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Ultraproducts of factorial W^*-bundles

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Abstract. This paper investigates factorial W^*-bundles and their ultraproducts. More precisely, a W^*-bundle is *factorial* if the von Neumann algebras associated to its fibers are all factors. Let $\mathcal{M}$ be the tracial ultraproduct of a family of factorial W^*-bundles over compact Hausdorff spaces with finite, uniformly bounded covering dimensions. We prove that, in this case, the set of limit traces in $\mathcal{M}$ is weak*-dense in the trace space $T(\mathcal{M})$. This in particular entails that $\mathcal{M}$ is factorial. We also provide, on the other hand, an example of an ultraproduct of factorial W^*-bundles which is not factorial. Finally, we obtain some results of model-theoretic nature: if A and B are exact, $\mathcal{Z}$-stable C^*-algebras, or if they both have strict comparison, then $A \equiv B$ implies that $T(A)$ is Bauer if and only if $T(B)$ is. If moreover both $T(A)$ and $T(B)$ are Bauer simplices and second countable, then the sets of extreme traces $\partial_e T(A)$ and $\partial_e T(B)$ have the same covering dimension.

1. Introduction

W^*-bundles, first introduced by Ozawa in [26] as tracial W^*-analogs of C^*-bundles and of $C(X)$-algebras, are C^*-algebras that arise as bundles over compact topological spaces and whose fibers are tracial von Neumann algebras (Definition 2.2).

Due to their hybrid nature, W^*-bundles generally find their main *raison d'être* in the role of bridge that they play between tracial von Neumann algebras and stably finite C^*-algebras. Indeed, Ozawa's foresighted intuition to isolate this class in [26] was prompted by the celebrated paper by Matui and Sato [24] and the subsequent series of work [20, 30, 31], investigating the Toms–Winter Conjecture for C^*-algebras whose trace space is a Bauer simplex. W^*-bundles arise as tracial completions (in the sense of [26]; see Definition 5.9)

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of such C^*-algebras, and they have been used systematically for arguments relying on approximations and properties where tracial 2-norms appear, with numerous applications also in the equivariant framework [7, 22, 23, 32].

In this note, we investigate W^*-bundles from a more abstract point of view, as a class in its own right, with an approach closer to that in [13, 12]. The main motivation for the present paper is the forthcoming work on tracially complete C^*-algebras [8] by Carrión, Castillejos, Evington, Gabe, Schafhauser, Tikuisis and White. We briefly pause to report some of the basic concepts and open problems considered in their project in order to give the proper context and motivation to our results. We would like to thank the authors of [8] for allowing us to include here some of the contents of their work, not yet publicly available at the time of writing this note.

The fundamental definition considered in [8] is that of *tracially complete C^*-algebra*, which provides an abstract framework to study tracial completions of C^*-algebras as defined in [26].

Definition 1.1 ([8])**.** Fix a C^*-algebra $\mathcal{M}$ and a nonempty set X of $T(\mathcal{M})$, the set of tracial states of $\mathcal{M}$. Consider the 2-semi-norm

$$\|a\|_{2,X} = \sup_{\tau \in X} \tau(a^*a)^{1/2}, \quad a \in \mathcal{M}.$$

A *tracially complete* C^*-algebra is a pair $(\mathcal{M}, X)$, where $\mathcal{M}$ is a unital[1] C^*-algebra and X is a compact, nonempty, convex subset of the trace space $T(\mathcal{M})$ of $\mathcal{M}$ such that $\|\cdot\|_{2,X}$ is a norm on $\mathcal{M}$ and such that the C^*-norm unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,X}$-complete. A tracially complete C^*-algebra $(\mathcal{M}, X)$ is *factorial* if moreover X is a closed face of $T(\mathcal{M})$.

The most elementary examples of tracially complete C^*-algebras are tracial von Neumann algebras $(\mathcal{M}, \tau)$, where τ is a faithful normal trace. This is the scenario where $X = \{\tau\}$. Moreover, W^*-bundles form another important class of tracially complete C^*-algebras (see § 2.1), and in the factorial case, they correspond precisely to those $(\mathcal{M}, X)$ for which X is a Bauer simplex (this is a consequence of [26, Thm. 3]; see Theorem 2.4).

A tracial von Neumann algebra $(\mathcal{M}, \tau)$ is factorial as a tracially complete C^*-algebra if and only if it is a factor, hence the name. Part of the motivation why the class of factorial tracially complete C^*-algebras has been isolated in [8] is that such algebras tend to be more manageable and tractable than general tracially complete C^*-algebras. This emerges both in technical and elementary facts (such as Lemma 5.2, Lemma 3.1 or Proposition 5.5), as well as in more ambitious and sophisticated results like the classification theorems announced in [8].

In this paper, we address the following question, which appeared in an early version of [8].

[1]The assumption of unitality is proved to be redundant in [8].

Question 1.2 ([8])**.** Let $((\mathcal{M}_i, X_i) \mid i \in I)$ be a sequence of factorial tracially complete C^*-algebras, let $\mathcal{M} = \prod^{\mathcal{U}} \mathcal{M}_i$ be the corresponding tracial ultraproduct, and let X be the weak*-closure of the set of all limit traces on $\mathcal{M}$. Is $(\mathcal{M}, X)$ factorial?

Limit traces on $\mathcal{M}$ are those that are obtained by taking $\mathcal{U}$-limits of sequences of traces $(\tau_i)_{i\in I} \in \prod_{i\in I} X_i$ (see § 2.5). The tracial ultraproduct $(\mathcal{M}, X)$ considered in Question 1.2 is the *right* notion of ultraproduct in the category of tracially complete C^*-algebras (see § 2.5 for the case of W^*-bundles and § 5.8 for general tracially complete C^*-algebras), so Question 1.2 is simply asking whether factoriality is preserved when passing to the ultraproduct. This is the case for tracial von Neumann algebras; indeed, it is well-known that the tracial ultraproduct of a family of finite factors (which in this case corresponds to the usual von Neumann ultraproduct) is again a finite factor (see *e.g.* [18]).

In [26, Thm. 8], Ozawa proved that, for ultraproducts of exact $\mathcal{Z}$-stable C^*-algebras, the set of limit traces is weak*-dense in the trace space. Rephrased in the framework of Question 1.2, what [26, Thm. 8] shows is that the ultraproduct $(\mathcal{M}, X)$ of a sequence of factorial tracially complete C^*-algebras arising as tracial completions of exact $\mathcal{Z}$-stable C^*-algebras satisfies $X = T(\mathcal{M})$; hence, in particular, it is factorial. An analog result is [9, Prop. 2.5], from which it can be deduced that if $(\mathcal{M}, X)$ is the tracial ultrapower of a factorial tracially complete C^*-algebra which is the tracial completion of separable C^*-algebra with *complemented partitions of unity*, then again $X = T(\mathcal{M})$, so $(\mathcal{M}, X)$ is factorial. The existence of complemented partitions of unity (usually referred to as *CPoU*) is a technical condition introduced in [10, Def. 3.1], which is automatic for instance in $\mathcal{Z}$-stable nuclear C^*-algebras [10, Thm. 3.8]. We finally refer to [2] for a recent and more general account, employing Cuntz semigroup techniques, on when limit traces are weak*-dense in the trace space of (C^*-norm) ultraproducts.

CPoU are an extremely powerful tool in the study of tracially complete C^*-algebras in [8], effectively dividing these algebras in two subclasses, a tamer one where the presence of CPoU allows to transfer numerous results and techniques from the theory of von Neumann algebras, and its complement, much less understood. This paper focuses on the latter, while restricting to W^*-bundles.

The following theorem shows that Question 1.2 has affirmative answer for ultraproducts of W^*-bundles, even without complemented partitions of unity, as long as their base spaces have bounded covering dimensions.

Theorem 1.3. *Let $(\mathcal{M}_i)_{i\in I}$ be a sequence of factorial W^*-bundles over compact Hausdorff spaces K_i. Suppose there is $d \in \mathbb{N}$ such that $\dim(K_i) \leq d$ for every $i \in I$. Let $\mathcal{M}$ be the corresponding ultraproduct, which is a W^*-bundle over the ultracoproduct $\sum^{\mathcal{U}} K_i$. Then the set of limit traces is weak*-dense in $T(\mathcal{M})$, and in particular, $\mathcal{M}$ is a factorial W^*-bundle.*

We also prove that if the uniform bound on the covering dimension of K_i is removed from Theorem 1.3, then its conclusion might fail. In fact, we show that Question 1.2 has negative answer in general, even when restricted to W^*-bundles.

Theorem 1.4. *There exists a sequence of factorial W^*-bundles whose ultraproduct is not factorial.*

The sequence we use for Theorem 1.4 dates back to [27], and it consists of 2-homogeneous C^*-algebras arising from certain vector bundles over finite-dimensional complex projective spaces. We remark that such family is the same as the one considered in [26] to give an example of an ultraproduct for which the set of limit traces is not weak*-dense in the whole trace space.

Note that the sequence considered in Theorem 1.4 is composed by W^*-bundles whose fibers are matrix algebras, hence type I. This is in contrast with the primary focus of [8] and of most applications of W^*-bundles and tracial completions in the literature, which mainly concerns tracially complete C^*-algebras whose fibers are infinite-dimensional. These are referred to as *type II_1 tracially complete C^*-algebras* in [8], and it would be interesting to know whether Question 1.2 has negative answer also for those algebras.

