

Holomorphic curves in the symplectizations of lens spaces: An elementary approach

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Abstract. We present an elementary computational scheme for the moduli spaces of rational pseudo-holomorphic curves in the symplectizations of 3-dimensional lens spaces, which are equipped with Morse–Bott contact forms induced by the standard Morse–Bott contact form on S^3 . As an application, we prove that, for p prime and $1 < q, q' < p - 1$, if there is a contactomorphism between lens spaces $L(p, q)$ and $L(p, q')$, where both spaces are equipped with their standard contact structures, then $q \equiv (q')^{\pm 1} \pmod{p}$. For the proof, we study the moduli spaces of a pair of pants with two non-contractible ends in detail and establish that the standard almost complex structure that is used is regular. Then the existence of a contactomorphism enables us to follow a neck-stretching process, by means of which we compare the homotopy relations encoded at the non-contractible ends of the pair of pants in the symplectizations of $L(p, q)$ and $L(p, q')$. Combining our proof with the result of Honda on the classification of universally tight contact structures on lens spaces, we provide a purely symplectic/contact topological proof of the diffeomorphism classification of lens spaces in the class mentioned above.

1. INTRODUCTION

The objects of the study in this paper are 3-dimensional lens spaces and the standard Reeb dynamics on them. These dynamical systems are given as finite quotients of the Hopf fibration on S^3 . To be more precise, we consider the *standard contact form*¹ on S^3 given by

$$(1) \quad \alpha_0(z)[w] = \operatorname{Im}\langle w, z \rangle_{\mathbb{C}},$$

where $z = (z_1, z_2) \in S^3 \hookrightarrow \mathbb{C}^2$, $w = (w_1, w_2) \in T_z S^3 \subset T_z \mathbb{C}^2 \cong \mathbb{C}^2$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the standard hermitian product. Then the *standard contact structure* ξ_0 at $z \in S^3$ is given by the hermitian complement of the complex line along $z \in \mathbb{C}^2$. The Reeb vector field and its flow are given by

$$R_0(z_1, z_2) = (iz_1, iz_2), \quad \phi_t(z_1, z_2) = (e^{it}z_1, e^{it}z_2).$$

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¹In order to match conventions, S^3 could be replaced by the sphere of radius $\sqrt{2}$.

We see that all Reeb orbits are closed, and we have 2π as the common minimal period. The fibration via Reeb orbits leads to the Hopf fibration

$$(2) \quad \pi : S^3 \rightarrow \mathbb{C}P^1, \quad (z_1, z_2) \mapsto (z_1 : z_2).$$

We consider the \mathbb{Z}_p -action on S^3 generated by

$$\sigma(z_1, z_2) = (e^{i\theta}z_1, e^{iq\theta}z_2), \quad \theta = 2\pi/p,$$

where $0 < q < p$ are integers with $(p, q) = 1$. This is a free action and gives rise to the lens space $L(p, q)$ as the quotient space, where

$$\mathbf{p} : S^3 \rightarrow L(p, q)$$

is the quotient/covering map. We note that, since \mathbf{p} is a universal covering,

$$\pi_1(L(p, q)) \cong H_1(L(p, q), \mathbb{Z}) \cong \mathbb{Z}_p.$$

For later purposes, we fix the integer $0 < v < p$ such that

$$vq \equiv 1 \pmod{p}.$$

Since σ is a complex linear map on \mathbb{C}^2 , it preserves the standard contact form. Hence we have an induced *standard contact form* α on $L(p, q)$, given by

$$\mathbf{p}^*\alpha = \alpha_0,$$

which defines the *standard contact structure*

$$\xi = \ker \alpha = \mathbf{p}_*\xi_0.$$

We also note that $\mathbf{p}_*R_0 = R$, where R is the Reeb field of α .

Remark 1.1. (i) In this paper, we restrict ourselves to the case where $1 < q < p - 1$. In the remaining case, the Reeb flow still induces an S^1 -bundle structure, and it is relatively less interesting; see [1].

(ii) We also assume that p is prime. This assumption simplifies the computational aspect of our work, but we claim that most of the statements we present in this paper can be generalized after some extra care.

(iii) In what follows, expressions like $k \equiv l$ and $k \not\equiv l$ should be understood in mod p .

The dynamics of the Reeb field R is of Morse–Bott type and is easy to describe since σ commutes with the flow of R_0 . There are two closed orbits of R_0 in S^3 which are invariant under the group action, namely

$$\gamma_\infty = \{(z_1, 0) \mid |z_1| = 1\} = \{(e^{it}, 0) \mid t \in [0, 2\pi]\},$$

$$\gamma_0 = \{(0, z_2) \mid |z_2| = 1\} = \{(0, e^{it}) \mid t \in [0, 2\pi]\}.$$

These orbits are mapped to the points ∞ and 0 in $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$ respectively via (2), and they project to p -fold covers of simple orbits $\overline{\gamma}_\infty$ and $\overline{\gamma}_0$ in $L(p, q)$. These simple orbits have period $2\pi/p$, and they represent generators of the fundamental group. We note that the simple orbits of R_0 besides γ_0 and γ_∞ are permuted by σ . In fact, we have an induced action on the orbit space

$$(3) \quad \sigma_b : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1, \quad (z_1 : z_2) \mapsto (e^{i\theta}z_1 : e^{iq\theta}z_2),$$

which gives an orbifold structure to the orbit spaces of R via the quotient map

$$p_b : \mathbb{C}P^1 \rightarrow \overline{\mathbb{C}P^1}.$$

The fixed points 0 and ∞ correspond to the (iterations of) orbits $\overline{\gamma}_0$ and $\overline{\gamma}_\infty$ respectively. The isotropy group of both singularities is \mathbb{Z}_p .

Our aim is to study punctured holomorphic curves in the symplectizations of lens spaces. These curves are the main objects studied in symplectic field theory (SFT); see [6]. As pointed above, the standard Reeb flow on S^3 is perfectly symmetric, and if one chooses an almost complex structure on ξ_0 , which is invariant under the Reeb flow, the resulting SFT-type almost complex structure on $\mathbb{R} \times S^3$ leads to a description of punctured curves in terms of the closed curves in the orbit space $\mathbb{C}P^1$ paired with meromorphic sections above them since $\mathbb{R} \times S^3$ may be viewed as a complex (in fact, holomorphic) line bundle without its zero section; see Section 2. This is a particular example of a phenomenon observed for the pre-quantization bundles; see [1].

If such a symmetric almost complex structure on $\mathbb{R} \times S^3$ is also invariant under the \mathbb{Z}_p -action, then it descends to $\mathbb{R} \times L(p, q)$, and one may study punctured curves in $\mathbb{R} \times L(p, q)$ explicitly in terms of their lifts to $\mathbb{R} \times S^3$, which are obtained after precomposing the curves in $\mathbb{R} \times L(p, q)$ with a suitable covering of their domains. This idea goes back to [9] and is executed in detail for lens spaces and their unit cotangent bundles in [19]. It turns out that it is easy to determine whether a given moduli space of rational curves is nonempty and has the correct dimension. We present a systematic treatment of this idea in the next section. For an application of the resulting computational scheme, we prove the following.

Theorem 1.2. *Let p be a prime number and $1 < q, q' < p - 1$. Let α and α' be the standard contact forms on $L(p, q)$ and $L(p, q')$ respectively. Suppose that there is a positive contactomorphism*

$$\varphi : (L(p, q), \xi = \ker \alpha) \rightarrow (L(p, q'), \xi' = \ker \alpha').$$

Then $q \equiv (q')^{\pm 1}$.

Here the *positivity* of φ means that $\varphi^* \alpha' = f \alpha$ for some positive smooth function f on $L(p, q)$. For the proof, we study the moduli spaces of a pair of pants with non-contractible positive ends in detail. We show that nonempty components of such moduli spaces are cut out transversally by comparing the Fredholm index of the pair of pants with the dimension of the equivariant pseudo-holomorphic perturbations of their lifts. This enables us to perturb the data and carry on a standard neck-stretching argument after the given contactomorphism. Noting the homotopy/homology relation $[\overline{\gamma}_\infty] = q[\overline{\gamma}_0]$, the result follows from the comparison of the homotopy relation that is encoded by the positive ends of a certain pair of pants in $\mathbb{R} \times L(p, q)$ with the corresponding relation encoded by the pair of pants in $\mathbb{R} \times L(p, q')$, which is obtained after stretching the neck; see Figure 4.

The lens spaces were introduced by W. Dyck in 1884, and since then, the problem of their classification had attracted many great mathematicians like

H. Tietze, H. Poincaré, J.W. Alexander, H. Seifert; see [22] for a more precise historical account. In 1935, the first complete classification of lens spaces was given by Reidemeister in the category of piecewise linear (PL) homeomorphisms [16], and later, in 1960, Brody showed that the homeomorphism classification coincides with the classification in PL category; see [3]. After many other contributions, we finally have the following classification statement for 3-dimensional lens spaces; see [5] for a modern proof.

Theorem 1.3. *The lens spaces $L(p, q)$ and $L(p, q')$ are*

- *homotopy equivalent if and only if*

$$qq' \equiv \pm a^2$$

for some $a \in \mathbb{N}$,

- *simple homotopy equivalent/homeomorphic/diffeomorphic if and only if*

$$q \equiv \pm (q')^{\pm 1}.$$

One of the important aspects of the above theorem is providing examples of manifolds which are homotopy equivalent but not simple homotopy equivalent/homeomorphic/diffeomorphic. We also note that if $q \equiv \pm (q')^{\pm 1}$ is assumed, then it is easy to define an explicit diffeomorphism between $L(p, q)$ and $L(p, q')$ for each case of the assumption. Hence the “only if” part of the second statement of Theorem 1.3 is the nontrivial part, and it requires sophisticated topological tools like the Reidemeister–Franz torsion, which was introduced by Reidemeister in [16] for his classification result and later generalized by Franz in [7].

We note that, from the topological point of view, Theorem 1.2 is a straightforward consequence of Theorem 1.3. If we orient lens spaces via the standard contact forms, then due to the dimensional reasons, any contactomorphism, independent of being positive or negative, is an orientation preserving diffeomorphism in the first place. Moreover, after paying more attention to the smooth classification, one realizes that the existence of an orientation preserving diffeomorphism between $L(p, q)$ and $L(p, q')$ implies that $q \equiv (q')^{\pm 1}$.

On the other hand, once combined with the classification of universally tight contact structure on 3-dimensional lens spaces given by Honda in [12], whose proof is purely contact topological and does not rely on any classification statement on lens spaces, Theorem 1.2 gives a pure symplectic/contact topological proof of the following classification statement.

Theorem 1.4. *Assume that p is prime and $1 < q, q' < p - 1$. If $L(p, q)$ and $L(p, q')$ are diffeomorphic, then $q \equiv \pm (q')^{\pm 1}$.*

Proof. We orient $L(p, q)$ and $L(p, q')$ via the volume forms $\alpha \wedge d\alpha$ and $\alpha' \wedge d\alpha'$ respectively, where α and α' are the standard contact forms. Let $\varphi : L(p, q) \rightarrow L(p, q')$ be an orientation preserving diffeomorphism. We put $\beta := \varphi^* \alpha'$. Since φ is orientation preserving, $\beta \wedge d\beta$ is a positive volume form on $L(p, q)$, and consequently, $(\ker \beta, d\beta)$ is a positive contact form in the sense of [12]. It is also clear that $\ker \beta$ is universally tight. By [12, Prop. 5.1], there are precisely

two universally tight positive contact structures on $L(p, q)$ since $q \neq p - 1$. We note that $(\xi, d\alpha)$ and $(\xi, -d\alpha)$ are two positive universally tight contact structures on $L(p, q)$, and they are not isotopic as oriented contact structures since they are distinguished by their Euler classes. In fact, an easy computation shows that the Poincaré dual of the Euler class of $(\xi, d\alpha)$ is given by $(q + 1)[\overline{\gamma}_0] \in H_1(L(p, q), \mathbb{Z})$. Note that the Poincaré dual of the Euler class of $(\xi, -d\alpha)$ is then given by $(-q - 1)[\overline{\gamma}_0]$ and $q + 1 \not\equiv -q - 1$ since $q \neq p - 1$. Hence $(\ker \beta, d\beta)$ is isotopic to either $(\xi, d\alpha)$ or $(\xi, -d\alpha)$ as positive contact structures. Hence we either have a positive contactomorphism

$$\psi^+ : (L(p, q), \xi = \ker \alpha) \rightarrow (L(p, q), \ker \beta)$$

or a positive contactomorphism

$$\psi^- : (L(p, q), \xi = \ker(-\alpha)) \rightarrow (L(p, q), \ker \beta).$$

In the first case, $\varphi \circ \psi^+$ is the desired positive contactomorphism. In the second case, we consider the diffeomorphism ψ of $L(p, q)$, which is induced by the map $(z_1, z_2) \rightarrow (\overline{z}_1, \overline{z}_2)$ on S^3 . It is easy to see that $\psi^*\alpha = -\alpha$, and therefore, $\varphi \circ \psi^+ \circ \psi$ is a positive contactomorphism. By Theorem 1.2, we get $q \equiv (q')^{\pm 1}$ in either case.

Now we assume that $\varphi : L(p, q) \rightarrow L(p, q')$ is an orientation reversing diffeomorphism. Then we consider the diffeomorphism $\psi : L(p, -q) \rightarrow L(p, q)$, which is induced by the map $(z_1, z_2) \rightarrow (z_1, \overline{z}_2)$ on S^3 . It is easy to see that ψ is orientation reversing. Hence $\varphi \circ \psi$ is an orientation preserving diffeomorphism. Applying the above argument to $\varphi \circ \psi$ leads to $-q \equiv (q')^{\pm 1}$. \square

The study of holomorphic curves in the symplectization of lens spaces with respect to the Morse–Bott data is not new. We note that these curves can be studied directly as orbicurves in the orbit space $\overline{\mathbb{C}P^1}$ paired with meromorphic sections of orbibundles. This leads to a formal relation between the orbifold Gromov–Witten potential of the orbit space $\overline{\mathbb{C}P^1}$ and the SFT hamiltonian of $L(p, q)$; see [18]. Compared to [18], our approach is rather elementary and serves us well since we do not aim to compute the SFT hamiltonian. We just provide a computational scheme, and applying this scheme to a small portion of the SFT hamiltonian, namely a pair of pants with positive non-contractible ends, gives us enough information to prove the classification statement.

Another work that aligns with our paper is presented in [4]. In [4], it is proven that if there is a directed symplectic cobordism, whose respective boundaries induce isomorphisms in integral homologies via their inclusions, between two lens spaces, which are equipped with the standard contact structures, then the cobordism is necessarily trivial. This statement is an application of the intersection theory for pseudo-holomorphic curves in 4-dimensional symplectic orbifolds, which is the main result of [4]. Given a symplectic homology cobordism between two lens spaces, one compactifies both ends of the symplectic cobordism in a particular way and studies a moduli space of holomorphic spheres with certain properties in the resulting closed symplectic orbifold. The moduli space turns out to be a closed 2-dimensional orbifold, and

one shows that, after suitable compactification, the total space of a particular line bundle over this moduli space is diffeomorphic to the compactification of the symplectic cobordism by means of an evaluation map. On the other hand, the diffeomorphism type of the compactified line bundle turns out to be the same as the compactification of the trivial cobordism over one of the given lens spaces. Although our treatment of holomorphic curves stay in the smooth category and we study punctured curves in the SFT setting, due to the Morse–Bott data, the punctured curves that appear in this work can be also thought as closed curves in suitable singular compactifications of the symplectizations or symplectic cobordisms, which are essentially different than the symplectic orbifold used in [4]. On the other hand, in order to achieve the necessary control on the neck-stretching process, we lift punctured curves to suitable coverings and apply the classical intersection theory to the extensions of these lifts, which are now smooth curves.

The organization of the paper. In Section 2, we describe the computational scheme for holomorphic curves in the symplectizations of lens spaces. In Section 3, we go over a simplified version of the computational scheme for curves with two non-contractible ends and describe the moduli space of a pair of pants in detail. We compute the dimension of the moduli space given by our computational scheme and compare it to the Fredholm index in order to show that the almost complex structure we use is generic. In Section 4, we prove Theorem 1.2. We note that a reader, who is solely interested in this proof may skip Sections 2.2 and 2.13 in the first reading.

2. HOLOMORPHIC CURVES IN THE SYMPLECTIZATIONS OF LENS SPACES: THE GENERAL SCHEME

Let (M, α) be a closed, $(2n - 1)$ -dimensional contact manifold, and let R_α be the associated Reeb vector field with the flow ϕ^t , which is of *Morse–Bott* type. Namely, the action spectrum of α is discrete, and for any action value (period) T , $N_T := \{x \in M \mid \phi^T(x) = x\} \subseteq M$ is a closed submanifold such that the rank of $d\alpha|_{N_T}$ is locally constant and $T_x N_T = \ker(d\phi^T - \text{id})_x$ for all $x \in N_T$. In the case that N_T consists of a single orbit $\gamma: \mathbb{R} \rightarrow M$, we say that γ is *non-degenerate*. We note that, in this case, $d\phi^T(\gamma(0))|_{\xi_{\gamma(0)}}$ does not admit 1 as an eigenvalue. We note that, for any period T , the Reeb flow defines an S^1 -action on N_T , and the resulting orbit space $S_T = N_T/S^1$ is an orbifold, which consists of a single point if N_T is geometrically a single orbit.

Given an almost complex structure J on $\xi = \ker \alpha$ that is compatible with $d\alpha$, one extends it to the *symplectization* $(\mathbb{R} \times M, d(e^a \lambda))$ in such a way that J is invariant under translations along ∂_a and $J\partial_a = R_\alpha$. An almost complex structure on $\mathbb{R} \times M$, which is defined as above, is called a *cylindrical* or an *SFT-type* almost complex structure.

Let (Σ, j) be a closed Riemann surface, and let $\Gamma \subset \Sigma$ be a finite ordered set of punctures. We consider the maps that satisfy the non-linear Cauchy–

Riemann equation, namely

$$(4) \quad u : \Sigma \setminus \Gamma \rightarrow \mathbb{R} \times M, \quad du \circ j = J \circ du.$$

We call such a map a (*punctured*) *pseudo-holomorphic curve* or a *J-holomorphic curve*.² Note that if $\gamma : \mathbb{R} \rightarrow M$ is a periodic orbit of R_α with period T , then the *trivial cylinder over γ* ,

$$(5) \quad u : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times M, \quad (s, t) \mapsto (Ts, \gamma(Tt)),$$

is a holomorphic curve, where $\mathbb{R} \times S^1 := \mathbb{C}/i\mathbb{Z}$. It turns out that finiteness of a suitable notion of energy forces a pseudo-holomorphic curve to behave as meromorphic objects in complex analysis. The *Hofer energy* of a holomorphic curve u is defined to be

$$E(u) := \sup_{f \in \mathcal{F}} \int_{\Sigma \setminus \Gamma} u^* d(e^f \alpha),$$

where $\mathcal{F} = \{f : \mathbb{R} \rightarrow (-1, 1) \mid f' > 0\}$. Under the Morse–Bott assumption, a holomorphic curve u with finite Hofer energy converges to a trivial cylinder near a puncture unless its image is bounded near that puncture, and in the latter case, u extends holomorphically over a puncture [1]. More precisely, for each honest puncture $z \in \Gamma$, one fixes a holomorphic coordinate chart identified with an open disk \mathbb{D} centered at z . The puncture set splits as $\Gamma = \Gamma^+ \cup \Gamma^-$, and one fixes the following cylindrical coordinates:

- $Z_+ := [0, +\infty) \times S^1 \rightarrow \mathbb{D} \setminus \{0\} \subset \Sigma \setminus \{z\}$, $(s, t) \mapsto e^{-2\pi(s+it)}$ if $z \in \Gamma^+$,
- $Z_- := (-\infty, 0] \times S^1 \rightarrow \mathbb{D} \setminus \{0\} \subset \Sigma \setminus \{z\}$, $(s, t) \mapsto e^{2\pi(s+it)}$ if $z \in \Gamma^-$.

Then, for every $z \in \Gamma^\pm$, there is a T -periodic Reeb orbit γ such that

$$u(s, t) = \exp_{(Ts, \gamma(Ts))} h(s, t), \quad (s, t) \in Z_\pm,$$

for $|z|$ large, where h is a vector field along the trivial cylinder (5) such that $h(s, \cdot) \rightarrow 0$ uniformly as $|s| \rightarrow \infty$, and \exp is the exponential map defined via an \mathbb{R} -invariant metric on $\mathbb{R} \times M$. We say that

- z is a *positive puncture* if $z \in \Gamma^+$. We write $u(z) = (+\infty, \gamma)$ if the asymptotic end is a non-degenerate orbit γ , and $u(z) \in \{+\infty\} \times S_T$ if the asymptotic end belongs to a nontrivial orbit space S_T .
- z is a *negative puncture* if $z \in \Gamma^-$. We write $u(z) = (-\infty, \gamma)$ if the asymptotic end is a non-degenerate orbit γ , and $u(z) \in \{-\infty\} \times S_T$ if the asymptotic end belongs to a nontrivial orbit space S_T .

Given a holomorphic curve u as (4) and a positive/negative puncture z with the asymptotic end γ_z , which can be a non-degenerate orbit or an element in a Morse–Bott orbit space, then the *action (of α) at³ the puncture z* is defined to be the integral

$$\int_{\gamma_z} \alpha,$$

²If it is not necessary to specify the almost complex structure, then we call such maps holomorphic curves or just curves in short.

