Münster J. of Math. **15** (2022), 167–220
DOI 10.17879/23049547771
urn:nbn:de:hbz:6-23049548218

Münster Journal of Mathematics

The stable rank of diagonal ASH algebras and crossed products by minimal homeomorphisms

Mihai Alboiu and James Lutley

(Communicated by Joachim Cuntz)

Abstract. We introduce a subclass of recursive subhomogeneous algebras, in which each of the pullback maps is diagonal in a suitable sense. We define the notion of a diagonal map between two such algebras and show that every simple inductive limit of these algebras with diagonal bonding maps has stable rank one. As an application, we prove that, for any infinite compact metric space T and minimal homeomorphism $h : T \to T$, the associated dynamical crossed product $C^*(\mathbb{Z}, T, h)$ has stable rank one. This affirms a conjecture of Archey, Niu, and Phillips. We also show that the Toms–Winter Conjecture holds for such crossed products.

1. Introduction

With the aim of formulating a notion of dimension for a C*-algebra, in [20], Rieffel introduced the concept of stable rank. The *stable rank* of a unital C*-algebra A is the least natural number n for which the set of all n-tuples of A that generate A as a left ideal is dense in A^n; if no such integer exists, the stable rank is said to be ∞. Of particular note is the instance when the stable rank is one. In [20, Prop. 3.1], it is shown that a unital C*-algebra has stable rank one if and only if the set of invertible elements is dense within the algebra. An important problem in the field has been to determine when a C*-algebra has stable rank one.

In [21], Rørdam supplied one of the first major results concerning stable rank. He showed that the tensor product of a simple unital stably finite C*-algebra and a UHF algebra has stable rank one. This was followed by a result of Dădărlat, Nagy, Némethi, and Pasnicu, who proved in [5] that a simple unital inductive limit of full matrix algebras (those of the form $C(X, M_n(\mathbb{C}))$ for a compact Hausdorff space X) always has stable rank one assuming there is a uniform upper bound on the dimensions of the base spaces in the finite stage algebras. Later, in [22], Rørdam also showed that every simple unital finite C*-algebra that absorbs the Jiang–Su algebra, $\mathcal{Z}$, tensorially has stable rank one.

The first author was supported by an NSERC CGS D grant.

Villadsen proved in [24] that the converse to the result in [5] does not hold by constructing a unital simple limit of full matrix algebras, whose base space dimensions were not uniformly bounded above, yet which nonetheless still had stable rank one. He went on, in [25], to construct a class of simple unital AH algebras—inductive limits of homogeneous C*-algebras (those whose irreducible representations all have the same dimension)—of arbitrary stable rank, thereby affirming the subtleties present in the problem of stable rank.

Given compact metric spaces X and Z, together with continuous functions $\lambda_1, \dots, \lambda_k : Z \to X$, there is a naturally induced *-homomorphism from $C(X, M_n(\mathbb{C}))$ to $C(Z, M_{nk}(\mathbb{C}))$ given by

$$f \mapsto \operatorname{diag}(f \circ \lambda_1, \dots, f \circ \lambda_k).$$

These induced maps between full matrix algebras are referred to as *diagonal*. They have been used to construct a rich class of examples in the field, including those of Goodearl in [11] and Villadsen in [24].

Just over a decade ago, another stable rank one result was obtained by Elliott, Ho, and Toms in [8]. Their paper, which stemmed from Ho's work in [12], showed that the condition of bounded dimension in [5] could be replaced with the assumption that all of the bonding maps in the inductive limit are diagonal.

In this present paper, we extend the AH stable rank one result of Elliott, Ho, and Toms in [8] to a suitable class of approximately subhomogeneous (ASH) algebras—inductive limits of subhomogeneous C*-algebras (those whose irreducible representations all have dimension at most some fixed integer).

The building-block algebras in the AH setting are full matrix algebras, whose primitive quotients are intrinsically matrix unit compatible. This internal compatibility is crucial to obtaining the stable rank one result in [8]. To achieve this for the subhomogeneous building blocks in the ASH setting, it is necessary, therefore, to consider only subhomogeneous algebras whose primitive quotients fit together in a compatible (*i.e.*, matrix unit compatible) way. We restrict our attention to a subclass of recursive subhomogeneous algebras.

Recursive subhomogeneous algebras are a particularly tractable class of unital subhomogeneous algebras introduced by Phillips in [17], which are iterative pullbacks of full matrix algebras. In order to ensure the aforementioned compatibility, it is necessary that all the pullback maps be diagonal in a suitable sense, and we call such algebras diagonal subhomogeneous (DSH) algebras. We are then able to define the notion of a diagonal map between two DSH algebras, which sends each point in the spectrum of the range algebra to an ordered list of eigenvalues in the domain algebra. It turns out that this set-up is enough to extend the results in [8]; more specifically, every simple inductive limit of DSH algebra with diagonal bonding maps has stable rank one (see Theorem 3.30).

DSH algebras arise naturally in the study of dynamical crossed products. The orbit-breaking subalgebras of crossed products introduced by Q. Lin in [13] (see also [14, 15]) following the work of Putnam in [19] are examples of DSH algebras. Using our stable rank one theorem for inductive limits and results of Archey and Phillips developed in [3], we are able to prove a conjecture of Archey, Niu, and Phillips stated in the same paper [3, Conj. 7.2]; namely, that, for an infinite compact metric space T and a minimal homeomorphism $h : T \to T$, the dynamical crossed product $C^*(\mathbb{Z}, T, h)$ has stable rank one (see Corollary 3.36). Using a result of Thiel in [23], we are also able to show that, for such crossed products, classifiability is determined solely by strict comparison, thereby affirming the Toms–Winter Conjecture for simple dynamical crossed products (see Corollary 3.37).

This paper is organized as follows. Section 2 is dedicated to structure and basic properties of DSH algebras. In Section 2.1, we formally introduce the class of DSH algebras, the notion of a diagonal map between two such algebras, and we prove some basic lemmas concerning this class, which are used throughout the remainder of the paper. The aim of Section 2.14 is to show that quotients of DSH algebras remain DSH, and that diagonal maps between two DSH algebras remain diagonal when passing to quotients; this allows one to assume that the bonding maps in Theorem 3.30 are injective. Finally, in Section 2.20, we show that every homogeneous DSH algebra is a full matrix algebra.

Section 3 contains the main results of the paper. The proof of Theorem 3.30 is quite technical and relies on several lemmas, which are established in Section 3.2 and Section 3.18. In Section 3.1, we outline the significance of these lemmas and illustrate how they come together to prove Theorem 3.30 in Section 3.29. Lastly, in Section 3.31, we discuss the significance of Theorem 3.30 in the setting of minimal dynamical crossed products, and we establish Corollaries 3.36 and 3.37.

Throughout the paper, we use $\mathbb{N}$ to denote the set of strictly positive integers and the symbol $\subset$ to denote non-strict set inclusion. Given a C^*-algebra A, we let $\hat{A}$ denote the set of equivalence classes of nonzero irreducible representations of A equipped with the hull-kernel topology. If A is unital, we use 1_A to denote the unit of A. For $n \in \mathbb{N}$, we use the shorthand M_n to refer to the matrix algebra $M_n(\mathbb{C})$. When speaking about a matrix $D \in M_n$, we denote the (i, j)-entry of D by $D_{i,j}$, and we let 1_n denote the identity matrix in M_n.

2. Diagonal subhomogeneous (DSH) algebras

In this section, we introduce the class of diagonal subhomogeneous (DSH) algebras that we deal with in this paper and examine their basic properties and structure. In Section 2.1, we define what a DSH algebra is and the notion of a diagonal map between two such algebras. We discuss some basic properties and notions concerning these algebras that are used throughout the remainder of the section and beyond. The chief purpose of Section 2.14 is to prove that, given any inductive limit of DSH algebras with diagonal bonding maps, one may always assume the maps in the sequence are injective.

In [17, Cor. 1.8], Phillips shows that every unital homogeneous C*-algebra (regardless of its Dixmier–Douady class) has a recursive subhomogeneous decomposition. This follows by using the pullback maps to appropriately adjoin various pieces of the spectrum over which the algebra is locally trivial. In the DSH setting, where the pullback maps preserve the matrix units of the primitive quotients in a very strong sense, such a bonding is possible only if the homogeneous algebra is in fact a full matrix algebra, as we show in Section 2.20.

2.1. **Introductory definitions and basic properties.** Let us start off by recalling the definition of a recursive subhomogeneous algebra.

Definition 2.2 ([17, Def. 1.1]). A *recursive subhomogeneous algebra* is a C*-algebra given by the following recursive definition.
(i) If X is a compact metric space and $n \geq 1$, then $C(X, M_n)$ is a recursive subhomogeneous algebra.
(ii) If A is a recursive subhomogeneous algebra, X is a compact metric space, $Y \subset X$ is closed, $\varphi : A \to C(Y, M_n)$ is a unital *-homomorphism, and $\rho : C(X, M_n) \to C(Y, M_n)$ is the restriction *-homomorphism, then the pullback

$$A \oplus_{C(Y,M_n)} C(X, M_n) := \{(a, f) \in A \oplus C(X, M_n) \mid \varphi(a) = \rho(f)\}$$

is a recursive subhomogeneous algebra.

Therefore, if A is a recursive subhomogeneous algebra, there are compact metric spaces $X_1, \ldots, X_l$ (the *base spaces* of A), closed subspaces $Y_1 := \varnothing$, $Y_2 \subset X_2, \ldots, Y_l \subset X_l$, positive integers $n_1, \ldots, n_l$, C*-algebras

$$A^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j, M_{n_j}) \quad \text{for } 1 \leq i \leq l,$$

and unital *-homomorphisms $\varphi_i : A^{(i)} \to C(Y_{i+1}, M_{n_{i+1}})$ for $1 \leq i \leq l-1$ such that
(i) $A^{(1)} = C(X_1, M_{n_1})$;
(ii) for all $1 \leq i \leq l-1$,

$$A^{(i+1)} = \{(a, f) \in A^{(i)} \oplus C(X_{i+1}, M_{n_{i+1}}) \mid \varphi_i(a) = f|_{Y_{i+1}}\};$$

(iii) $A = A^{(l)}$.
Simply put,

$$A = \big[\cdots[[C_1 \oplus_{C_2'} C_2] \oplus_{C_3'} C_3] \cdots\big] \oplus_{C_l'} C_l,$$

where $C_i := C(X_i, M_{n_i})$, $C_i' := C(Y_i, M_{n_i})$, and the maps $\varphi_1, \ldots, \varphi_{l-1}$ are used in the pullback. In this case, we say the length of the composition sequence is l. As shown in [17], the decomposition of a recursive subhomogeneous is highly non-unique. We make the same tacit assumption adopted in that paper: unless otherwise specified, every recursive subhomogeneous algebra comes equipped with a decomposition of the form given above. In particular, we refer to the number l above as the *length of A*.

Since, for all $1 \leq i \leq l$, we have $A^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j, M_{n_j})$, we can view each element $f \in A^{(i)}$ as $(f_1, \dots, f_i)$, where $f_j \in C(X_j, M_{n_j})$ for all $1 \leq j \leq i$. For $1 \leq i \leq l$ and $x \in X_i$, we have the usual *evaluation map* $\mathrm{ev}_x : A \to M_{n_i}$ given by $\mathrm{ev}_x(f) := f_i(x)$ for all $f \in A$. We let $\mathfrak{s}(A) := \min\{n_1, \dots, n_l\}$ and $\mathfrak{S}(A) := \max\{n_1, \dots, n_l\}$.