The final part of the paper has a model-theoretic flavor, investigating how the first-order theory of a C^*-algebra can determine the topological properties of its trace space. A precursor to the result below can be found in [16, Sec. 3.5], where it is proved that, for classes of C^*-algebras where the Cuntz–Pedersen nullset is definable (in the sense of [16, Chap. 3]), being monotracial is preserved by elementary equivalence.

Theorem 1.5. *Let A, B be two unital C^*-algebras which are exact and $\mathcal{Z}$-stable, or which have strict comparison, or which belong to any other class where the Cuntz–Pedersen nullset is definable. Suppose that A is elementarily equivalent to B. Then $T(A)$ is a Bauer simplex if and only if $T(B)$ is. If moreover both $T(A)$ and $T(B)$ are Bauer simplices and second countable, then $\partial_e T(A)$ and $\partial_e T(B)$ have the same covering dimension.*

The proof of Theorem 1.5 uses a tracial analog of Dixmier's averaging property for factorial W^*-bundles (Proposition 5.5), which permits us to show that the center of the ultraproduct of a sequence of factorial W^*-bundles is isomorphic to the ultraproduct of the centers (Theorem 5.7).

The paper is structured as follows. In § 2, we present definitions and preliminaries needed in later sections, § 3 is devoted to the proof of Theorem 1.3, while § 4 is where Theorem 1.4 is proved. Finally, § 5 contains the background and proofs needed to show Theorem 1.5.

2. Preliminaries

Given a C^*-algebra A, denote by A_1, A_{sa} and A_+ respectively the set of all contractions, selfadjoint and positive elements in A. We let $Z(A)$ denote the center of A, and given $a, b \in A$, we abbreviate the commutator $ab - ba$ as $[a, b]$.

The *trace space* $T(A)$ of A is the set of all tracial states on A, which we refer to simply as *traces*. For $\tau \in T(A)$, define the 2-semi-norm $\|\cdot\|_{2,\tau}$ on A as

$$\|a\|_{2,\tau} = \tau(a^*a)^{1/2} \quad \text{for every } a \in A.$$

Given a nonempty $X \subseteq T(A)$, the 2-semi-norm on A associated to X is

$$\|a\|_{2,X} = \sup_{\tau \in X} \|a\|_{2,\tau} \quad \text{for every } a \in A.$$

Note that $\|\cdot\|_{2,X} = \|\cdot\|_{2,\overline{\mathrm{conv}}(X)}$, where $\overline{\mathrm{conv}}(X)$ is the closed convex hull of X. Given a convex set X, we let $\partial_e X$ denote the set of extreme points of X. The equality $\|\cdot\|_{2,X} = \|\cdot\|_{2,\partial_e X}$ follows by the Krein–Milman Theorem.

Given a unital C^*-algebra A, the trace space $T(A)$, as well as any other of its closed, convex subsets X, is a *Choquet simplex* [1, Sec. 3]. In particular, for every $x \in X$, there exists a unique boundary measure μ_x (in the sense of [1, Prop. I.4.5]) such that $f(x) = \int_X f(t)\, d\mu_x$ for every continuous affine function $f : X \to \mathbb{R}$. A *Bauer simplex* X is a Choquet simplex such that $\partial_e X$ is closed.

2.1. W^*-bundles.

Definition 2.2 ([26, Sec. 5])**.** A *W^*-bundle over (a compact Hausdorff space) K* is a unital C^*-algebra $\mathcal{M}$, with a unital embedding of $C(K)$ in the center of $\mathcal{M}$ and a faithful unital conditional expectation $E : \mathcal{M} \to C(K)$ such that
(i) E is tracial, that is, $E(ab) = E(ba)$ for all $a, b \in \mathcal{M}$,
(ii) the C^*-norm unit ball of $\mathcal{M}$ is complete with respect to the norm $\|\cdot\|_{2,K}$ induced by E, defined as $\|a\|_{2,K} = \|E(a^*a)\|^{1/2}$ for all $a \in \mathcal{M}$.

Let $\mathcal{M}$ be a W^*-bundle over K with conditional expectation $E : \mathcal{M} \to C(K)$, and fix $\lambda \in K$. Throughout the paper, we shall denote by τ_λ the trace $\mathrm{ev}_\lambda \circ E$ and by π_λ the GNS-representation corresponding to τ_λ. The von Neumann algebra $\pi_\lambda(\mathcal{M})''$ is the *fiber* corresponding to λ.

More generally, every regular Borel probability measure μ over K naturally induces a trace τ_μ defined for $a \in \mathcal{M}$ as

$$\tau_\mu(a) = \int_K E(a)\, d\mu.$$

Let $X = \{\tau_\mu \mid \mu \in \mathrm{Prob}(K)\}$. Faithfulness of E entails that $\|\cdot\|_{2,X}$ is a norm on $\mathcal{M}$, while item (ii) of Definition 2.2 implies that $(\mathcal{M}, X)$ is a tracially complete C^*-algebra since $\|\cdot\|_{2,X} = \|\cdot\|_{2,\partial_e X} = \|\cdot\|_{2,K}$. We sometimes identify X and $\partial_e X$ with $\mathrm{Prob}(K)$ and K respectively. Note that the notations $\|\cdot\|_{2,\partial_e X}$ and $\|\cdot\|_{2,K}$ are consistent with this identification. We always implicitly consider the W^*-bundle $\mathcal{M}$ as a tracially complete C^*-algebra; in particular, we say that $\mathcal{M}$ is *factorial* if the pair $(\mathcal{M}, \mathrm{Prob}(K))$ is a factorial tracially complete C^*-algebra.

The following proposition isolates some useful reformulations of factoriality. Its statement, as well as its proof, originates from some analog statements appearing in an early version of [8].

Proposition 2.3. *Let $(\mathcal{M}, X)$ be a tracially complete C^*-algebra. The following conditions are equivalent.*
(i) $(\mathcal{M}, X)$ *is factorial.*
(ii) $\partial_e X \subseteq \partial_e T(\mathcal{M})$.
(iii) $\pi_\lambda(\mathcal{M})''$ *is a factor for every* $\lambda \in \partial_e X$.

Proof. (1) $\Rightarrow$ (2) is true since every extreme point of a face must be extreme in the simplex itself.

For (2) $\Rightarrow$ (1), let $F = \operatorname{conv}(\partial_e X)$ be the convex hull of $\partial_e X$. Given $x \in F$, then $x \in \operatorname{conv}(x_1, \dots, x_n)$ for some $x_1, \dots, x_n \in \partial_e X$. The set $\operatorname{conv}(x_1, \dots, x_n)$ is the (closed) convex hull of a compact subset of $\partial_e X \subseteq \partial_e T(\mathcal{M})$; hence it is a face of $T(\mathcal{M})$ by [19, Cor. 11.1.19]. It follows that if $x = \lambda y + (1-\lambda)z$ for $\lambda \in (0,1)$ and $y, z \in T(\mathcal{M})$, then $y, z \in \operatorname{conv}(x_1, \dots, x_n) \subseteq F$; hence F is a face of $T(\mathcal{M})$. Finally, $X = \overline{F}$ is face by [29, Prop. 4.4].

(2) $\Leftrightarrow$ (3) follows by the well-known fact that, for $\tau \in T(\mathcal{M})$ and π_τ the corresponding GNS-representation, $\pi_\tau(\mathcal{M})''$ is a factor if and only if $\tau \in \partial_e T(\mathcal{M})$. □

Summarizing, a W^*-bundle $\mathcal{M}$ over K gives rise to a tracially complete C^*-algebra $(\mathcal{M}, X)$, where X is a Bauer simplex whose boundary is homeomorphic to K. The converse also holds in the factorial case: every factorial tracially complete C^*-algebra whose base space is a Bauer simplex can be naturally endowed with a W^*-bundle structure. This fact is direct consequence of [26, Thm. 3], and it is stated and proved below in a form which is due to [8].

Theorem 2.4 ([26, Thm. 3], [8])**.** *Let $(\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra such that X is a Bauer simplex. Then there exists an embedding $\theta : C(\partial_e X) \to Z(\mathcal{M})$ such that*

$$\tau(\theta(f)a) = \int_{\partial_e X} f(\sigma)\sigma(a)\, d\mu_\tau(\sigma) \quad \text{for every } \tau \in X,\ f \in C(\partial_e X),\ a \in \mathcal{M}.$$

Moreover, $(\mathcal{M}, X)$ can be endowed with the structure of a W^*-bundle over $\partial_e X$ with the conditional expectation $E : \mathcal{M} \to C(\partial_e X)$ defined as $E(a)(\tau) = \tau(a)$ for $\tau \in \partial_e X$.