³We also say the *action at a given positive/negative end* if the puncture itself is not needed to be mentioned.

which is simply the period of the orbit γ_z . From the definition of a cylindrical almost complex structure, it follows that the $d\alpha$ -energy of u ,

$$\int_{\Sigma \setminus \Gamma} u^* d\alpha,$$

is nonnegative and satisfies

$$\int_{\Sigma \setminus \Gamma} u^* d\alpha = \sum_{z \in \Gamma^+} \int_{\gamma_z} \alpha - \sum_{z \in \Gamma^-} \int_{\gamma_z} \alpha.$$

In particular, the total action at positive ends of u is greater than or equal to the total action at its negative ends.

In general, one studies the moduli space of holomorphic curves for a fixed asymptotic data. One picks a collection of orbit spaces S_1^\pm, \dots, S_n^\pm and defines

$$(6) \quad \mathcal{M} = \{(\Sigma, j, \Gamma, u) \mid u(z_i^\pm) \in \{\pm\infty\} \times S_i^\pm \text{ for all } z_i^\pm \in \Gamma^\pm\} / \sim$$

to be the moduli space of equivalence classes $[\Sigma, j, \Gamma, u]$ of holomorphic curves, where $(\Sigma, j, \Gamma, u) \sim (\Sigma', j', \Gamma', u')$ if there exists a biholomorphism $h : \Sigma \rightarrow \Sigma'$ such that h restricts to an ordering preserving bijection on the corresponding puncture sets and $u = u' \circ h$. It turns out that, once it is cut out transversally, the space (6) admits smooth structure (of finite dimension) as the zero set of the Cauchy–Riemann operator

$$(7) \quad \bar{\partial}_J(u) = du + J \circ du \circ j,$$

which is defined on a suitable function space. In general, it is hard to carry out a hands-on study of moduli spaces of holomorphic curves. Under nice circumstances, a generic choice of J on ξ leads to a smooth structure on the moduli space. But such a generic choice makes it hard to grasp the moduli space itself even in the Morse–Bott case. But in certain perfectly symmetric Morse–Bott situations like pre-quantization bundles, these moduli spaces can be described as rather elementary objects [1].

Remark 2.1. In what follows, we will be mostly interested in moduli spaces of rational curves, namely the case $\Sigma = \mathbb{C}P^1$, with at least three punctures. We note that, due to the uniqueness of the complex structure on $\mathbb{C}P^1$, when $\Sigma = \mathbb{C}P^1$ and $|\Gamma| = 3 + k, k \geq 0$, the above moduli space \mathcal{M} given by (6) is identified with the space of pairs $(u, (z_1, \dots, z_k))$, where

$$u : \mathbb{C}P^1 \setminus \{0, 1, \infty, z_1, \dots, z_k\} \rightarrow \mathbb{R} \times M$$

is a holomorphic curve with prescribed sign of punctures and asymptotic ends.

The aim of this section is to study holomorphic curves in $\mathbb{R} \times L(p, q)$ explicitly, where the cylindrical almost complex structure is a quotient of the cylindrical almost complex structure on $\mathbb{R} \times S^3$ associated to the tautological line bundle over $\mathbb{C}P^1$. Our strategy is to study curves in $\mathbb{R} \times L(p, q)$ through their lifts to $\mathbb{R} \times S^3$. We want to describe spaces of solutions of the Cauchy–Riemann equation on $\mathbb{R} \times L(p, q)$ as the subspaces of equivariant solutions of lifted problems on $\mathbb{R} \times S^3$ on which one has a very good understanding.

First we describe the cylindrical almost complex structure on $\mathbb{R} \times S^3$. In real coordinates $z_j = x_j + iy_j$, (1) reads as

$$\alpha_0 = -y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2.$$

The symplectic form $d\alpha_0$ on ξ_0 is given by

$$d\alpha_0 = 2(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)|_{\xi_0}.$$

The Hopf fibration is the S^1 -bundle associated to the tautological line bundle

$$(8) \quad L \rightarrow \mathbb{C}P^1, \quad L_{(z_1:z_2)} = \text{span}_{\mathbb{C}}\{(z_1, z_2)\} \subset \mathbb{C}^2$$

and the hermitian metric on L , which is induced by the standard hermitian metric on \mathbb{C}^2 . Hence the Euler class of (2) is given by

$$e(\pi) = c_1(L) = -[\pi^{-1}\omega_{\text{FS}}],$$

where ω_{FS} is the Fubini–Study form on $\mathbb{C}P^1$ such that $\langle \omega_{\text{FS}}, [\mathbb{C}P^1] \rangle = \pi$ and

$$\pi^*\omega_{\text{FS}} = (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)|_{S^3}.$$

Since the S^1 -action is generated by the Reeb field R_0 and the period is 2π , being the associated contact form, α_0 satisfies $\mathcal{L}_{R_0}\alpha_0 = 0$ and $\alpha_0(R_0) = 1$. So α_0 is a connection 1-form, and from the equality above, we have $\pi^*\omega_{\text{FS}} = \frac{1}{2}d\alpha_0$, that is, $2\omega_{\text{FS}}$ is the curvature form.

We pull the standard complex structure on $\mathbb{C}P^1$ back to the contact distribution ξ_0 as an S^1 -invariant complex structure via the map (2), and we extend it to the symplectization as described above. We call this almost complex structure the *standard almost complex structure* and denote it by J_0 . We note that the extension $\pi : \mathbb{R} \times S^3 \rightarrow \mathbb{C}P^1$ of the Hopf fibration is J_0 -holomorphic. We consider the diffeomorphism

$$\Phi : \mathbb{R} \times S^3 \rightarrow L^*, \quad (a, (z_1, z_2)) \mapsto e^a(z_1, z_2),$$

where L^* is the total space of (8), where the zero section is removed. We note that, once conjugated by Φ , J_0 coincides with the complex structure on the fibers of L . Since Φ also covers holomorphic bundle projections on both its domain and target, it is a biholomorphism.

For any punctured curve

$$u : \Sigma \setminus \Gamma \rightarrow \mathbb{R} \times S^3$$

with finite energy, the holomorphic map $c = \pi \circ u$ extends over the punctures and gives a closed holomorphic curve in $\mathbb{C}P^1$. The map u then corresponds to a meromorphic section f of the holomorphic line bundle $c^*L \rightarrow \Sigma$; see [1]. The positive ends of u correspond to the poles of f , and negative ends of u correspond to the zeros of f since the complex structure on the symplectization fits to the complex structure on the fibers of L . Note that the first Chern number of the bundle c^*L is given by $-d$, where $d \geq 0$ is the degree of the map c . If $\Sigma = \mathbb{C}P^1$, then a necessary and sufficient condition for the existence

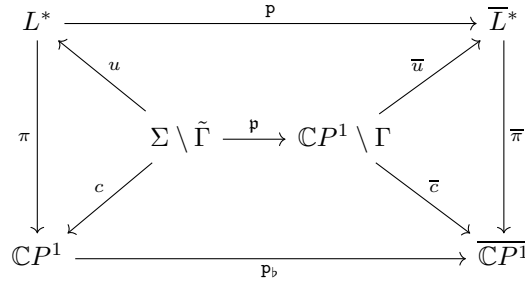


FIGURE 1. Lifting diagram for a holomorphic curve in $\mathbb{R} \times L(p, q)$.

of a meromorphic section f is that both the divisor of the section and the bundle have the same degree, namely

$$\#f^{-1}(0) - \#f^{-1}(\infty) = -d.$$

We note that the above formula says that there is no punctured curve with only negative ends, which is consistent with the maximum principle.

We now want to add the \mathbb{Z}_p action into the setting. This action is free on L^* but has two fixed points on the zero section $\mathbb{C}P^1$; see (3). Note that the standard almost complex structure J_0 on $\mathbb{R} \times S^3$ is \mathbb{Z}_p -invariant, so we identify $\mathbb{R} \times L(p, q)$ with $\bar{L}^* := L^*/\mathbb{Z}_p$, where the former space is equipped with the quotient almost complex structure denoted by J_α . We call J_α the *standard almost complex structure on $\mathbb{R} \times L(p, q)$* .

We consider a rational J_α -holomorphic curve

$$\bar{u} : \mathbb{C}P^1 \setminus \Gamma \rightarrow \mathbb{R} \times L(p, q) \cong \bar{L}^*.$$

Such a curve lifts to the cover L^* if and only if $\bar{u}_* = 0$ on $\pi_1(\mathbb{C}P^1 \setminus \Gamma)$. We note that $\mathbb{C}P^1 \setminus \Gamma$ is homotopy equivalent to the bouquet of $(\#\Gamma - 1)$ circles, so its fundamental group is the free group with $(\#\Gamma - 1)$ generators. Hence the image of the fundamental group is trivial if and only if the image of each generator is trivial, that is, if all of the asymptotics are contractible. In this paper, our main concern are the curves with non-contractible ends. But since a curve \bar{u} with a non-contractible end does not lift to the cover immediately, we need to precompose the map \bar{u} with a suitable covering map

$$(9) \quad \mathfrak{p} : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{C}P^1 \setminus \Gamma.$$

Once we pick a suitable cover (9), we have the commutative diagram given by Figure 1. The problem is then to study the lifted J_0 -holomorphic curves

$$u : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{R} \times S^3 \cong L^*,$$

which are equivariant with respect to the action of the group of Deck transformations G on the punctured surface $\Sigma \setminus \tilde{\Gamma}$ and \mathbb{Z}_p -action on L^* .

2.2. Equivariant curves: The necessary conditions for the existence.

In this section, our aim is to determine a suitable minimal covering (9) and establish a correspondence between a given moduli problem in \bar{L}^* and a lifted moduli problem in L^* , which is determined by (9).

We fix a set of punctures $\Gamma \subset \mathbb{C}P^1$, which partitions as follows:

- $\Gamma = \Gamma_{nc} \cup \Gamma_c$ with cardinalities n_{nc} and n_c ,
- $\Gamma_{nc} = \Gamma_0 \cup \Gamma_\infty$ with cardinalities n_0 and n_∞ so that $n_{nc} = n_0 + n_\infty$,
- $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$ with cardinalities n_0^+ and n_0^- so that $n_0 = n_0^+ + n_0^-$,
- $\Gamma_\infty = \Gamma_\infty^+ \cup \Gamma_\infty^-$ with cardinalities n_∞^+ and n_∞^- so that $n_\infty = n_\infty^+ + n_\infty^-$,
- $\Gamma_c = \Gamma_c^+ \cup \Gamma_c^-$ with cardinalities n_c^+ and n_c^- so that $n_c = n_c^+ + n_c^-$,
- $\Gamma_0^\pm = \{z_1^{0,\pm}, \dots, z_{n_0^\pm}^{0,\pm}\}$, $\Gamma_\infty^\pm = \{z_1^{\infty,\pm}, \dots, z_{n_\infty^\pm}^{\infty,\pm}\}$ and $\Gamma_c^\pm = \{w_1^\pm, \dots, w_{n_c^\pm}^\pm\}$.

Suppose that we have a holomorphic curve

$$\bar{u} : \mathbb{C}P^1 \setminus \Gamma \rightarrow \bar{L}^* \cong \mathbb{R} \times L(p, q)$$

with asymptotics

- (a1) $\bar{u}(z_i^{0,\pm}) = (\pm\infty, k_i^{0,\pm}\bar{\gamma}_0)$, where $k_i^{0,\pm} \neq 0$ for $i = 1, \dots, n_0^\pm$, that is, \bar{u} has a positive/negative puncture at $z_i^{0,\pm}$ with positive/negative non-contractible asymptotic end $k_i^{0,\pm}\bar{\gamma}_0$ for $i = 1, \dots, n_0^\pm$,
- (a2) $\bar{u}(z_i^{\infty,\pm}) = (\pm\infty, k_i^{\infty,\pm}\bar{\gamma}_\infty)$, where $k_i^{\infty,\pm} \neq 0$ for $i = 1, \dots, n_\infty^\pm$, that is, \bar{u} has a positive/negative puncture at $z_i^{\infty,\pm}$ with positive/negative non-contractible asymptotic end $k_i^{\infty,\pm}\bar{\gamma}_\infty$ for $i = 1, \dots, n_\infty^\pm$,
- (a3) $\bar{u}(w_i^\pm) \in \{\pm\infty\} \times S_{k_i^\pm}^\pm$ for $i = 1, \dots, n_c^\pm$, that is, \bar{u} has a positive/negative puncture at w_i^\pm with positive/negative contractible asymptotic end, which lies in the orbit space $S_{k_i^\pm}^\pm$ of orbits with action $2\pi k_i^\pm$.

We fix a point $z \in \mathbb{C}P^1 \setminus \Gamma$ and consider the map

$$\bar{u}_* : \pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \rightarrow \pi_1(\bar{L}^*, x_{\bar{u}}) \cong \mathbb{Z}_p,$$

where $x_{\bar{u}} = \bar{u}(z)$. Since p is prime, it is clear that \bar{u}_* is surjective if and only if $n_{nc} \geq 1$. We know that $K := \ker \bar{u}_*$ is a normal subgroup of $\pi_1(\mathbb{C}P^1 \setminus \Gamma, z)$ and there exists a covering space

$$\mathfrak{p} : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{C}P^1 \setminus \Gamma,$$

where $\Sigma \setminus \tilde{\Gamma}$ is smooth punctured surface and $\mathfrak{p}_*(\pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z})) = K$, where \tilde{z} is a fixed lift of z . This covering is (up to isomorphism) determined by K , and the group of Deck transformations G is given by

$$G \cong \pi_1(\mathbb{C}P^1 \setminus \Gamma, z)/K \cong \mathbb{Z}_p.$$

We endow $\Sigma \setminus \tilde{\Gamma}$ with the pullback complex structure and get a punctured Riemann surface so that \mathfrak{p} is holomorphic. By the removal of singularities theorem, \mathfrak{p} extends to a holomorphic branched covering $\mathfrak{p} : \Sigma \rightarrow \mathbb{C}P^1$.

We note that $\bar{u} \circ \mathfrak{p} : \Sigma \setminus \tilde{\Gamma} \rightarrow \bar{L}^*$ satisfies

$$(\bar{u} \circ \mathfrak{p})_*(\pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z})) = \bar{u}_* \mathfrak{p}_*(\pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z})) = \{0\} = p_*(\pi_1(L^*, \tilde{x}_{\bar{u}})),$$

where $\tilde{x}_{\bar{u}}$ is a fixed lift of $x_{\bar{u}}$. Hence we have the unique lift

$$(10) \quad u := \widetilde{\bar{u} \circ \mathfrak{p}} : \Sigma \setminus \tilde{\Gamma} \rightarrow L^*, \quad u(\tilde{z}) = \tilde{x}_{\bar{u}}.$$

We note that u is equivariant with respect to the action of G on $\Sigma \setminus \tilde{\Gamma}$ and the action of \mathbb{Z}_p on L^* . We fix a generator τ_b of G such that

$$(11) \quad u \circ \tau_b = \sigma \circ u.$$

Now we take a closer look at the branched covering $\mathfrak{p} : \Sigma \rightarrow \mathbb{C}P^1$. It is clear that $\tilde{\Gamma} = \mathfrak{p}^{-1}(\Gamma)$. We put $\tilde{\Gamma}_{nc} := \mathfrak{p}^{-1}(\Gamma_{nc})$ and $\tilde{\Gamma}_c = \mathfrak{p}^{-1}(\Gamma_c)$.

Lemma 2.3. *Each point in Γ_{nc} is a branch point with exactly one preimage and each point in Γ_c is regular and has exactly p preimages. Further, $\tilde{\Gamma}_{nc}$ forms the fixed point set of the extended action of G over Σ . Moreover, the genus of Σ is given by*

$$(12) \quad g = \frac{(p-1)(n_{nc}-2)}{2}.$$

Proof. For any $z_j^{i,\pm} \in \Gamma_{nc}$, the cardinality of the set $\mathfrak{p}^{-1}(z_j^{i,\pm})$ is either p or 1 . In fact, any element in $\mathfrak{p}^{-1}(z_j^{i,\pm})$ has a local isotropy group with respect to the extended action of G on Σ , which is a subgroup of $G \cong \mathbb{Z}_p$. Since p is prime, this subgroup is either trivial or \mathbb{Z}_p . We note also that at least one point in $\mathfrak{p}^{-1}(z_j^{i,\pm})$ is a puncture of u with an asymptotic, which descends to a non contractible asymptotic \bar{u} at $z_j^{i,\pm}$. Hence at least one point in $\mathfrak{p}^{-1}(z_j^{i,\pm})$ has isotropy group \mathbb{Z}_p , and therefore, $\mathfrak{p}^{-1}(z_j^{i,\pm})$ consists of a single point. We conclude that \mathfrak{p} branches points over Γ_{nc} so that each branch point has a single preimage with ramification number p . Similarly, for each $w_j^\pm \in \Gamma_c$, there is a contractible end of \bar{u} , which lifts to p contractible ends of u . Hence the punctures corresponding to these ends are precisely the preimages of w_j^\pm . Applying the Riemann–Hurwitz formula, we get

$$2 - 2g = 2p - n_{nc}(p-1) \implies g = \frac{(p-1)(n_{nc}-2)}{2}.$$

It is clear that $\tilde{\Gamma}_{nc}$ is the fixed point set of the extended G -action. □

We abuse the notation and denote preimages of $z_i^{0,\pm}, z_i^{\infty,\pm}$ under \mathfrak{p} by the same letters. For the preimages of w_i^\pm , which is given by $G = \langle \tau_b \rangle$ -orbit of any point in the preimage, we write $w_{i,j}^\pm, j = 1, \dots, p$. Now we have an equivariant curve $u : \Sigma \setminus \tilde{\Gamma} \rightarrow L^*$ with asymptotics

- (la1) $u(z_i^{0,\pm}) = (\pm\infty, k_i^{0,\pm}\gamma_0)$ for $i = 1, \dots, n_0^\pm$,
- (la2) $u(z_i^{\infty,\pm}) = (\pm\infty, k_i^{\infty,\pm}\gamma_\infty)$ for $i = 1, \dots, n_\infty^\pm$,
- (la3) $u(w_{i,j}^\pm) \in \{\pm\infty\} \times S_{k_i^\pm}^\pm$ for $i = 1, \dots, n_c^\pm, j = 1, \dots, p$,

where now $S_{k_i^\pm}^\pm$ denotes the space of orbits in S^3 with the action $2\pi k_i^\pm$.

Remark 2.4. We note the lift u depends on the choice of the lift $\tilde{x}_{\bar{u}}$. In fact, imposing that the lift maps \tilde{z} to $\sigma^k \tilde{x}_{\bar{u}}$ leads to the unique lift, say u' , which satisfies $u' = u \circ \tau_b^k$. More precisely, there are p lifts of \bar{u} , which are distinct as maps, given by $u \circ \tau_b^k, k = 0, \dots, p-1$ or alternatively by $\sigma^k \circ u, k = 0, \dots, p-1$. We note that any such lift u' satisfies $u' \circ \tau_b = \sigma \circ u'$. In particular, the image of any lift is invariant under σ and the images of all lifts coincide.

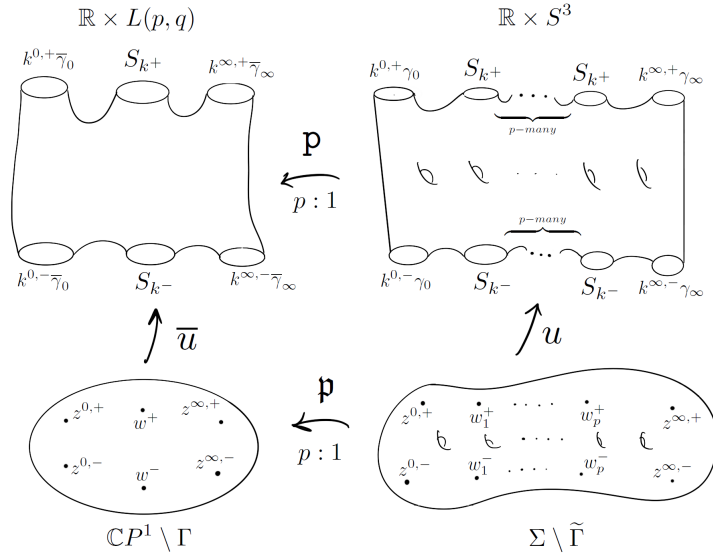


FIGURE 2. An example of a curve in $\mathbb{R} \times L(p, q)$ and its lift to $\mathbb{R} \times S^3$.

Now we want to understand to what extent the above description depends on \bar{u} .

Remark 2.5. By (12), the topology of Σ depends only on the number of the non-contractible ends of \bar{u} .

Since $\pi_1(\bar{L}^*, *)$ is abelian, the homomorphism \bar{u}_* is invariant under conjugations and therefore depends only on non-contractible asymptotic ends of \bar{u} . In fact, given any representation of $\pi_1(\mathbb{C}P^1 \setminus \Gamma, z)$ via loops, each being the concatenation of a small simple loop around a puncture and a path joining a point on it to the base point z , the image of a generator (associated to a puncture) under \bar{u}_* is equal to the image of the generator (associated to the same puncture) represented by the small simple loop itself, which comes from a representation where the base point z is taken on that very same loop. Hence \bar{u}_* and therefore K are determined up to conjugacy by the data (a1) and (a2). In particular, if \bar{u} is contained in a continuous family of parametrized curves, one can fix the same subgroup K for the whole family. We also note that the generator τ_b of G can be chosen constantly for such a family of curves.