The chief reasons for working with recursive subhomogeneous algebras are that they are very convenient computationally and they also allow us to carry forward much of the structure intrinsic to a full matrix algebra. There is, however, no restriction on the pullback maps used to join together the full matrix algebras in the recursive decomposition. In particular, the pullback maps need not piece together the matrix units of the various primitive quotients in a compatible way. Therefore, in order to harness the internal matrix unit compatible structure of a full matrix algebra, one must ensure that the pullback maps used in the recursive decomposition preserve the matrix units of each full matrix algebra. An effective way to achieve this is to require the pullback maps to be diagonal in an appropriate sense, which we now make clear.

Definition 2.3 (DSH algebras)**.** A C*-algebra A is a *diagonal subhomogeneous (DSH) algebra* (of length l) provided that it is a recursive subhomogeneous algebra (of length l) (with a decomposition as described above), and for all $1 \leq i \leq l-1$ and $y \in Y_{i+1}$, there is a list of points $x_1 \in X_{i_1} \setminus Y_{i_1}, \dots, x_t \in X_{i_t} \setminus Y_{i_t}$ such that, for all $f \in A^{(i)}$,

$$\varphi_i(f)(y) = \mathrm{diag}\big(f_{i_1}(x_1), \dots, f_{i_t}(x_t)\big).$$

We say y *decomposes into* $x_1, \dots, x_t$, that each x_j is a point *in the decomposition of* y, and that x_j *begins at index* $1 + n_{i_1} + \cdots + n_{i_{j-1}}$ *down the diagonal of* y. Given $1 \leq j \leq i$ and $y' \in Y_j$, we say that y' is *in the decomposition of* y if there exists a $1 \leq k \leq n_i$ with the property that, for all $f \in A^{(i)}$, there are matrices $P \in M_{k-1}$ and $Q \in M_{n_i - n_j - (k-1)}$ such that $f_i(y) = \mathrm{diag}(P, f_j(y'), Q)$.

Whenever we work with a DSH algebra of length l, we adopt, unless otherwise specified, the same notation for the decomposition used above. Thus, if A is a DSH algebra of length l, we can view A as the set of all $f := (f_1, \dots, f_l) \in \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$ such that, for all $1 \leq i < l$ and $y \in Y_{i+1}$,

$$f_{i+1}(y) = \mathrm{diag}\big(f_{i_1}(x_1), \dots, f_{i_t}(x_t)\big).$$

As is shown in Lemma 2.7 below, the decomposition of y is unique up to the reindexing of identical points; that is, if y decomposes into $x_1, \dots, x_t$ and $z_1, \dots, z_s$, then $s = t$ and, for $1 \leq j \leq s$, $x_j = z_j$.

Definition 2.4 (Diagonal maps between DSH algebras)**.** Given two DSH algebras A_1 and A_2 of lengths l_1 and l_2 and with base spaces $X_1^1, \dots, X_{l_1}^1$ and $X_1^2, \dots, X_{l_2}^2$, respectively, we say that a *-homomorphism $\psi : A_1 \to A_2$ is *diagonal* provided that, for all $1 \leq i \leq l_2$ and $x \in X_i^2$, there are points $x_1, \dots, x_t$

with $x_j \in X^1_{i_j}$ such that, for all $f \in A_1$,

$$\psi(f)_i(x) = \mathrm{diag}\big(f_{i_1}(x_1), \ldots, f_{i_t}(x_t)\big).$$

We say that x *decomposes into* $x_1, \ldots, x_t$.

Note that if $Y^1_1 \subset X^1_1, \ldots, Y^1_{l_1} \subset X^1_{l_1}$ and $Y^2_1 \subset X^2_1, \ldots, Y^2_{l_2} \subset X^2_{l_2}$ are the corresponding closed subsets of the base spaces in Definition 2.4, then, owing to the decomposition structure of A_1 and A_2, we get an equivalent definition by replacing X^2_i and $X^1_{i_j}$ above with $X^2_i \setminus Y^2_i$ and $X^1_{i_j} \setminus Y^1_{i_j}$, respectively. Note, too, that, by definition, diagonal maps are automatically unital.

For the remainder of Section 2.1, let us assume that A is a DSH algebra of length l. The following lemma provides us with a description of the spectrum of A.

Lemma 2.5 ([17, Lem. 2.1])**.** *The map $x \mapsto [\mathrm{ev}_x]$ defines a continuous bijection*

$$\bigsqcup_{i=1}^{\ell} (X_i \setminus Y_i) \to \hat{A},$$

(where, recall, $Y_1 := \varnothing$) whose restriction to each $X_i \setminus Y_i$ is a homeomorphism onto its image. In particular, every irreducible representation of A is unitarily equivalent to ev_x for some $x \in \bigsqcup_{i=1}^{\ell}(X_i \setminus Y_i)$.

We often tacitly refer to a given irreducible representation ev_x simply as x since we view such an element both as an irreducible representation and as a point in X_i.

Remark 2.6. A subset $D \subset X_i \setminus Y_i$ can be viewed as a subset both of X_i and of $\hat{A}$. We denote by $\overline{D}^{X_i}$ the closure of D with respect to the topology on X_i. With one or two exceptions, when speaking about open and closed subsets of X_i in this paper, we mean with respect to the topology on X_i; such subsets could, in general, include points in Y_i, in which case they would not even be a subset of the spectrum. In any case, for subsets of $X_i \setminus Y_i$, the topology is always made explicit.

Lemma 2.7. *Suppose $2 \le i \le l$ and $y \in Y_i$. If y decomposes into $x_1, \ldots, x_t$ and $z_1, \ldots, z_s$, then $s = t$ and, for $1 \le j \le s$, $x_j = z_j$.*

Proof. By Lemma 2.5, A is liminary and $x_1 = z_1$ if and only if $\mathrm{ev}_{x_1} = \mathrm{ev}_{z_1}$. Hence, if $x_1 \neq z_1$, [7, Prop. 4.2.5] furnishes a function $f \in A$ such that $\mathrm{ev}_{x_1}(f)$ and $\mathrm{ev}_{z_1}(f)$ are the 0 and identity matrix of appropriate sizes, respectively. This contradicts the assumption that

$$\mathrm{diag}\big(\mathrm{ev}_{x_1}(f), \ldots, \mathrm{ev}_{x_t}(f)\big) = f_i(y) = \mathrm{diag}\big(\mathrm{ev}_{z_1}(f), \ldots, \mathrm{ev}_{z_s}(f)\big). \tag{1}$$

Therefore, $x_1 = z_1$. Continuing inductively, we see that if $s < t$, $s > t$, or $x_j \neq z_j$ for some j, then equation (1) is violated. Hence, Lemma 2.7 follows. ∎

By Lemma 2.5 and Definition 2.3, given $y \in \bigsqcup_{i=1}^{l} X_i$, either ev_y is an irreducible representation of A or, if y is in some Y_i, ev_y splits up into irreducible representations of A. The following definition categorizes the elements in the base spaces depending on the indices at which these irreducible representations occur.

Definition 2.8. Given $1 \leq i \leq l$ and $1 \leq k \leq n_i$, we define $B_{i,k}$ to be the set of points in X_i that have an irreducible representation beginning at index k down their diagonal. For $k \leq 0$, we set $B_{i,k} := \varnothing$.

The following rudimentary observations about the $B_{i,k}$'s defined above will be very helpful in the proofs of the lemmas in Section 3.2 and Section 3.18.

Lemma 2.9.
(i) $B_{i,1} = X_i$ *for all* $1 \leq i \leq l$.
(ii) *If* $1 \leq i \leq l$ *and* $k > 1$, *then* $B_{i,k} \subset Y_i$. *In particular,* $B_{1,k} = \varnothing$ *for* $k > 1$.
(iii) *If* $2 \leq i \leq l$ *and* $y \in Y_i$ *decomposes into* $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$, *then* $y \in B_{i,k}$ *if and only if* $k = 1 + n_{i_1} + \cdots + n_{i_{j-1}}$ *for some* $1 \leq j \leq t$. *In particular,* $B_{i,k} = \varnothing$ *for all* $n_i - (\mathfrak{s}(A) - 1) < k \leq n_i$, *where, recall,* $\mathfrak{s}(A) := \min\{n_j \mid 1 \leq j \leq l\}$.

Proof. Fix $1 \leq i \leq l$, and suppose $y \in X_i$. If y belongs to Y_i and decomposes into $x_1, \ldots, x_t$, then by Definition 2.3, x_1 begins at index 1 down the diagonal of y, and so $y \in B_{i,1}$. If $y \notin Y_i$, then by Lemma 2.5, y is irreducible and trivially begins at index 1 down its diagonal and, moreover, cannot have any irreducible representation beginning at any index $k \geq 2$. This proves (i) and (ii). To prove (iii), suppose y belongs to Y_i and decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots,$ $x_t \in X_{i_t} \setminus Y_{i_t}$. By Definition 2.3, $y \in B_{i,k}$ if $k = 1 + n_{i_1} + \cdots + n_{i_{j-1}}$ for some $1 \leq j \leq t$. Since Lemma 2.7 shows that the decomposition of y is unique, it follows that y cannot belong to any other $B_{i,k}$, which establishes (iii). $\square$

The following lemma allows us to approximate any point in some Y_i by irreducible representations in $X_i \setminus Y_i$ (with respect to the topology on X_i).

Lemma 2.10. *For each* $2 \leq i \leq l$, *we may assume* $\operatorname{int}(Y_i) = \varnothing$.

Proof. Fix $1 \leq i \leq l-1$. Let $Y'_{i+1} := Y_{i+1} \setminus \operatorname{int}(Y_{i+1})$, $X'_{i+1} := X_{i+1} \setminus \operatorname{int}(Y_{i+1})$. Then we have the following commutative diagram of restriction *-homomorphisms:

$$\begin{array}{ccc} C(X_{i+1}, M_{n_{i+1}}) & \xrightarrow{\rho} & C(Y_{i+1}, M_{n_{i+1}}) \\ \lambda\big\downarrow & & \big\downarrow\tau \\ C(X'_{i+1}, M_{n_{i+1}}) & \xrightarrow{\rho'} & C(Y'_{i+1}, M_{n_{i+1}}). \end{array}$$

Let

$$B^{(i+1)} := A^{(i)} \oplus_{C(Y'_{i+1}, M_{n_{i+1}})} C(X'_{i+1}, M_{n_{i+1}}),$$

where the connecting *-homomorphism is $\varphi'_i := \tau \circ \varphi_i : A^{(i)} \to C(Y'_{i+1}, M_{n_{i+1}})$. Let us show that $A^{(i+1)}$ is isomorphic to $B^{(i+1)}$. Given $a \in A^{(i)}$ and $f \in C(X_{i+1}, M_{n_{i+1}})$ with $(a, f) \in A^{(i+1)}$, define $\Gamma : A^{(i+1)} \to B^{(i+1)}$ by $\Gamma((a, f)) := (a, \lambda(f))$. Note that $\varphi'_i(a) = \tau(\varphi_i(a)) = \tau(\rho(f)) = \rho'(\lambda(f))$ so that Γ is well defined. It is easy to see that Γ is a *-homomorphism. To see that Γ is injective, suppose $(a, f) \in A^{(i+1)}$ with $(a, \lambda(f)) = \Gamma((a, f)) = (0, 0)$. Then $a = 0$, and so $f|_{Y_{i+1}} = \varphi_i(a) = 0$, which, together with the fact that $\lambda(f) = 0$, yields that $f = 0$. For surjectivity, suppose $a \in A^{(i)}$ and $g \in C(X'_{i+1}, M_{n_{i+1}})$ with $(a, g) \in B^{(i+1)}$. Then $\varphi_i(a)|_{Y'_{i+1}} = g|_{Y'_{i+1}}$ so that the function $h : X_{i+1} \to M_{n_{i+1}}$ defined to be $\varphi_i(a)(x)$ for $x \in Y_{i+1}$ and $g(x)$ for $x \in X'_{i+1}$ is well defined and continuous. Moreover, $\varphi_i(a) = h|_{Y_{i+1}}$, which implies $(a, h) \in A^{(i+1)}$ and $\Gamma((a, h)) = (a, \lambda(h)) = (a, g)$, proving surjectivity. □

The lemma following guarantees that a function in A will be invertible provided it is an invertible matrix in every primitive quotient of A.