Proof. The existence of an embedding of $\theta : C(\partial_e X) \to Z(\mathcal{M})$ as claimed in the statement is a consequence of [26, Thm. 3]. The latter result is proved for X a metrizable face (which appears as S in the notation of [26]), where metrizability is required only to make sure that $\partial_e X$ is Borel. In our context, this is automatic since $\partial_e X$ is assumed to be closed. Given $a \in \mathcal{M}$, let $\hat{a} \in C(\partial_e X)$ be defined as $\hat{a}(\tau) = \tau(a)$ for every $\tau \in \partial_e X$. Let E be the conditional expectation $E(a) = \theta(\hat{a})$. It is immediate to check that the 2-norm induced by E is the same as $\|\cdot\|_{2,\partial_e X} = \|\cdot\|_{2,X}$. By assumption, $\mathcal{M}_1$ is $\|\cdot\|_{2,X}$-complete, making $\mathcal{M}$ a W^*-bundle over $\partial_e X$. □

2.5. **Ultrapowers and ultraproducts of W^*-bundles.** Fix an infinite index set I and a free ultrafilter $\mathcal{U}$ over I. Let $((A_i, X_i) \mid i \in I)$ be a sequence of pairs, where A_i is a unital C^*-algebra and X_i is a nonempty subset of $T(A_i)$ for every $i \in I$. The *tracial ultraproduct* of such sequence is the C^*-algebra

$$\prod^{\mathcal{U}}(A_i, X_i) = \frac{\prod_{i\in I} A_i}{\{(a_i)_{i\in I} \in \prod_{i\in I} A_i \mid \lim_{i\to\mathcal{U}}\|a_i\|_{2,X_i} = 0\}}.$$

In case of a constant sequence $(A_i, X_i) = (A, X)$, we use the notation $A^{\mathcal{U}}_X$ and refer to this C^*-algebra as the *tracial ultrapower* of (A, X). We write $A^{\mathcal{U}}$ if $X = T(A)$. Throughout this paper, we shall only consider cases where $\|\cdot\|_{2,X_i}$ is a norm for every $i \in I$.

We drop X_i from the notation when it is clear from the context and simply write $\prod^{\mathcal{U}} A_i$. For instance, let $(\mathcal{M}_i)_{i\in I}$ be W^*-bundles over K_i. Then, by $\prod^{\mathcal{U}} \mathcal{M}_i$, we mean the tracial ultraproduct of the sequence $((\mathcal{M}_i, K_i) \mid i \in I)$.

Given $((A_i, X_i) \mid i \in I)$, every sequence of traces $\bar\tau = (\tau_i)_{i\in I} \in \prod_{i\in I} X_i$ determines a trace on $\prod^{\mathcal{U}} A_i$ defined on each representing sequence as

$$\bar\tau((a_i)_{i\in I}) = \lim_{i\to\mathcal{U}} \tau_i(a_i).$$

We denote by $\prod_{\mathcal{U}} X_i$ the set of traces which arise in this manner, and we refer to them as *limit traces*. Moreover, $\prod_{\mathcal{U}} X_i$ corresponds to the set-theoretic ultraproduct of $(X_i)_{i\in I}$. This is a convex, not necessarily closed, subset of the trace space of $\prod^{\mathcal{U}} A_i$. Let $\sum^{\mathcal{U}} X_i$ denote its weak*-closure. In case $X_i = X$ for every $i \in I$, we abbreviate such closure as $X^{\mathcal{U}}$. When every X_i is compact (*e.g.* in the case of W^*-bundles), the space $\sum^{\mathcal{U}} X_i$ can also be obtained as the *ultracoproduct* of the sequence $(X_i)_{i\in I}$, namely the compact Hausdorff space such that $C(\sum^{\mathcal{U}} X_i) \cong \prod_{\mathcal{U}} C(X_i)$, where the latter (with $\mathcal{U}$ in subscript) is the canonical C^*-norm ultraproduct.

The unit ball of the tracial ultraproduct $\prod^{\mathcal{U}} A_i$ is complete with respect to the 2-norm $\|\cdot\|_{\sum^{\mathcal{U}} X_i}$ (see § 5.8); in particular, for ultraproducts of W^*-bundles, we have the following.

Proposition 2.6 ([7, Prop. 3.9]). *Let $(\mathcal{M}_i)_{i\in I}$ be a sequence of W^*-bundles over K_i with conditional expectations $E_i : \mathcal{M}_i \to C(K_i)$. The tracial ultraproduct $\prod^{\mathcal{U}} \mathcal{M}_i$ is a W^*-bundle over $\sum^{\mathcal{U}} K_i$, with the conditional expectation*

$$E^{\mathcal{U}} : \prod^{\mathcal{U}} \mathcal{M}_i \to C\Big(\sum^{\mathcal{U}} K_i\Big), \quad (a_i)_{i\in I} \mapsto (E_i(a_i))_{i\in I}$$

inducing the norm $\|\cdot\|_{\sum^{\mathcal{U}} K_i}$.

3. W^*-BUNDLES WITH FINITE-DIMENSIONAL BASE SPACE

Despite employing different techniques, both [26, Thm. 8] and [9, Prop. 2.5] are proved by showing first that elements which are small with respect to all traces can be approximated with sums of commutators, with the number of summands not changing, or at least being kept under control, as the precision of the approximation varies. Theorem 1.3 is no exception and is based on the following lemma.

Lemma 3.1. *Let $\mathcal{M}$ be a factorial W^*-bundle over the space K. Suppose that K has finite covering dimension d, and let $a \in \mathcal{M}_{\mathrm{sa}}$ be a contraction such that $E(a) = 0$. Then, for every $\varepsilon > 0$, there exist contractions $w_i, z_i \in \mathcal{M}$ for $i = 1, \dots, 10d$ such that*

$$\Big\| a - 24 \sum_{i \leq 10d} [w_i, z_i] \Big\|_{2,K} < \varepsilon.$$

Proof. Let $\lambda \in K$. Recall that τ_λ denotes the trace $\mathrm{ev}_\lambda \circ E$ on $\mathcal{M}$ and that π_λ denotes the GNS-representation corresponding to τ_λ. Fix $a \in \mathcal{M}_{\mathrm{sa}}$ as in the statement. As $\mathcal{M}$ is factorial, by Proposition 2.3, the von Neumann algebra $\pi_\lambda(\mathcal{M})''$ is a factor. The assumption $E(a)(\lambda) = 0$ thus entails that $\pi_\lambda(a)$ is mapped to zero by the unique trace on $\pi_\lambda(\mathcal{M})''$. By [14, Thm. 2.3], there exist contractions $\tilde{w}_1^\lambda, \dots, \tilde{w}_{10}^\lambda, \tilde{z}_1^\lambda, \dots, \tilde{z}_{10}^\lambda \in \pi_\lambda(\mathcal{M})''$ such that

$$\pi_\lambda(a) = 24 \sum_{k \leq 10} [\tilde{w}_k^\lambda, \tilde{z}_k^\lambda].$$

By the Kaplansky Density Theorem, for every $k \leq 10$, there are contractions $w_k^\lambda, z_k^\lambda \in \mathcal{M}$ approximating $\tilde{w}_k^\lambda$ and $\tilde{z}_k^\lambda$ well enough so that

$$\Big\| a - 24 \sum_{k \leq 10} [w_k^\lambda, z_k^\lambda] \Big\|_{2,\tau_\lambda} = \Big\| \pi_\lambda(a) - 24 \sum_{k \leq 10} [\pi_\lambda(w_k^\lambda), \pi_\lambda(z_k^\lambda)] \Big\|_{2,\tau_\lambda} < \varepsilon.$$

By continuity of the 2-norm, for every $\lambda \in K$, there exists an open neighborhood U_λ of λ such that

$$\Big\| a - 24 \sum_{k \leq 10} [w_k^\lambda, z_k^\lambda] \Big\|_{2,\tau_{\lambda'}} < \varepsilon \quad \text{for every } \lambda' \in U_\lambda. \tag{1}$$

By compactness of K, there exists a finite open cover $\mathcal{V}$ of K where, for each $U \in \mathcal{V}$, there is $\lambda_U \in K$ such that $U = U_{\lambda_U}$. Moreover, as K has covering dimension equal to d, $\mathcal{V}$ can be partitioned as $\mathcal{V}_0 \sqcup \dots \sqcup \mathcal{V}_d$ so that the elements of each $\mathcal{V}_j$ are pairwise disjoint [6, Lem. 3.2].

Let $\{f_U\}_{U \in \mathcal{V}} \subseteq C(K) \subseteq \mathcal{M}$ be a partition of the unity on K such that $\mathrm{supp}(f_U) \subseteq U$ for every $U \in \mathcal{V}$. For every $j = 0, \dots, d$ and $k = 1, \dots, 10$, define the following elements of $\mathcal{M}$:

$$w_{k+10j} = \sum_{U \in \mathcal{V}_j} f_U^{1/2} w_k^{\lambda_U},$$
$$z_{k+10j} = \sum_{U \in \mathcal{V}_j} f_U^{1/2} z_k^{\lambda_U}.$$

Note that, as the functions $\{f_U\}_{U \in \mathcal{V}_j}$ have pairwise disjoint support, the elements defined above are still contractions and verify the equality

$$[w_{k+10j}, z_{k+10j}] = \sum_{U \in \mathcal{V}_j} f_U [w_k^{\lambda_U}, z_k^{\lambda_U}] \tag{2}$$

for every $j = 0, \dots, d$ and $k = 1, \dots, 10$.