Remark 2.6. We note that the set of punctures Γ is fixed so far, but in general, one needs to let punctures move. Although the covering \mathbf{p} topologically depends only on n_{nc} , the complex structure on Σ does depend on the positions of the punctures in $\mathbb{C}P^1$.

Now we want to understand the picture for the *base curves*. We fix a lift u given by (10) and get a holomorphic map

$$c := \pi \circ u : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{C}P^1;$$

see Figure 1. After extending c over $\tilde{\Gamma}$, we have a closed curve

$$(13) \quad c : \Sigma \rightarrow \mathbb{C}P^1 \quad \text{such that} \quad c \circ \tau_b = \sigma_b \circ c$$

in the view of (3) and (11). We call such a curve c a *lifted base curve*. Since c is equivariant, one can pull back the action on the total space of $L \rightarrow \mathbb{C}P^1$ uniquely to the total spaces of $c^*L \rightarrow \Sigma$ in such a way that the bundle map $c^*L \rightarrow L$, which covers the map c , is equivariant. We fix a generator τ of this \mathbb{Z}_p -action on c^*L so that u corresponds to a meromorphic section $u : \Sigma \rightarrow c^*L$ such that

$$(14) \quad u \circ \tau_b = \tau \circ u.$$

Moreover, the section u has poles/zeros at $z_i^{0,\pm}$'s, $z_i^{\infty,\pm}$'s and $w_{i,j}^{\pm}$'s. We note that

$$\begin{aligned} c(z_i^{0,\pm}) &= (0 : 1) = 0 \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}, \\ c(z_i^{\infty,\pm}) &= (1 : 0) = \infty \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \end{aligned}$$

for all i , which explains the notation for $z_i^{0/\infty,\pm}$.

We want to understand the consequences of (14) in terms of the multiplicities $k_i^{0/\infty,\pm}$. To this end, we need to understand the $\langle \tau \rangle$ -action on c^*L around the fixed points of $\langle \tau_b \rangle$ -action on Σ . We note that, since we do not know much about the global behavior of the map c , it is hard to describe $\langle \tau_b \rangle$ -action globally. But since c behaves as a monomial around any point, which is determined by the ramification number at that point, a local description is still easy to get.

We first fix local trivializations of L and compute the $\langle \sigma \rangle$ -action on these trivializations.

- *Near* $0 = (0 : 1)$, we fix the local trivialization

$$(15) \quad \tilde{\phi}_0 : \mathbb{C} \times \mathbb{C} \rightarrow L, \quad (z, \lambda) \mapsto \left(\lambda \frac{z}{\sqrt{1+|z|^2}}, \lambda \frac{1}{\sqrt{1+|z|^2}} \right),$$

which covers the chart $\phi_0 : \mathbb{C} \rightarrow \mathbb{C}P^1, z \mapsto (z : 1)$. Hence σ reads as

$$(16) \quad \sigma_0(z, \lambda) := (\tilde{\phi}_0^{-1} \sigma \tilde{\phi}_0)(z, \lambda) = (e^{i(1-q)\theta} z, e^{iq\theta} \lambda).$$

- *Near* $\infty = (1 : 0)$, we fix the local trivialization

$$(17) \quad \tilde{\phi}_\infty : \mathbb{C} \times \mathbb{C} \rightarrow L, \quad (z, \lambda) \mapsto \left(\lambda \frac{1}{\sqrt{1+|z|^2}}, \lambda \frac{z}{\sqrt{1+|z|^2}} \right),$$

which covers the chart $\phi_\infty : \mathbb{C} \rightarrow \mathbb{C}P^1, z \mapsto (1 : z)$. Hence σ reads as

$$(18) \quad \sigma_\infty(z, \lambda) := (\tilde{\phi}_\infty^{-1} \sigma \tilde{\phi}_\infty)(z, \lambda) = (e^{i(q-1)\theta} z, e^{i\theta} \lambda).$$

We recall that $\tilde{\Gamma}_{nc}$ is the fixed point set of the G action on Σ . So, using the above trivializations, we may determine the local behavior of an equivariant meromorphic section of c^*L near $\tilde{\Gamma}_{nc}$.

- Around $z_j^{0,\pm}$, we take holomorphic coordinates on Σ centered at $z_j^{0,\pm}$ so that $c(z) = z^r$ for some r , and we trivialize c^*L above this coordinate neighborhood using (15). Locally, $\tau_b = e^{im\theta}$ for some $0 < m < p$. By (13) and (16), we have

$$c(\tau_b(z)) = \sigma_b c(z) \implies e^{imr\theta} z^r = e^{i(1-q)\theta} z^r \implies m \equiv (1 - q)r^{-1}.$$

Hence, by (16), τ reads locally as

$$\tau(z, \lambda) = (e^{im\theta} z, e^{iq\theta} \lambda).$$

For a local section $z \mapsto (z, f(z))$ of c^*L to be equivariant, it has to satisfy

$$f(e^{im\theta} z) = e^{iq\theta} f(z).$$

Writing the meromorphic section f as $f(z) = \sum_k a_k z^k$ around $z_j^{0,\pm}$, we get

$$\sum_k a_k e^{imk\theta} z^k = \sum_k a_k e^{iq\theta} z^k,$$

and we note that

$$a_k \neq 0 \iff mk \equiv q \iff k \equiv qm^{-1} \iff k \equiv rq(1 - q)^{-1}.$$

- Around $z_j^{\infty,\pm}$, similarly, we have $c(z) = z^r$ for some r , and we put $\tau_b = e^{im\theta}$ for some $0 < m < p$. By the equivariance of c and (18), we have

$$c(\tau_b(z)) = \sigma_b c(z) \implies e^{imr\theta} z^r = e^{i(q-1)\theta} z^r \implies m \equiv (q - 1)r^{-1}$$

and locally

$$\tau(z, \lambda) = (e^{im\theta} z, e^{i\theta} \lambda).$$

Hence, for an equivariant meromorphic section $f = \sum_k a_k z^k$, one gets

$$a_k \neq 0 \iff mk \equiv 1 \iff k \equiv m^{-1} \iff k \equiv r(q - 1)^{-1}.$$

The above observations lead to the following lemma.

Lemma 2.7. *Let u be a lift of \bar{u} , and let $c : \Sigma \rightarrow \mathbb{C}P^1$ be the corresponding lifted base curve. Let $r_i^{0/\infty,\pm}$ denote the local degree of c at $z_i^{0/\infty,\pm} \in \bar{\Gamma}_{nc}$, and let $m_i^{0/\infty,\pm}$ denote the local representative of τ_b at $z_i^{0/\infty,\pm}$, that is $\tau_b = e^{im_i^{0/\infty,\pm}\theta}$ near $z_i^{0/\infty,\pm}$. Then we have the following relations.*

- (i) *For the positive punctures $z_i^{0/\infty,+}$, we have*
 - $m_i^{0,+} \equiv (1 - q)(r_i^{0,+})^{-1}$ and $k_i^{0,+} \equiv r_i^{0,+}(1 - v)^{-1}$,
 - $m_i^{\infty,+} \equiv (q - 1)(r_i^{\infty,+})^{-1}$ and $k_i^{\infty,+} \equiv r_i^{\infty,+}(1 - q)^{-1}$.
- (ii) *for the negative punctures $z_i^{0/\infty,-}$, we have*
 - $m_i^{0,-} \equiv (1 - q)(r_i^{0,-})^{-1}$ and $k_i^{0,-} \equiv r_i^{0,-}(v - 1)^{-1}$,
 - $m_i^{\infty,-} \equiv (q - 1)(r_i^{\infty,-})^{-1}$ and $k_i^{\infty,-} \equiv r_i^{\infty,-}(q - 1)^{-1}$.

Proof. As a meromorphic section of c^*L , u satisfies $u \circ \tau_b = \tau \circ u$. We write u locally as $(z, f(z))$ so that f has poles at $z_i^{0,+}/z_i^{\infty,+}$ of order $k_i^{0,+}/k_i^{\infty,+}$ and has zeros at $z_i^{0,-}/z_i^{\infty,-}$ of order $k_i^{0,-}/k_i^{\infty,-}$. Hence, at any $z_i^{0,+}$ or $z_i^{\infty,+}$, the Laurent expansions terminates at degrees $-k_i^{0,+}$ and $-k_j^{\infty,+}$ respectively, while at any $z_i^{0,-}$ and $z_j^{\infty,-}$ at degrees $k_i^{0,-}$ and $-k_j^{\infty,-}$ respectively. Making these adjustments in above local descriptions, we get the required relations. \square

Remark 2.8. We note that the relations in the above lemma depend only on the multiplicities of non-contractible ends of \bar{u} . Hence, given a moduli space of curves (parametrized or unparametrized) with fixed non-contractible ends, the multiplicities of these ends determine, in mod p , local degrees of lifted base curves at fixed points of τ_b on Σ and the behavior of τ_b around these fixed points.

We point out two immediate necessary conditions for the existence of \bar{u} in terms of the multiplicities of its asymptotic ends.

Lemma 2.9. *Given \bar{u} as with asymptotics (a1), (a2) and (a3), one has*

$$-d = \sum_{i=1}^{n_0^-} k_i^{0,-} + \sum_{i=1}^{n_\infty^-} k_i^{\infty,-} + p \sum_{i=1}^{n_c^-} k_i^- - \sum_{i=1}^{n_0^+} k_i^{0,+} - \sum_{i=1}^{n_\infty^+} k_i^{\infty,+} - p \sum_{i=1}^{n_c^+} k_i^+,$$

where d is the degree of c .

Proof. Let u be a lift of \bar{u} as in (10). Then it has the asymptotics (la1)–(la3). Viewing u as a meromorphic section of the bundle c^*L leads to the desired equation since the left-hand side is the degree of the bundle c^*L and the right-hand side is the degree of the divisor of the section u . \square

Lemma 2.10. *Given \bar{u} as with asymptotics (a1), (a2) and (a3), one has*

$$\sum_{i=1}^{n_0^-} k_i^{0,-} + q \sum_{i=1}^{n_\infty^-} k_i^{\infty,-} - \sum_{i=1}^{n_0^+} k_i^{0,+} - q \sum_{i=1}^{n_\infty^+} k_i^{\infty,+} \equiv 0.$$

Proof. Since \bar{u} is a rational curve, the sum of the homotopy classes of positive ends is equal to the sum of the homotopy classes of negative ends. Writing all homotopy classes in terms of $[\bar{\gamma}_0]$ leads to the above equation since $[\bar{\gamma}_\infty] = q[\bar{\gamma}_0]$ in π_1 . \square

We now have a closer look at the *singular base curve*

$$\bar{c} := \bar{\pi} \circ \bar{u} : \mathbb{C}P^1 \setminus \Gamma \rightarrow \overline{\mathbb{C}P^1},$$

where $\bar{\pi} : \mathbb{R} \times L(p, q) \rightarrow \overline{\mathbb{C}P^1}$ is the projection along the Reeb orbits. Using the extension of c , we get the orbicurve $\bar{c} : \mathbb{C}P^1 \rightarrow \overline{\mathbb{C}P^1}$. We remove singularities $\{\bar{0}, \bar{\infty}\} \subset \overline{\mathbb{C}P^1}$ and consider the map

$$(19) \quad \bar{c} : \mathbb{C}P^1 \setminus P \rightarrow \overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{\infty}\}, \quad P := \bar{c}^{-1}(\{\bar{0}, \bar{\infty}\}),$$

which is a holomorphic branched covering. One can biholomorphically identify $\overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{\infty}\}$ with \mathbb{C}^* , and \bar{c} can be viewed as a holomorphic branched covering of \mathbb{C}^* . Extending over P , we get a holomorphic branched covering

$$(20) \quad \hat{c} : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1,$$

which we call the *smoothened base curve* of \bar{u} . We note that $P = \hat{c}^{-1}(\{0, \infty\})$ and P is in general larger than Γ_{nc} . We put $\tilde{P} := c^{-1}(\{0, \infty\}) = \mathfrak{p}^{-1}(P) \subset \Sigma$.

Lemma 2.11. *The degrees of the maps (19), (20) and (13) coincide.*

Proof. It is clear that degrees of \bar{c} and \hat{c} coincide. The last part of the statement follows since $\bar{c} \circ \mathfrak{p} = \mathfrak{p}_b \circ c$ and both \mathfrak{p} and \mathfrak{p}_b are $p : 1$ coverings. \square

Lemma 2.12. *The ramification profiles of \hat{c} and c satisfy the following.*

- (i) *For all i , the ramification number of \hat{c} at $z_i^{0/\infty, \pm} \in \Gamma_{nc}$ coincides with the ramification number $r_i^{0/\infty, \pm}$ of c at $z_i^{0/\infty, \pm} \in \tilde{\Gamma}_{nc}$ (see Lemma 2.7).*
- (ii) $|\tilde{P} \setminus \tilde{\Gamma}_{nc}| = p|P \setminus \Gamma_{nc}|$.
- (iii) *For any $z \in P \setminus \Gamma_{nc}$, the ramification numbers of c at p preimages of z are all the same.*
- (iv) *For any $z \in P \setminus \Gamma_{nc}$, the ramification number of \hat{c} at z is p times the ramification number of c at any preimage of z .*

Proof. We note that the ramification number of \hat{c} at any $z \in P$ corresponds to the local covering number of (19) around z . Hence the remaining statements follow from the local description of the equation $\bar{c} \circ \mathfrak{p} = \mathfrak{p}_b \circ c$. \square

2.13. Equivariant curves: The sufficient conditions for the existence.

In this section, we discuss the sufficient conditions for the existence of equivariant meromorphic sections of L . It turns out that, when $n_{nc} = 2$, due to Lemma 2.3 the necessary conditions given above are also sufficient. But if $n_{nc} \geq 3$, then there may be an *a priori* obstruction due to the genus of Σ . We postpone the treatment of the first case to the next section, and concerning the second case, we discuss the problem for $n_{nc} = 3$ for the sake of presentation. We further assume $n_c = 0$ since contractible ends do not essentially change the problem and we can omit the ambiguity between parametrized or unparametrized curves; see Remark 2.1. At the end of the section, we point out the necessary modifications for more general types of curves.

Let \mathcal{M} denote the moduli space of J_α -holomorphic curves

$$(21) \quad \bar{u} : \mathbb{C}P^1 \setminus \Gamma \rightarrow \bar{L}^*, \quad \Gamma = \{z^{0,+}, z^{\infty,+}, z^{0,-}\}$$

with the asymptotics

- $\bar{u}(z^{0,+}) = (+\infty, k^{0,+}\bar{\gamma}_0)$,
- $\bar{u}(z^{\infty,+}) = (+\infty, k^{\infty,+}\bar{\gamma}_\infty)$,
- $\bar{u}(z^{0,-}) = (-\infty, k^{0,-}\bar{\gamma}_0)$,

where the necessary conditions given by Lemma 2.10 and Lemma 2.9 are satisfied, namely

$$d := k^{0,+} + k^{\infty,+} - k^{0,-} > 0, \quad k^{0,+} + qk^{\infty,+} - k^{0,-} \equiv 0.$$

Now, given the above data, we define $\mathcal{C}_\mathcal{M}$ to be the moduli space of curves

$$\hat{c} : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1; \quad \hat{c}(z^{0,+}) = 0, \quad \hat{c}(z^{\infty,+}) = \infty, \quad \hat{c}(z^{0,-}) = 0$$

such that

- (cm1) the degree of \hat{c} is d ,
- (cm2) the ramification numbers of \hat{c} satisfies
 - at $z^{0,+}$, $r^{0,+} \equiv k^{0,+}(1 - v)$,
 - at $z^{0,-}$, $r^{0,-} \equiv k^{0,-}(v - 1)$,

$$\begin{array}{ccc}
 \Sigma \setminus \tilde{P} & \xrightarrow{c} & \mathbb{C}P^1 \setminus \{0, \infty\} \\
 \downarrow \mathfrak{p} & & \downarrow \mathfrak{p}_b \\
 \mathbb{C}P^1 \setminus \bar{c}^{-1}(\{\bar{0}, \bar{\infty}\}) & \xrightarrow{\bar{c}} & \overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{\infty}\}
 \end{array}$$

FIGURE 3. Lifting diagram for \bar{c} .

- at $z^{\infty,+}$, $r^{\infty,+} \equiv k^{\infty,+}(1 - q)$,
- for any $z \in P \setminus \Gamma$, the ramification number at z is divisible by p , where $P := \hat{c}^{-1}\{0, \infty\}$.

Given the moduli problem \mathcal{M} , we fix the covering $(\Sigma, \tilde{\Gamma}, \mathfrak{p})$, which exists even if \mathcal{M} is empty; see Remark 2.5. We note that, for each $\bar{u} \in \mathcal{M}$, there is a corresponding smoothed base curve \hat{c} , and due to Lemmas 2.7, 2.2 and 2.12, we know that $\hat{c} \in \mathcal{C}_{\mathcal{M}}$. Now the question is to determine which curves in $\mathcal{C}_{\mathcal{M}}$ provide a curve in \mathcal{M} . Hence we need to reverse the procedure given in the previous section. Now, given $\hat{c} \in \mathcal{C}_{\mathcal{M}}$, it corresponds to a non-singular branched covering

$$\bar{c} : \mathbb{C}P^1 \setminus P \rightarrow \overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{\infty}\},$$

where $P := \hat{c}^{-1}(\{0, \infty\})$. We first need to construct the lifted base curve c . Namely, we should check the diagram in Figure 3 is valid.

Lemma 2.14. *For any $\hat{c} \in \mathcal{C}_{\mathcal{M}}$, the corresponding (non-singular) branched covering \bar{c} lifts through*

$$\mathfrak{p} : \Sigma \setminus \tilde{P} \rightarrow \mathbb{C}P^1 \setminus P,$$

where $\tilde{P} := \mathfrak{p}^{-1}(P)$.

Proof. As in the previous section, we fix $z \in \mathbb{C}P^1 \setminus P$ and put $w := \bar{c}(z)$. We have the induced homomorphism

$$\bar{c}_* : \pi_1(\mathbb{C}P^1 \setminus P, z) \rightarrow \pi_1(\overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{\infty}\}, w) \cong \mathbb{Z},$$

where we fix the generator $\eta := \{we^{it} : t \in [0, \theta]\}$ for the latter group. The covering

$$\mathfrak{p}_b : \mathbb{C}P^1 \setminus \{0, \infty\} \rightarrow \overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{\infty}\}$$

induces the monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$, $1 \rightarrow p$ on the fundamental group, where we fix a generator of the former group as a lift of $p\eta$. Let

$$\rho : \mathbb{Z} \rightarrow \mathbb{Z} / \text{im}(\mathfrak{p}_b)_* = \text{coker}(\mathfrak{p}_b)_* \cong \mathbb{Z}_p = \langle [\eta] \rangle$$

denote the quotient homomorphism. Then $\bar{c} \circ \mathfrak{p}$ lifts if and only if

$$(22) \quad \rho \circ \bar{c}_* \circ \mathfrak{p}_* : \pi_1(\Sigma \setminus \tilde{P}, \tilde{z}) \rightarrow \mathbb{Z}_p$$

is trivial for some $\tilde{z} \in \mathfrak{p}^{-1}(z)$. We note that, by the last statement of (cm2) and the fact that \mathbb{Z}_p is abelian, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathbb{C}P^1 \setminus P, z) & \xrightarrow{\iota_*} & \pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \\ & \searrow \rho \circ \bar{c}_* & \downarrow \rho \circ \bar{c}_* \\ & & \mathbb{Z}_p, \end{array}$$

where the upper horizontal arrow is induced by the inclusion $\iota : \mathbb{C}P^1 \setminus P \hookrightarrow \mathbb{C}P^1 \setminus \Gamma$. Combining this with the with the commutative diagram induced by

$$\begin{array}{ccc} \Sigma \setminus \tilde{P} & \xrightarrow{\tilde{\iota}} & \Sigma \setminus \tilde{\Gamma} \\ \mathfrak{p} \downarrow & & \downarrow \mathfrak{p} \\ \mathbb{C}P^1 \setminus P & \xrightarrow{\iota} & \mathbb{C}P^1 \setminus \Gamma, \end{array}$$

we conclude that if

$$(23) \quad \rho \circ \bar{c}_* \circ \mathfrak{p}_* : \pi_1(\Sigma \setminus \tilde{\Gamma}, \tilde{z}) \rightarrow \mathbb{Z}_p$$

vanishes, then (22) vanishes as well. Once we fix an isomorphism $\varphi : \pi_1(\bar{L}^*) \rightarrow \text{coker}(\mathfrak{p}_b)_* \cong \mathbb{Z}_p$ such that $\varphi([\bar{\gamma}_0]) = [\eta]$, then it is not hard to see that, by (cm2), $\rho \circ \bar{c}_*$ coincides with the homomorphism $\pi_1(\mathbb{C}P^1 \setminus \Gamma, z) \rightarrow \pi_1(\bar{L}^*)$ determined by \mathcal{M} (see Remark 2.5) up to multiplication by $(1 - q)$. But the kernel of the latter homomorphism is precisely the image of \mathfrak{p}_* . Hence (23) is trivial. \square

Remark 2.15. As in Remark 2.4, there are p lifts of given \bar{c} . Once we fix a lift c , the other lifts are given by $\sigma_b^k \circ c, k = 1, \dots, p - 1$.