Lemma 2.11. *Suppose $f \in A$ and that, for all $1 \le i \le l$ and $x \in X_i \setminus Y_i$, the matrix $f_i(x)$ is invertible in M_{n_i}. Then f is invertible in A.*

Proof. Owing to the diagonal decomposition at points in Y_i, we may assume that $f_i(x)$ is an invertible matrix for all $1 \le i \le l$ and $x \in X_i$. Define $g \in \bigoplus_{i=1}^l C(X_i, M_{n_i})$ to be $(g_1, \ldots, g_l)$, where for $1 \le i \le l$ and $z \in X_i$, $g_i(z) := f_i(z)^{-1}$. Since g is the inverse of f in $\bigoplus_{i=1}^l C(X_i, M_{n_i})$, to prove the lemma, we need only to verify that $g \in A$. Suppose $1 \le i \le l-1$ and that $y \in Y_{i+1}$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$. Then

$$\begin{aligned} g_{i+1}(y) = f_{i+1}(y)^{-1} &= \operatorname{diag}\big(f_{i_1}(x_1), \ldots, f_{i_t}(x_t)\big)^{-1} \\ &= \operatorname{diag}\big(f_{i_1}(x_1)^{-1}, \ldots, f_{i_t}(x_t)^{-1}\big) \\ &= \operatorname{diag}\big(g_{i_1}(x_1), \ldots, g_{i_t}(x_t)\big) \end{aligned}$$

so that $g \in A$. □

This next lemma illustrates a particular circumstance in which a set which is open in one of the base spaces of A is open when viewed as a subset of the spectrum.

Lemma 2.12. *Suppose $1 \le i \le \ell$. If $U \subset X_i \setminus Y_i$ is open with respect to the topology on X_i and has the property that no point in U appears in the decomposition of any point in Y_j for any $j > i$, then U is open with respect to the hull-kernel topology on $\hat{A}$.*

Proof. Let $x \in U$ be arbitrary. Put $g_j \equiv 0$ for $j < i$, and define $g_i \in C(X_i, M_{n_i})$ to be any function such that $g_i(x) \neq 0$ and $g_i|_{X_i \setminus U} \equiv 0$. Since $Y_i \subset X_i \setminus U$, g_i vanishes on Y_i so that $\varphi_{i-1}((g_1, \ldots, g_{i-1})) = 0 = g_i|_{Y_i}$; thus, $(g_1, \ldots, g_i) \in A^{(i)}$. For $j > i$, set $g_j \equiv 0$. Since no point in U is in the decomposition of any point in Y_j for any $j > i$, it follows inductively that $g := (g_1, \ldots, g_\ell) \in A$. This proves that U is open in $\hat{A}$. □

The final lemma in this subsection shows that if a point $x \in X_i$ is not in the decomposition of any point in some Y_j, then there must be an open neighborhood of x in X_i consisting only of points which also do not appear in the decomposition of any point in Y_j.

Lemma 2.13. *Suppose $1 \leq i < j \leq \ell$, and let $F \subset X_i$ denote the set of points that are in the decomposition of a point in Y_j. Then F is closed in X_i.*

Proof. Suppose $(z_n)_n$ is a sequence of points in F converging to $x \in X_i$. For each $n \in \mathbb{N}$, there is a $y_n \in Y_j$ with the property that z_n is in the decomposition of y_n. Since Y_j is compact, we may pass to a subsequence to conclude that there is a $y \in Y_j$ such that $y_n \to y$. Passing to a further subsequence, we may assume that there is a $1 \leq k \leq n_j$ such that, for all $n \in \mathbb{N}$, the representation ev_{z_n} begins at index k down the diagonal of ev_{y_n}. Suppose y decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_t \in X_{i_t} \setminus Y_{i_t}$. Let us show that $x \in F$ by proving it is in the decomposition of y. Let $f \in A$ be arbitrary. For each $n \in \mathbb{N}$, there are matrices $P_n \in M_{k-1}$ and $Q_n \in M_{n_j - n_i - (k-1)}$ such that $f_j(y_n) = \mathrm{diag}(P_n, f_i(z_n), Q_n)$. Since $\lim_{n\to\infty} f_j(y_n) = f_j(y)$, there are matrices $P \in M_{k-1}$ and $Q \in M_{n_j - n_i - (k-1)}$ such that

$$f_j(y) = \mathrm{diag}(P, f_i(x), Q). \tag{2}$$

If $x \in Y_i$, it follows by definition that x is in the decomposition of y. If $x \in X_i \setminus Y_i$ and x is not in the decomposition of y, then we may use [7, Prop. 4.2.5] to find a function $g \in A$ that is nonzero at x, but vanishes at all points in the decomposition of y, which implies that $g_j(y) = 0$ and contradicts equation (2). Thus, x is in the decomposition of y and, hence, $x \in F$. □

2.14. **Quotients of DSH algebras.** In [17, Prop. 3.1], Phillips shows that the class of (separable) recursive subhomogeneous algebras is closed under the taking of quotients. The recursive decomposition of the quotient is not explicitly constructed from that of the original algebra, but rather is furnished using a characterization of (separable) recursive subhomogeneous algebras (see [17, Thm. 2.16]).

We show in this subsection that, associated to any quotient B of a DSH algebra A, there is a DSH algebra (see Proposition 2.17) whose decomposition is canonically obtained from the decomposition of A, and which is isomorphic to B (see Proposition 2.18). We are then able to prove (see Proposition 2.19) that the diagonality of maps between two DSH algebras is preserved when passing to quotients, thus allowing us to assume that the bonding maps in Theorem 3.30 are injective.

Let A be a DSH algebra of length l. Suppose we have a nonzero C*-algebra B and a surjective *-homomorphism $\rho : A \to B$. This yields an injective single-valued map $\hat{\rho} : \hat{B} \to \hat{A}$ given by $\hat{\rho}([\pi]) := [\pi \circ \rho]$. For $1 \leq i \leq l$, define $X_i' := X_i \cap \hat{\rho}(\hat{B})^{X_i}$ and $Y_i' := X_i' \cap Y_i$. Recall that these definitions make sense by Lemma 2.5.

Lemma 2.15. *$\hat{\rho}(\hat{B})$ is closed in $\hat{A}$.*

Proof. Suppose $[\pi] \in \overline{\hat{\rho}(\hat{B})}$. Then

$$\ker \pi \supset \bigcap_{[\sigma] \in \hat{\rho}(\hat{B})} \ker \sigma = \bigcap_{[\tau] \in \hat{B}} \ker \hat{\rho}([\tau]) = \bigcap_{[\tau] \in \hat{B}} \ker(\tau \circ \rho).$$

Note that $a \in \bigcap_{[\tau]\in\hat{B}} \ker(\tau\circ\rho)$ if and only if $\rho(a) \in \bigcap_{[\tau]\in\hat{B}} \ker\tau$ if and only if $\rho(a) = 0$. Hence, $\ker\pi \supset \ker\rho$. Thus, the irreducible representation τ of B given by $\tau(b) := \pi(a)$, where a is any lift of b under ρ, is well defined. Therefore, $[\pi] = [\tau\circ\rho] = \hat{\rho}([\tau]) \in \hat{\rho}(\hat{B})$ so that $\hat{\rho}(\hat{B}) \subset \hat{\rho}(\hat{B})$. □

Lemma 2.16. *Suppose $1 \le i \le l$ and $y \in Y_i'$. If $1 \le j < i$ and $x \in X_j \setminus Y_j$ is in the decomposition of y, then $x \in X_j \cap \hat{\rho}(\hat{B}) \subset X_j'$.*

Proof. Since $y \in Y_i'$, we have $y \in X_i' = \overline{X_i \cap \hat{\rho}(\hat{B})}^{X_i}$. Choose a sequence $(z_n)_n$ in $X_i \cap \hat{\rho}(\hat{B})$ such that $z_n \to y$ with respect to the topology on X_i. Let us show that $(\mathrm{ev}_{z_n})_n \to \mathrm{ev}_x$, with respect to the hull-kernel topology on $\hat{A}$. Suppose U is an open set in $\hat{A}$ containing ev_x. Then there is a function $f \in A$ that is nonzero at x, but vanishes at each point in $\hat{A}\setminus U$. Since x is in the decomposition of y, this implies that $f_i(y) \neq 0$. Since $z_n \to y$ in X_i and since f_i is continuous, there is an n_0 such that, for all $n \ge n_0$, $f_i(z_n) \neq 0$. In particular, this means that, for all $n \ge n_0$, $\mathrm{ev}_{z_n} \in U$. Therefore, $\mathrm{ev}_{z_n} \to \mathrm{ev}_x$ in $\hat{A}$. Now, by Lemma 2.15, $\hat{\rho}(\hat{B})$ is closed and, hence, what we have shown implies that $\mathrm{ev}_x \in \hat{\rho}(\hat{B})$. Therefore, $x \in X_j \cap \hat{\rho}(\hat{B}) \subset X_j'$. □

In the following lemma, we construct a DSH algebra from A over the base spaces X_i', where the pullback maps are just restrictions of the pullback maps in the definition of A (the φ_i's). We show afterwards (see Proposition 2.18) that this new DSH algebra is isomorphic to the quotient B.

Proposition 2.17. *There is a DSH algebra D of length l with the following properties:*

(i) $D^{(1)} = C(X_1', M_{n_1})$;

(ii) *for all $1 \le i \le l$, $D^{(i)} \subset \bigoplus_{j=1}^{i} C(X_j', M_{n_j})$;*

(iii) *for all $1 \le i < l$, the pullback map $\tau_i : D^{(i)} \to C(Y_{i+1}', M_{n_{i+1}})$ is given by $\tau_i(f)(y) := \mathrm{diag}(f_{i_1}(x_1), \dots, f_{i_t}(x_t))$, where $x_1, \dots, x_t$ are the points in the decomposition of y coming from the definition of A;*

(iv) *for $1 \le i < l$, $D^{(i+1)} = D^{(i)} \oplus_{C(Y_{i+1}', M_{n_{i+1}})} C(X_{i+1}', M_{n_{i+1}})$ with pullback map τ_i;*

(v) *for all $1 \le i \le l$, if $(f_1, \dots, f_i) \in D^{(i)}$, there is a $(g_1, \dots, g_i) \in A^{(i)}$ such that, for all $1 \le j \le i$, $g_j|_{X_j'} = f_j$.*

Proof. Let us proceed by induction on i. Define $D^{(1)} := C(X_1', M_{n_1})$ so that (i) holds. Since X_1' is closed in X_1, we may extend a function in $D^{(1)}$ to a function in $A^{(1)} = C(X_1, M_{n_1})$ so that (v) holds when $i = 1$. Now, fix $1 \le i \le l-1$, and assume that we have defined $D^{(1)}, \dots, D^{(i)}$ and $\tau_1, \dots, \tau_{i-1}$ satisfying conditions (i) to (v). Let us show how to define τ_i and $D^{(i+1)}$. Given $(f_1, \dots, f_i) \in D^{(i)}$, use (v) to get $(g_1, \dots, g_i) \in A^{(i)}$ such that $g_j|_{X_j'} = f_j$ for $1 \le j \le i$. Define $\tau_i : D^{(i)} \to C(Y_{i+1}', M_{n_{i+1}})$ by $\tau_i((f_1, \dots, f_i)) := \varphi_i((g_1, \dots, g_i))|_{Y_{i+1}'}$.