We claim that $\{w_i, z_i\}_{i\leq 10d}$ are the desired elements. Indeed, for any $\lambda \in K$ and $0 \leq j \leq d$, there is at most one $U_j \in \mathcal{V}_j$ such that $\lambda \in U_j$ (if there is none, simply pick a random $U_j \in \mathcal{V}_j$); hence

$$\begin{aligned}
&\Big\|a - 24 \sum_{i\leq 10d} [w_i, z_i]\Big\|_{2,\tau_\lambda} \\
&\overset{(2)}{=} \Big\|\sum_j \sum_{U\in\mathcal{V}_j} f_U(\lambda)a - 24 \sum_{k,j} \sum_{U\in\mathcal{V}_j} f_U(\lambda)[w_k^{\lambda_U}, z_k^{\lambda_U}]\Big\|_{2,\tau_\lambda} \\
&= \Big\|\sum_j f_{U_j}(\lambda)a - 24 \sum_{k,j} f_{U_j}(\lambda)[w_k^{\lambda_{U_j}}, z_k^{\lambda_{U_j}}]\Big\|_{2,\tau_\lambda} \\
&\leq \sum_j f_{U_j}(\lambda)\Big\|a - 24 \sum_k [w_k^{\lambda_{U_j}}, z_k^{\lambda_{U_j}}]\Big\|_{2,\tau_\lambda} \\
&\overset{(1)}{<} \sum_j f_{U_j}(\lambda)\varepsilon \leq \varepsilon.
\end{aligned}$$

We conclude that

$$\Big\|a - 24 \sum_{i\leq 10d} [w_i, z_i]\Big\|_{2,K} = \sup_{\lambda\in K}\Big\|a - 24 \sum_{i\leq 10d} [w_i, z_i]\Big\|_{2,\tau_\lambda} < \varepsilon. \qquad \square$$

We first prove Theorem 1.3 for the case $I = \mathbb{N}$, to make it more accessible for readers not well acquainted with model theory. An elementary model-theoretic argument (which we defer to § 5.8) shows that its conclusion holds for ultraproducts over arbitrary sets of indices.

Theorem 3.2. *Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of factorial W^*-bundles over compact Hausdorff spaces K_n. Suppose there is $d \in \mathbb{N}$ such that $\dim(K_n) \leq d$ for every $n \in \mathbb{N}$. Let $\mathcal{M} = \prod^{\mathcal{U}} \mathcal{M}_n$ be the corresponding ultraproduct. Then the set of limit traces is weak*-dense in $T(\mathcal{M})$, and in particular, $\mathcal{M}$ is a factorial W^*-bundle.*

Proof. By Proposition 2.6, the ultraproduct $\mathcal{M} = \prod^{\mathcal{U}} \mathcal{M}_n$ is a W^*-bundle over $K = \sum^{\mathcal{U}} K_n$. Arguing as in [10, Lem. 4.4] and [26, Thm. 8], by an application of the Hahn–Banach Theorem, it is sufficient to show that the following equality holds for every $a \in \mathcal{M}_{\mathrm{sa}}$:

$$\sup_{\lambda\in K}|\tau_\lambda(a)| = \sup_{\tau\in T(\mathcal{M})} |\tau(a)|. \tag{3}$$

Fix thus a contraction $a \in \mathcal{M}_{\mathrm{sa}}$, and suppose then that $\sup_{\lambda\in K}|\tau_\lambda(a)| \leq \delta$ for some $\delta \geq 0$. The equality in (3) follows if we can provide $c, w_1, \dots, w_{10d}, z_1, \dots, z_{10d} \in \mathcal{M}$ such that $\|c\| \leq \delta$ and

$$a - c = \sum_{i\leq 10d} [w_i, z_i]. \tag{4}$$

Let $(a_n)_{n\in\mathbb{N}}$ be a representative sequence of selfadjoint contractions for a. Up to a rescaling of a_n, we can assume that $\sup_{\lambda\in K_n}|\tau_\lambda(a_n)| \leq \delta$ for all $n \in \mathbb{N}$.

This gives

$$\|E_n(a_n)\| \leq \delta \quad \text{for every } n \in \mathbb{N}. \tag{5}$$

Moreover, by Lemma 3.1, there exist $w_{1,n}, \ldots, w_{10d,n}, z_{1,n} \ldots, z_{10d,n} \in \mathcal{M}_n$ of norm bounded by 48 (since $\|a_n - E_n(a_n)\| \leq 2$) such that

$$\Big\| a_n - E_n(a_n) - \sum_{i \leq 10d} [w_{i,n}, z_{i,n}] \Big\|_{2,K_n} < \frac{1}{n} \quad \text{for every } n \in \mathbb{N}.$$

The elements $c = (E_n(a_n))_{n\in\mathbb{N}}$, $w_i = (w_{i,n})_{n\in\mathbb{N}}$ and $z_i = (z_{i,n})_{n\in\mathbb{N}}$ satisfy the equality in (4), and $\|c\| \leq \delta$ by (5) as desired. □

4. A non-factorial ultraproduct

The example we provide for Theorem 1.4 is an ultraproduct of homogeneous C^*-algebras. The fact that unital homogeneous C^*-algebras can be endowed with a structure of W^*-bundle over their spectrum is proved in Proposition 4.1, and it is a direct consequence of the Dauns–Hoffman Theorem [28, Thm. A.34].

A C^*-algebra is *n-homogeneous* if every irreducible representation has dimension n. A C^*-algebra is *homogeneous* if it is n-homogeneous for some $n \in \mathbb{N}$. Homogeneous C^*-algebras have continuous trace [5, Prop. IV.1.4.14], and in the unital case, their spectrum, namely the set of all irreducible representations up to unitary equivalence, is compact and Hausdorff with the Jacobson topology. For a C^*-algebra A, let $\hat{A}$ denote its spectrum. When A has continuous-trace, the spectrum $\hat{A}$ is homeomorphic to the *primitive ideal space* of A. In what follows, we shall thus identify every $t \in \hat{A}$ with the corresponding primitive ideal on A. Given $a \in A$, we denote by $a(t)$ the class of a in the quotient A/t.

Proposition 4.1. *Let A be a unital n-homogeneous C^*-algebra. Then A is a factorial W^*-bundle over $\hat{A}$ with conditional expectation $E : A \to C(\hat{A})$ defined as $E(a)(t) = \mathrm{tr}_n(a(t))$, where tr_n is the normalized trace on $n \times n$ matrices.*

Proof. Since A has continuous trace and $A/t \cong M_n$ for every $t \in \hat{A}$, the function $\hat{a}$ mapping $t \in \hat{A}$ to $\mathrm{tr}_n(a(t))$ is continuous over $\hat{A}$ for every $a \in A$. By the Dauns–Hoffman Theorem [28, Thm. A.34], there exists an isomorphism θ of $C(\hat{A})$ onto the center $Z(A)$ of A such that

$$(\theta(f)a)(t) = f(t)a(t) \quad \text{for every } t \in \hat{A},\ f \in C(\hat{A}),\ a \in A. \tag{6}$$

As a consequence of these facts, the map

$$E : A \to C(\hat{A}),$$
$$a \mapsto \hat{a}$$

is a tracial conditional expectation of A onto $Z(A)$ (up to θ). The map E is moreover faithful since tr_n is faithful and, for every $a \in A_+$, there exists some $t \in \hat{A}$ such that $a(t) > 0$.

We claim next that every extremal trace on A is of the form $\mathrm{ev}_t \circ E$ for some $t \in \hat{A}$. To see this, fix $\tau \in \partial_e T(A)$, and let π_τ be the corresponding GNS-representation. The center $Z(A)$ is mapped by π_τ into $Z(\pi_\tau(A)'')$ which, as τ is extremal, only consists of scalars. The restriction of π_τ to $\theta(C(\hat{A}))$ is hence a point evaluation, meaning that there is $s \in \hat{A}$ such that $\pi_\tau(\theta(f)) = 0$ whenever $f \in C(\hat{A})$ verifies $f(s) = 0$. This fact can be used to show that π_τ factors through the quotient map $A \to A/s$. Indeed, given $a \in A$ such that $a(s) = 0$ and $\varepsilon > 0$, we can find an open neighborhood U of s in $\hat{A}$ such that $\|a(t)\| < \varepsilon$ for every $t \in U$ (see [28, Lem. 5.2.b]). Let next $g \in C(\hat{A})$ of norm 1 be such that $g(s) = 0$ and $g \restriction \hat{A} \setminus U \equiv 1$. It follows that

$$\|a - \theta(g)a\| \overset{(6)}{=} \sup_{t \in \hat{A}} \|a(t) - g(t)a(t)\| < 2\varepsilon.$$

Since $g(s) = 0$, it follows that $\theta(g)a \in \ker \pi_\tau$. We have thus showed that a can be approximated with elements in $\ker \pi_\tau$, so $\pi_\tau(a) = 0$, and both π_τ and τ factor through A/s. Since the latter admits a unique trace, we conclude that $\tau(a) = E(a)(s)$ for every $a \in A$.

In order to conclude that A is a W^*-bundle, we need to prove that the unit ball of A is complete with respect to the 2-norm induced by E. Note that, since every extremal trace on A is captured by E, then the 2-norm induced by E is equal to $\|\cdot\|_{2,T(A)}$. As A is n-homogeneous, the C^*-norm $\|\cdot\|$ and $\|\cdot\|_{2,T(A)}$ are equivalent; in fact, we have

$$n^{-1/2}\|a\| \leq \|a\|_{2,T(A)} \leq \|a\| \quad \text{for every } a \in A.$$

The inequality $\|a\|_{2,T(A)} \leq \|a\|$ is always verified. For the other inequality, as $A/t \cong M_n$ for every $t \in \hat{A}$, we have

$$\|a(t)\| \leq n^{1/2} \operatorname{tr}_n(a^*a(t))^{1/2} = n^{1/2}\widehat{a^*a}(t)^{1/2} \quad \text{for every } a \in A.$$

We thus conclude that, for every $a \in A$,

$$\|a\| = \sup_{t \in \hat{A}} \|a(t)\| \leq n^{1/2}\|E(a^*a)\|^{1/2} = n^{1/2}\|a\|_{2,T(A)}.$$

This implies that A is $\|\cdot\|_{2,T(A)}$-complete.