Given $\hat{c} \in \mathcal{C}_M$, we consider the extension $c : \Sigma \rightarrow \mathbb{C}P^1$ of a lift of \bar{u} such that $c \circ \tau_b = \sigma_b \circ c$, where τ_b is a fixed generator of the group G acting on Σ . The question is now to determine whether there is a meromorphic section $u : \Sigma \rightarrow c^*L$ such that $u \circ \tau_b = \tau \circ u$, where τ is the corresponding generator of the \mathbb{Z}_p action on c^*L and zeros and poles of u are given by the moduli problem \mathcal{M} . Namely, we ask for a pole at $z^{0,+}$ of order $k^{0,+}$, a pole at $z^{\infty,+}$ of order $k^{\infty,+}$ and a zero at $z^{0,-}$ of order $k^{0,-}$ (as in the previous section, $z^{i,\pm}$ denotes the punctures in $\tilde{\Gamma}_{nc}$ corresponding to punctures in Γ_{nc}).

Lemma 2.16. *Assume that there is meromorphic section u of c^*L with zeros and poles determined by \mathcal{M} , and c is equivariant. Then u is also equivariant.*

Proof. Given fixed zeros and poles, one has a \mathbb{C}^* -family of meromorphic sections given by $\lambda u, \lambda \in \mathbb{C}^*$. Note that, since the scaling along fibers commutes with τ -action, if there is some equivariant meromorphic section u , then all meromorphic sections of the form λu are equivariant.

Let u be a meromorphic section of c^*L with fixed zeros and poles, where c is equivariant. Then $u' := \tau^{-1} \circ u \circ \tau_b$ is also a meromorphic section of c^*L having same zeros and poles with u . Hence there is some $\lambda \in \mathbb{C}^*$ such that $u' = \lambda u$. In fact, $\lambda \in S^1$ since all the actions we have are unitary. Both u and u'

have a pole at $z^{\infty,+}$ of order $k^{\infty,0}$. We know that the ramification number of c at $z^{\infty,+}$ coincides with \hat{c} and satisfies

$$(24) \quad k^{\infty,+} \equiv r^{\infty,+}(1 - q)^{-1}$$

by the definition of $\mathcal{C}_{\mathcal{M}}$. We take the local trivialization of c^*L around $z^{\infty,+} \in \Sigma$ as in the previous section (see Lemma 2.7) so that, around $z^{\infty,+}$, we have $\tau(z, \lambda) = (e^{im\theta}z, e^{i\theta}\lambda)$ and $\tau_b(z) = e^{im\theta}z$, where

$$(25) \quad r^{\infty,+}m \equiv (q - 1).$$

With respect to the above trivialization, we write $u(z) = (z, f(z))$ and $u'(z) = (z, f'(z))$, where f and f' are meromorphic functions. With polar coordinates $z = \rho e^{it}$, we have

$$\lim_{\rho \rightarrow 0} \frac{f(\rho e^{it})}{|f(\rho e^{it})|} = e^{-ik^{\infty,+}t}, \quad \lim_{\rho \rightarrow 0} \frac{f'(\rho e^{it})}{|f'(\rho e^{it})|} = \lambda e^{-ik^{\infty,+}t}.$$

On the other hand, definition of u' gives

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{f'(\rho e^{it})}{|f'(\rho e^{it})|} &= \lim_{\rho \rightarrow 0} \frac{\tau^{-1} \circ f(\tau_b(\rho e^{it}))}{|\tau^{-1} \circ f(\tau_b(\rho e^{it}))|} = \lim_{\rho \rightarrow 0} \frac{e^{-i\theta} f(\rho e^{it+im\theta})}{|f(\rho e^{it+im\theta})|} \\ &= e^{-i\theta} (e^{-ik^{\infty,+}(t+m\theta)}) = e^{-ik^{\infty,+}t} e^{-i(k^{\infty,+}m+1)\theta} = e^{-ik^{\infty,+}t}, \end{aligned}$$

where the last equation follows from (24) and (25). Hence $\lambda = 1$, and therefore, $\tau \circ u = u \circ \tau_b$. □

We want to understand the moduli space $\mathcal{C}_{\mathcal{M}}$ and possible obstructions on the existence of meromorphic sections over the lifts of singular curves, which correspond to the elements of $\mathcal{C}_{\mathcal{M}}$. To this end, we recall some basic notions in the theory of Riemann surfaces; see [14] for details.

Given a closed Riemann surface Σ with genus g and $[c] \in H_1(X, \mathbb{Z})$, there is a functional on the (vector) space $\Omega^1(\Sigma)$ consisting of holomorphic 1-forms on Σ ,

$$\int_{[c]} : \Omega^1(\Sigma) \rightarrow \mathbb{C}, \quad \omega \mapsto \int_c \omega,$$

which is well-defined since any holomorphic 1-form on Σ is necessarily closed. An element of $\Omega^1(\Sigma)^*$ is called a *period* if it is of the above form. The *Jacobian* of Σ is the quotient space

$$J(\Sigma) := \Omega^1(\Sigma)^* / \Lambda,$$

where Λ is the space of periods. It turns out that $J(\Sigma)$ is isomorphic to the complex torus of dimension g .

Let z_0 be a point in Σ . We consider the map

$$A : \Sigma \rightarrow \Omega^1(\Sigma)^*; \quad A(z)(\omega) := \int_{\gamma_z} \omega, \quad \omega \in \Omega^1(\Sigma),$$

where γ_z is some path connecting z_0 to z . Although this map depends on γ_z , it descends to the so-called *Abel map* $A : \Sigma \rightarrow J(\Sigma)$ which is independent of the choice of γ_z . Note that the Abel map naturally extends to a group

homomorphism over the group $\text{Div}(\Sigma)$ of divisors via $A(\sum n_z z) := \sum n_z A(z)$. It turns out that, once restricted to the subgroup $\text{Div}_0(\Sigma) = \ker \text{deg}$, where $\text{deg} : \text{Div}(\Sigma) \rightarrow \mathbb{Z}$ is the degree map, the Abel map is independent of the point z_0 .

$\text{Div}_0(\Sigma)$ has a special subgroup $\text{PDiv}(\Sigma)$ consisting of *principal divisors*, namely the divisors given by meromorphic functions. Given a meromorphic function f on Σ , then its divisor is given by $D(f) = D_0(f) - D_\infty(f)$, where

$$D_0(f) := \sum_{f(z)=0} o_z(f)z, \quad D_\infty(f) := \sum_{f(z)=\infty} o_z(f)z$$

and $o_z(f) > 0$ stands for the order of zeros/poles. It turns out that the map

$$A : \text{Div}_0(\Sigma) \rightarrow J(\Sigma)$$

is surjective and its kernel is given by $\text{PDiv}(\Sigma)$. Hence one gets the isomorphism

$$\text{Pic}(\Sigma) = \text{Div}_0(\Sigma) / \text{PDiv}(\Sigma) \cong J(\Sigma),$$

where the *Picard group* $\text{Pic}(\Sigma)$ is the group of isomorphism classes of degree zero line bundles over Σ . Moreover, if $A([D]) = 0$ for some $[D] \in \text{Pic}(\Sigma)$, then the divisor D defines the trivial bundle and we have the following consequence. If D_1 and D_2 are two divisors, which define a line bundle of the same degree, then these bundles are isomorphic if and only if $A(D_1 - D_2) = 0$, and this is trivially the case if $D_1 - D_2 = 0$. We let $\mathfrak{S}^n(X)$ denote the n -fold symmetric product of the set X .

Setting the ground, we first give a description of $\mathcal{C}_{\mathcal{M}}$, which is adopted from [15].

Lemma 2.17. $\mathcal{C}_{\mathcal{M}}$ is biholomorphic to

$$((\mathfrak{S}^{n_0}(\mathbb{C}P^1 \setminus \{z^{\infty,+}\}) \times \mathfrak{S}^{n_\infty}(\mathbb{C}P^1 \setminus \{z^{0,-}, z^{0,+}\})) \setminus \Delta) \times \mathbb{C}^*,$$

where Δ is the subset of pairs (D_0, D_∞) in $\mathfrak{S}^{n_0}(\mathbb{C}P^1 \setminus \{z^{\infty,+}\}) \times \mathfrak{S}^{n_\infty}(\mathbb{C}P^1 \setminus \{z^{0,-}, z^{0,+}\})$ with at least one common point and

$$n_0 := \frac{d - \bar{r}^{0,+} - \bar{r}^{0,-}}{p}, \quad n_\infty := \frac{d - \bar{r}^{\infty,+}}{p}.$$

Proof. The moduli space $\mathcal{C}_{\mathcal{M}}$ itself can be seen as the space of meromorphic functions on $\mathbb{C}P^1$, which has a particular distribution of its zeros and poles. We note that, in this case, the Abel map vanishes identically, that is, any degree zero divisor defines a \mathbb{C}^* -family of meromorphic functions. Hence it is enough to characterize the set of divisors satisfying conditions (cm1) and (cm1).

Condition (cm1) tells us that $\text{deg}(D_0(\hat{c})) = \text{deg}(D_\infty(\hat{c})) = d$. Namely, we have complex d -dimensional freedom to choose the zeros or poles. The conditions given by (cm2) translate as follows. We have two zeros $z^{0,-}, z^{0,+}$ and a pole $z^{\infty,+}$ whose orders, which correspond to the ramification numbers, are fixed in mod p . We fix

$$\bar{r}^{0,+}, \bar{r}^{\infty,-}, \bar{r}^{0,+} \in \{1, \dots, p - 1\}$$

such that

$$\bar{r}^{0,+} \equiv r^{0,+}, \quad \bar{r}^{\infty,+} \equiv r^{\infty,+}, \quad \bar{r}^{0,-} \equiv r^{0,-}.$$

Now we can perturb away $(r^{0,+} - \bar{r}^{0,+})$ zeros at $z^{0,+}$, but any zero different than $z^{0,\pm}$ has to have order divisible by p . The same reasoning applies to $z^{0,+}$ and $z^{\infty,+}$. Hence, for zeros, we have $(d - \bar{r}^{0,+} - \bar{r}^{0,-})/p$ choices among the points in $\mathbb{C}P^1 \setminus \{z^{\infty,+}\}$, and for poles, we have $(d - \bar{r}^{\infty,+})/p$ choices among the points in $\mathbb{C}P^1 \setminus \{z^{0,+}, z^{0,-}\}$. We note that $d \equiv \bar{r}^{0,+} + \bar{r}^{0,-} \equiv \bar{r}^{\infty,+}$. After taking suitable symmetric products of our domain and removing collections of points which do not result in divisors, we get the above description. \square

We note that

$$\mathfrak{S}^{n_0}(\mathbb{C}P^1 \setminus \{z^{\infty,+}\}) \times \mathfrak{S}^{n_\infty}(\mathbb{C}P^1 \setminus \{z^{0,-}, z^{0,+}\})$$

is a complex manifold and Δ is an irreducible subvariety of (complex) codimension one [15]. In particular, the dimension of \mathcal{C}_M is given by

$$\dim_{\mathbb{R}} \mathcal{C}_M = 2 + 2n_0 + 2n_\infty = 2 + \frac{2}{p}(2d - \bar{r}^{0,+} - \bar{r}^{0,-} - \bar{r}^{\infty,+}).$$

Now the question is that, given $\hat{c} \in \mathcal{C}_M$, does c^*L admit a meromorphic section with zeros and poles determined by \mathcal{M} for some lift c of corresponding \bar{c} . It turns out that, by the very nature of the equivariant picture, there is no obstruction to the existence of such meromorphic sections.

Proposition 2.18. *Given $\hat{c} \in \mathcal{C}_M$, there exists a meromorphic section of c^*L , which leads to a punctured curve in \mathcal{M} , where c is a lift of the singular curve \bar{c} corresponding to \hat{c} .*

Proof. We define the divisor D_M of Σ by

$$D_M = k^{0,-}z^{0,-} - k^{0,+}z^{0,+} - k^{\infty,+}z^{\infty,+}.$$

We note that the isomorphism class of the line bundle L is determined by the isomorphism class of divisors on $\mathbb{C}P^1$, whose degree is -1 . We fix a divisor in this class of the form

$$D_L := l \cdot 0 - (l + 1) \cdot \infty, \quad l > 0,$$

where l is to be chosen. Since the phase shift on \mathcal{C}_M is not relevant for our problem, we ignore it in what follows. The lifting scheme above provides the following continuous embedding:

$$\mathcal{T} : \mathcal{C}_M \rightarrow \text{Div}^0(\Sigma), \quad \hat{c} \mapsto D_M - c^*D_L,$$

where c is (up to phase shift) the unique lift of \hat{c} corresponding to \hat{c} satisfying

$$c \circ \tau_b = \sigma_b \circ c$$

and c^*D_L is the pullback divisor. We note that $\text{deg}(D_M) = \text{deg}(c^*D_L) = d$ so that the map is well-defined. Now we want to show that there exists some $l \geq 0$ such that $A \circ \mathcal{T}(\hat{c}) = 0$ for a curve \hat{c} satisfying

$$D_0(\hat{c}) = r^{0,+}z^{0,+} + r^{0,-}z^{0,-}, \quad D_\infty(\hat{c}) = r^{\infty,+}z^{\infty,+}.$$

In this case,

$$D_{\mathcal{M}} - c^*D_L = (k^{0,-} - lr^{0,-})z^{0,-} + (-k^{0,+} - lr^{0,+})z^{0,+} + ((l + 1)r^{\infty,+} - k^{\infty,+})z^{\infty,+}.$$

First we want to find l such that all the coefficients above are zero in mod p , and by (cm2), we see that this holds for $l \equiv (1 - v)^{-1}$. We choose some $l > 0$ such that $l \equiv (1 - v)^{-1}$. Then

$$D_{\mathcal{M}} - c^*D_L = p(m^{0,-}z^{0,-} + m^{0,+}z^{0,+} + m^{\infty,+}z^{\infty,+})$$

for some $m^{0/\infty,\pm} \in \mathbb{Z}$. We note that the divisor

$$m^{0,-}z^{0,-} + m^{0,+}z^{0,+} + m^{\infty,+}z^{\infty,+}$$

descends to a divisor on $\mathbb{C}P^1$, where $z^{0/\infty,\pm}$ stands for the images of $z^{0/\infty,\pm}$ under \mathfrak{p} . Moreover, this divisor has degree 0. Then we know that there is a meromorphic function, say f on $\mathbb{C}P^1$ which realizes this divisor. Then $D_{\mathcal{M}} - c^*D_L$ is precisely the divisor for the meromorphic function $f \circ \mathfrak{p}$ on Σ . Hence we have $A(D_{\mathcal{M}} - c^*D_L) = 0$. □

Corollary 2.19. *We have $\mathcal{M} \cong \mathcal{C}_{\mathcal{M}} \times \mathbb{C}^*$, and in particular,*

$$\dim_{\mathbb{R}} \mathcal{M} = 4 + \frac{2}{p}(2d - \bar{r}^{0,+} - \bar{r}^{0,-} - \bar{r}^{\infty,+}).$$

Remark 2.20. Note that, given the moduli problem (21), one reads of the degree d and the quantities $\bar{r}^{0/\infty,\pm}$ from the multiplicities and immediately gets the dimension of the moduli space. Then one can check the Fredholm regularity of J_{α} -holomorphic curves in \mathcal{M} immediately by comparing the above dimension with the virtual dimension of the moduli space; see Section 3.6 for the definitions. Such a comparison will be carried out for a pair of pants in Section 3.6, and the arguments used for the pair of pants immediately generalizes to other configurations. In fact, we claim that moduli spaces of rational J_{α} -holomorphic curves are cut out transversally.

3. COMPUTATIONS: PAIR OF PANTS, CYLINDERS AND OTHERS

As we saw above, if $n_{nc} = 2$, then one has $g = 0$. In this case, the covering \mathfrak{p} can be studied more explicitly and one can study all possible lifted base curves, which lead to the different nonempty components of the moduli space. In this section, we study the pair of pants with two positive non-contractible ends in detail and comment on other kinds of moduli problems with $n_{nc} = 2$ as well.

After determining the nonempty moduli spaces of a pair of pants, we compute the dimensions of these moduli spaces in terms of the dimensions of equivariant moduli spaces in the lift (see Remark 2.20) and compare them with the virtual dimension of these moduli spaces given by the well-known index formula (31). The observation is that the dimension of the equivariant moduli space coincides with the index of the problem, and this establishes the regularity of the almost complex structure J_{α} ; see Remark 2.20.

3.1. The moduli space of pair of pants. We consider the following moduli problem \mathcal{M} of J_α -holomorphic curves:

$$(26) \quad \bar{u} : \mathbb{C}P^1 \setminus \{0, 1, \infty\} \rightarrow \mathbb{R} \times L(p, q)$$

with asymptotics

$$(27) \quad \bar{u}(0) = (+\infty, k^0 \bar{\gamma}_0), \quad \bar{u}(\infty) = (+\infty, k^\infty \bar{\gamma}_\infty), \quad \bar{u}(1) \in \{-\infty\} \times S_k,$$

where $k^0, k^\infty \neq 0$ and S_k denotes the orbit space of contractible orbits of action $2\pi k$.

In order to work with simpler terms, we choose a model for the domain of (26) as follows. We consider the $G \cong \mathbb{Z}_p$ -action on $\mathbb{C}P^1$ given by

$$\tau_b((z : 1)) = (e^{im\theta} z : 1),$$

where $m \in \{1, \dots, p - 1\}$ and $\theta = 2\pi/p$. The quotient map leads to the covering map

$$p : \mathbb{C}P^1 \setminus \{0, 1, w_1, \dots, w_{p-1}, \infty\} \rightarrow \overline{\mathbb{C}P^1} \setminus \{\bar{0}, \bar{1}, \bar{\infty}\}.$$

We identify the quotient space above with the our domain $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$.

Now we need to determine lifted base curves, namely the equivariant holomorphic maps $c : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$, where the $\langle \sigma_b \rangle$ -action on the range is given by

$$\sigma_b((z_1 : z_2)) = (e^{i(1-q)\theta} z_1 : z_2).$$

Once we parametrize the domain and the range via $z \mapsto (z : 1)$, any nontrivial holomorphic map c is given by $c(z) = \lambda g(z)/h(z)$, where $\lambda \in \mathbb{C}^*$ and g and h are monic polynomials without a common root. Imposing the equivariance leads to the following characterization.

Lemma 3.2. *A nontrivial holomorphic map c satisfies $c \circ \tau_b = \sigma_b \circ c$ if and only if it has the following form:*

$$c(z) = \lambda z^r g(z)/h(z),$$

where $mr \equiv 1 - q$ and

$$g(z) = \prod_{s=1}^n (z^p - a_s)^{k_s}, \quad h(z) = \prod_{t=1}^m (z^p - b_t)^{l_t}$$

such that $\lambda \in \mathbb{C}^*$, $r \in \{\mp 1, \mp 2, \dots, \mp(p - 1)\}$, $k_s, l_t \in \mathbb{N}$ and $a_s, b_t \in \mathbb{C}$.

Proof. It is clear that such a map is equivariant. For the other direction, one sees that if c admits $w \in \mathbb{C}P^1 \setminus \{0, \infty\}$ as a zero or pole, then it must admit $p - 1$ distinct zeros/poles which are given by the orbit of w . Hence these polynomials have to factorize through terms like $(z^p - a_s)^{a_s}$ and $(z^p - b_t)^{b_t}$, where a_s and b_t are any nonzero complex numbers. The only thing that requires attention then is the case where we have zero or infinity as zero or pole. Checking the equivariance, one gets the relation above between m and r . □

Now we restate the equivariance conditions given by Lemma 2.7 in this special case. By (27), we have $c(0) = 0$ and $c(\infty) = \infty$, and by (15) and (17), an equivariant section u locally looks like $(z, f(z))$, where

$$\begin{aligned} f(e^{im\theta} z) &= e^{iq\theta} \quad \text{around } 0, \\ f(e^{-im\theta} z) &= e^{i\theta} \quad \text{around } \infty. \end{aligned}$$

Now, if $f(z) = \sum a_n z^n$ around 0, we have

$$a_n \neq 0 \implies n \equiv m^{-1}q.$$

Similarly, if $f(z) = \sum b_n z^n$ around ∞ , we have

$$b_n \neq 0 \implies n \equiv -m^{-1}.$$

Since we require that u has poles of order k^0 at 0 and of order k^∞ at ∞ , we get $-k^0 \equiv m^{-1}q$ and $-k^\infty \equiv -m^{-1}$. This means that

$$(28) \quad k^0 \equiv -m^{-1}q \equiv rq(q-1)^{-1} \quad \text{and} \quad k^\infty \equiv m^{-1} \equiv r(1-q)^{-1}.$$

Since $c(0) = 0$ and $c(\infty) = \infty$, we have $r > 0$ and $r + p \sum k_s > \sum l_t$ so that the degree of c is given by $\deg(c) = r + p \sum k_s$. We note that

$$-k^0 - k^\infty \equiv m^{-1}q - m^{-1} \equiv -m^{-1}(1-q) \equiv -r \equiv -\deg(c) \equiv \deg(c^*L).$$

We also require that u has p zeros of order k . Hence, for suitable choice of k , we have

$$(29) \quad pk - k^0 - k^\infty = -r - p \sum k_s = \deg(c^*L)$$

so that the divisor of u has the correct degree. Therefore, in the lift, we have a curve with two positive ends asymptotic to $k^0\gamma_0$ and $k^\infty\gamma_\infty$ and p negative ends with multiplicity k . Passing to the quotient, we get a pair of pants in the moduli space we look for.