To see that τ_i is a well-defined *-homomorphism satisfying (iii), suppose $(h_1, \dots, h_i) \in A^{(i)}$ also restricts coordinate-wise to $(f_1, \dots, f_i)$. If $y \in Y_{i+1}'$

decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \dots, x_t \in X_{i_t} \setminus Y_{i_t}$, then by Lemma 2.16, we have $x_1 \in X'_{i_1}, \dots, x_t \in X'_{i_t}$. Hence,

$$\begin{aligned}\varphi_i\big((g_1,\dots,g_i)\big)(y) &= \operatorname{diag}\big(g_{i_1}(x_1),\dots,g_{i_t}(x_t)\big)\\ &= \operatorname{diag}\big(f_{i_1}(x_1),\dots,f_{i_t}(x_t)\big)\\ &= \operatorname{diag}\big(h_{i_1}(x_1),\dots,h_{i_t}(x_t)\big)\\ &= \varphi_i\big((h_1,\dots,h_i)\big)(y).\end{aligned}$$

Therefore, τ_i satisfies (iii) and is independent of the choice of extension used. Moreover, $\tau_i((f_1,\dots,f_i))$ is continuous, being the restriction of a continuous function. Thus, τ_i is well defined, and it is clearly a *-homomorphism since φ_i is.

Next, define $D^{(i+1)} := D^{(i)} \oplus_{C(Y'_{i+1},M_{n_{i+1}})} C(X'_{i+1}, M_{n_{i+1}})$, using τ_i as the pullback map. This ensures that (ii) and (iv) hold, and so we just need to verify (v). Suppose $(d,f) \in D^{(i+1)}$, where $d \in D^{(i)}$ and $f \in C(X'_{i+1}, M_{n_{i+1}})$. By the inductive hypothesis, we may apply (v) to d to obtain a $b \in A^{(i)}$ such that $b_j|_{X'_j} = d_j$ for all $1 \le j \le i$. Let $g := \varphi_i(b) \in C(Y_{i+1}, M_{n_{i+1}})$. If $y \in X'_{i+1} \cap Y_{i+1} = Y'_{i+1}$, then $g(y) = \varphi_i(b)(y) = \tau_i(d)(y) = f(y)$. Thus, since X'_{i+1} and Y_{i+1} are both closed in X_{i+1} and since f and g agree on their intersection, they have a common extension $h \in C(X_{i+1}, M_{n_{i+1}})$. Since $\varphi_i(b) = g = h|_{Y_{i+1}}$, we have $(b,h) \in A^{(i+1)}$, and since $h|_{X'_{j+1}} = f$, it follows that (v) holds. ☐

Proposition 2.18. *Let $D = D^{(l)}$ be the DSH algebra constructed in Proposition 2.17. There is a *-isomorphism $\Gamma : B \to D$ given coordinate-wise by $\Gamma(b)_i := a_i|_{X'_i}$ for $1 \le i \le l$, where $a \in A$ is any lift of b under ρ. In particular, the quotient B is a DSH algebra.*

Proof. Let us first show that $\Gamma(b)$ is independent of the choice of lift. Fix $1 \le i \le l$, and suppose $g, h \in A$ satisfy $\rho(g) = \rho(h)$. We must show that $g_i|_{X'_i} = h_i|_{X'_i}$. Note that $\overline{X'_i \setminus Y'_i}^{X_i} = X'_i$. Indeed, X'_i is closed with respect to the topology on X_i, and so the fact that $\overline{X'_i \setminus Y'_i}^{X_i}$ is a subset of X'_i is clear; for the reverse inclusion, if $z \in X'_i$, there is a sequence $(z_n)_n \subset \hat\rho(\hat B) \cap X_i \subset X'_i \setminus Y_i \subset X'_i \setminus Y'_i$ that converges to z in X_i. Hence, by continuity, it suffices to show that $g_i|_{X'_i\setminus Y'_i} = h_i|_{X'_i\setminus Y'_i}$. To this end, suppose $x \in X'_i \setminus Y'_i$. Then

$$x \in \overline{X_i \cap \hat\rho(\hat B)}^{X_i} = \overline{(X_i \setminus Y_i) \cap \hat\rho(\hat B)}^{X_i} \quad \text{and} \quad x \notin Y_i.$$

By Lemma 2.15 and Lemma 2.5, $(X_i \setminus Y_i) \cap \hat\rho(\hat B)$ is closed in $X_i \setminus Y_i$ in the subspace topology coming from X_i. Thus,

$$\begin{aligned}x \in \overline{(X_i \setminus Y_i) \cap \hat\rho(\hat B)}^{X_i} \cap (X_i \setminus Y_i) &= \overline{(X_i \setminus Y_i) \cap \hat\rho(\hat B)}^{X_i\setminus Y_i}\\ &= (X_i \setminus Y_i) \cap \hat\rho(\hat B) \subset \hat\rho(\hat B).\end{aligned}$$

Therefore, there is a $[\pi] \in \hat B$ such that $[\pi \circ \rho] = \hat\rho([\pi]) = [\mathrm{ev}_x]$. But this implies that $g - h \in \ker \mathrm{ev}_x$ since $g - h \in \ker \rho$. Hence, $g_i(x) = h_i(x)$, as desired. Moreover, $\Gamma(b)_i$ belongs to $C(X'_i, M_{n_i})$, being the restriction of a continuous function. To see that $\Gamma(b)$ respects the decomposition structure of D, suppose $y \in Y_i'$ decomposes into $x_1 \in X_{i_1}' \setminus Y_{i_1}', \dots, x_t \in X_{i_t}' \setminus Y_{i_t}'$. Then

$$\begin{aligned}\Gamma(b)_i(y) = a_i(y) &= \operatorname{diag}\big(a_{i_1}(x_1), \dots, a_{i_t}(x_t)\big)\\ &= \operatorname{diag}\big(\Gamma(b)_{i_1}(x_1), \dots, \Gamma(b)_{i_t}(x_t)\big).\end{aligned}$$

Therefore, Γ is well defined and it is straight-forward to check that it is a *-homomorphism. We have left only to check that it is a bijection.

To see that Γ is injective, suppose $b \in B$ and $a \in A$ is such that $\rho(a) = b$. Assume that $\Gamma(b) = 0$. Let π be an arbitrary irreducible representation of B. To show that $b = 0$, it suffices to show that $\pi(b) = 0$. Note that $[\pi \circ \rho] = \hat{\rho}([\pi]) \in \hat{\rho}(\hat{B})$. Thus, for some $1 \leq i \leq l$, there is an $x \in (X_i \setminus Y_i) \cap \hat{\rho}(\hat{B}) \subset X_i'$ such that $[\pi \circ \rho] = [\mathrm{ev}_x]$. Since $\mathrm{ev}_x(a) = a_i(x) = \Gamma(b)_i(x) = 0$, it follows that $\pi(b) = \pi(\rho(a)) = 0$. Thus, Γ is injective.

To see that Γ is surjective, suppose $d \in D$. By property (v) in Proposition 2.17, there is a $g \in A$ such that $g_i|_{X_i'} = d_i$ for all $1 \leq i \leq l$. Let $h = \rho(g) \in B$, and observe that, for all $1 \leq i \leq l$, we have $\Gamma(h)_i = g_i|_{X_i'} = d_i$. Thus, $\Gamma(h) = d$, so Γ is surjective.

We have shown that Γ is a *-isomorphism, from which it follows that B is a DSH algebra. □

Proposition 2.19. *Given an inductive limit*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

of DSH algebras with diagonal maps, there exist DSH algebras $D_1, D_2, \dots$ and injective diagonal maps $\psi_i' : D_i \to D_{i+1}$ such that

$$D_1 \xrightarrow{\psi_1'} D_2 \xrightarrow{\psi_2'} D_3 \xrightarrow{\psi_3'} \cdots \longrightarrow A.$$

Proof. For $n \in \mathbb{N}$, let $\mu_n : A_n \to A$ denote the map in the construction of the inductive limit, and consider the surjective map $\kappa_n : A_n \to A_n/\ker \mu_n =: B_n$. The induced map $\nu_n : B_n \to B_{n+1}$ given by $\nu_n(\kappa_n(a)) := \kappa_{n+1}(\psi_n(a))$ for all $a \in A_n$ is well defined and injective. Furthermore, we have $\varinjlim(B_n, \{\nu_n\}_n) = A$. Let $X_1^n, \dots, X_{l(n)}^n$ denote the base spaces of A_n, and let $Y_1^n, \dots, Y_{l(n)}^n$ denote the corresponding closed subsets. Let D_n denote the DSH algebra given by Proposition 2.17 and isomorphic to B_n (with base spaces $X_i^n \cap \hat{\kappa}_n(\hat{B}_n)^{X_i^n} =: Z_i^n$ and corresponding closed subsets $Z_i^n \cap Y_i^n =: W_i^n$ for $1 \leq i \leq l(n)$). By Proposition 2.18, the injective map ν_n drops down to an injective map $\psi_n' : D_n \to D_{n+1}$ given by $\psi_n'(d)_i := \psi_n(a)_i|_{Z_i^{n+1}}$ for all $1 \leq i \leq l(n+1)$, where $a \in A_n$ is any coordinate-wise extension of d. Moreover, $\varinjlim(D_n, \{\psi_n'\}_n) = A$.

We need to check that ψ_n' is diagonal. Fix $1 \leq i \leq l(n+1)$, and suppose $x \in Z_i^{n+1} \setminus W_i^{n+1} \subset X_i^{n+1} \setminus Y_i^{n+1}$ decomposes into $x_1 \in X_{i_1}^n \setminus Y_{i_1}^n, \dots, x_t \in X_{i_t}^n \setminus Y_{i_t}^n$ under the diagonal map ψ_n. We need to show that $x_j \in Z_{i_j}^n \setminus W_{i_j}^n$ for all $1 \leq j \leq t$. Since $\mathrm{ev}_x \circ \psi_n'$ is a *-representation of D_n, it is unitarily equivalent to a finite direct sum of irreducible representations

$$\mathrm{ev}_{z_1}, \dots, \mathrm{ev}_{z_k} \in \bigsqcup_{s=1}^{l(n)} (Z_s^n \setminus W_s^n) \subset \hat{A}_n.$$

Fix $1 \leq j \leq t$. If $x_j \notin \{z_1, \dots, z_k\}$, then by [7, Prop. 4.2.5], there is a function $a \in A_n$ such that $\mathrm{ev}_{z_s}(a) = 0$ for all $1 \leq s \leq k$, but $\mathrm{ev}_{x_j}(a) \neq 0$. Since x_j is in the decomposition of x under ψ_n, this implies that $\mathrm{ev}_x(\psi_n(a))$ is both zero and nonzero simultaneously. Therefore, it must be that $x_j \in \{z_1, \dots, z_k\}$ and, thus, that $x_j \in Z_{i_j}^n \setminus W_{i_j}^n$, as desired. □

2.20. **Homogeneous DSH algebras.** Suppose A is an n-homogeneous DSH algebra. We show in this subsection that there is a compact metric space X such that A is isomorphic to $C(X, M_n)$.