Finally, we verify the third condition of Proposition 2.3 to prove that A is factorial. Every trace τ of the form $\mathrm{ev}_t \circ E$, for $t \in \hat{A}$, annihilates on the primitive ideal t. Further, A/t is simple; hence $\pi_\tau(A) \cong A/t \cong M_n$, which in turn implies that $\pi_\tau(A)'' \cong M_n$ is a factor and thus that the trace τ is extremal. □

Using the arguments in the previous proof, one can furthermore deduce that the map from $\hat{A}$ to $\partial_e T(A)$ sending $t \mapsto \mathrm{ev}_t \circ E$ is a homeomorphism.

The sequence of factorial W^*-bundles that we consider to prove Theorem 1.4 goes back to [27]. We briefly recall their definition here, and we refer to the discussion preceding [27, Lem. 3.5] and to [4, Sec. 2] for all the missing details.

Given $n \in \mathbb{N}$, let A_n be the C^*-algebra of continuous sections of the following vector bundle over the n-dimensional complex projective space $\mathbb{C}P^n$:

$$B_n = \left\{ \left(x, \begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{pmatrix} \right) \;\middle|\; a, d \in \mathbb{C},\, \mathbf{b} \in x,\, \bar{\mathbf{c}} \in x \right\},$$

where $\bar{\mathbf{c}} = \overline{(c_1, \dots, c_{n+1})} = (\bar{c}_1, \dots, \bar{c}_{n+1})$, with multiplication and adjoint defined pointwise as

$$\begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{pmatrix} \begin{pmatrix} a' & \mathbf{b}' \\ \mathbf{c}' & d' \end{pmatrix} = \begin{pmatrix} aa' + \mathbf{b} \cdot \mathbf{c}' & a\mathbf{b}' + d'\mathbf{b} \\ a'\mathbf{c} + d\mathbf{c}' & dd' + \mathbf{b}' \cdot \mathbf{c} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & \bar{\mathbf{c}} \\ \bar{\mathbf{b}} & \bar{d} \end{pmatrix}.$$

All fibers of this bundle are isomorphic to M_2 (see *e.g.* [27, p. 201]); hence each A_n is a 2-homogeneous C^*-algebra.

We recall that the C^*-norm ultraproduct of $\{A_n\}_{n\in\mathbb{N}}$ is defined as

$$\prod_{\mathcal{U}} A_n = \frac{\prod_n A_n}{\{(a_n)_{n\in\mathbb{N}} \in \prod_n A_n \mid \lim_{n\to\mathcal{U}} \|a_n\| = 0\}}.$$

As pointed out by Ozawa before stating [26, Thm. 8], the C^*-norm ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} A_n$ is an example where the set of limit traces is not weak*-dense in $T(\mathcal{M})$. The reason for this is the existence, for every $n \in \mathbb{N}$, of nonzero projections $p_n, q_n \in A_n$ such that $p_n - q_n$ can be approximated by a finite sum of commutators of the form $a^*a - aa^*$, but it cannot be approximated by sums of less than $n+1$ such commutators (see [27, Lem. 3.5] or [4, Thm. 2.1]). As a consequence, the projections $p = (p_n)_{n\in\mathbb{N}}$ and $q = (q_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ are such that $p - q$ is evaluated as zero on every limit trace, but on the other hand, it cannot be approximated by a finite sum of commutators in $\mathcal{M}$. That is saying that $p - q$ does not belong to the *Cuntz–Pedersen nullset* $\mathcal{M}_0$ of $\mathcal{M}$ (see [11] and also § 5.8); hence, by [11, Prop. 2.7], the weak*-closure of the set of limit traces does not exhaust $T(\mathcal{M})$.

This set-up, combined with some elementary arguments, provides an answer to Question 1.2, even in the setting of W^*-bundles.

Corollary 4.2. *There exists a sequence of factorial W^*-bundles whose ultraproduct is not factorial.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be the sequence of C^*-algebras discussed above. Since A_n is unital and 2-homogeneous for every $n \in \mathbb{N}$, by Proposition 4.1, each A_n can be naturally endowed with a structure of factorial W^*-bundle over $\partial_e T(A_n) \cong \hat{A}_n \cong \mathbb{C}P^n$. The tracial ultraproduct $\prod^{\mathcal{U}} A_n$ is a W^*-bundle over the space $K = \sum^{\mathcal{U}} \hat{A}_n$ by Proposition 2.6. Every A_n is 2-homogeneous; hence the quotient map from the C^*-norm ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} A_n$ onto the tracial ultraproduct $\prod^{\mathcal{U}} A_n$ is an isomorphism. Indeed, the kernel of the quotient map is

$$\left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{\mathcal{U}} A_n \;\middle|\; \lim_{n\to\mathcal{U}} \|a_n\|_{2,T(A_n)} = 0 \right\},$$

which in this case is equal to the set of those $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\mathcal{U}}\|a_n\| = 0$, since, for every n, arguing as in the proof of Proposition 4.1, we have

$$\|a\|_{2,T(A_n)} \leq \|a\| \leq 2^{1/2}\|a\|_{2,T(A_n)} \quad \text{for every } a \in A_n.$$

This also entails that the C^*-norm and $\|\cdot\|_{2,K} = \lim_{n\to\mathcal{U}}\|\cdot\|_{2,T(A_n)}$ on $\mathcal{M}$ are equivalent; in particular,

$$\|a\|_{2,K} \leq \|a\| \leq 2^{1/2}\|a\|_{2,K} \quad \text{for every } a \in \mathcal{M}. \tag{7}$$

Identify K with a subspace of $T(\mathcal{M})$, and let X be its closed convex hull, namely the weak*-closure of the set of all limit traces on $\mathcal{M}$ (the unconvinced reader may take a look at the discussion leading to equation (11)). To show that $(\mathcal{M}, X)$ is not factorial, we argue by contradiction and suppose that X is a closed face of $T(\mathcal{M})$. Let $\sigma \in T(\mathcal{M}) \setminus X$, whose existence is guaranteed by the discussion preceding the statement of the corollary. By [1, Cor. II.5.20] and [20, Lem. 6.2], there exists $a \in \mathcal{M}_+$ such that
(i) $\tau(a) < 1/4$ for every $\tau \in X$,
(ii) $\sigma(a) > 1/2$.
Then $\|a^{1/2}\|_{2,X}^2 < 1/4$; thus, by (7), it follows that $\|a^{1/2}\|^2 < 1/2$, which is a contradiction since, by item (ii) above,

$$\|a^{1/2}\|^2 = \|a\| \geq \sigma(a) > 1/2. \qquad \square$$

5. Center of the ultraproduct and consequences in model theory

We start this section with a preliminary result (Theorem 5.7), a generalization of [17, Cor. 4.3], where we show that the center of the ultraproduct of a family of factorial W^*-bundles is the ultraproduct of the centers. We then proceed to prove Theorem 1.5.

5.1. Center of the ultraproduct.

Theorem 5.7 is a consequence Proposition 5.5, stating that factorial W^*-bundles verify a tracial analog of the *strong Dixmier's averaging property* [5, Def. III.2.5.16]. This fact is neither explicitly stated nor proved in [26], but it follows from arguments analog to those appearing in that paper and was known to the author, who uses it to show [26, Thm. 15]. We give a full proof for the reader's convenience, starting with some preliminary lemmas. The argument for the first one is due to the authors of [8].

Lemma 5.2 ([8]). *Let A be a unital C^*-algebra and let $X \subseteq T(A)$ be nonempty. Set $\pi = \bigoplus_{\tau\in X}\pi_\tau$ and let $\sigma \in T(\pi(A)'')$. Then the trace $\sigma \circ \pi$ on A belongs to the closed face generated by X in $T(A)$.*

Proof. Let F be the closed face generated by X in $T(A)$. It is sufficient to prove the lemma for normal traces since F is closed and the set of normal traces is weak*-dense in $T(\pi(A)'')$. Let then $\sigma \in T(\pi(A)'')$ be normal and suppose that $\sigma \circ \pi \notin F$. By [1, Cor. II.5.20], there exists a continuous affine function

$f : T(A) \to [0,1]$ such that $f(\sigma \circ \pi) = 1$ and $f \restriction F \equiv 0$. By [20, Lem. 6.2], there is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A_+$ such that

$$\lim_{n\to\infty} \sup_{\tau\in T(A)} |\tau(a_n) - f(\tau)| = 0.$$

This means that $\tau(a_n) \to 0$ for every $\tau \in F$ which, as $a_n \geq 0$ for every $n \in \mathbb{N}$, implies that $\pi(a_n)$ converges to zero in the strong topology. The trace σ is normal; hence $\sigma(\pi(a_n))$ also converges to zero, which contradicts the fact

$$\lim_{n\to\infty} \sigma(\pi(a_n)) = f(\sigma \circ \pi) = 1. \qquad \square$$

Lemma 5.3. *Let A be a unital C^*-algebra and let $X \subseteq T(A)$ be a nonempty closed face. Let $\pi = \bigoplus_{\tau\in X} \pi_\tau$ and let* $\mathrm{ctr} : \pi(A)'' \to Z(\pi(A)'')$ *be the center-valued trace on $\pi(A)''$. Then*

$$\|a\|_{2,X} = \|\mathrm{ctr}(\pi(a^*a))\|^{1/2} \quad \textit{for all } a \in A.$$

Proof. By [5, Thm. III.2.5.7], the map from the state space of $Z(\pi(A)'')$ to $T(\pi(A)'')$ sending φ to $\varphi \circ \mathrm{ctr}$ is a bijection; hence

$$\|\mathrm{ctr}(b^*b)\|^{1/2} = \|b\|_{2,T(\pi(A)'')} \quad \text{for every } b \in \pi(A)''.$$