Remark 3.3. We remark that, in terms of Lemma 2.7, we have $m^{0,+} = m$ and $m^{\infty,+} = -m$. Moreover, $r > 0$ implies that the ramification numbers $r^{0/\infty,+}$ given in Lemma 2.7 coincide with r in mod p and (28) coincides with the conditions given in Lemma 2.7.

Remark 3.4. In the previous section, we constructed the covering $(\Sigma, \tilde{\Gamma}, \mathfrak{p})$ together with a fixed generator $\tau_{\mathfrak{b}}$ for the Deck group so that, given the moduli space with fixed asymptotics, the equivariant curves are given by $u \circ \tau_{\mathfrak{b}} = \sigma \circ u$. In the above treatment, the representation m of $\tau_{\mathfrak{b}}$ determines r and hence the homotopy classes of positive ends. Hence different choices of $\tau_{\mathfrak{b}}$, equivalently r , lead to different homotopy classes of non-contractible ends and therefore different components of the moduli space.

We want to compute the dimension of the moduli space \mathcal{M} using the above description. We note that the formula given in Corollary 2.19 immediately applies here. Nevertheless, we repeat this computation by directly looking at the lifted base curves.

Let c be a curve as in Lemma 3.2. As a lifted base curve, c contributes to the dimension of the moduli spaces in only two ways. One is the freedom of perturbing the roots of g and h , and the other is perturbing the constant λ . In fact, if one moves the “roots” of z^r , then the resulting nonzero root has to appear p times, but this is not possible since this makes the degree of the map jump. The contribution of moving “roots” of g and h is a little delicate. Given a term like $(z^r - a_s)^{k_s}$ in the factorizations of these polynomials, one has to move roots of a_s simultaneously to keep invariance, that is, we only have the freedom of moving a_s 's and b_t 's. More precisely, if d is the degree of a given base curve c , then the contribution of the base curve to the dimension of the moduli space is $4\lfloor d/p \rfloor + 2$.

For the problem \mathcal{M} given by (26) and (27), the lifted base curves satisfy $r > 0$ and $r + p \sum k_s > \sum l_t$ so that the degree of c is given by $r + p \sum k_s$ and therefore $\lfloor d/p \rfloor = \sum k_s$. Hence, for fixed $a > 0$ and $d^I := \lfloor d/p \rfloor \geq 0$, the dimension of \mathcal{M} is given by (compare to Corollary 2.19)

$$(30) \quad 4 + 4d^I,$$

where we add 2-dimensional freedom of rescaling sections.

Remark 3.5 (Components of the moduli space). The description given above makes it easy to determine the components of the moduli space of pants with two non-contractible ends in general. For the moduli space of curves given by (26) and (27), when d^I and the multiplicity of the contractible end are fixed, we have $p - 1$ components, corresponding to each value of $0 < r < p$ and therefore to each nontrivial homotopy class of ends. We note that the minimal dimension for the moduli space of pair of pants is 4, and it corresponds to the case $d^I = 0$. If $d^I = 0$, then each component may be identified with $\mathbb{C}^* \times \mathbb{C}^*$, where one factor stands for the freedom of perturbing λ and the other corresponds to the rescaling of the meromorphic section.

We note that, once d^I is fixed, one may increase multiplicities of all ends simultaneously in such a way that the degree condition is satisfied and the dimension is unchanged. Hence, once we fix the dimension, the moduli space of the pair of pants, whose asymptotics at punctures $0, \infty$ are fixed geometrically as above, can be written as

$$\bigcup_{d^I \geq 0} \bigcup_{k \geq 0} \bigcup_{a=1}^{p-1} \mathcal{M}_{d^I, k, a}, \quad \dim \mathcal{M}_{d^I, k, a} = 4 + 4d^I,$$

where k is the multiplicity of the contractible end.

3.6. The index of pair of pants. In the previous section, we studied the moduli space of a pair of pants with a hands-on approach, and at the end, we endow the moduli space with a smooth atlas so that it gets the structure of a smooth manifold with a prescribed dimension. Since we later want to perturb the almost complex structure J_α , we need to establish that the moduli space of a J_α -holomorphic pair of pants is cut out transversally as the zero set of the associated Cauchy–Riemann operator (7).

To this end, we first recall the generalities on the virtual dimension of moduli spaces and sufficient conditions for endowing them with smooth structure without going into functional analytic details. Consider a general moduli space \mathcal{M} given by (6), where $(M, \xi = \ker \alpha)$ is a 3-dimensional Morse–Bott contact manifold. As in the non-degenerate case, the virtual dimension of a moduli space is determined by the Conley–Zehnder indices of the asymptotic ends, where the Conley–Zehnder index is suitably generalized to the Morse–Bott situation. We consider the generalization given in [17], which is axiomatically described as follows in dimension two [8].

Let $\Sigma(1)$ be the space of paths $\varphi : [0, 1] \rightarrow \text{Sp}(1)$ with $\varphi(0) = I$, where $\text{Sp}(1)$ is the space of 2-by-2 symplectic matrices and I is the identity matrix. The *Conley–Zehnder index* is a unique map $\mu : \Sigma(1) \rightarrow \frac{1}{2}\mathbb{Z}$ characterized by the following axioms.

(CZ1) μ is constant on homotopies $\varphi_s \in \Sigma(1)$ for which $\dim \ker(\varphi_s(1) - I)$ is constant.

(CZ2) If $\varphi \in \Sigma(1)$ and $\psi : \mathbb{R}/\mathbb{Z} \rightarrow \text{Sp}(1)$ is a loop, then

$$\mu(\psi\varphi) = \mu(\varphi) + 2m(\psi),$$

where $m(\psi)$ is the Maslov index of ψ .

(CZ3) If $\varphi \in \Sigma(1)$ and $\varphi^{-1} \in \Sigma(1)$ is the corresponding path of inverses, then

$$\mu(\varphi) + \mu(\varphi^{-1}) = 0.$$

(CZ4) $\mu(e^{i\pi t}) = 1$ and if $\varphi(t) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$, then $\mu(\varphi) = \frac{1}{2}$.

Recall that, for any $x \in M$ and $t \in \mathbb{R}$, the linearization of the Reeb flow ϕ_t leads to a symplectic map

$$d\phi_t(x) : (\xi_x, (d\alpha)_x) \rightarrow (\xi_{\phi_t(x)}, (d\alpha)_{\phi_t(x)}).$$

Let γ be a closed Reeb orbit with period $T > 0$. We fix a symplectic trivialization

$$\Phi : S^1 \times \mathbb{R}^2 \rightarrow \gamma(T \cdot)^*\xi,$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. Then the Conley–Zehnder index of the orbit γ with respect to Φ is given by

$$\mu^\Phi(\gamma) := \mu(\{t \mapsto \Phi^{-1}(t) \circ d\phi_{Tt}(\tilde{\gamma}(0)) \circ \Phi(0)\}).$$

Let $C = [\Sigma, j, \Gamma, u] \in \mathcal{M}$. We pick a collection $\{\Phi_{z_i^\pm}\}$ of trivializations for the asymptotic ends $\{\gamma_i^\pm\}$. Then we consider the complex line bundle $(u^*\xi, J) \rightarrow \Sigma \setminus \Gamma$. Let $U \subset \Sigma$ be an open neighborhood of the puncture set Γ , consisting of disks centered at each puncture. We endow each such disk with cylindrical coordinates via

$$[0, +\infty) \times S^1 \rightarrow \mathbb{D}, \quad (s, t) \mapsto e^{-2\pi(s+it)},$$

and we extend $\{\Phi_z\}$ to a complex trivialization $\Phi : U \times \mathbb{C} \rightarrow (u^*\xi, J)|_U$ and define the *first Chern number of u relative to Φ* to be the signed count

$$c_1^\Phi(u) := \#s^{-1}(0),$$

where s is a generic section of $u^*\xi$ such that $\Phi(s) = 1$. Having all the ingredients at hand, the *Fredholm index* of $C = [\Sigma, j, \Gamma, u] \in \mathcal{M}$ (or the *index of C* for short) given in [1] reads as⁴

$$(31) \quad \begin{aligned} \text{ind}(C) = 2c_1^\Phi(u) + \sum_{z_i^+ \in \Gamma^+} \mu^\Phi(\gamma_i^+) - \sum_{z_i^- \in \Gamma^-} \mu^\Phi(\gamma_i^-) \\ + \frac{1}{2} \sum_{z_i^\pm \in \Gamma} \dim S_i^\pm + \#\Gamma - \chi(\Sigma). \end{aligned}$$

We recall that $\text{ind}(C)$ is precisely the index of the Fredholm operator associated to the curve C so that, once this Fredholm operator is surjective, an open neighborhood of $C \in \mathcal{M}$ is cut out transversally as the zero set of the Cauchy–Riemann operator (7). Hence \mathcal{M} gains a smooth structure (of an orbifold in general) near C , and the kernel of the Fredholm operator defines the tangent space $T_C\mathcal{M}$. In such a circumstance, we say that the J -holomorphic curve C is (*Fredholm*) *regular*. We note that the Fredholm index is constant on a component of the moduli space. In this sense, the Fredholm index is the *virtual dimension* of a given component of the moduli space, which turns out to be the actual dimension once the component is cut out transversally.

Now we want to compute the index given by (31) for any curve in the moduli space \mathcal{M} of a pair of pants given by (26) and (27). We note that this makes sense due to Remark 2.1. Our strategy is to utilize the lifting procedure here as well.

We first fix “trivializations” of ξ_0 and ξ . Using (1), we define a non-vanishing section

$$s : S^3 \rightarrow \mathbb{C}^2; \quad (z_1, z_2) \mapsto (\bar{z}_2, -\bar{z}_1)$$

of ξ_0 and get a global complex trivialization

$$(32) \quad \Phi_0 : S^3 \times \mathbb{C} \rightarrow \xi_0, \quad ((z_1, z_2), \lambda) \mapsto \lambda s(z_1, z_2).$$

Note that, on ξ_0 , the standard almost complex structure J_0 coincides with i . As mentioned in the introduction, $\xi \rightarrow L(p, q)$ is nontrivial. Nevertheless, it is convenient to fix a section of it which vanishes along a mild subset. We construct such a section as a quotient of a section of ξ_0 as follows. We define a section

$$k(z_1, z_2) := f(z_1, z_2)s(z_1, z_2), \quad f(z_1, z_2) := z_1^{q+1} + z_2^{v+1}.$$

An easy computation shows that the section $k : S^3 \rightarrow \xi_0 \subset TS^3$ is equivariant in the sense that

$$k(\sigma(z_1, z_2)) = d\sigma_{(z_1, z_2)}[k(z_1, z_2)],$$

where $\langle d\sigma \rangle$ is the induced action on TS^3 for which ξ_0 is invariant. We note that k vanishes along a torus knot $K \subset S^3$, which does not intersect with γ_0

⁴Although we cite different sources for the Conley–Zehnder index and the Fredholm index, the formulas we used are taken from [21]. This choice is made in order to avoid possible inconsistencies due to the conventions used by other sources.

or γ_∞ . It is not hard to see that K descends to a curve that is homologous to $(q + 1)[\overline{\gamma}_0]$ in $L(p, q)$. We define another trivialization of ξ_0 away from K ,

$$(33) \quad \Phi : (S^3 \setminus K \times \mathbb{C}) \rightarrow \xi_0, \quad ((z_1, z_2), \lambda) \mapsto \lambda k(z_1, z_2),$$

so that Φ induces a trivialization of ξ away from $\mathfrak{p}(K)$ given by

$$\overline{\Phi} : (L(p, q) \setminus \mathfrak{p}(K)) \times \mathbb{C} \rightarrow \xi \quad \text{such that} \quad \overline{\Phi} \circ (\mathfrak{p} \times \text{id}) = d\mathfrak{p} \circ \Phi.$$

We use this trivialization along the orbits which are away from the vanishing set $\mathfrak{p}(K)$.

We first compute the Conley–Zehnder indices of the positive ends given by (27). Due to the definition of $\overline{\Phi}$, instead of considering the flow of R along $k^\infty \overline{\gamma}_\infty$, we consider the flow of R_0 along the lifted Reeb arc

$$[0, k^\infty \theta] \rightarrow S^3, \quad t \mapsto (e^{it}, 0); \quad \theta = 2\pi/p,$$

together with the trivialization Φ . The resulting symplectic arc $\varphi_\infty(t)$ is given by

$$\varphi_\infty(t) = \Phi^{-1}(e^{itk^\infty \theta}, 0) \circ d\phi_{(tk^\infty \theta)}(1, 0) \circ \Phi(1, 0).$$

Viewing $\varphi_\infty(t) \in \mathbb{C}$, we compute

$$\begin{aligned} \varphi_\infty(t)\Phi(e^{itk^\infty \theta}, 0) &= d\phi_{(tk^\infty \theta)}(1, 0)[f(1, 0)(0, -1)], \\ \varphi_\infty(t)f(e^{itk^\infty \theta}, 0)s(e^{itk^\infty \theta}, 0) &= (0, -e^{itk^\infty \theta}), \\ \varphi_\infty(t)e^{it(q+1)k^\infty \theta}(0, -e^{-itk^\infty \theta}) &= (0, -e^{itk^\infty \theta}) \end{aligned}$$

and get

$$\varphi_\infty(t) = e^{it2\pi \frac{k^\infty(1-q)}{p}}.$$

A similar computation for $k^0 \overline{\gamma}_0$ leads to the arc

$$\varphi_0(t) = e^{it2\pi \frac{k^0(1-v)}{p}}.$$

We note that the orbits $k \overline{\gamma}_0/\infty$ are non-degenerate if and only if $k \not\equiv 0$ since $v, q > 1$. Hence we can compute the Conley–Zehnder indices of $k_{0/\infty} \overline{\gamma}_0/\infty$ using a standard computational recipe; see [8]. We note that

$$\det_{\mathbb{R}}(I - \varphi_{0/\infty}(1)) > 0,$$

that is, $\varphi_{0/\infty}(1)$ and $-I = e^{i\pi}$ are in the same component of non-degenerate symplectic matrices. We connect

$$\varphi_\infty(1) = e^{i2\pi \frac{k^\infty(1-q)}{p}} \quad \text{to} \quad -I = e^{i2\pi(\lfloor \frac{k^\infty(1-q)}{p} \rfloor + \frac{1}{2})}$$

by a rotation that does not hit the Maslov cycle. Squaring the resulting loop and computing the degree leads to

$$\mu_{\overline{\Phi}}(k^\infty \overline{\gamma}_\infty) = 2 \left\lfloor \frac{k^\infty(1-q)}{p} \right\rfloor + 1.$$

A similar argument gives

$$\mu_{\overline{\Phi}}(k^0 \overline{\gamma}_0) = 2 \left\lfloor \frac{k^0(1-v)}{p} \right\rfloor + 1.$$

Lemma 3.7. *The Fredholm index of a pair of pants given by (26) and (27) is given by*

$$(34) \quad \text{ind}(\bar{u}) = \mu^{\bar{\Phi}}(k^\infty \bar{\gamma}_\infty) + \mu^{\bar{\Phi}}(k^0 \bar{\gamma}_0) + \frac{2}{p}(d + k^0 v + k^\infty q) - 2k + 2,$$

where $d = k^0 + k^\infty - pk$.

Proof. Let \bar{u} be given by (26) and (27). We can safely assume that $\bar{u}(1) = (-\infty, k\bar{\gamma})$ and $\bar{\gamma}$ is away from the vanishing set of $\bar{\Phi}$. Then, by (31), the index of \bar{u} is given by

$$(35) \quad \text{ind}(\bar{u}) = 2c_1^{\bar{\Phi}}(\bar{u}) + \mu^{\bar{\Phi}}(k^0 \bar{\gamma}_0) + \mu^{\bar{\Phi}}(k^\infty \bar{\gamma}_\infty) - \mu^{\bar{\Phi}}(k\bar{\gamma}) + 2.$$

Now, instead of directly computing the unknown terms, we utilize the lifting procedure again. Let u be a lift of \bar{u} . We consider u as a representative of an unparametrized curve C and compute the index via formula (31) and the trivialization Φ given by (33). We get

$$\begin{aligned} \text{ind}(C) &= 2c_1^\Phi(u) + \mu^\Phi(k^0 \gamma_0) + \mu^\Phi(k^\infty \gamma_\infty) \\ &\quad - \sum_{i=1}^p \mu^\Phi(k\gamma_i) + \frac{1}{2} \sum_{i=1}^{p+2} \dim S_i + (p+2) - 2 \\ &= 2c_1^\Phi(u) + \mu^\Phi(k^0 \gamma_0) + \mu^\Phi(k^\infty \gamma_\infty) - \sum_{i=1}^p \mu^\Phi(k\gamma_i) + 2p + 2, \end{aligned}$$

where $\gamma_1, \dots, \gamma_p$ are p -distinct lifts of the simple contractible orbit $\bar{\gamma}$. Due to the equivariance, we have $\mu^\Phi(k\gamma_i) = \mu^{\bar{\Phi}}(k\bar{\gamma})$ for all i and $c_1^{\bar{\Phi}}(\bar{u}) = c_1^\Phi(u)/p$. Hence we have

$$(36) \quad 2c_1^{\bar{\Phi}}(\bar{u}) - \mu^{\bar{\Phi}}(k\bar{\gamma}) = \frac{1}{p}(\text{ind}(C) - \mu^\Phi(k^0 \gamma_0) - \mu^\Phi(k^\infty \gamma_\infty) - 2p - 2).$$

Now we compute the right-hand side of the above equation. A computation similar to the one carried out for $k_{0/\infty} \bar{\gamma}_{0/\infty}$ leads to the symplectic paths

$$\psi_\infty(t) = e^{it2\pi k^\infty(1-q)}, \quad \psi_0(t) = e^{it2\pi k^0(1-v)}$$

for $k^\infty \gamma_\infty$ and $k^0 \gamma_0$ respectively. We first observe that the constant path $\mathcal{I}(t) = I$ leads to

$$\mu(\mathcal{I}) = \frac{1}{2}(\mu(\mathcal{I}) + \mu(\mathcal{I})) = \frac{1}{2}(\mu(\mathcal{I}) + \mu(\mathcal{I}^{-1})) = 0$$

by (CZ3). For a general symplectic path of the form $\varphi = e^{itk2\pi}$, $k \in \mathbb{Z}$, viewing the inverse path as a loop and using (CZ2), we get

$$0 = \mu(\mathcal{I}) = \mu(\varphi^{-1}\varphi) = \mu(\varphi) + 2m(\varphi^{-1}) = \mu(\varphi) - 2k \implies \mu(\varphi) = 2k.$$

Hence we get

$$\mu^\Phi(k^\infty \gamma_\infty) = 2k^\infty(1 - q), \quad \mu^\Phi(k^0 \gamma_0) = 2k^0(1 - v).$$

Now, for the term $\text{ind}(C)$, we use the trivialization Φ_0 given by (32). Note that, since it is induced by a non-vanishing section, we have $c_1^{\Phi_0}(u) = 0$. For any closed orbit

$$k\gamma(t) = (e^{it}z_1, e^{it}z_2), \quad t \in [0, 2\pi k],$$

the associated symplectic path reads as

$$\begin{aligned} \varphi(t)\Phi_0(e^{itk2\pi}z_1, e^{itk2\pi}z_2) &= d\phi_{(tk2\pi)}(z_1, z_2)[(\bar{z}_2, -\bar{z}_1)], \\ \varphi(t)(e^{-itk2\pi}\bar{z}_1, e^{-itk2\pi}\bar{z}_2) &= (e^{itk2\pi}\bar{z}_2, e^{itk2\pi} - \bar{z}_1), \\ \varphi(t) &= e^{it(2k)2\pi}. \end{aligned}$$

Hence we get $\mu^{\Phi_0}(k\gamma) = 4k$, and therefore,

$$\begin{aligned} \text{ind}(C) &= 2c_1^{\Phi_0}(u) + \mu^{\Phi_0}(k^0\gamma_0) + \mu^{\Phi_0}(k^\infty\gamma_\infty) \\ &\quad - \sum_{i=1}^p \mu^{\Phi_0}(k\gamma_i) + \frac{1}{2} \sum_{i=1}^{p+2} S_i + (p+2) - 2 \\ &= 4k^0 + 4k^\infty - p4k + 2p + 2. \end{aligned}$$

Then (36) leads to

$$\begin{aligned} 2c_1^{\bar{\Phi}}(\bar{u}) - \mu^{\bar{\Phi}}(k\bar{\gamma}) &= \frac{1}{p}(4k^0 + 4k^\infty - p4k - 2k^0(1-v) - 2k^\infty(1-q)) \\ &= \frac{2}{p}(d + k^0v + k^\infty q) - 2k. \end{aligned}$$

Substituting the above formula in (35) leads to formula (34). □

Lemma 3.8. *For any \bar{u} given by (26) and (27), we have*

$$\text{ind}(\bar{u}) = 4 + 4d^I,$$

where $d^I = \lfloor d/p \rfloor$.