Proposition 2.21. *Let X_1, X_2 be compact metric spaces. Let Y_2 be a closed subset of X_2. Let $\varphi : C(X_1, M_n) \to C(Y_2, M_n)$ be a unital *-homomorphism, and suppose that the associated pullback $C(X_1, M_n) \oplus_{C(Y_2, M_n)} C(X_2, M_n)$ is a DSH algebra. Then there exists a compact metric space Z^* such that*

$$C(X_1, M_n) \oplus_{C(Y_2, M_n)} C(X_2, M_n)$$

is isomorphic to $C(Z^*, M_n)$.

Proof. For a given $y \in Y_2$, we know by Lemma 2.7 that the point it decomposes into is unique; alternatively, note that if there were two distinct points in the decomposition of y under φ, then these two points could not be separated by any function in $C(X_1, M_n)$. Denote this unique point by $\tau(y)$. We claim that $\tau : Y_2 \to X_1$ is a closed and continuous map.

To see that τ is continuous, suppose that $(y_n)_n$ is a sequence in Y_2 converging to a point y, and let $f \in C(X_1, M_{n_1})$ be arbitrary. As $\varphi(f)$ is continuous,

$$\lim_n f(\tau(y_n)) = \lim_n \varphi(f)(y_n) = \varphi(f)(y) = f(\tau(y)),$$

which proves that $(\tau(y_n))_n$ converges to $\tau(y)$ since functions in $C(X_1, M_{n_1})$ separate points. Thus, τ is continuous.

To see that τ is closed, fix a closed subset F of Y_2, and suppose that $(x_n)_n$ is a sequence in $\tau(F)$ converging to a point $x \in X_1$. Choose, for each n, a point $y_n \in F$ with $\tau(y_n) = x_n$. Since F is compact, we may assume (by passing to a subsequence) that $(y_n)_n$ converges to a point y in F. Letting $f \in C(X_1, M_{n_1})$ be arbitrary, it follows that

$$\begin{aligned} f(x) &= \lim_n f(x_n) = \lim_n f(\tau(y_n)) \\ &= \lim_n \varphi(f)(y_n) = \varphi(f)(y) = f(\tau(y)). \end{aligned}$$

Since this holds for all $f \in C(X_1, M_{n_1})$, it must be that $x = \tau(y) \in \tau(F)$, which proves that τ is closed.

Now, let $Z := X_1 \sqcup X_2$. Then Z is a compact metric space. Given $z \in Z$, we define $[z]$ as follows:

$$[z] := \begin{cases} \{z\} & \text{if } z \in X_2 \setminus Y_2, \\ \{z\} \cup \tau^{-1}(z) & \text{if } z \in X_1, \\ \{\tau(z)\} \cup \tau^{-1}(\tau(z)) & \text{if } z \in Y_2. \end{cases}$$

Let $Z^* := \{[z] \mid z \in Z\}$, and let $p : Z \to Z^*$ denote the canonical surjection $p(z) := [z]$. Then Z^* is a collection of sets that partition Z. We equip it with the quotient topology induced by p; that is, a set $U \subset Z^*$ is open in Z^* if and only if $p^{-1}(U)$ is open in Z. Since Z is compact, so is Z^*. To establish that Z^* is in fact a metric space, it suffices to ensure that it is Hausdorff. Indeed, letting $w(Y)$ denote the smallest cardinality of a basis for a given topological space Y, it follows by [10, Thm. 3.1.22] that $w(Z^*) \leq w(Z)$. Since a compact Hausdorff space is metrizable if and only if it has a countable basis, showing that Z^* is Hausdorff would guarantee that it is also metrizable.

To this end, let us now verify that Z^* is Hausdorff. Suppose $z_1, z_2 \in Z$ with $[z_1] \neq [z_2]$. Let us show that $[z_1]$ and $[z_2]$ can be separated by open sets in Z^*. Without loss of generality, we must be in one of the following four cases.

Case one: $z_1, z_2 \in (X_2 \setminus Y_2) \cup (X_1 \setminus \tau(Y_2))$. In this case, it is easy to see (since Y_2 and $\tau(Y_2)$ are closed) that we may choose open sets $U_1 \ni z_1$ and $U_2 \ni z_2$ in Z that are disjoint and such that $U_i \subset X_2 \setminus Y_2$ if $z_i \in X_2 \setminus Y_2$ and $U_i \subset X_1 \setminus \tau(Y_2)$ if $z_i \in X_1 \setminus \tau(Y_2)$. Since $p|_{(X_2 \setminus Y_2) \cup (X_1 \setminus \tau(Y_2))}$ is a bijection, the sets $p(U_1)$ and $p(U_2)$ are open in Z^*, disjoint, and contain $[z_1]$ and $[z_2]$, respectively.

Case two: $z_1 \in X_1 \setminus \tau(Y_2)$ and $z_2 \in Y_2 \cup \tau(Y_2)$. Choose disjoint sets $U_1 \ni z_1$ and $U_2 \supset \tau(Y_2)$ that are open in X_1. Let $V_1 := p(U_1) \ni [z_1]$ and $V_2 := p(U_2 \cup X_2) \ni [z_2]$, and note that $V_1 \cap V_2 = \varnothing$. Since $p^{-1}(V_1) = U_1$ and $p^{-1}(V_2) = U_2 \cup X_2$ are both open in Z, it follows that V_1 and V_2 are open in Z^*.

Case three: $z_1 \in X_2 \setminus Y_2$ and $z_2 \in Y_2 \cup \tau(Y_2)$. Choose disjoint sets $U_1 \ni z_1$ and $U_2 \supset Y_2$ that are open in X_2. Let $V_1 := p(U_1) \ni [z_1]$ and $V_2 := p(X_1 \cup U_2) \ni [z_2]$, and note that $V_1 \cap V_2 = \varnothing$. Since $p^{-1}(V_1) = U_1$ and $p^{-1}(V_2) = X_1 \cup U_2$ are both open in Z, it follows that V_1 and V_2 are open in Z^*.

Case four: $z_1, z_2 \in Y_2 \cup \tau(Y_2)$. We may assume without loss of generality that $z_1, z_2 \in \tau(Y_2)$. Choose sets $U_1 \ni z_1$ and $U_2 \ni z_2$, which are open in X_1 and satisfy $\overline{U_1} \cap \overline{U_2} = \varnothing$. Since τ is continuous, there are open subsets V_1 and V_2 of X_2 such that $\tau^{-1}(U_1) = V_1 \cap Y_2$ and $\tau^{-1}(U_2) = V_2 \cap Y_2$. Choose disjoint open subsets W_1 and W_2 of X_2 containing $\tau^{-1}(\overline{U_1})$ and $\tau^{-1}(\overline{U_2})$, respectively. Put $\mathcal{O}_1 := V_1 \cap W_1$ and $\mathcal{O}_2 := V_2 \cap W_2$, and let $\mathcal{E}_1 := p(\mathcal{O}_1 \cup U_1)$ and $\mathcal{E}_2 := p(\mathcal{O}_2 \cup U_2)$. Note that $[z_1] \in \mathcal{E}_1$ and $[z_2] \in \mathcal{E}_2$. Let us show that $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint and open in Z^*. Suppose $t_1 \in \mathcal{O}_1 \cup U_1$ and $t_2 \in \mathcal{O}_2 \cup U_2$. Assume first that $t_1 \in U_1$. If $t_2 \in U_2 \cup (X_2 \setminus Y_2)$, then $[t_1] \neq [t_2]$ since $U_1 \cap U_2 = \varnothing$. If instead $t_2 \in \mathcal{O}_2 \cap Y_2$, then $\tau(t_2) \in U_2$ so that we again have $[t_1] \neq [t_2]$. A symmetric analysis shows that $[t_1] \neq [t_2]$ if $t_2 \in U_2$. Thus, we may assume $t_1 \in \mathcal{O}_1$ and $t_2 \in \mathcal{O}_2$. If either t_1 or t_2 is in $X_2 \setminus Y_2$, then $[t_1] \neq [t_2]$ since $t_1 \neq t_2$, as $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint. If instead $t_1 \in \mathcal{O}_1 \cap Y_2$ and $t_2 \in \mathcal{O}_2 \cap Y_2$, then $\tau(t_1) \in U_1$,

$\tau(t_2) \in U_2$, and once again, $[t_1] \neq [t_2]$. It follows that $\mathcal{E}_1 \cap \mathcal{E}_2 = \varnothing$. It remains to be shown that $\mathcal{E}_1$ and $\mathcal{E}_2$ are open in Z^*. Owing to the symmetry of the set-up, we only show that $\mathcal{E}_1$ is open in Z^*, and to do this, it is sufficient to prove that $p^{-1}(\mathcal{E}_1) \cap X_1 = U_1$ and $p^{-1}(\mathcal{E}_1) \cap X_2 = \mathcal{O}_1$. Assume we are given $t \in p^{-1}(\mathcal{E}_1) \cap X_1$. Thus, $[t] \in p(\mathcal{O}_1) \cup p(U_1)$. If there is an $s \in U_1$ such that $[t] = [s]$, then, since both t and s lie in X_1, it must be that $t = s$. If instead there is an $s \in \mathcal{O}_1$ such that $[t] = [s]$, then it follows that $s \in Y_2$ and, hence, that $\tau(s) \in U_1$. Thus, $[t] = [s] = [\tau(s)]$, from which we deduce as before that $t = \tau(s) \in U_1$. Therefore, we may conclude that $p^{-1}(\mathcal{E}_1) \cap X_1 \subset U_1$, and hence, $p^{-1}(\mathcal{E}_1) \cap X_1 = U_1$. Now, suppose that we are given $t \in p^{-1}(\mathcal{E}_1) \cap X_2$. As before, $[t] \in p(\mathcal{O}_1) \cup p(U_1)$. Suppose first that there is an $s \in \mathcal{O}_1$ such that $[t] = [s]$. If either t or s is in $X_2 \setminus Y_2$, then $t = s \in \mathcal{O}_1$; otherwise, it must be that $s, t \in Y_2$ and, in particular, $\tau(t) = \tau(s) \in U_1$. Therefore, $t \in V_1 \cap W_1 = \mathcal{O}_1$. If instead there is an $s \in U_1$ such that $[t] = [s]$, then it must be that $t \in Y_2$ and $\tau(t) = s \in U_1$, which implies (as above) that $t \in \mathcal{O}_1$. Thus, $p^{-1}(\mathcal{E}_1) \cap X_2 \subset \mathcal{O}_1$, and hence, $p^{-1}(\mathcal{E}_1) \cap X_2 = \mathcal{O}_1$. Therefore, by our analysis, this implies that $\mathcal{E}_1$ and $\mathcal{E}_2$ are open in Z^*. This completes the proof that Z^* is Hausdorff and, hence, a compact metric space.