The conclusion of the lemma with $b = \pi(a)$ for $a \in A$ follows by Lemma 5.2. $\square$

Lemma 5.4. *Let $\mathcal{M}$ be a factorial W^*-bundle over K with conditional expectation $E : \mathcal{M} \to C(K)$. Let $\pi = \bigoplus_{\mu\in\mathrm{Prob}(K)} \pi_{\tau_\mu}$, let $\mathcal{N} = \pi(\mathcal{M})''$, and denote by* ctr *the center-valued trace of $\mathcal{N}$. Then, for every $a \in \mathcal{M}$,*

$$\pi(E(a)) = \mathrm{ctr}(\pi(a)).$$

Proof. Suppose $\pi(E(a)) \neq \mathrm{ctr}(\pi(a))$ for some $a \in \mathcal{M}$. By [5, Thm. III.2.5.7], there exists $\tau \in T(\mathcal{N})$ such that $\tau(\pi(E(a))) \neq \tau(\mathrm{ctr}(\pi(a)))$. As $\mathcal{M}$ is factorial, by Lemma 5.2, there is $\mu \in \mathrm{Prob}(K)$ such that $\tau \circ \pi = \tau_\mu$. In particular, $\tau_\mu \circ E = \tau_\mu$; hence, on the one hand, we have

$$\tau(\pi(E(a))) = \tau_\mu(E(a)) = \tau_\mu(a).$$

On the other hand,

$$\tau(\mathrm{ctr}(\pi(a))) = \tau(\pi(a)) = \tau_\mu(a),$$

which is a contradiction. $\square$

Proposition 5.5. *Let $\mathcal{M}$ be a factorial W^*-bundle over K with conditional expectation $E : \mathcal{M} \to C(K)$. For every $a \in \mathcal{M}$ and $\varepsilon > 0$, there are unitaries $u_1, \ldots, u_k \in \mathcal{M}$ such that*

$$\Big\|E(a) - \frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*\Big\|_{2,K} < \varepsilon.$$

Proof. The proof follows closely the one of [26, Thm. 3]. Fix $a \in \mathcal{M}$ and $\varepsilon > 0$. Let π be the direct sum $\bigoplus_{\mu \in \mathrm{Prob}(K)} \pi_{\tau_\mu}$ and set $\mathcal{N} = \pi(\mathcal{M})''$. Denote by $\mathrm{ctr} : \mathcal{N} \to Z(\mathcal{N})$ the center-valued trace of $\mathcal{N}$. By the Dixmier Averaging Theorem [5, Thm. III.2.5.19] and Lemma 5.4, the element $\pi(E(a))$ belongs to the norm closure of the convex hull of $\{u\pi(a)u^* \mid u \in \mathcal{U}(\mathcal{N})\}$.

Let $C = \{\pi(uau^*) \mid u \in \mathcal{U}(\mathcal{M})\}$. By the Kaplansky Density Theorem, every $u \in \mathcal{U}(\mathcal{N})$ is a strong limit of a net of unitaries from $\pi(\mathcal{M})$. Since the adjoint operation is strongly continuous on normal operators [25, Thm. 4.3.1] and multiplication is strongly continuous on bounded sets [25, Rem. 4.3.1], there exists a net $\{b_\lambda\}_\lambda$, where every b_λ belongs to convex hull of C, which converges to $b = \pi(E(a))$ in the strong topology. This in turn entails that the net $\{(b_\lambda - b)^*(b_\lambda - b)\}_\lambda$ converges to 0 in the weak operator topology and thus, being a bounded net, in the ultraweak topology. Since the center-valued trace ctr is ultraweakly continuous, this in particular implies

$$\varphi\big(\mathrm{ctr}((b_\lambda - b)^*(b_\lambda - b))\big) \to 0 \quad \text{for every } \varphi \in Z(\mathcal{N})_*, \tag{8}$$

where $Z(\mathcal{N})_*$ denotes the set of normal functionals on $Z(\mathcal{N})$.

We claim that (8) implies

$$\varphi\big(\mathrm{ctr}((b_\lambda - b)^*(b_\lambda - b))\big) \to 0 \quad \text{for every } \varphi \in Z(\mathcal{N})^*. \tag{9}$$

Indeed, let $\varphi \in Z(\mathcal{N})^*$. Then $\varphi \circ \mathrm{ctr} \in T(\mathcal{N})$; hence, by Lemma 5.2, there is $\mu \in \mathrm{prob}(K)$ such that $\varphi \circ \mathrm{ctr} \restriction \pi(\mathcal{M}) = \tau_\mu$. The latter extends to a normal trace on $\mathcal{N}$; hence there is $\varphi' \in Z(\mathcal{N})_*$ such that $\varphi \circ \mathrm{ctr} \restriction \pi(\mathcal{M}) = \varphi' \circ \mathrm{ctr} \restriction \pi(\mathcal{M})$. We conclude that φ and φ' are equal on $\{\mathrm{ctr}((b_\lambda - b)^*(b_\lambda - b))\}_\lambda$ by Lemma 5.4, and therefore (9) follows.

By the Hahn–Banach Theorem, there are thus finitely many $\alpha_j > 0$ with $\sum_j \alpha_j = 1$ such that

$$\Big\|\sum_j \alpha_j \,\mathrm{ctr}((b_{\lambda_j} - b)(b_{\lambda_j} - b)^*)\Big\| < \varepsilon. \tag{10}$$

Set $c = \sum_j \alpha_j b_{\lambda_j}$ and note that

$$c = [\alpha_1^{1/2} \ \dots \ \alpha_m^{1/2}] \begin{bmatrix} \alpha_1^{1/2} b_{\lambda_1} \\ \vdots \\ \alpha_m^{1/2} b_{\lambda_m} \end{bmatrix} =: rs.$$

Hence $c^*c \le s^*r^*rs \le \|r\|^2 s^*s = \sum_j \alpha_j b_{\lambda_j}^* b_{\lambda_j}$, which in turn gives

$$\begin{aligned} \mathrm{ctr}((c-b)^*(c-b)) &= \mathrm{ctr}(c^*c - b^*c - c^*b + b^*b) \\ &\le \mathrm{ctr}\Big(\sum_j \alpha_j b_{\lambda_j}^* b_{\lambda_j} - b^* \sum_j \alpha_j b_{\lambda_j} - \sum_j \alpha_j b_{\lambda_j}^* b + b^*b\Big) \\ &= \mathrm{ctr}\Big(\sum_j \alpha_j (b_{\lambda_j} - b)^*(b_{\lambda_j} - b)\Big). \end{aligned}$$

As a consequence, by (10), we have $\|\mathrm{ctr}((c-b)^*(c-b))\| < \varepsilon$. Summarizing, there are unitaries $u_1, \dots, u_\ell \in \mathcal{M}$ and $\beta_i \ge 0$ with $\sum_{i \le \ell} \beta_i = 1$ such that

$c = \pi(\sum_{i\leq\ell} \beta_i u_i a u_i^*)$, and by Lemma 5.3 and the inequalities above, it follows that

$$\Big\|E(a) - \sum_{i\leq\ell} \beta_i u_i a u_i^*\Big\|_{2,K}^2 = \|\mathrm{ctr}((b-c)^*(b-c))\| < \varepsilon. \qquad \square$$

We need one last lemma before showing Theorem 5.7 (see also [17, Lem. 4.2]).

Lemma 5.6. *Let $\mathcal{M}$ be a factorial W^*-bundle over K with conditional expectation $E : \mathcal{M} \to C(K)$. Then, for every $a \in \mathcal{M}$, the following holds:*

$$\|a - E(a)\|_{2,K} \leq \sup_{b\in\mathcal{M}_1} \|[a,b]\|_{2,K} \leq 2\|a - E(a)\|_{2,K}.$$

Proof. Let $a \in \mathcal{M}$ and $b \in \mathcal{M}_1$. The right-hand side inequality in the statement follows from the computation below:

$$\begin{aligned}\|ab - ba\|_{2,K} &\leq \|ab - E(a)b\|_{2,K} + \|bE(a) - ba\|_{2,K}\\ &\leq \|a - E(a)\|_{2,K}\|b\| + \|b\|\|E(a) - a\|_{2,K}\\ &\leq 2\|E(a) - a\|_{2,K}.\end{aligned}$$

For the other inequality, given $\varepsilon > 0$, by Proposition 5.5, there exist unitaries $u_1, \ldots, u_k \in \mathcal{M}$ such that

$$\Big\|E(a) - \frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*\Big\|_{2,K} < \varepsilon.$$

We thus have

$$\begin{aligned}\|a - E(a)\|_{2,K} &< \Big\|a - \frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*\Big\|_{2,K} + \varepsilon\\ &\leq \frac{1}{k}\sum_{i=1}^{k}\|au_i - u_i a\|_{2,K} + \varepsilon\\ &\leq \sup_{b\in\mathcal{M}_1}\|[a,b]\|_{2,K} + \varepsilon. \qquad \square\end{aligned}$$

Theorem 5.7. *Let $(\mathcal{M}_i)_{i\in I}$ be a sequence of factorial W^*-bundles. Then*

$$Z\Big(\prod^{\mathcal{U}} \mathcal{M}_i\Big) = \prod^{\mathcal{U}} Z(\mathcal{M}_i).$$

Proof. The inclusion $\prod^{\mathcal{U}} Z(\mathcal{M}_i) \subseteq Z(\prod^{\mathcal{U}} \mathcal{M}_i)$ always holds. For the reverse inclusion, let $(a_i)_{i\in I} \in Z(\prod^{\mathcal{U}} \mathcal{M}_i)$. It suffices to show that $(a_i)_{i\in I} = (E_i(a_i))_{i\in I}$. For every $i \in I$, let $b_i \in (\mathcal{M}_i)_1$ be such that

$$2\|[a_i, b_i]\|_{2,K_i} \geq \sup_{c\in(\mathcal{M}_i)_1} \|[a_i, c]\|_{2,K_i}.$$

Lemma 5.6 gives

$$\|a_i - E_i(a_i)\|_{2,K_i} \leq \sup_{c\in(\mathcal{M}_i)_1} \|[a_i, c]\|_{2,K_i} \leq 2\|[a_i, b_i]\|_{2,K_i} \quad \text{for every } i \in I.$$

The right-most term goes to zero as $i \to \mathcal{U}$ since $(a_i)_{i\in I} \in Z(\prod^{\mathcal{U}} \mathcal{M}_i)$; hence $(a_i)_{i\in I} = (E_i(a_i))_{i\in I}$. $\square$

5.8. **Consequences in model theory.** Before proving Theorem 1.5, we set up a model-theoretic framework suitable for tracially complete C^*-algebras.