Proof. We write

$$k^\infty(1-q) = l_\infty + n_\infty p, \quad k^0(1-v) = l_0 + n_0 p, \quad 0 < l_\infty, l_0 < p,$$

so that

$$\mu(k^\infty\bar{\gamma}_\infty) = 2n_\infty + 1, \quad \mu(k^0\bar{\gamma}_0) = 2n_0 + 1.$$

We note that, by (28),

$$k^\infty(1-q) \equiv k^0(1-v) \equiv k^\infty + k^0.$$

By (29), we also have $k^\infty + k^0 = r + p(d^I + k)$, where $0 < r < p$. In particular, $k^\infty + k^0 \equiv r$. Hence we conclude that $r = l_1 = l_2$. Combining all these, we get

$$\begin{aligned} \text{ind}(\bar{u}) &= \frac{2}{p}(k_1q + k_2u + d) + \mu^{\bar{\Phi}}(k^\infty\bar{\gamma}_\infty) + \mu^{\bar{\Phi}}(k^0\bar{\gamma}_0) - 2k + 2 \\ &= \frac{2}{p}(k^\infty - r - n_\infty p + k^0 - r - n_0 p + r + pd^I) \\ &\quad + (2n_\infty + 1) + (2n_0 + 1) - 2k + 2 \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{p}(k^0 + k^\infty - r + pd^I - (n_0 + n_\infty)p) \\
&\quad + 2(n_0 + n_\infty) - 2k + 4 \\
&= \frac{2}{p}(r + pd^I + pk - r + pd^I - (n_0 + n_\infty)p) \\
&\quad + 2(n_0 + n_\infty) - 2k + 4 \\
&= \frac{2}{p}(2pd^I + pk - (n_0 + n_\infty)p) + 2(n_0 + n_\infty) - 2k + 4 \\
&= (4d^I + 2k - 2(n_0 + n_\infty)) + 2(n_0 + n_\infty) - 2k + 4 \\
&= 4d^I + 4. \quad \square
\end{aligned}$$

We list the outcomes of the above discussion below.

Remark 3.9 (Regularity of J_α -holomorphic pair of pants). By the above discussion, we conclude that any nonempty component of the moduli space (26)–(27) admits a smooth structure as the zero set of the Cauchy–Riemann operator. More precisely, Lemma 3.8 shows that any J_α -holomorphic pair of pants \bar{u} with the asymptotics (26)–(27) is Fredholm regular. Note that the J_α -holomorphic perturbations of \bar{u} are in one-to-one correspondence with the equivariant holomorphic perturbations of the lift u . The dimension of these perturbations, which is the dimension of \mathcal{M} with the smooth structure given in Section 3.1, is given precisely by (30) and it coincides with $\text{ind}(\bar{u})$ by the above lemma. Hence the kernel of the corresponding Fredholm operator has the dimension equal to the index of \bar{u} . This implies that the Fredholm operator is surjective at \bar{u} .

3.10. Other configurations. The discussion above can be repeated with minor modifications for other curves with only two non-contractible ends.

More contractible ends. Adding more contractible ends to the pair of pants configuration does not require any essential change. To be more specific, let us consider the moduli space \mathcal{M} of unparametrized curves with two non-contractible positive ends, which are fixed geometrically as above, s^+ contractible positive ends and $(s^- + 1)$ negative contractible ends. As noted in Remark 2.1, we interpret \mathcal{M} as the set of tuples

$$(\bar{u}, (z_1, \dots, z_{s^+}), (w_1, \dots, w_{s^-})),$$

where \bar{u} has the positive ends at 0 asymptotic to $k_2\bar{\gamma}_0$ and at ∞ asymptotic to $k_1\bar{\gamma}_\infty$, a negative end at 1 of multiplicity k , positive ends at z_1, \dots, z_{s^+} of multiplicities $l_1^+, \dots, l_{s^+}^+$, negative ends at w_1, \dots, w_{s^-} of multiplicities $l_1^-, \dots, l_{s^-}^-$. That is, we fix tree punctures and let the rest of the punctures move. Given such an object, we have a lifted base curve c with $r > 0$ and $\deg(c) = r + pd^I$ and

$$-k^\infty - k^0 - p \sum l_i^+ + pk + p \sum l_j^- = -(r + pd^I).$$

The multiplicities k_i are determined in mod p as before. The dimension of the component of $(\bar{u}, (z_1, \dots, z_{s^+}), (w_1, \dots, w_{s^-}))$ is then

$$4 + 4d^I + 2s^+ + 2s^-$$

since we added the freedom of choosing the places of zeros and poles of the meromorphic section other than $0, \infty$ and the lifts of 1 . Yet, in the cover, lifts of the remaining zeros and poles should be distributed invariantly. Moreover, an analysis similar to above shows that

$$\text{ind}(\bar{u}, (z_1, \dots, z_{s^+}), (w_1, \dots, w_{s^-})) = 4 + 4d^I + 2s^+ + 2s^-.$$

In order to describe the components of the moduli space, one simply adds free zeros and poles to the pair of pants configurations, and the multiplicities of the non-contractible ends are adjusted until the degree condition is satisfied. For fixed values of d^I, a and k as before, each component of the moduli space may be identified with

$$\mathcal{M}_{d^I, k, a} \times \mathfrak{S}^{(s^+ + s^-)}(\mathbb{C}P^1 \setminus \{0, 1, \infty\}),$$

where $\mathcal{M}_{d^I, k, a}$ is given in Remark 3.5.

Cylinders. For the cylinders, we first consider the parametrized cylinders and then mode out biholomorphisms that fix the punctures. In our case, this corresponds to removing the contractible end from the configurations of the pair of pants with one positive and one negative non-contractible end and killing the freedom in the domain. More concretely, lets consider the moduli space of cylinders with positive end asymptotic to $k^0\bar{\gamma}_0$ and negative end asymptotic to $k^\infty\bar{\gamma}_\infty$. The equivariant index of parametrized cylinders is again $4 + 4d^I$, and moding out reparametrization means we kill the freedom of moving λ in the definition of the lifted curve c ; see Lemma 3.2. Hence the equivariant index reads as $2 + 4d^I$. On the other hand, the Fredholm index reads as

$$\text{ind}([\bar{u}]) = 2c_1^{\bar{\Phi}}(\bar{u}) + \mu^{\bar{\Phi}}(k^0\bar{\gamma}_0) - \mu^{\bar{\Phi}}(k^\infty\bar{\gamma}_\infty).$$

We know that a lift u lies above a closed curve c so that $k^\infty - k^0 = -\text{deg}(c)$, and similar to Lemma 3.7, we get

$$c_1(\bar{u}) = k^0v - k^\infty q + k^\infty - k^0.$$

Combining all these, one can show that $\text{ind}([\bar{u}]) = 4d^I + 2$, and therefore, we have transversality for cylinders as well. Concerning the big moduli space of fixed dimension, we have two parameters, namely d^I and a , that may be used as indices of the components.

4. THE APPLICATION

In this section, we carry out a neck-stretching procedure, which is initiated by a positive contactomorphism. We perturb a J_α -holomorphic pair of pants with non-contractible positive ends in $\mathbb{R} \times L(p, q)$, which come from a particular component of the moduli space, and at the end, we would like to obtain a very particular holomorphic building; see Figure 4.

In general, it is very unlikely to achieve such a picture since there are the common issues of transversality and compactness. It turns out that the study of the index behavior of multiples of non-contractible Reeb orbits does not give enough control to handle these issues. The crucial observation is that one can

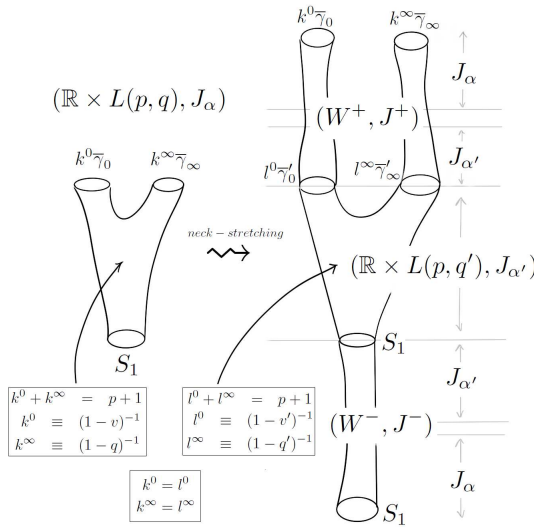


FIGURE 4. One of the two possible outcomes of the neck-stretching procedure, which leads to $q \equiv q'$.

apply well-established 4-dimensional methods to the lifts of punctured curves in the symplectic cobordisms that show up along the way, after extending them to closed curves in certain completions of these cobordisms. The outcome of this observation is that, given any holomorphic curve in a cobordism, the total action of its positive ends is greater than or equal to the total action of its negative ends. We note that this property holds in general for symplectizations and for cobordisms where the contact forms at ends are suitably rescaled. But here, we achieve this property for the cobordisms for which contact forms are not rescaled, and this provides a strong control over the components of the limiting buildings that emerge along the neck-stretching procedure. As a result, we prove the following.

Theorem 4.1. *Let p be prime and $1 < q, q' < p - 1$. Suppose that there is a positive contactomorphism*

$$(37) \quad \varphi : (L(p, q), \xi = \ker \alpha) \rightarrow (L(p, q'), \xi' = \ker \alpha').$$

Then $q \equiv (q')^{\pm 1} \pmod{p}$.

4.2. The proof of Theorem 4.1. We take two lens spaces $L(p, q)$ and $L(p, q')$, where p, q and q' satisfy the assumptions of Theorem 4.1, with the contact structures induced by contact forms α and α' , which are quotients of α_0 ; see the introduction.

We consider the moduli space of J_α -holomorphic pair of pants with non-contractible positive ends in the symplectization of $L(p, q)$, which has the minimal index. By the consideration of equivariant curves in $\mathbb{R} \times S^3 \cong L^*$ (see

Section 3), we know that the minimal index for this problem is 4, and we have a moduli space with countably many components that can be collected in $p - 1$ groups, each group being determined by the degree of the underlying closed curve; see Remark 3.5.

We consider the component, denoted by $\mathcal{M}(J_\alpha)$, associated to the underlying closed curve with degree one and minimal multiplicity of non-contractible ends. More precisely, $\mathcal{M}(J_\alpha)$ is the moduli space of a pair of pants

$$u = (a, v) : \mathbb{C}P^1 \setminus \Gamma \rightarrow \mathbb{R} \times L(p, q) \cong \overline{L}^*$$

with punctures $\Gamma = \{0, 1, \infty\}$ and asymptotics

$$(38) \quad u(0) = (+\infty, k^0 \overline{\gamma}_0), \quad u(\infty) = (+\infty, k^\infty \overline{\gamma}_\infty), \quad u(1) \in \{-\infty\} \times S_1,$$

where S_1 is the orbit space of the contractible orbits of action 2π . We know from the previous section that

$$(39) \quad k^0 \equiv (1 - v)^{-1}, \quad k^\infty \equiv (1 - q)^{-1}, \quad k^0 + k^\infty = p + 1.$$

Next we impose a 4-dimensional constraint on \mathcal{M} as follows. We pick a point $x_0 \in L(p, q)$ away from the non-contractible orbits and with the property that $\varphi(x_0)$ is also away from the non-contractible orbits in $L(p, q')$. Given the contactomorphism in (37), we have a positive function

$$(40) \quad f : L(p, q) \rightarrow (0, +\infty) \quad \text{such that} \quad \varphi^* \alpha' = f \alpha.$$

Now consider the evaluation map

$$(41) \quad \text{ev} : \mathcal{M} \rightarrow \mathbb{R} \times L(p, q), \quad \text{ev}(u) = u(2).$$

We cut out a 0-dimensional submanifold of \mathcal{M} via

$$\mathcal{M}^0(J_\alpha) := \text{ev}^{-1}((\log f(x_0), x_0)).$$

By Remark 3.9 and the description of the curves in $\mathcal{M}(J_\alpha)$ given in Section 3.1, it is easy to see that $\mathcal{M}^0(J_\alpha)$ is cut out transversely and consists of a single curve. In fact, one notes that x_0 lies on some contractible orbit, say γ , and considering the equivariant lifts of the curves in $\mathcal{M}(J_\alpha)$, γ as a point in the orbit space is the image of the point 2 under the underlying base curve, and this choice fixes the parameter $\lambda \in \mathbb{C}$ given in Lemma 3.2. Then the choice of $x_0 \in \gamma$ and the quantity $f(x_0)$ fix the freedom over the equivariant meromorphic section, which is due to the \mathbb{C}^* -action on L .

Remark 4.3. The conditions on the point x_0 are easily satisfied and play a significant role in ruling out certain unpleasant configurations at the end of the neck-stretching argument; see Lemma 4.7 and Lemma 4.10.

We consider the following exact symplectomorphism:

$$(42) \quad \Phi : (\mathbb{R} \times L(p, q'), d(e^t \alpha')) \rightarrow (\mathbb{R} \times L(p, q), d(e^s \alpha)),$$

$$(t, x) \mapsto (t + \log f \circ \varphi^{-1}(x), \varphi^{-1}(x)).$$

Let $\Sigma := \Phi(\{0\} \times L(p, q'))$ be the contact type hypersurface in $\mathbb{R} \times L(p, q)$. Then we have

$$N^+ \cup N^- := (\mathbb{R} \times L(p, q)) \setminus \Sigma,$$

where N^+ is the upper and N^- is the lower connected component. We fix $\varepsilon > 0$ and put

$$U := \Phi([-\varepsilon, \varepsilon] \times L(p, q')) \subset \mathbb{R} \times L(p, q).$$

We fix an open subset $V \subset \mathbb{R} \times L(p, q)$ such that $\overline{U} \subset V$. For later purposes, we set

$$V^+ \sqcup V^- := V \setminus \overline{U},$$

where V^+ and V^- are the upper and lower components respectively. Let n be a positive integer. Following [10], we construct the symplectic cobordism W^n as follow. We remove $\Phi((-\varepsilon/2, \varepsilon/2) \times L(p, q'))$ from $\mathbb{R} \times L(p, q)$, and we glue $[-\varepsilon - n, n + \varepsilon] \times L(p, q)$ in the middle via the following identifications:

$$(43) \quad \begin{aligned} [-\varepsilon - n, -n - \varepsilon/2] \times L(p, q') \ni (t, x) &\sim \Phi(t + n, x) \in U, \\ [n + \varepsilon/2, n + \varepsilon] \times L(p, q') \ni (t, x) &\sim \Phi(t - n, x) \in U. \end{aligned}$$

We consider a smooth function $\phi_n : [-\varepsilon - n, n + \varepsilon] \rightarrow [-\varepsilon, \varepsilon]$ such that

- $\phi'_n > 0$,
- $\phi_n(t) = t + n$ for $t \in [-\varepsilon - n, -\varepsilon/2 - n]$,
- $\phi_n(t) = t - n$ for $t \in [n + \varepsilon/2, n + \varepsilon]$,
- $\phi_n(0) = 0$.

Such a function leads to a diffeomorphism

$$(44) \quad \Phi_n : W^n \rightarrow \mathbb{R} \times L(p, q),$$

where $\Phi_n = \text{id}$ on $(\Phi((-\varepsilon/2, \varepsilon/2) \times L(p, q)))^c$ and

$$\Phi_n(t, x) = (\phi_n(t) + \log f \circ \varphi^{-1}(x), \varphi^{-1}(x))$$

on $[-\varepsilon - n, n + \varepsilon] \times L(p, q')$. For later purposes, we note that

$$(45) \quad \begin{aligned} \Phi_n(0, \varphi(x_0)) &= (\phi_n(0) + \log f \circ \varphi^{-1}(\varphi(x_0)), \varphi^{-1}(\varphi(x_0))) \\ &= (\log f(x_0), x_0). \end{aligned}$$

We consider the exact symplectic form

$$\omega_n := \Phi_n^* d(e^s \alpha) = d(\Phi_n^*(e^s \alpha))$$

on W^n , which reads as

$$\omega_n = \begin{cases} d(e^s \alpha) & \text{on } U^c, \\ d(\phi_n \alpha') & \text{on } [-\varepsilon - n, n + \varepsilon] \times L(p, q'). \end{cases}$$

We note that the standard almost complex structure $J_{\alpha'}$ is compatible with the symplectic form $d(\phi_n \alpha')$. Next we consider an almost complex structure J_n on $\mathbb{R} \times L(p, q)$ with the following properties:

- (Jn1) $J_n = J_\alpha$ on V^c ;
- (Jn2) $(d\Phi_n)^{-1} \circ J_n \circ d\Phi_n = J_{\alpha'}$ on $[-\varepsilon - n, n + \varepsilon] \times L(p, q')$;
- (Jn3) J_n is compatible with $d(e^s \alpha)$;

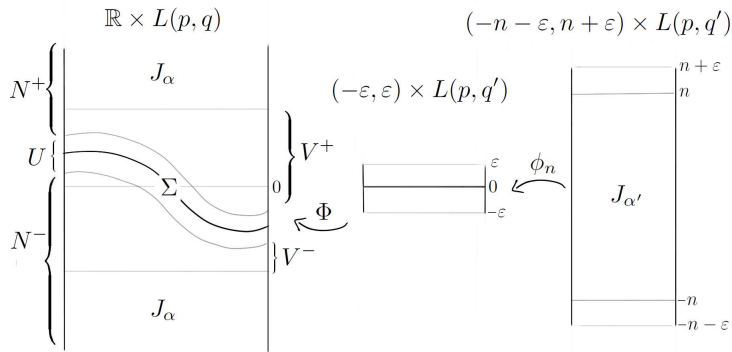


FIGURE 5. The construction of (W^n, J_n) .

(Jn4) any relevant simple⁵ J_n -holomorphic curve passing through the open subset $V^+ \sqcup V^-$ and satisfying

$$(46) \quad \text{ev}(u) \in \{(\log f(x_0), x_0)\} \subset \mathbb{R} \times L(p, q)$$

is regular.

It is well-known that, in the space of almost complex structures that satisfy (Jn1)–(Jn3), the almost complex structures that also satisfy (Jn4) for curves with a fixed asymptotic profile form a dense subset. This statement has a nature of a folklore fact in the study of holomorphic curves and may be attributed to many authors in different settings of problems. For a clean exposition of the case of curves with non-degenerate asymptotics, we refer to [23, Thm. 7.2]. We note that the proof of [23, Thm. 7.2] applies to the Morse–Bott setting once the functional analytic set-up is suitably modified since (Jn4) is achieved by a generic perturbation away from the ends of the symplectization. Moreover, via an argument similar to the one in [20, Lem. 2.5], one can show that the universal moduli space with the condition (46) is smooth. We also note that the curves relevant to us are the pair of pants with the asymptotics given by (38) and (39) and simple curves that appear in the proof of Lemma 4.7. The configurations that appear in the proof of Lemma 4.7 are finitely many and *a priori* determined.⁶ Hence we have finitely many configurations of curves for which we need regularity so that J_n can be chosen from a finite intersection of dense sets.

Now we consider the moduli space $\mathcal{M}^0(J_n)$ of a J_n -holomorphic pair of pants with the asymptotics given by (38) and (39), satisfying (46).

⁵A holomorphic curve $u : (\Sigma \setminus \Gamma, j) \rightarrow (\mathbb{R} \times M, J)$ is said to be *multiply covered* if there exists another holomorphic curve $v : \Sigma' \setminus \Gamma' \rightarrow \mathbb{R} \times M$ and a holomorphic branched covering $\phi : (\Sigma, j) \rightarrow (\Sigma', j')$ such that $u = v \circ \phi$ and $\deg(\phi) > 1$. We say a holomorphic curve is *simple* if it is not multiply covered.

⁶See the moduli problems given by (47), (48), (49) and (50).

Lemma 4.4. *Any J_n -holomorphic pair of pants satisfying (38) and (39) is simple.*

Proof. We note that, since the negative end of a such a pair of pants u has action 2π , it is a simple orbit unless, without loss of generality, it is $p\bar{\gamma}_0$. Assume that u factors through a branched covering ψ and a simple curve v . By the Riemann–Hurwitz formula, v is rational. Since p is prime, the degree of ψ is p , and the negative end of v is $\bar{\gamma}_0$. It is clear that v has at most two positive ends. Note that v cannot be a cylinder since the positive ends of u are geometrically distinct. Hence v is also a pair of pants. In particular, ψ maps punctures of u to punctures of v respectively. Hence the ramification number of ψ at each puncture of u has to be p . In particular, p divides both k^0 and k^∞ . But this is not possible since $k^0 + k^\infty = p + 1$. \square

By the lemma above and (Jn4), we conclude that any curve in $\mathcal{M}^0(J_n)$ is regular and $\mathcal{M}^0(J_n)$ is a 0-dimensional manifold. We know that same property holds for $\mathcal{M}^0(J_\alpha)$. Now we choose a generic homotopy $(J_t)_{t \in [0,1]}$ of almost complex structures, which connects J_α to J_n such that $J_t = J_\alpha$ on V^c for all t , and J_t is compatible with $d(e^s\alpha)$ for all t . Here the genericity means that, for any $t_0 \in [0, 1]$, the index of a relevant simple J_{t_0} -holomorphic curve with the constraint (46) is at least -1 . The existence of such a homotopy follows from the parametric version of the geometric transversality statement used for J_n ; see [23, Rem. 7.4]. As in the choice of J_n , we also safely assume that $(J_t)_{t \in [0,1]}$ is generic for simple curves that appear in the proof of Lemma 4.7.

