ii).

Corollaries 8.7. *Let X be a path connected, locally path connected, semilocally simply connected, Hausdorff and locally compact space. Then the following hold.*
(i) *The action of $\Pi_1(X)$ on the simply connected covering space is proper.*
(ii) *The action of $\Pi_1(X)$ on X is proper if and only if the fundamental group of X is finite.*
(iii) *For each finite subgroup G of $\pi_1(X)$, the associated covering space $X_G \to X$ is a proper $\Pi_1(X)$-space, and these are the only proper $\Pi_1(X)$-spaces in the action category $\mathcal{E}_{\Pi_1(X)}$.*

Acknowledgments. We are grateful to Prahlad Vaidyanathan and Angshuman Bhattacharya discussions with whom led to this article. We thank Suliman Albandik for his patience and beneficial discussions. We thank Ralf Meyer, Jean Renault and Atreyee Bhattacharya for fruitful discussions.

REFERENCES

[1] P. F. Baum, A. Connes, and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras, in *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, 240–291, Contemp. Math. 167, Amer. Math. Soc., Providence, RI, 1994. MR1292018

[2] N. Bourbaki, *Integration. II. Chapters 7–9*, translated from the 1963 and 1969 French originals by Sterling K. Berberian, Elements of Mathematics (Berlin), Springer, Berlin, 2004. MR2098271

[3] J. Brazas, The fundamental group as a topological group, Topology Appl. **160** (2013), no. 1, 170–188. MR2995090

[4] J. Brazas and P. Fabel, On fundamental groups with the quotient topology, J. Homotopy Relat. Struct. **10** (2015), no. 1, 71–91. MR3313636

[5] N. Brodskiy, J. Dydak, B. Labuz, and A. Mitra, Topological and uniform structures on universal covering spaces, arXiv:1206.0071v1 (2012)

[6] R. Brown and G. Danesh-Naruie, The fundamental groupoid as a topological groupoid, Proc. Edinburgh Math. Soc. (2) **19** (1974/75), 237–244. MR0413096

[7] K. R. Davidson, *C^*-algebras by example*, Fields Inst. Monogr. 6, Amer. Math. Soc., Providence, RI, 1996. MR1402012

[8] A. Deitmar, On Haar systems for groupoids, Z. Anal. Anwend. **37** (2018), no. 3, 269–275. MR3826616

[9] M. P. do Carmo, *Differential geometry of curves and surfaces*, translated from the Portuguese, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1976. MR0394451

[10] J. Dugundji, *Topology*, Allyn and Bacon Ser. Adv. Math., Allyn and Bacon, Inc., Boston, MA, 1978. MR0478089

[11] T. A. Edwards, On function spaces with the compact-open topology, New Zealand J. Math. **28** (1999), no. 2, 185–192. MR1775062

[12] C. Ehresmann, Catégories topologiques et catégories différentiables, in *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, 137–150, Librairie Universitaire, Louvain, 1959. MR0116360

[13] H. Emerson and R. Meyer, Dualities in equivariant Kasparov theory, New York J. Math. **16** (2010), 245–313. MR2740579

[14] P. Fabel, Topological fundamental groups can distinguish spaces with isomorphic homotopy groups, Topology Proc. **30** (2006), no. 1, 187–195. MR2280667

[15] P. Fabel, Multiplication is discontinuous in the Hawaiian earring group (with the quotient topology), Bull. Pol. Acad. Sci. Math. **59** (2011), no. 1, 77–83. MR2810974

[16] H. Fischer and A. Zastrow, Generalized universal covering spaces and the shape group, Fund. Math. **197** (2007), 167–196. MR2365886

[17] G. B. Folland, *A course in abstract harmonic analysis*, Stud. Adv. Math., CRC, Boca Raton, FL, 1995. MR1397028

[18] A. E. Hatcher, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2002. MR1867354

[19] R. D. Holkar, Composition of topological correspondences, J. Operator Theory **78** (2017), no. 1, 89–117. MR3679607

[20] M. Khalkhali, *Basic noncommutative geometry*, second edition, EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2013. MR3134494

[21] A. Kumjian, P. S. Muhly, J. N. Renault, and D. P. Williams, The Brauer group of a locally compact groupoid, Amer. J. Math. **120** (1998), no. 5, 901–954. MR1646047

[22] J. M. Lee, *Introduction to smooth manifolds*, second edition, Grad. Texts in Math. 218, Springer, New York, 2013. MR2954043

[23] P. Muhly, *Coordinates in operator algebra*, in CMBS Conference Lecture Notes (Texas Christian University, 1990), 1999.

[24] P. S. Muhly, J. N. Renault, and D. P. Williams, Equivalence and isomorphism for groupoid C^*-algebras, J. Operator Theory **17** (1987), no. 1, 3–22. MR0873460

[25] J. R. Munkres, *Topology: a first course*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975. MR0464128

[26] P. O'Meara, On paracompactness in function spaces with the compact-open topology, Proc. Amer. Math. Soc. **29** (1971), 183–189. MR0276919

[27] A. Pakdaman and F. Shahini, The fundamental groupoid as a topological groupoid: Lasso topology, Topology Appl. **302** (2021), Paper No. 107838, 8 pp. MR4310565

[28] A. Pakdaman and F. Shahini, Fundamental groupoid and whisker topology, Math. Res. **9** (2023), 55–71.

[29] B. L. Reinhart, *Differential geometry of foliations*, Ergeb. Math. Grenzgeb. (3) 99, Springer, Berlin, 1983. MR0705126

[30] J. N. Renault, Représentation des produits croisés d'algèbres de groupoïdes, J. Operator Theory **18** (1987), no. 1, 67–97. MR0912813

[31] W. Rudin, *Real and complex analysis*, third edition, McGraw-Hill, New York, 1987. MR0924157

[32] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR0210112

[33] J.-L. Tu, Non-Hausdorff groupoids, proper actions and K-theory, Doc. Math. **9** (2004), 565–597. MR2117427

[34] D. P. Williams, Haar systems on equivalent groupoids, Proc. Amer. Math. Soc. Ser. B **3** (2016), 1–8. MR3478528

[35] D. P. Williams, *A tool kit for groupoid C^*-algebras*, Math. Surveys Monogr. 241, Amer. Math. Soc., Providence, RI, 2019. MR3969970

[36] A. Zabrodsky, Covering spaces of paracompact spaces, Pacific J. Math. **14** (1964), 1489–1503. MR0173229

Received August 23, 2023; accepted October 30, 2024

Rohit Dilip Holkar
Department of Mathematics, Indian Institute of Science Education and Research Bhopal, Bhopal Bypass Road, Bhauri, Bhopal 462 066, Madhya Pradesh, India
E-mail: `rohit.d.holkar@gmail.com`

Md Amir Hossain
Department of Mathematics, Indian Institute of Science Education and Research Bhopal, Bhopal Bypass Road, Bhauri, Bhopal 462 066, Madhya Pradesh, India

Current affiliation: The Institute of Mathematical Sciences,
A CI of Homi Bhabha National Institute,
4th cross street, CIT Campus, Taramani, Chennai, 600113, India
E-mail: `mdamirhossain18@gmail.com`

Dheeraj Kulkarni
Department of Mathematics, Indian Institute of Science Education and Research Bhopal, Bhopal Bypass Road, Bhauri, Bhopal 462 066, Madhya Pradesh, India
E-mail: `dheeraj@iiserb.ac.in`