We refer to [16] for all the necessary background concerning continuous model theory of C^*-algebras (see also [3] for a more general approach beyond the context of operator algebras). In [16], unital C^*-algebras are presented as multi-sorted structures in the language $\mathcal{L}_{C^*} = \{\|\cdot\|, +, \cdot, ^*, \{z\}_{z\in\mathbb{C}}, 0, 1\}$ whose sorts, interpreted as the closed balls, represent the domains of quantification. This language is not handy for studying tracially complete C^*-algebras, whose behavior is closer to that of tracial von Neumann algebras (see *e.g.* [18]).

Anticipating the model-theoretic analysis of tracially complete C^*-algebras which will be presented in [15], let $\mathcal{L}_2 = \{\|\cdot\|_2, +, \cdot, ^*, \{z\}_{z\in\mathbb{C}}, 0, 1\}$ be a language with a single sort and countably many domains D_k, with two constant symbols 0 and 1, symbols for the algebraic operations $+$, $\cdot$, and *, a symbol z for each $z \in \mathbb{C}$ and a symbol for the tracial norm $\|\cdot\|_2$. The moduli of continuity assigned to the algebraic operations are chosen in the natural fashion. This is similar to the language considered in [18] for von Neumann algebras with the exception of the predicate tr, interpreted on tracial von Neumann algebras as the trace.

A tracially complete C^*-algebra $(\mathcal{M}, X)$ is an $\mathcal{L}_2$-structure with the symbol $\|\cdot\|_2$ interpreted as $\|\cdot\|_{2,X}$, the operation symbols interpreted in the obvious way, and D_k interpreted as the k-ball in the operator norm.

Given a sequence $((\mathcal{M}_i, X_i) \mid i \in I)$ of tracially complete C^*-algebras, it is possible to check that the tracial ultraproduct $\prod^{\mathcal{U}} \mathcal{M}_i$ introduced in § 2.5 corresponds to the standard ultraproduct of metric structures (see [3, Sec. 5]) obtained when considering each $(\mathcal{M}_i, X_i)$ as an $\mathcal{L}_2$-structure. In fact, the ultraproduct of the norms $\|\cdot\|_{2,X_i}$ can be verified to be precisely $\|\cdot\|_{2,\sum^{\mathcal{U}} X_i}$, that is,

$$\|(a_i)_{i\in I}\|_{2,\sum^{\mathcal{U}} X_i} = \lim_{i\to\mathcal{U}} \|a_i\|_{2,X_i} \quad \text{for every } (a_i)_{i\in I} \in \prod^{\mathcal{U}} \mathcal{M}_i.$$

This implies in particular that the unit ball of $\prod^{\mathcal{U}} \mathcal{M}_i$ is $\|\cdot\|_{2,\sum^{\mathcal{U}} X_i}$-complete (see [3, Prop. 5.3]), so it automatically follows that

$$\Big(\prod^{\mathcal{U}} \mathcal{M}_i, \sum^{\mathcal{U}} X_i\Big)$$

is a tracially complete C^*-algebra. Through this section, we stress the fact that we consider $\prod^{\mathcal{U}} \mathcal{M}_i$ as an $\mathcal{L}_2$-structure with the 2-norm induced by $\sum^{\mathcal{U}} X_i$ by saying that the ultraproduct of the sequence $((\mathcal{M}_i, X_i) \mid i \in I)$ is the pair $(\prod^{\mathcal{U}} \mathcal{M}_i, \sum^{\mathcal{U}} X_i)$.

In case the sequence $((\mathcal{M}_i, X_i) \mid i \in I)$ is composed of W^*-bundles, with $X_i = \mathrm{Prob}(K_i)$, then the W^*-bundle structure induced on $\prod^{\mathcal{U}} \mathcal{M}_i$ over $K = \sum^{\mathcal{U}} K_i$ as in Proposition 2.6 makes the pair $(\prod^{\mathcal{U}} \mathcal{M}_i, \mathrm{Prob}(K))$ a tracially complete C^*-algebra (see the discussion preceding Proposition 2.3). On the other hand, the ultraproduct of $((\mathcal{M}_i, X_i) \mid i \in I)$ as $\mathcal{L}_2$-structures is $(\prod^{\mathcal{U}} \mathcal{M}_i, \sum^{\mathcal{U}} X_i)$.

These are two presentations of the same object; in fact,

$$\sum^{\mathcal{U}} X_i = \mathrm{Prob}(K).$$

Indeed, given $(a_i)_{i\in I} \in \prod^{\mathcal{U}} \mathcal{M}_i$, we have

$$\begin{aligned}(11)\qquad \sup_{\tau\in\sum^{\mathcal{U}} X_i} |\tau((a_i)_{i\in I})| &= \lim_{i\to\mathcal{U}} \sup_{\tau\in X_i} |\tau(a_i)| = \lim_{i\to\mathcal{U}} \sup_{\tau\in K_i} |\tau(a_i)| \\ &= \sup_{\tau\in K} |\tau((a_i)_{i\in I})| = \sup_{\tau\in\mathrm{Prob}(K)} |\tau((a_i)_{i\in I})|.\end{aligned}$$

An application of the Hahn–Banach Theorem as in [10, Lem. 4.4] then yields $\sum^{\mathcal{U}} X_i = \mathrm{Prob}(K)$ since they are both convex and closed. In particular, in this case, $\partial_e(\sum^{\mathcal{U}} X_i) = \sum^{\mathcal{U}} \partial_e X_i$.

Given this model-theoretic set-up, a direct application of Łoś's Theorem gives Theorem 1.3 for ultrapowers over sets of indices different from $\mathbb{N}$.

Proof of Theorem 1.3. Let $(\mathcal{M}_i)_{i\in I}$ be a sequence of factorial W^*-bundles over K_i, and suppose that there is $d \in \mathbb{N}$ such that $\dim(K_i) \le d$ for every $i \in I$. Then $\mathcal{M} = \prod^{\mathcal{U}} \mathcal{M}_i$ is a W^*-bundle over $K = \sum^{\mathcal{U}} K_i$, and as in the proof of Theorem 3.2, it is sufficient to show that, for every $a \in \mathcal{M}_{\mathrm{sa}}$,

$$(12)\qquad \sup_{\lambda\in K} |\tau_\lambda(a)| = \sup_{\tau\in T(\mathcal{M})} |\tau(a)|.$$

Consider the formula

$$\varphi(x,y) = \inf_{\substack{w_1,\dots,w_{10d}\\ z_1,\dots,z_{10d}}} \Big\| x - y - 48 \sum_{i\le 10d} [w_i, z_i] \Big\|_2,$$

where the inf ranges over the sort corresponding to the unit ball. Fix a contraction $a = (a_i)_{i\in I} \in \mathcal{M}$. Lemma 3.1 entails that $\varphi^{(\mathcal{M}_i,\mathrm{Prob}(K_i))}(a_i, E_i(a_i)) = 0$ for every $i \in I$. Then, by Łoś's Theorem [3, Thm. 5.4], we also have

$$\varphi^{(\mathcal{M},\mathrm{Prob}(K))}(a, E^{\mathcal{U}}(a)) = 0.$$

In case $\sup_{\lambda\in K} |\tau_\lambda(a)| \le \delta$ for some $\delta \ge 0$, then $\|E^{\mathcal{U}}(a)\| \le \delta$; hence, arguing as in the proof of Theorem 3.2, it follows that

$$\sup_{\tau\in T(\mathcal{M})} |\tau(a)| \le \delta.$$

The equality in (12) follows since δ was chosen arbitrarily. □

The C^*-algebras A and B considered in the statement of Theorem 1.5 are assumed to be elementarily equivalent as $\mathcal{L}_{C^*}$-structures. In order to use the tools developed in the previous sections, we would like to be able to compare their tracial completions, as defined in [26].