We recall that J_α -holomorphic pair of pants we consider, and Lemma 4.4 applies to J_t . Hence the moduli space

$$\bigcup_{t \in [0,1]} \mathcal{M}^0(J_t)$$

of a J_t -holomorphic pair of pants with asymptotics given by (38) and (39) together with the evaluation condition is a 1-dimensional cobordism between \mathcal{M}^0 and $\mathcal{M}^0(J_n)$. We note that if the cobordism $\bigcup \mathcal{M}^0(J_t)$ is compact, then $\mathcal{M}^0(J_n)$ is not empty since \mathcal{M}^0 consists of an odd number of points.

Compactness of the cobordism. Let $t_0 \in [0, 1]$, and let (u_n) be a sequence of J_{t_n} -holomorphic pairs of pants such that t_n converges to t_0 . Then, by the SFT compactness theorem, there is a subsequence, again denoted by (u_n) , converging to a holomorphic building u_∞ . *A priori*, the holomorphic building u_∞ is a collection of curves that lie in $\mathbb{R} \times L(p, q)$, which are either J_α -holomorphic or J_{t_0} -holomorphic. These components fit together along their asymptotic ends and lead to the level structure of the building; see [1, 2].

Since the complex structure on the domain of u_n 's is fixed, the components of the building u_∞ emerge only out of bubbling off. In particular, there is a finite set $P \subset \mathbb{C}P^1 \setminus \Gamma$ such that, on $\mathbb{C}P^1 \setminus (\Gamma \cup P)$, the sequence (u_n) has a uniform gradient bound. Hence there exists a component

$$u_0 : \mathbb{C}P^1 \setminus (\Gamma \cup P) \rightarrow \mathbb{R} \times L(p, q)$$

of u_∞ such that one of the followings hold:

- u_0 is J_α -holomorphic and u_n converges to u_0 in C_{loc}^∞ on $\mathbb{C}P^1 \setminus (\Gamma \cup P)$ after a sequence of shifts in \mathbb{R} -direction;
- u_0 is J_{t_0} -holomorphic and u_n converges to u_0 in C_{loc}^∞ on $\mathbb{C}P^1 \setminus (\Gamma \cup P)$.

We note that the signs of the punctures of u_0 that are in Γ do not have to match with the signs of the punctures of u_n 's, but the homotopy classes of corresponding asymptotics do have to match. In particular, u_0 is non-constant. The remaining structure of the building is given by a so-called bubble tree. Instead of describing the *a priori* structure of the bubble tree, we immediately utilize a particular control over the components, which is due to the following fact.

Proposition 4.5. *Given any component of the limiting building, the total action of its positive ends is greater than or equal to the total action of its negative ends.*

We note that, for the components of u_∞ that lie in upper or lower translation invariant levels, the above statement is trivial; see Section 2. The nontrivial part of the statement is about J_{t_0} -holomorphic components, namely the ones in the middle level, and the proof is given in Section 4.11.3.

The first implication of Proposition 4.5 is the absence of holomorphic planes. We note that a holomorphic plane requires a positive end of action at least 2π . Together with the action of the very bottom end of the building, a finite energy plane forces the total action of the very top end of the building to be at least 4π . But we know that the total action at the top is $2\pi(1 + 1/p)$.

Now, *a priori*, there may be components of the building that are bubbled off at points in P . But any collection of such components associated to a given bubble point in P must contain a finite energy plane. Since such planes are ruled out, we conclude that $P = \emptyset$. Next we consider the components that are bubbled off at the punctures in Γ and note that any such component is cylindrical since the domain of the building has arithmetic genus zero and there are no holomorphic planes. Hence we have the essential component with the puncture set Γ (possibly with different signs compared to u_n 's), and the remaining components are cylindrical (possibly with two positive ends), which can be grouped into collections associated to the punctures in Γ .

Concerning the signs of the punctures of components, we first note that the puncture 1 has to stay negative since turning it into a positive puncture requires a cylindrical component associated to the puncture 1, which has two negative punctures. Consequently, only one of the punctures among 0 and ∞ may change sign. But it is easy to see that, by Proposition 4.5, such a configuration is not possible. Hence the building consists of the essential component u_0 , which is an honest pair of pants, and we have honest cylinders.

Concerning the level structure, we note the following. Since there is no bubbling off at the marked point 2, we have C_{loc}^∞ -convergence of u_n to u_0 . Combining this with the fact that u_n 's satisfy the evaluation condition (41), we conclude that the u_0 lies in the middle layer.

Now we need to show that there are no nontrivial cylinders in upper and lower levels. To this end, we let \mathcal{A}^\pm denote the total action of the positive/negative ends in the middle layer. We note that the minimal action of a closed Reeb orbit is $2\pi/p$ if the orbit is non-contractible and 2π if the orbit is contractible. By Proposition 4.5, we have the following possibilities regarding \mathcal{A}^\pm :

- (i) $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi$,
- (ii) $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi(1 + 1/p)$,
- (iii) $\mathcal{A}^+ = 2\pi$ and $\mathcal{A}^- = 2\pi$.

We first assume that $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi$. In this case, any cylinder in an upper or lower level has trivial $d\alpha$ -energy. Hence any such cylinder is trivial. Now it remains to rule out the last two cases.

Lemma 4.6. *The case of $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi(1 + 1/p)$ is not possible.*

Proof. In this case, there is no nontrivial cylinder in the upper levels, and there has to be a nontrivial cylinder in a lower level. But such a cylinder has to have a contractible positive end with action $2\pi(1 + 1/p)$, and this is not possible. \square

Lemma 4.7. *The case of $\mathcal{A}^+ = 2\pi$ and $\mathcal{A}^- = 2\pi$ is not possible.*

Proof. We note that there is no nontrivial component in the lower level and there is only one nontrivial cylinder in the upper level since the minimal period of Reeb orbits is $2\pi/p$, and this is precisely the action difference between \mathcal{A}^+ and the total action at the top of the building.

Without loss of generality, we assume that the positive end of the nontrivial cylinder is $k^0\bar{\gamma}_0$. Then the negative end of the nontrivial cylinder is necessarily $(k^0 - 1)\bar{\gamma}_\infty$, and the positive ends of u_0 are given by $(k^0 - 1)\bar{\gamma}_\infty$ and $k^\infty\bar{\gamma}_\infty$. Note that the nontrivial cylinder and u_0 are adjacent at a non-degenerate orbit $(k^0 - 1)\bar{\gamma}_\infty$. Hence the indices of these curves must add up to 0. We note also that the index of the nontrivial cylinder is 2, and therefore, the index of u_0 is -2 .

We first assume that u_0 is simple. We consider the pair of pants with asymptotics

$$(47) \quad u(0) = (+\infty, (k^0 - 1)\bar{\gamma}_\infty), \quad u(\infty) = (+\infty, k^\infty\bar{\gamma}_\infty), \quad u(1) \in S_1$$

and the constraint

$$(48) \quad u(2) = (\log f(x_0), x_0).$$

Note that there is no J_α -holomorphic pair of pants that satisfies (47) and (48). In fact, any such curve is necessarily the multiple cover of a trivial cylinder over $\bar{\gamma}_\infty$, but such a cylinder misses the point $(\log f(x_0), x_0)$ by the definition of x_0 . Hence any J_α -holomorphic pair of pants that satisfies (47) and (48) is regular tautologically. By the choice of J_n , we assume that any simple J_n -holomorphic pair of pants that satisfies (47) and (48) is regular. Hence we may assume that $(J_t)_{t \in [0,1]}$ is a generic homotopy for a simple pair of pants satisfying (47) and (48). In particular, the index of u_0 is at least -1 , and this is a contradiction.

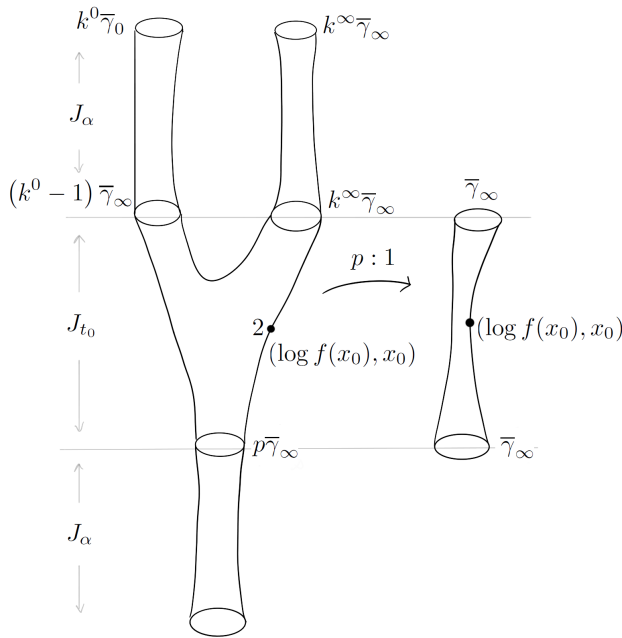


FIGURE 6. The case of u_0 being multiply covered in Lemma 4.7.

Now we assume that u_0 is multiply covered. Hence the negative end of u_0 is either $p\bar{\gamma}_0$ or $p\bar{\gamma}_\infty$. Now let

$$v_0 : \Sigma \setminus \Gamma \rightarrow \mathbb{R} \times L(p, q)$$

be the underlying simple curve and $\psi : \mathbb{C}P^1 \rightarrow \Sigma$ the branched covering so that $u_0 = v_0 \circ \psi$ and $\psi^{-1}(\Gamma) = \{0, 1, \infty\}$. Let $N > 1$ be the degree of ψ . By the Riemann–Hurwitz formula, Σ is a sphere. Since u_0 has only one negative end, so does v_0 , and this negative end has multiplicity p/N . Since p is prime, we get $N = p$. It is clear that v_0 has at most two positive ends.

Suppose that v_0 has two positive ends. In this case, the ramification number of the points 0 and ∞ are both p . Hence p divides both $k^0 - 1$ and k^∞ . But this is not possible since $k^0 + k^\infty - 1 = p$.

Next we assume that v_0 has one positive end, namely $l\bar{\gamma}_\infty$. Let $r_{0/\infty}$ be the ramification number of ψ at the point $0/\infty$. Then we get $k^0 - 1 = r_0 l$ and $k^\infty = r_\infty l$. Since $k^0 - 1 + k^\infty = r_0 + r_\infty = p$, we get $l = 1$. By Proposition 4.5 and the fact that $q \not\equiv \pm 1$, the negative end of v is also $\bar{\gamma}_\infty$. Let z^+ and z^- be the positive and negative punctures of v_0 . We have $\psi(0) = \psi(\infty) = z^+$ and $\psi(1) = z^-$. We put $z_0 := \psi(2)$ and get $v_0(z_0) = (\log f(x_0), x_0)$.

Now we consider parametrized cylinders

$$(49) \quad v : \mathbb{C}P^1 \setminus \{0, \infty\} \rightarrow \mathbb{R} \times L(p, q); \quad v(0) = (-\infty, \bar{\gamma}_\infty), \quad v(\infty) = (+\infty, \bar{\gamma}_\infty)$$

satisfying

$$(50) \quad v(1) = (\log f(x_0), x_0).$$

As noted above, a J_α -holomorphic cylinder satisfying (49) and (50) does not exist. Hence we have *a priori* regularity. We may safely assume that a J_n -holomorphic cylinder with the above conditions is also regular since such a cylinder is simple. Hence we may assume that $(J_t)_{t \in [0,1]}$ is a generic homotopy for the moduli problem given by (49) and (50). But the index of this moduli problem is -2 . In fact, the index of parametrized cylinders without the constraint (50) is 2. Since we have a point constraint in a 4-dimensional manifold, the index with the constraint is -2 . By genericity of the homotopy, we conclude that there is no J_{t_0} -holomorphic parametrized cylinder satisfying (49) and (50). But this is a contradiction since reparametrizing v_0 such that $\infty \mapsto z^+$, $0 \mapsto z^-$ and $1 \mapsto z_0$ leads to such a cylinder. \square

Hence we conclude that the only nontrivial component of the limiting building u_∞ is a pair of pants u_0 having the asymptotics of (u_n) and satisfying the evaluation condition. This finishes the proof of the compactness of the cobordism.

Stretching the neck. Knowing that the cobordism $\bigcup_{t \in [0,1]} \mathcal{M}^0(J_t)$ is compact, we get a J_n -holomorphic pair of pants u_n for each n . Now we look at the limit of the sequence (u_n) as $n \rightarrow \infty$. We know that a subsequence of (u_n) , again denoted by (u_n) , converges to a holomorphic building u_∞ . *A priori*, the limiting building has J_α -holomorphic components in upper and lower levels, $J_{\alpha'}$ -holomorphic components in middle levels and finally some components in the upper and the lower connecting levels. The connecting levels have the following description. Note that the hypersurface $\Sigma = \Phi(\{0\} \times L(p, q'))$ divides $\mathbb{R} \times L(p, q)$ into two components N^+ and N^- , each admitting Σ as its boundary. The upper connecting level can be seen as the manifold

$$W^+ := N^+ \cup (-\infty, \varepsilon) \times L(p, q'),$$

where the neighborhood $\Phi([0, \varepsilon) \times L(p, q'))$ of $\Sigma \subset N^+$ is identified with $[0, \varepsilon) \times L(p, q')$ via the symplectomorphism Φ that is given by (42). Moreover, W^+ is endowed with an almost complex structure J^+ such that $J^+ = J_\alpha$ on $N^+ \setminus V^+$ and $J^+ = J_{\alpha'}$ on $(-\infty, 0] \times L(p, q')$. Similarly, the lower connecting level is given by

$$W^- := N^- \cup (-\varepsilon, +\infty) \times L(p, q')$$

together with the almost complex structure J^- such that $J^- = J_\alpha$ on $N^- \setminus V^-$ and $J^- = J_{\alpha'}$ on $[0, +\infty) \times L(p, q')$.

As before, since the domains of u_n 's are fixed, we have a bubble tree structure on the limiting building. We have an essential component u_0 with domain $\mathbb{P} \setminus (\Gamma \cup P)$, which in this case can be J_α -holomorphic or $J_{\alpha'}$ -holomorphic or J^\pm -holomorphic. In order to rule out unpleasant components of the building, we want to argue in terms of the actions of the asymptotic ends as before.

Similar to Proposition 4.5, we have an *a priori* control over the actions of asymptotic ends of J^\pm -holomorphic components.

Proposition 4.8. *For any component in (W^\pm, J^\pm) , the total action at its positive ends is greater than or equal to the total action at its negative ends.*

We postpone the proof of this statement to Section 4.11 and continue with the proof of the main statement. We note that, replacing Proposition 4.5 with Proposition 4.8, the previous arguments apply to u_∞ word by word since, in $L(p, q')$, the minimal action of the Reeb orbits is $2\pi/p$ and the action of a contractible orbit is 2π . Hence we have a pair of pants u_0 and a bunch of cylinders in the building. Moreover, u_n converges to u_0 in C_{loc}^∞ near the marked point 2, and in light of equation (45), u_0 lies in a middle layer since u_n 's satisfy (46). Note that, since the contactomorphism (37) induces an isomorphism on the fundamental group, u_0 has two positive non-contractible ends and one negative contractible end.

Due to the action window, u_0 is the only nontrivial component in the middle layer. We let \mathcal{A}^+ be the total action of the positive ends of u_0 , and \mathcal{A}^- the total action of the negative end of u_0 . By Proposition 4.8, we have the following possibilities regarding \mathcal{A}^\pm :

- (i) $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi$,
- (ii) $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi(1 + 1/p)$,
- (iii) $\mathcal{A}^+ = 2\pi$ and $\mathcal{A}^- = 2\pi$.

We first discuss the unpleasant cases.

Lemma 4.9. *The case of $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi(1 + 1/p)$ is not possible.*

Proof. Note that the negative end of u_0 has action $2\pi(1 + 1/p)$, but such an orbit cannot be contractible. □

Lemma 4.10. *The case of $\mathcal{A}^+ = 2\pi$ and $\mathcal{A}^- = 2\pi$ is not possible.*

Proof. In this case, u_0 has trivial $d\alpha$ -energy, and this is possible only if u_0 is a cover of a trivial cylinder. In this case, the negative end of u_0 is either $p\overline{\gamma}'_0$ or $p\overline{\gamma}'_\infty$. Without loss of generality, we assume that it is $p\overline{\gamma}'_0$. Then u_0 is the p -fold cover of the trivial cylinder over $p\overline{\gamma}'_0$. But this is not possible since the image of u_0 misses the point $(0, \varphi(x_0))$ in $\mathbb{R} \times L(p, q')$ due to the choice of the point x_0 ; see Figure 7. □

Now consider the case $\mathcal{A}^+ = 2\pi(1 + 1/p)$ and $\mathcal{A}^- = 2\pi$. In this case, u_0 is a $J_{\alpha'}$ -holomorphic pair of pants. Moreover, we have cylinders C_1 and C_2 in W^+ such that C_1 has the positive end $k^0\overline{\gamma}_0$ and C_2 has the positive end $k^\infty\overline{\gamma}_\infty$. Following the discussions given in the previous sections, we conclude that u_0 is a pair of pants with a base curve of degree 1. We have two possible profiles for the positive ends of u_0 :

- (i) $u_0(0) = (+\infty, l^0\overline{\gamma}'_0)$ and $u_0(\infty) = (+\infty, l^\infty\overline{\gamma}'_\infty)$,
- (ii) $u_0(0) = (+\infty, l^\infty\overline{\gamma}'_\infty)$ and $u_0(\infty) = (+\infty, l^0\overline{\gamma}'_0)$.

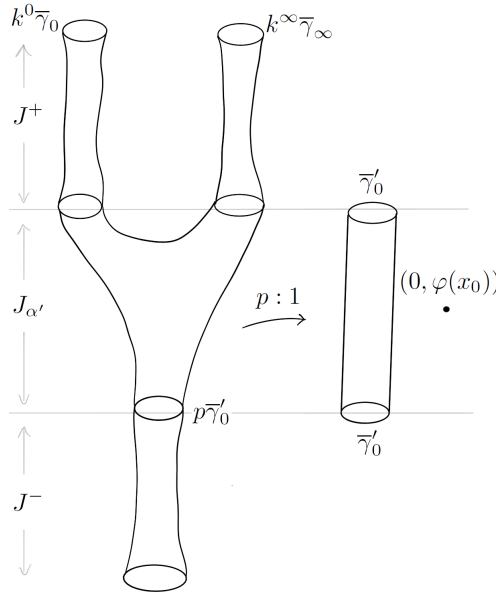


FIGURE 7. The case of u_0 being multiply covered in Lemma 4.10.

In both cases, we have $l^0 + l^\infty = p + 1$. Following Lemma 2.7, we have

$$l^0 \equiv (1 - v')^{-1} \quad \text{and} \quad l^\infty \equiv (1 - q')^{-1}$$

in the first case. Here the integer v' is given by $1 < v' < p - 1$ and $v'q' \equiv 1$. Now the negative end of C_1 is $l^0 \bar{\gamma}'_0$, and the negative end of C_2 is $l^\infty \bar{\gamma}'_\infty$. We apply Proposition 4.8 to C_1 and C_2 and use the fact that $l^0 + l^\infty = k^0 + k^\infty = p + 1$ to conclude $l^0 = k^0$ and $l^\infty = k^\infty$. We finally get

$$(1 - v')^{-1} \equiv l^0 = k^0 \equiv (1 - v)^{-1} \implies v' \equiv v \implies q' \equiv q.$$

In the second case, we have

$$l^0 \equiv (1 - q')^{-1} \quad \text{and} \quad l^\infty \equiv (1 - v')^{-1}.$$

Considering the cylinder C_1 again, we get

$$(1 - q')^{-1} \equiv l^0 = k^0 \equiv (1 - v)^{-1} \implies q' \equiv v \implies q' \equiv q^{-1}.$$

This concludes the proof the theorem.

4.11. Action control via 4-dimensional tools. In this section, we prove Proposition 4.5 and Proposition 4.8. The method is to apply intersection theory for closed holomorphic curves to the lifts of relevant punctured curves in symplectic cobordisms associated to lens spaces. Once these curves are lifted to the cobordisms associated to S^3 , we compactify these cobordisms and extend the punctured curves to closed curves in order to apply the intersection theory.

4.11.1. *Basics of intersection theory.* We briefly recall the basics of the intersection theory of closed holomorphic curves. For the details and proofs of the following statements, we refer to [11, 13, 24].

Let W be a closed oriented 4-manifold, let Σ and Σ' be closed oriented surfaces, and let $u : \Sigma \rightarrow W$, $v : \Sigma' \rightarrow W$ be smooth maps. An intersection $u(z) = v(w) = p$ is *transverse* if $du(T_z\Sigma) \oplus dv(T_w\Sigma') = T_pW$. We say this intersection is *positive* if the direct sum of the orientations of the surfaces coincides with the orientation of W and say *negative* otherwise. We define the *local intersection index* $\iota(u, z; v, w)$ as $+1$ if the intersection is positive and -1 otherwise. We note that if an intersection is transverse, then it is isolated. Hence, if all intersections of u and v are transverse, there are finitely many of them, and we can define the *total intersection number*

$$[u] \cdot [v] = \sum_{u(z)=v(w)} \iota(u, z; v, w).$$

It turns out that $[u] \cdot [v]$ depends only on the homology classes $[u], [v] \in H_2(W)$. Moreover, it defines a bilinear symmetric form on $H_2(W)$, which is non-degenerate.