Now, define $\Lambda : C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n) \to C(Z^*, M_n)$ by

$$\Lambda((f,g))([z]) := \begin{cases} f(z) & \text{if } z \in X_1, \\ g(z) & \text{if } z \in X_2. \end{cases}$$

To conclude the proof, let us show that Λ is a well-defined *-isomorphism. To see that Λ is well defined, suppose $z_1, z_2 \in Z$ and that $[z_1] = [z_2]$. Unless $z_1 = z_2$, this implies that one of the two points is in the decomposition of the other. Assume without loss of generality that $\tau(z_2) = z_1$. Then, for all $(f, g) \in C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n)$, we have $g(z_2) = \varphi(f)(z_2) = f(\tau(z_2)) = f(z_1)$. This shows that Λ is well defined. It is clear that Λ is an injective *-homomorphism. To see surjectivity, suppose $h \in C(Z^*, M_n)$, and define $f := h \circ p|_{X_1} \in C(X_1, M_n)$ and $g := h \circ p|_{X_2} \in C(X_2, M_n)$. Given $y \in Y_2$, we have

$$g(y) = h([y]) = h([\tau(y)]) = f(\tau(y)) = \varphi(f)(y)$$

so that $(f, g) \in C(X_1, M_n) \oplus_{C(Y_2,M_n)} C(X_2, M_n)$. Moreover, $\Lambda((f, g)) = h$, proving that Λ is surjective. The proof of Proposition 2.21 is now complete. □

Applying Proposition 2.21 inductively, we obtain the following corollary.

Corollary 2.22. *Every n-homogeneous DSH algebra is isomorphic to a full matrix algebra, i.e., isomorphic to $C(X, M_n)$ for some compact metric space X.*

3. Stable rank

This section focuses on simple inductive limits of DSH algebras with diagonal bonding maps. Section 3.29 contains the principal result, which states that every limit algebra of this type necessarily has stable rank one (see Theorem 3.30). In Section 3.31, Theorem 3.30 is applied to obtain two results about simple dynamical crossed products. Given an infinite compact metric space T and a minimal homeomorphism $h : T \to T$, we show that every orbit-breaking subalgebra of the induced dynamical crossed product $C^*(\mathbb{Z}, T, h)$ associated to any non-isolated point is a simple inductive limit of DSH algebras with diagonal maps (see Theorem 3.35). Consequently, we are able to show that $C^*(\mathbb{Z}, T, h)$ has stable rank one (see Corollary 3.36) and that $\mathcal{Z}$-stability is determined for such an algebra by strict comparison of positive elements (see Corollary 3.37).

The proof of Theorem 3.30 is quite technical and requires several lemmas, which are developed in Section 3.2 and Section 3.18. In Section 3.2, facts concerning continuous paths of unitary matrices are established. These results are used in Section 3.18 to construct certain unitary elements in DSH algebras that are needed to prove Theorem 3.30. Before formulating these lemmas, in Section 3.1, we provide a more detailed overview of how they come together to prove Theorem 3.30, and we compare and contrast our approach to that used by Elliott, Ho, and Toms in [8].

3.1. **Outline of the proof of the main theorem.** Section 3.18 consists of all of the lemmas that are used in the proof of Theorem 3.30 in Section 3.29, with the following dependency diagram:

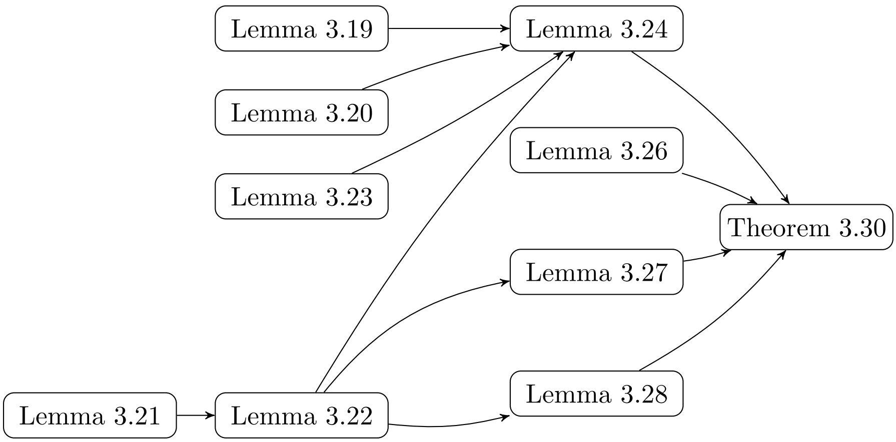

Figure 1. Dependency chart for the main lemmas used in the proof of Theorem 3.30.

Let us now outline the importance of each of these lemmas and give a brief overview of how they are used to prove Theorem 3.30.

Our general strategy for proving that a simple inductive limit of DSH algebras with diagonal maps has stable rank one is essentially the one in [8]. We start with a given element f in the limit algebra A, which may be assumed to lie in some finite-stage building block A_j. If f is invertible, then there is nothing to prove, and so we may assume that f is not invertible. The goal is then to show that the image $\psi_{j',j}(f)$ of f in a later stage algebra $A_{j'}$ is close to an invertible in $A_{j'}$.

If we approximate $\psi_{j',j}(f)$, multiply this approximation by unitaries, approximate again, multiply the new approximation by unitaries, and show that an element thus obtained is close to an invertible, then, upon unpacking the approximations, it follows that $\psi_{j',j}(f)$ is close to an invertible in $A_{j'}$. Finally, as Rørdam observed in [21], every nilpotent element of a unital C*-algebra is close to an invertible. Therefore, it suffices to show that an element, obtained from $\psi_{j',j}(f)$ as above, is nilpotent.

To execute this strategy, we proceed as follows. In Lemma 3.20, we first use Lemma 2.11 to show that there is a point in one of the base spaces X_i of A_j at which f_i is a non-invertible matrix. After multiplying by unitary matrices on the left and right, we obtain a new matrix whose first row and column contain only zeros (or one that has a *zero cross at index* 1 (see Definition 3.3)). We show that, after perturbing f slightly, we may multiply this perturbation f' on the left and right by unitaries $w, v \in A_j$ so that $wf'v$ has a zero cross at index 1 not just at one point, but over a whole open subset of the spectrum of A_j.

By Proposition 2.19, we may assume that the maps in the given sequence are injective. Hence, in Lemma 3.24, we may apply our simplicity criterion (Lemma 3.19) with the open subset of the spectrum obtained above to conclude that, in some later stage algebra $A_{j'}$, the diagonal image $\psi_{j',j}(wf'v)$ has "many" (see the following paragraphs) zero crosses at every point in each base space of $A_{j'}$; because of simplicity and the fact that the maps in the sequence are diagonal, this "many" may be taken (using Lemma 3.23) to be as large as desired. We are then able to construct unitaries $V, V' \in A_{j'}$ that organize the location of these zero crosses so that the element $f'' = V\psi_{j',j}(f')V'$ has "many" zero crosses occurring at tractable locations at each point in every base space of $A_{j'}$.

We use Lemma 3.26 to approximate f'' by a function $g \in A_{j'}$ that preserves the zero crosses of f'' at each point and, in addition, extends the block-diagonal structure of the algebra to neighborhoods of the closed subsets of the base spaces (the Y_i's in the definition of $A_{j'}$). This allows us, in Lemma 3.27, to conjugate g by a unitary $W \in A_{j'}$ so that, in the resulting conjugation $g' = WgW^*$, the zero crosses of g are grouped together into block zero crosses at every point in each of the base spaces of $A_{j'}$.

The unitaries V, V', and W above are constructed in such a way that, at every point in each base space, the *bandwidth*, which measures how far a nonzero entry can occur from the diagonal in a matrix (see Definition 3.4), of g' at that point is bounded above by a quantity independent of j'. Thus, by ensuring that the "many" above is at least as large as this upper bound, we are able to construct a unitary W' in Lemma 3.28 that shifts the block zero cross mentioned above so that $g'W'$ is strictly lower triangular at each point. This ensures that $g'W'$ is nilpotent and yields the desired result.

The unitaries V, V', W, and W' above are all defined using continuous paths of unitaries between permutation matrices (see Definitions 3.5 and 3.11). In Lemma 3.21, we construct certain indicator-function-like elements of DSH algebras, the final versions of which (Lemma 3.22) help to define V, V', W, and W' by allowing us to implement the continuous paths of unitary matrices constructed in Section 3.2 in the DSH framework. Their job is to tell the continuous paths used in defining these unitaries which rows and columns to shift around, so as to ensure that they respect the decomposition structure of the algebra and that the zero crosses are achieved in the target locations.

The proof of Theorem 3.30 shares many similarities with the original AH proof of Elliott, Ho, and Toms found in [8]. In particular, in the case that all of the DSH algebras in the context of Theorem 3.30 are homogeneous (hence, by Corollary 2.22, full matrix algebras), the unitaries V, V', W, and W' constructed above essentially reduce the those constructed in [8]. For a more in-depth analysis of this, see [1, Sec. 5.1], where it is also observed that the AH proof does not require the full matrix algebra building blocks in the inductive limit to be separable. In our ASH setting, however, separability is necessary since the indicator-function-like elements constructed in Lemma 3.22, which are not required in the AH case, rely on the assumption that the base spaces of any given DSH algebra are metrizable.

3.2. **Preliminary lemmas.** The purpose of this subsection is to introduce some continuous paths of unitary matrices and prove certain facts about them. These paths will be used in the sequel to construct the unitaries in the DSH algebras used in the proof of the main result.

Definition 3.3 (Zero cross)**.** Given a matrix $D \in M_n$ and $1 \leq k \leq n$, we say that D has a *zero cross at index k* provided that each entry in the kth row and column of D is 0.

Definition 3.4 (Bandwidth of a matrix)**.** Given a matrix $D \in M_n$, we let

$$\mathfrak{r}(D) := \min\{m \geq 0 \mid D_{i,j} = 0 \text{ whenever } |i-j| \geq m\}$$

if it exists, and $\mathfrak{r}(D) := n$ otherwise, and we call this number the *bandwidth* of D.

Definition 3.5 (see [8])**.** Given $n \in \mathbb{N}$ and a permutation $\pi \in S_n$, let $U[\pi]$ denote the permutation unitary in M_n obtained from the identity matrix by moving the ith row to the $\pi(i)$th row. If we are given a transposition $(i\ \ j) \in S_n$, let $u_{(i\ j)} : [0,1] \to \mathcal{U}(M_n)$ denote a continuous path of unitaries with the following properties:
(i) $u_{(i\ j)}(0) = 1_n$;
(ii) $u_{(i\ j)}(1) = U[(i\ \ j)]$;
(iii) for all $0 \leq \theta \leq 1$, $u_{(i\ j)}(\theta)$ may only differ from the identity matrix at entries (i,i), (i,j), (j,i), and (j,j).

Lemma 3.6. *Let $n, M, l \in \mathbb{N}$ with $l + M - 1 \leq n$, and let $(\xi_l, \dots, \xi_{l+M-1})$ be a vector in $[0,1]^M$. Put*

$$U := \prod_{t=1}^{M-1} u_{(l\ l+t)}(\xi_{l+t}) \in \mathcal{U}(M_n),$$

where each $u_{(l\ l+t)} : [0,1] \to M_n$ is a connecting path of unitaries as described in Definition 3.5.