Definition 5.9. Given a unital C^*-algebra A with nonempty trace space $T(A)$, its *tracial completion* is the C^*-algebra

$$\overline{A}^{T(A)} = \frac{\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid (a_n)_{n\in\mathbb{N}} \text{ is a } \|\cdot\|_{2,T(A)}\text{-Cauchy sequence}\}}{\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid \lim_{n\to\infty} \|a_n\|_{2,T(A)} = 0\}}.$$

Every trace $\tau \in T(A)$ canonically extends to a trace $\bar{\tau}$ on $\overline{A}^{T(A)}$, defined on each representing Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ as

$$\bar{\tau}((a_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} \tau(a_n).$$

We can thus identify $T(A)$ with a subset of the trace space of $\overline{A}^{T(A)}$, and it is immediate to check that $(\overline{A}^{T(A)}, T(A))$ is a factorial tracially complete C^*-algebra.

Lemma 5.10 below is a straight-forward generalization of [16, Prop. 3.5.1], and it shows that the first-order $\mathcal{L}_{C^*}$-theory of a C^*-algebra completely determines the $\mathcal{L}_2$-theory of its tracial completion for classes where the Cuntz–Pedersen nullset is definable, in the sense of [16, Chap. 3]. Given a C^*-algebra A, its *Cuntz–Pedersen nullset* A_0, introduced in [11], is the norm-closure of the linear span of the set of selfadjoint commutators $[a, a^*]$. Theorem 2.9 in [11] tightly relates the 2-norm $\|\cdot\|_{2,T(A)}$ of an element with its distance from A_0; more precisely, it shows that

$$\|a\|^2_{2,T(A)} = \sup_{\tau\in T(A)} \tau(a^*a) = d(a^*a, A_0) \quad \text{for all } a \in A. \tag{13}$$

In [16, Thm. 3.5.5], various classes where the Cuntz–Pedersen nullset is $\mathcal{L}_{C^*}$-definable are listed. Examples are the set of exact $\mathcal{Z}$-stable C^*-algebras, and the collection of C^*-algebras with strict comparison, to which the following lemma and Theorem 1.5 apply.

Lemma 5.10. *Let A and B be C^*-algebras belonging to a class where the Cuntz–Pedersen nullset is definable. If $A \equiv B$ as $\mathcal{L}_{C^*}$-structures, then*

$$(\overline{A}^{T(A)}, T(A)) \equiv (\overline{B}^{T(B)}, T(B))$$

as $\mathcal{L}_2$-structures.

Proof. This fact is a consequence of [16, Prop. 3.5.1] in case both $T(A)$ and $T(B)$ are singletons. More generally, given a C^*-algebra A, definability of A_0 implies that the norm $\|\cdot\|_{2,T(A)}$ is also a definable predicate, by [16, Thm. 3.2.2] and (13).

It follows that if A is a C^*-algebra belonging to a class where the Cuntz–Pedersen nullset is definable, then all interpretations of $\mathcal{L}_2$-formulas on A are $\mathcal{L}_{C^*}$-definable predicates. This can be proved arguing by induction on the complexity of formulas, with the atomic case being covered since $\|\cdot\|_{2,T(A)}$ is a definable predicate and the quantifier case since A_1 is dense in $(\overline{A}^{T(A)})_1$. □

The following lemma shows that isomorphisms between tracially complete C^*-algebras preserving the 2-norm also induce affine homeomorphisms between the sets of traces inducing the 2-norms. The argument, an application of the Hahn–Banach Theorem, is due to the authors of [8].

Lemma 5.11 ([8]). *Let $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ be two tracially complete C^*-algebra and let $\theta : \mathcal{M} \to \mathcal{N}$ be an isomorphism such that $\|a\|_{2,X} = \|\theta(a)\|_{2,Y}$ for every $a \in \mathcal{M}$. Then $\theta^*(Y) = X$.*

Proof. Suppose that $\theta^*(Y) \neq X$ and that there is $\tau \in X \setminus \theta^*(Y)$ (the case where $\tau \in \theta^*(Y) \setminus X$ is analog). As $\theta^*(Y)$ is closed and convex, by the Hahn–Banach Theorem, there exist an affine, positive, continuous function f on $T(\mathcal{M})$ and $\alpha \in \mathbb{R}$ such that $f(\tau) > \alpha$ and $f(\sigma) < \alpha$ for every $\sigma \in \theta^*(Y)$. By [20, Lem. 6.2], we can approximate f arbitrarily well with evaluations on positive elements of $\mathcal{M}$; hence there exists $a \in \mathcal{M}_+$ such that

$$\sup_{\sigma \in \theta^*(Y)} \sigma(a) < \tau(a).$$

This gives $\|\theta(a^{1/2})\|_{2,Y} < \|a^{1/2}\|_{2,X}$, which contradicts the assumption on θ. □

In the next proof, we repeatedly use the fact, a consequence of a standard density argument, that the tracial ultraproduct $A^{\mathcal{U}}$ of a unital C^*-algebra A is equal to the ultrapower of its tracial completion $(\overline{A}^{T(A)})^{\mathcal{U}}_{T(A)}$.

Theorem 5.12. *Let A, B be two unital C^*-algebras belonging to any class where the Cuntz–Pedersen nullset is definable. Suppose that $A \equiv B$ (as $\mathcal{L}_{C^*}$-structures). Then $T(A)$ is a Bauer simplex if and only if $T(B)$ is. If moreover both $T(A)$ and $T(B)$ are Bauer simplices and second countable, then $\partial_e T(A)$ and $\partial_e T(B)$ have the same covering dimension.*

Proof. Fix A and B as in the statement. By Lemma 5.10, their tracial completions are elementarily equivalent as $\mathcal{L}_2$-structures; thus, by [3, Thm. 5.7], there are an index set I and an ultrafilter $\mathcal{U}$ over I such that

$$(A^{\mathcal{U}}, T(A)^{\mathcal{U}}) \cong (B^{\mathcal{U}}, T(B)^{\mathcal{U}})$$

with an isomorphism which is $\|\cdot\|_{2,T(A)^{\mathcal{U}}} - \|\cdot\|_{2,T(B)^{\mathcal{U}}}$-isometric. Suppose that $T(A)$ is a Bauer simplex. By Theorem 2.4 and Proposition 2.6, the pair $(A^{\mathcal{U}}, T(A)^{\mathcal{U}})$ can be endowed with a W^*-bundle structure over $(\partial_e T(A))^{\mathcal{U}} = \partial_e(T(A)^{\mathcal{U}})$, and thus $T(A)^{\mathcal{U}}$ is a Bauer simplex (see also the discussion preceding the computation in (11)). This in turn implies, by Lemma 5.11, that $T(B)^{\mathcal{U}}$ is a Bauer simplex.

Fix $\tau \in \overline{\partial_e T(B)}$. We prove that the canonical extension of τ to $B^{\mathcal{U}}$ (which we still denote τ) belongs to the boundary of $T(B)^{\mathcal{U}}$. This is sufficient to conclude that $\tau \in \partial_e T(B)$ since any nontrivial convex decomposition of τ in $T(B)$ induces a nontrivial convex decomposition of τ as an element of $T(B)^{\mathcal{U}}$. The simplex $T(B)^{\mathcal{U}}$ is Bauer; hence it is enough to prove that τ is in the closure of $\partial_e(T(B)^{\mathcal{U}})$. To this end, pick $\varepsilon > 0$ and $a_1, \dots, a_m \in B^{\mathcal{U}}$. Fix a representing sequence $(a_{j,i})_{i\in I}$ of a_j for all $j \leq m$. For every $i \in I$, let $\sigma_i \in \partial_e T(B)$ be such that

$$\max_{j\leq m}\{|\sigma_i(a_{j,i}) - \tau(a_{j,i})|\} < \varepsilon.$$

Notice that the limit trace $\sigma = (\sigma_i)_{i\in I}$ is extremal in $T(B^{\mathcal{U}})$. This is the case since, by uniqueness of the GNS representation, the identity map on $B^{\mathcal{U}}$ induces a surjective isomorphism

$$\pi_\sigma(B^{\mathcal{U}})'' \to \prod^{\mathcal{U}} \pi_{\sigma_i}(B)''$$

between the von Neumann algebra $\pi_\sigma(B^{\mathcal{U}})''$ generated by the GNS-representation corresponding to σ, and the tracial ultraproduct of those corresponding to σ_i. Each σ_i is extremal; hence every $\pi_{\sigma_i}(B)''$ is factor, and so is their ultraproduct. We conclude that $\pi_\sigma(B^{\mathcal{U}})''$ is a factor and therefore that σ is an extremal point in $T(B^{\mathcal{U}})$ approximating τ.

We rely on Theorem 5.7 to prove the second part of the statement. More in detail, the completion $\overline{A}^{T(A)}$ is a factorial W^*-bundle over $\partial_e T(A)$ by Theorem 2.4, and its center is equal to $C(\partial_e T(A))$ by factoriality. The same holds for B. The relation $A^{\mathcal{U}} \cong B^{\mathcal{U}}$ implies $Z(A^{\mathcal{U}}) \cong Z(B^{\mathcal{U}})$; hence Theorem 5.7 gives $C(\partial_e T(A))^{\mathcal{U}} \cong C(\partial_e T(B))^{\mathcal{U}}$. This in turn gives $C(\partial_e T(A)) \equiv C(\partial_e T(B))$ as $\mathcal{L}_{C^*}$-structures since, for abelian C^*-algebras, the tracial ultrapower is equal to the C^*-norm ultrapower. The covering dimension of X is equal to the decomposition rank of $C(X)$ (see [21, Prop. 3.3]; this is the only step where the fact that $T(A)$ and $T(B)$ are second countable is used), and the latter is definable by uniform families of formulas [16, Thm. 5.7.3]; therefore, the conclusion follows. □

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