If an intersection $u(z) = v(w) = p$ is not transverse but still isolated, one can still define a local intersection index as follows. One localizes the intersection via closed disks D_z and D_w around z and w . Then one picks C^∞ -small perturbation u_ε of u so that, when restricted to D_z and D_w , u_ε and v have only transverse intersections and $u_\varepsilon(\partial D_z) \cap v(D_w) = \emptyset$. Then one defines

$$\iota(u, z; v, w) = \sum_{u_\varepsilon(z')=v(w')} \iota(u_\varepsilon, z'; v, w'),$$

where the sum is taken for $(z', w') \in D_z \times D_w$.

Now we assume that W is equipped with an almost complex structure J and it is oriented via J . We also assume that Σ and Σ' carry complex structures j and j' respectively, and they are oriented via j and j' . Finally, we assume that u and v are closed J -holomorphic curves, that is, $du \circ j = J \circ du$ and $dv \circ j' = J \circ dv$. A well-known fact is that any intersection $u(z) = v(w) = p$ of two such curves is either isolated or there are neighborhoods $z \in U$ and $w \in V$ such that $u(U) = v(V)$. We note that this phenomenon is independent of the dimension of W .

The special features of the case $\dim W = 4$ are as follows. It is clear that, for any transverse intersection $u(z) = v(w) = p$, we have $\iota(u, z; v, w) = +1$. The nontrivial fact is that if an intersection is isolated, then $\iota(u, z; v, w) \geq 1$, with equality if and only if the intersection is transverse. This phenomenon is referred as *local positivity of intersections* and has the following global consequence. By the principle of unique continuation, one can show that if u and v have infinitely many intersections, then $\text{im}(u) = \text{im}(v)$, that is, either one is a reparametrization of the other or they are multiple covers of the same simple curve. Hence, if $\text{im}(u) \neq \text{im}(v)$, there are finitely many intersections, and we

have

$$[u] \cdot [v] \geq \#\{(z, w) \in \Sigma \times \Sigma' \mid u(z) = v(w)\},$$

with equality if and only if all the intersections are transverse. In particular, $[u] \cdot [v] = 0$ if and only if $\text{im}(u) \cap \text{im}(v) = \emptyset$. We refer to this fact as *global positivity of intersections*.

Finally, we want to address the question of self-intersections of a holomorphic curve. Let $u : \Sigma \rightarrow W$ be a closed J -holomorphic curve. If u is not simple, then it is clear that it has infinitely many double points. But if u is simple, one has only finitely many self-intersections. For a simple curve u , one defines the integer

$$\delta(u) = \frac{1}{2} \sum_{u(z)=u(w), z \neq w} i(u, z; u, w) + \sum_{du(z)=0} \delta(u, z),$$

where $\delta(u, z)$ is the local singularity index of u at a “critical point” z ; see [24] for the definition. It turns out that the local singularity index is the right correction term for the count of self-intersections, and consequently, $\delta(u)$ depends only on the homology class $[u]$ via the *adjunction formula*

$$[u] \cdot [u] = 2\delta + c_N(u),$$

where the *normal Chern number* $c_N(u)$ is defined as $c_N(u) = c_1([u]) - \chi(\Sigma)$. We note that $c_N(u) = c_1(N_u)$ when u is immersed, where N_u is the normal bundle of u . Moreover, the positivity of intersections phenomenon manifests itself by the fact that

$$\delta(u) \geq 0 \text{ and } \delta(u) = 0 \text{ if and only if } u \text{ is embedded.}$$

4.11.2. Intersections in the completions of the symplectizations and symplectic cobordisms. We recall the biholomorphic identification between $(\mathbb{R} \times S^3, J_0)$ and L^* , where the latter space is the total space of the tautological line bundle without its zero section. Let

$$u : \Sigma \setminus (\Gamma^+ \cup \Gamma^-) \rightarrow \mathbb{R} \times S^3 \cong L^*$$

be a finite energy J_0 -holomorphic curve, where k_i^+ 's/ k_j^- 's are the multiplicities of positive/negative ends of u . We recall from the previous sections that

$$c := \pi \circ u : \Sigma \setminus (\Gamma^+ \cup \Gamma^-) \rightarrow \mathbb{C}P^1$$

extends to a closed curve $c : \Sigma \rightarrow \mathbb{C}P^1$ with degree d , and u is identified with a meromorphic section of the bundle $c^*L \rightarrow \Sigma$, zeros corresponding the negative ends and poles corresponding to the positive ends, with

$$\#\text{zeros} - \#\text{poles} = -d.$$

In particular, $\sum_i k_i^+ - \sum_j k_j^- = d \geq 0$. We note that this inequality holds for any punctured curve in $\mathbb{R} \times S^3$ that is holomorphic with respect to a translation

invariant almost complex structure induced by α_0 due to the nonnegativity of $d\alpha$ -energy. Namely, one has

$$\sum_i 2\pi k_i^+ - \sum_j 2\pi k_j^- \geq 0.$$

Now we want to interpret this property in terms of positivity of intersections so that it generalizes to the settings with almost complex structures that coincide with J_0 near the ends.

We now consider the completion \hat{L} of L , where we compactify each fiber of L by turning it into $\mathbb{C}P^1$. We get the complex manifold \hat{L} with the complex structure \hat{J}_0 that extends J_0 . In fact, \hat{L} is nothing but $\mathbb{C}P^2$ blown up at one point, but we do not need this description in what follows, so we stick with our notation. Note that \hat{L} is a sphere bundle over $\mathbb{C}P^1$ such that each fiber is holomorphic. Moreover, \hat{L} contains two holomorphically embedded spheres S_0 and S_∞ , the first being the zero section of L and the second being “the section at infinity”. It is easy to see that

$$H_2(\hat{L}, \mathbb{Z}) = \mathbb{Z} \cdot [S_0] \oplus \mathbb{Z} \cdot [S_\infty].$$

We observe that $[S_0] \cdot [S_0] = c_1(N_0) = -1$. Here N_0 is the normal bundle of S_0 , and it can be identified with the bundle L . Similarly, $[S_\infty] \cdot [S_\infty] = c_1(N_\infty) = 1$ since the normal bundle N_∞ of S_∞ can be identified with the dual bundle of L . It is also clear that $[S_0] \cdot [S_\infty] = 0$.

Now we extend u to \hat{L} by extending the corresponding meromorphic section over the zeros and poles using the local holomorphic coordinates. We note that the extension is unique, and we end up with a closed curve, denoted by \hat{u} in \hat{L} . We observe that $[\hat{u}] \cdot [S_0] = K^-$ and $[\hat{u}] \cdot [S_\infty] = K^+$, where $K^- := \sum k_j^-$ and $K^+ := \sum k_i^+$. In fact, if u has a negative end with multiplicity k at a puncture z and c has the ramification number r at z , then one can locally write $u(z) = (z^r, z^k)$. In order to compute the intersection number with $[S_0]$ at z , one needs to compute the local intersection number between u and $v(w) = (w, 0)$. Since the intersection at 0 is not transverse, we perturb u and put $u_\varepsilon(z) = (z^r, z^k + \varepsilon)$. We see that an intersection (z, w) is a solution of the system $(z^r, z^k + \varepsilon) = (w, 0)$, and the second coordinates produce k -distinct roots z_1, \dots, z_k of $-\varepsilon$, and for each z_i , we have $w_i = z_i^r$. We note that $v'(w_i) = (1, 0)$, and $u'(z_i) = (z_i^r, kz_i^{k-1})$ has a non-vanishing second coordinate, and hence the intersections are transversal. Applying the same argument at each isolated intersection that appears at each negative end leads to the claim. Note that the same argument applies to the positive ends.

We write the homology class $[\hat{u}] = m[S_0] + n[S_\infty]$ for some $m, n \in \mathbb{Z}$ and compute

$$\begin{aligned} K^- &= [\hat{u}] \cdot [S_0] = (m[S_0] + n[S_\infty]) \cdot [S_0] = m[S_0] \cdot [S_0] + n[S_\infty] \cdot [S_0] = -m, \\ K^+ &= [\hat{u}] \cdot [S_\infty] = (m[S_0] + n[S_\infty]) \cdot [S_\infty] = m[S_0] \cdot [S_\infty] + n[S_\infty] \cdot [S_\infty] = n. \end{aligned}$$

Hence we get

$$(51) \quad [\hat{u}] = -K^-[S_0] + K^+[S_\infty],$$

and therefore,

$$[\hat{u}] \cdot [\hat{u}] = (-K^-)^2[S_0] \cdot [S_0] + (K^+)^2[S_\infty] \cdot [S_\infty] = (K^+)^2 - (K^-)^2.$$

We also note that

$$\begin{aligned} c_1([S_\infty]) &= [S_\infty] \cdot [S_\infty] + \chi(S_\infty) = 1 + 2 = 3, \\ c_1([S_0]) &= [S_0] \cdot [S_0] + \chi(S_0) = -1 + 2 = 1. \end{aligned}$$

Hence we get

$$c_1([\hat{u}]) = -K^-c_1([S_0]) + K^+c_1([S_\infty]) = 3K^+ - K^-.$$

Next we assume that u is simple. In this case, \hat{u} is simple as well, and by the adjunction formula, we get

$$(K^+)^2 - (K^-)^2 = 2\delta(\hat{u}) + 3K^+ - K^- - (2 - 2g),$$

where g is the genus of Σ . In particular,

$$(K^+)^2 - (K^-)^2 - 3K^+ + K^- + 2 = 2\delta(\hat{u}) + 2g \geq 0.$$

We note that $K^+ \geq 1$. This is due to the fact that the almost complex structure is tamed by an exact symplectic form, and therefore, a curve without positive ends cannot exist. Consequently, $-K^+ \geq -3K^+ + 2$ and

$$(K^+ - K^-)(K^+ + K^- - 1) = (K^+)^2 - (K^-)^2 - K^+ + K^- \geq 0.$$

Hence if $K^+ + K^- > 1$, then $K^+ \geq K^-$, and if $K^+ + K^- = 1$, then $K^- = 0$ and $K^+ \geq K^-$.

Now if u is not simple and say v is the underlying simple curve with L^+/L^- being the total multiplicity of its positive/negative ends, then the above argument says that $L^+ \geq L^-$. It is easy to see that $K^+ = NL^+$ and $K^- = NL^-$, where N is the degree of the underlying branched covering. Hence we get $K^+ \geq K^-$.

We claim that the above discussion applies if one considers an almost complex structure with cylindrical ends. In fact, if J is an almost complex structure on $\mathbb{R} \times S^3$ which coincides with J_0 outside of a compact set, then it extends to an almost complex structure \hat{J} on \hat{L} , and we have embedded \hat{J} -holomorphic spheres S_0 and S_∞ with the properties that $[S_0] \cdot [S_0] = -1$, $[S_\infty] \cdot [S_\infty] = 1$ and $[S_0] \cdot [S_\infty] = 0$. Now let u be a finite energy J -holomorphic curve and z a negative puncture with the asymptotic end of multiplicity k . Note that the projection $L \rightarrow \mathbb{C}P^1$ is not anymore J -holomorphic. Nevertheless, we consider a punctured disk neighborhood D^* of z in Σ and consider $u : D^* \rightarrow L^*$. This map is J_0 -holomorphic and leads to a holomorphic map $c : D^* \rightarrow \mathbb{C}P^1$. Since u has finite energy, c extends over the origin, and one gets a section $f : D^* \rightarrow c^*L$. Clearly, the section f is holomorphic and with an isolated zero of order k at the origin due to the asymptotic behavior of u . A similar argument applies to the positive punctures, and we conclude that u extends to a closed curve \hat{u}

in (\hat{L}, \hat{J}) . Moreover, near the intersection points, the local models of \hat{u} and $S_{0/\infty}$ are as before, and we get $[\hat{u}] \cdot [S_\infty] = K^+$ and $[\hat{u}] \cdot [S_0] = K^-$, where K^+/K^- is the sum of the multiplicities of the positive/negative ends. Consequently, (51) holds, and the rest of the above computation goes through.

4.11.3. *Proofs of Proposition 4.5 and Proposition 4.8.* We want to apply the above discussion to the lifts of curves that are given in Proposition 4.5 and Proposition 4.8. To this end, we need to specify coverings of the cobordisms at hand. We fix an equivariant lift $\tilde{\varphi} : S^3 \rightarrow S^3$ of the contactomorphism (37). We put $\beta_0 := \tilde{f}\alpha_0 = \tilde{\varphi}^*\alpha_0$. Here $\tilde{f} = f \circ \mathbf{p} : S^3 \rightarrow (0, +\infty)$ is the lift of (40). We let

$$\tilde{\Phi} : \mathbb{R} \times S^3 \rightarrow \mathbb{R} \times S^3$$

be the corresponding lift of (42). Here the actions of σ and σ' are extended to be invariant under the translation along \mathbb{R} -directions, and $\tilde{\Phi}$ is also equivariant. We put $\tilde{\Sigma} := \tilde{\Phi}(\{0\} \times S^3)$. We note that, for any n , the construction of W_n given by (43) lifts via $\tilde{\Phi}$. Namely, we have the covering space $\tilde{W}_n \rightarrow W_n$, where the group of Deck transformations is \mathbb{Z}_p , and for each n , we have an equivariant diffeomorphism

$$\tilde{\Phi}_n : \tilde{W}_n \rightarrow \mathbb{R} \times S^3$$

that lifts (44). Consequently, J_n lifts to an invariant almost complex structure \tilde{J}_n on \tilde{W}_n . Looking at the picture on the other side, \tilde{J}_n is a σ -invariant almost complex structure on $\mathbb{R} \times S^3$, and we have a homotopy of invariant almost complex structures $(\tilde{J}_t)_{t \in [0,1]}$ on $\mathbb{R} \times S^3$, which connects J_0 and \tilde{J}_n and lifts the path $(J_t)_{t \in [0,1]}$.

Let J_{t_0} be the almost complex structure on $\mathbb{R} \times L(p, q)$ given in Proposition 4.5, and let $\bar{u} : \mathbb{C}P^1 \setminus \Gamma \rightarrow \mathbb{R} \times L(p, q)$ be a J_{t_0} -holomorphic curve. First we assume that \bar{u} has a non-contractible end and the asymptotic profile of \bar{u} is as follows:

- positive ends:
 - non-contractible ones: $k_i^{0,+}\bar{\gamma}_0$, where $k_i^{0,+} \neq 0$ for $i = 1, \dots, n_0^+$, and $k_i^{\infty,+}\bar{\gamma}_\infty$, where $k_i^{\infty,+} \neq 0$ for $i = 1, \dots, n_\infty^+$,
 - contractible ones: in the orbit spaces $S_{k_i^+}^+$ of action $2\pi k_i^+$, $i = 1, \dots, n_c^+$,
- negative ends:
 - non-contractible ones: $k_i^{0,-}\bar{\gamma}_0$, where $k_i^{0,-} \neq 0$ for $i = 1, \dots, n_0^-$, and $k_i^{\infty,-}\bar{\gamma}_\infty$, where $k_i^{\infty,-} \neq 0$ for $i = 1, \dots, n_\infty^-$,
 - contractible ones: in the orbit spaces $S_{k_i^-}^-$ of action $2\pi k_i^-$, $i = 1, \dots, n_c^-$.

Then, using the scheme given in previous section, which is purely topological, after precomposing it with a suitable p -fold covering $\mathbf{p} : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{C}P^1 \setminus \Gamma$, we get a lifted \tilde{J}_{t_0} -holomorphic curve $u : \Sigma \setminus \tilde{\Gamma} \rightarrow \mathbb{R} \times S^3$. Recall that, once \mathbf{p} is extended over the punctures, the punctures in Γ with non-contractible ends are precisely the branch points of \mathbf{p} , and each of these branch points has a unique preimage in $\tilde{\Gamma}$. Hence, at each such preimage, the ramification number is p . On the other hand, the punctures in Γ with contractible ends are all regular

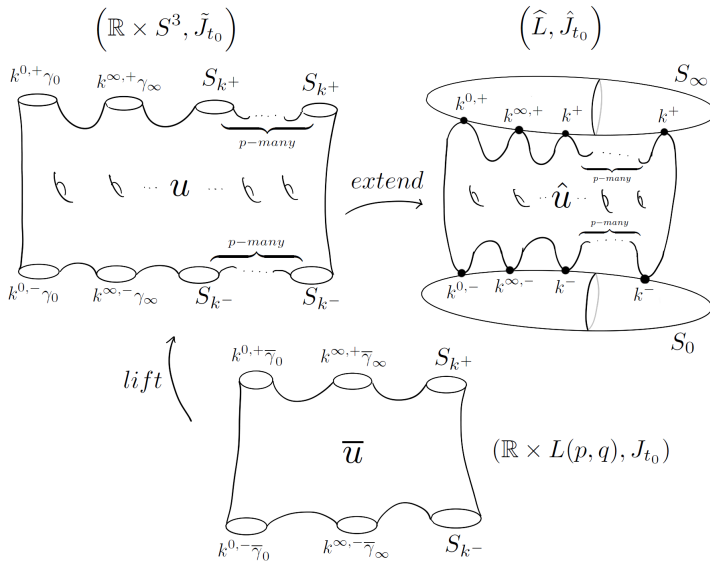


FIGURE 8. Intersections after lifting and extending.

points; hence each such puncture has precisely p preimages. As a result, the asymptotics of u are given as follows:

- positive ends:
 - $k_i^{0,+}\gamma_0$ for $i = 1, \dots, n_0^+$, and $k_i^{\infty,+}\gamma_\infty$ for $i = 1, \dots, n_\infty^+$,
 - for each $i \in \{1, \dots, n_c^+\}$, p positive ends in the orbit space $S_{k_i^+}^+$ of multiplicity k_i^+ .
- negative ends:
 - $k_i^{0,-}\gamma_0$ for $i = 1, \dots, n_0^-$, and $k_i^{\infty,-}\gamma_\infty$ for $i = 1, \dots, n_\infty^-$,
 - for each $i \in \{1, \dots, n_c^-\}$, p negative ends in the orbit space $S_{k_i^-}^+$ of multiplicity k_i^- .

Since \tilde{J}_{t_0} coincides with J_0 near the ends, the discussion above applies to u , and we get

$$(52) \quad \sum_{i=1}^{n_0^+} k_i^{0,+} + \sum_{i=1}^{n_\infty^+} k_i^{\infty,+} + p \sum_{i=1}^{n_c^+} k_i^+ \geq \sum_{i=1}^{n_0^-} k_i^{0,-} + \sum_{i=1}^{n_\infty^-} k_i^{\infty,-} + p \sum_{i=1}^{n_c^-} k_i^-.$$

Recall that the action of the orbit $\bar{\gamma}_{0/\infty}$ is given by $2\pi/p$. Hence dividing both sides of (52) by p and multiplying with 2π shows that the total action at the positive ends of \bar{u} is not less than the total action at its negative ends. Now if \bar{u} does not have any non-contractible end, it lifts without being pre-composed with a covering, and the result again follows. This concludes the proof of Proposition 4.5.

For Proposition 4.8, we repeat the same reasoning for the lifts of corresponding curves. More precisely, we let $\tilde{W}^+ \rightarrow W^+$ denote the covering space, which

is determined by the lifting scheme given above, and let \tilde{J}^+ be the lift of J^+ . Let \bar{u} be a given J^+ -holomorphic component in W^+ . Then the asymptotic ends of \bar{u} are given as above, where, for the negative non-contractible ends, $\bar{\gamma}_{0/\infty}$'s are replaced with $\bar{\gamma}'_{0/\infty}$'s. Since the contactomorphism φ induces an isomorphism on the fundamental groups, $\bar{\gamma}'_{0/\infty}$'s also generate the fundamental group of W^+ and our lifting scheme applies the covering $\tilde{W}^+ \rightarrow W^+$ as well. Consequently, we have a \tilde{J}^+ -holomorphic lift $u : \Sigma \setminus \tilde{\Gamma} \rightarrow \tilde{W}^+$ with the asymptotics given as before.

Now we extend u to a closed curve as follows. Topologically, we view \tilde{W}^+ as the upper connected component \tilde{N}^+ of $(\mathbb{R} \times S^3) \setminus \tilde{\Sigma}$. We compactify \tilde{W}^+ by collapsing its positive end, namely S^3 , to the 2-sphere S_∞ via the Reeb flow of α_0 and by collapsing its negative end, namely $\tilde{\Sigma}$, to the 2-sphere S_0 via the Reeb flow of β_0 . When we view \tilde{W}^+ as an almost complex manifold equipped with \tilde{J}^+ , the negative end is given by S^3 , and it is collapsed via the Reeb flow of α_0 since \tilde{J}^+ coincides with J_0 near the negative end. We denote the resulting almost complex manifold by (\hat{W}^+, \hat{J}^+) . It is not hard to see that $H_2(\hat{W}^+)$ is generated again by two spherical classes $[S_0]$ and $[S_\infty]$, and clearly, the representatives S_0 and S_∞ are \hat{J}^+ -holomorphic. Since \hat{J}^+ coincides with \hat{J}_0 also on a tubular neighborhood of S_∞ in \hat{W}^+ , we get the same intersection properties of S_0 and S_∞ , and due to the asymptotic behavior of u , equation (51) holds, and the rest of the computation goes through. Hence the total multiplicities of the positive ends of u are greater than or equal to the total multiplicities of the negative ends of u . Relating this inequality to total actions of positive and negative ends of \bar{u} leads to the claim.

We note that the same argument applies to the components in (W^-, J^-) , and this finishes the proof of Proposition 4.8.

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