(a) *Suppose $D \in M_n$, $\xi \in [0,1]$, and $(l_1\ \ l_2) \in S_n$. If D has a zero cross at index $l \neq l_1, l_2$, then so does $u_{(l_1\ l_2)}(\xi) D u_{(l_1\ l_2)}(\xi)^*$.*

(b) *Suppose $D \in M_n$. If D has a zero cross at index $l' \in \{1, \dots, n\} \setminus \{l, \dots, l + M - 1\}$, then so does UDU^*.*

(c) *Suppose $D \in M_n$ is such that, for all $l \leq l' \leq l + M - 1$, D has a zero cross at index l' whenever $\xi_{l'} > 0$. If at least one of $\xi_l, \dots, \xi_{l+M-1}$ is 1, then UDU^* has a zero cross at index l.*

Proof. Let us start by proving (a). Suppose D has a zero cross at index $l \neq l_1, l_2$. By property (iii) of Definition 3.5, the l_1th and l_2th columns of $Du_{(l_1\ l_2)}(\xi)^*$ are linear combinations of the l_1th and l_2th columns of D, while every other column is identical to its corresponding column in D. Since $l \neq l_1, l_2$ and since every entry in the lth row of D is zero, it follows that $Du_{(l_1\ l_2)}(\xi)^*$ has a zero cross at index l. A similar analysis involving rows shows that $u_{(l_1\ l_2)}(\xi) D u_{(l_1\ l_2)}(\xi)^*$ has a zero cross at index l, which proves (a). Looking at the definition of U, we see that (b) follows from $M - 1$ applications of (a).

Let us now prove (c). Suppose that D has a zero cross at index l' whenever $\xi_{l'} > 0$ and that at least one of $\xi_l, \dots, \xi_{l+M-1}$ is 1. Let

$$T := \{l + 1 \leq q \leq l + M - 1 \mid \xi_q > 0\}.$$

If $\xi_{l+t} = 0$, we have $u_{(l\ l+t)}(\xi_{l+t}) = 1_n$. Hence,

$$U := \begin{cases} u_{(l\ l_1)}(\xi_{l_1}) \cdots u_{(l\ l_r)}(\xi_{l_r}) & \text{if } T = \{l_1 < \dots < l_r\}, \\ 1_n & \text{if } T = \varnothing. \end{cases}$$

If $T = \varnothing$, then, since at least one of $\xi_l, \dots, \xi_{l+M-1}$ is 1, it must be that $\xi_l = 1$. Hence, $UDU^* = D$ has a zero cross at index l in this case by the assumption in the lemma. Thus, we may assume $T \neq \varnothing$ so that D has zero crosses at indices $l_1, \dots, l_r$. We consider two cases.

Case one: $\xi_{l_s} < 1$ for all $1 \leq s \leq r$. In this case, as we argued above, it must be that D has a zero cross at index l. When conjugating D by $u_{(l\ l_r)}(\xi_{l_r})$, we can see by property (iii) of Definition 3.5 that $u_{(l\ l_r)}(\xi_{l_r})$ is only acting on two zero crosses (the one at index l and the one at index l_r) of D and, hence,

$$u_{(l\ l_r)}(\xi_{l_r}) D u_{(l\ l_r)}(\xi_{l_r})^* = D.$$

From this, we can inductively see that $UDU^* = D$, which has a zero cross at index l.

Case two: $\xi_{l_s} = 1$ for some $1 \leq s \leq r$. Let

$$D' := \Big(\prod_{p=s+1}^{r} u_{(l\ l_p)}(\xi_{l_p})\Big) D \Big(\prod_{p=s+1}^{r} u_{(l\ l_p)}(\xi_{l_p})\Big)^*.$$

Then $r-s$ applications of (a) show that D' has zero crosses at indices $l_1, \ldots, l_s$. Note that

$$\begin{aligned} UDU^* &= \Big(\prod_{p=1}^{s} u_{(l\ l_p)}(\xi_{l_p})\Big) D' \Big(\prod_{p=1}^{s} u_{(l\ l_p)}(\xi_{l_p})\Big)^* \\ &= \Big(\prod_{p=1}^{s-1} u_{(l\ l_p)}(\xi_{l_p})\Big) E \Big(\prod_{p=1}^{s-1} u_{(l\ l_p)}(\xi_{l_p})\Big)^*, \end{aligned}$$

where $E := U[(l\ \ l_s)]D'U[(l\ \ l_s)]^*$. Since D' has a zero cross at index l_s, conjugating it by $U[(l\ \ l_s)]$ brings this zero cross to index l so that the matrix E has zero crosses at indices $l, l_1, \ldots, l_{s-1}$. Hence, as in the argument used in case one, the matrix E is unaltered when conjugated by $\prod_{p=1}^{s-1} u_{(l\ l_p)}(\xi_{l_p})$. Therefore, $UDU^* = E$, which has a zero cross at index l. This proves (c) and establishes the lemma. □

Definition 3.7. Let $n \in \mathbb{N}$. For $1 \leq i \leq j \leq n$, let $\delta_j^i : [0,1] \to [0,1]$ be given by the following definition:

$$\delta_j^i(\xi) := \begin{cases} 0 & \text{if } 0 \leq \xi \leq \frac{i-1}{j}, \\ \text{linear} & \text{if } \frac{i-1}{j} \leq \xi \leq \frac{i}{j}, \\ 1 & \text{if } \frac{i}{j} \leq \xi \leq 1. \end{cases}$$

Moreover, for $1 \leq i < j \leq n$, let $w_j^i \in C([0,1], M_n)$ be the unitary defined by

$$w_j^i(\xi) := u_{(i\ i+1)}(\delta_{j-i}^{j-i}(\xi)) u_{(i+1\ i+2)}(\delta_{j-i}^{j-i-1}(\xi)) \cdots u_{(j-1\ j)}(\delta_{j-i}^{1}(\xi)),$$

where the unitaries $u_{(k\ k+1)} : [0,1] \to M_n$ are those of Definition 3.5, and set $w_i^i \equiv 1_n$. In particular,

$$w_j^i(1) = u_{(i\ i+1)}(1) \cdots u_{(j-1\ j)}(1) = U[(i\ \ i+1\ \ \cdots\ \ j)]. \tag{3}$$

Lemma 3.8. *Suppose that $D \in M_n$ has a zero cross at index j. Then*

$$\begin{aligned} &\mathfrak{r}(w_j^1(1) D w_j^1(1)^*) \leq \mathfrak{r}(D), \\ &\mathfrak{r}(w_j^i(1) D w_j^i(1)^*) \leq \mathfrak{r}(D) + 1 \quad \textit{for } 2 \leq i \leq j. \end{aligned}$$

Proof. By equation (3), $w_j^1(1) = U[(1\ \ 2\ \ \cdots\ \ j)]$. Consider the matrix D broken up into the four regions created by the zero cross at j, together with the matrix $w_j^1(1) D w_j^1(1)^*$:

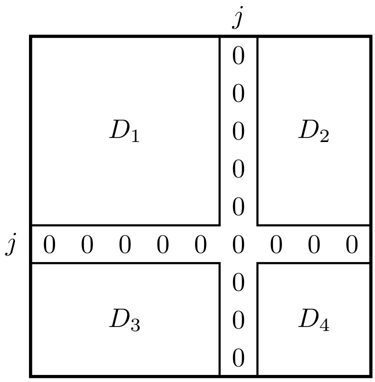

FIGURE 2. The matrix D

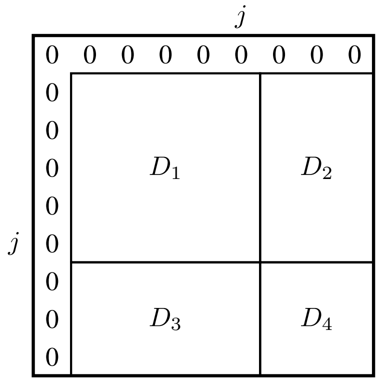

FIGURE 3. The matrix $w_j^1(1)Dw_j^1(1)^*$

Since no nonzero entry gets shifted away from the diagonal, it follows that $\mathfrak{r}(w_j^1(1)Dw_j^1(1)^*) \leq \mathfrak{r}(D)$.

Suppose now that $2 \leq i \leq j$. If $i = j$, then the desired inequality is trivial, so we may assume that $i < j$. By equation (3), $w_j^i(1) = U[(i \;\; i+1 \;\; \cdots \;\; j)]$. Consider the matrix D broken up into the following nine regions created by the zero cross at j and the ith row and column, together with the matrix $w_j^i(1)Dw_j^i(1)^*$:

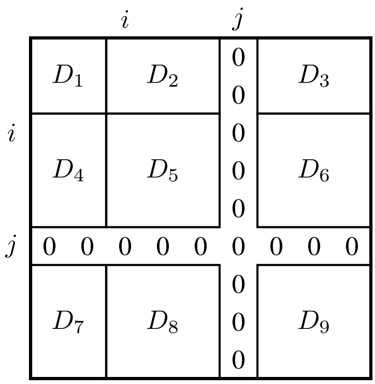

FIGURE 4. The matrix D

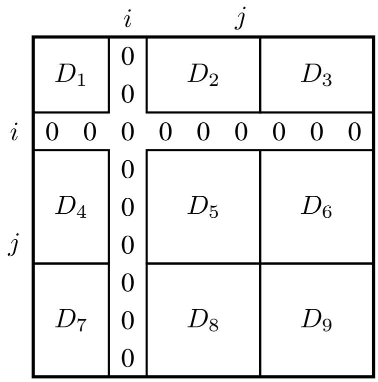

FIGURE 5. The matrix $w_j^i(1)Dw_j^i(1)^*$

With the exception of D_2 and D_4, which get shifted one unit away from the diagonal, no entry in the other seven regions is moved away from the diagonal. Therefore, $\mathfrak{r}(w_j^i(1)Dw_j^i(1)^*) \leq \mathfrak{r}(D) + 1$, which proves Lemma 3.8. $\square$

Lemma 3.9. *Suppose $D \in M_n$ has a zero cross at index j. If $1 \leq i \leq j$ and $\xi \in [0,1]$, then $\mathfrak{r}(w_j^i(\xi)Dw_j^i(\xi)^*) \leq \mathfrak{r}(D) + 2$.*

Proof. Fix $1 \leq i \leq j$ and $\xi \in [0,1]$. If $\xi = 0$ or if $i = j$, then $w_j^i(\xi) = 1_n$ and the result is trivial. Hence, we may assume $i < j$ and $\xi \in (0,1]$. Let $1 \leq k \leq j-i$ be the unique integer such that $\xi \in \big(\frac{k-1}{j-i}, \frac{k}{j-i}\big]$. Then

$$\begin{aligned}(4)\qquad w_j^i(\xi) &= u_{(i\ i+1)}(0)\cdots u_{(j-k-1\ j-k)}(0)\\ &\qquad \cdot u_{(j-k\ j-k+1)}(\delta_{j-i}^k(\xi))u_{(j-k+1\ j-k+2)}(1)\cdots u_{(j-1\ j)}(1)\\ &= u_{(j-k\ j-k+1)}(\delta_{j-i}^k(\xi))w_j^{j-k+1}(1).\end{aligned}$$

Let $D' := w_j^{j-k+1}(1)Dw_j^{j-k+1}(1)^*$. By Lemma 3.8, $\mathfrak{r}(D') \leq \mathfrak{r}(D) + 1$. Now, consider the conjugation of D' by $u_{(j-k\ j-k+1)}(\delta_{j-i}^k(\xi))$, which we denote by E. The entries of D' affected by this conjugation lie in one of the following three shaded regions:

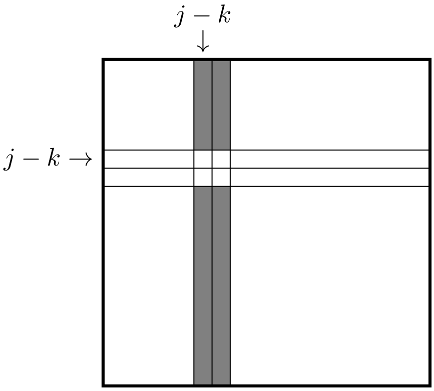

FIGURE 6. Region A

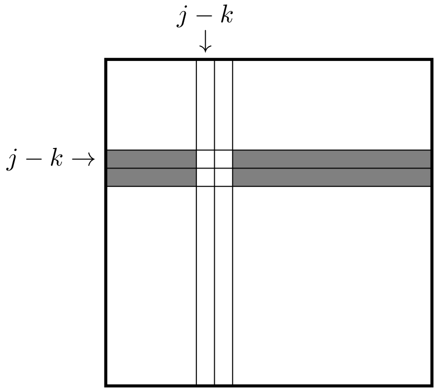

FIGURE 7. Region B

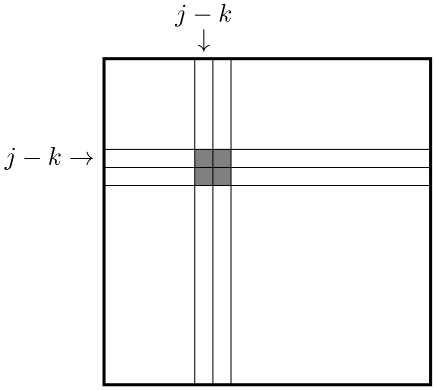

FIGURE 8. Region C

We make the following observations:

- an entry in E lying in Figure 6 will be a linear combination of the two corresponding shaded entries in D' lying in the same row;
- an entry in E lying in Figure 7 will be a linear combination of the two corresponding shaded entries in D' lying in the same column;
- an entry in E lying in Figure 8 will be a linear combination of the four shaded entries in D' lying in Figure 8.

We see that, in all instances, a nonzero entry in E never appears more than one unit further away from the diagonal than a nonzero entry in D'. Thus,

$$\mathfrak{r}\big(u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi))D'u_{(j-k\ j-k+1)}(\delta^k_{j-i}(\xi))^*\big) \leq \mathfrak{r}(D') + 1 \leq \mathfrak{r}(D) + 2,$$

which proves Lemma 3.9. □

Lemma 3.10. *Suppose $n \in \mathbb{N}$ and $1 \leq z_1 < z_2 < \cdots < z_m \leq n$. There is a unitary $W \in C([0,1], M_n)$ with the following properties:*

(a) $W(0) = 1_n$*;*

(b) *if $T \in M_n$ has zero crosses at indices $z_1, \dots, z_m$, then $W(1)TW(1)^*$ has zero crosses at indices $1, 2, \dots, m$ and $\mathfrak{r}(W(\xi)TW(\xi)^*) \leq \mathfrak{r}(T) + 2$ for all $\xi \in [0,1]$;*

(c) *if $b \in \mathbb{N}$, $T \in M_{n\times b}$, and the rows of T at indices $z_1, \dots, z_m$ consist entirely of zeros, then the first m rows of $W(1)T$ consist entirely of zeros and, for all $\xi \in [0,1]$,*

$$\mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & W(\xi)T \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & T \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right);$$

(d) *if $b \in \mathbb{N}$, $T \in M_{b\times n}$, and the columns of T at indices $z_1, \dots, z_m$ consist entirely of zeros, then the first m columns of $TW(1)^*$ consist entirely of zeros and, for all $\xi \in [0,1]$,*

$$\mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & 0_{n\times b} \\ TW(\xi)^* & 0_{b\times b}\end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & 0_{n\times b} \\ T & 0_{b\times b}\end{pmatrix}\right).$$

Proof. For $1 \leq i \leq j \leq n$, let δ^i_j and $w^i_j \in C([0,1], M_n)$ be given as in Definition 3.7. Define

$$W := (w^1_{z_m} \circ \delta^m_m) \cdots (w^1_{z_1} \circ \delta^1_m),$$

which is a unitary in $C([0,1], M_n)$.

By Definitions 3.5 and 3.7, $W(0) = w^1_{z_m}(0) \cdots w^1_{z_1}(0) = 1_n$ so that (a) holds. Since (d) follows immediately from (c) by taking adjoints, only (b) and (c) remain to be verified. Let σ denote the permutation

$$(1\ \ 2\ \ \cdots\ \ z_m)(1\ \ 2\ \ \cdots\ \ z_{m-1})\cdots(1\ \ 2\ \ \cdots\ \ z_1) \in S_n,$$

and note that $\sigma(z_k) = m - k + 1$ for $1 \leq k \leq m$. Hence, if T is any matrix in M_n (resp. $M_{n\times b}$) with zero crosses (resp. rows) at indices $z_1, \dots, z_m$, then $U[\sigma]TU[\sigma]^*$ (resp. $U[\sigma]T$) has zero crosses (resp. rows) at indices $1, \dots, m$. Since $W(1) = U[\sigma]$ by equation (3), this proves the first half of (b) and (c).

Let us now establish the bandwidth approximations in (b) and (c). Fix $\xi \in [0,1]$. If $\xi = 0$, the results are trivial, and so we may assume $\xi \in (0,1]$. Let $1 \leq k \leq m$ be the unique integer such that $\xi \in \big(\frac{k-1}{m}, \frac{k}{m}\big]$. Then we may write

$$\begin{aligned}W(\xi) &= w^1_{z_m}(0)\cdots w^1_{z_{k+1}}(0) w^1_{z_k}(\delta^k_m(\xi)) w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)\\ &= w^1_{z_k}(\delta^k_m(\xi)) w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1).\end{aligned}$$

Suppose first that $T \in M_n$ has zero crosses at indices $z_1, \ldots, z_m$. Then, by Lemma 3.8, $\mathfrak{r}(w^1_{z_1}(1) T w^1_{z_1}(1)^*) \leq \mathfrak{r}(T)$ as T has a zero cross at index z_1. Moreover, $z_1 - 1$ applications of part (a) of Lemma 3.6 show that $w^1_{z_1}(1) T w^1_{z_1}(1)^*$ has a zero cross at indices $z_2, \ldots, z_m$ since $z_2, \ldots, z_m$ are not among the indices affected by the conjugation. Hence, we may apply Lemma 3.8 again to conclude that

$$\mathfrak{r}\big(w^1_{z_2}(1) w^1_{z_1}(1) T w^1_{z_1}(1)^* w^1_{z_2}(1)^*\big) \leq \mathfrak{r}\big(w^1_{z_1}(1) T w^1_{z_1}(1)^*\big) \leq \mathfrak{r}(T).$$

Continuing inductively in this way, it follows that $\mathfrak{r}(D) \leq \mathfrak{r}(T)$, where

$$D = w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1) T w^1_{z_1}(1)^* \cdots w^1_{z_{k-1}}(1)^*$$

and, moreover, D has a zero cross at indices $z_k, \ldots, z_m$. Thus, by Lemma 3.9,

$$\mathfrak{r}\big(W(\xi) T W(\xi)^*\big) = \mathfrak{r}\big(w^1_{z_k}(\delta^k_m(\xi)) D w^1_{z_k}(\delta^k_m(\xi))^*\big) \leq \mathfrak{r}(D) + 2 \leq \mathfrak{r}(T) + 2,$$

which yields the approximation in (b).

To complete the proof of (c), suppose $T \in M_{n\times b}$ and that the rows of T at indices $z_1, \ldots, z_m$ consist entirely of zeros. Following the lines of the proof of Lemma 3.8, we have

$$\mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)T \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & T \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right)$$

by equation (3) since only rows of zeros are shifted up when multiplying T on the left by $w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)$, while nonzero entries remain in place or are shifted down towards the diagonal. Similar reasoning to that used when deducing equation (4) shows that there exist $\beta \in [0,1]$ and $1 \leq p \leq z_k - 1$ such that

$$w^1_{z_k}(\delta^k_m(\xi)) = u_{(z_k - p\ \ z_k - p + 1)}(\beta) w^{z_k - p + 1}_{z_k}(1).$$

Since the z_kth row of a given matrix remains unchanged when multiplying on the left by $w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)$, the z_kth row of $w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)T$ contains only zeros. Hence, multiplying this given matrix on the left by $w^{z_k-p+1}_{z_k}(1) = U[(z_k - p + 1\ \ \cdots\ \ z_k)]$ shifts the zero row from index z_k to index $z_k - p + 1$, while shifting the rows $z_k - p + 1, \ldots, z_k - 1$ down by one towards the diagonal. Thus,

$$\begin{aligned}\mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & E \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right) &\leq \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & w^1_{z_{k-1}}(1)\cdots w^1_{z_1}(1)T \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right)\\ &\leq \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & T \\ 0_{b\times n} & 0_{b\times b}\end{pmatrix}\right),\end{aligned}$$

where $E := U[(z_k - p + 1 \ \cdots \ z_k)] w^1_{z_{k-1}}(1) \cdots w^1_{z_1}(1) T$. Now, the matrices E and $u_{(z_k-p \ z_k-p+1)}(\beta)E$ may differ only on rows $z_k - p$ and $z_k - p + 1$, where these two rows of the latter matrix are linear combinations of the same two rows of E. From this and the fact that the $z_k - p + 1$ row of E consists only of zeros, it is clear that, for a given column λ, the $(z_k - p, \lambda)$- or $(z_k - p + 1, \lambda)$-entry of $u_{(z_k-p \ z_k-p+1)}(\beta)E$ can be nonzero only if the $(z_k - p, \lambda)$ entry of E is nonzero. Hence,

$$\mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & u_{(z_k-p \ z_k-p+1)}(\beta)E \\ 0_{b\times n} & 0_{b\times b} \end{pmatrix}\right) \le \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & E \\ 0_{b\times n} & 0_{b\times b} \end{pmatrix}\right) \le \mathfrak{r}\left(\begin{pmatrix} 0_{n\times n} & T \\ 0_{b\times n} & 0_{b\times b} \end{pmatrix}\right).$$

Since $W(\xi)T = u_{(z_k-p \ z_k-p+1)}(\beta)E$, this establishes the bandwidth approximation in (c), thus completing the proof of the lemma. □

Definition 3.11. Let $N \in \mathbb{N}$. For $j, k, n \in \mathbb{N}$ satisfying $N \le j \le k \le n$, we define

$$\sigma^n_{j,k} := (j - N + 1 \ \ k - N + 1) \cdots (j \ \ k) \in S_n.$$

We define $u^n_{j,k} : [0, 1] \to M_n$ to be the unitary

$$u^n_{j,k}(\xi) := u_{(j-N+1 \ k-N+1)}(\xi) \cdots u_{(j \ k)}(\xi),$$

where $u_{(i \ i')} : [0, 1] \to M_n$ is a continuous path of unitaries defined as in Definition 3.5.

Remark 3.12. Note that, in the definition above, if $j \le n - N$, then $\sigma^n_{j,n}$ is the permutation in S_n that interchanges $j - N + 1, \ldots, j$ and $n - N + 1, \ldots, n$; moreover, in this case, all of the factors in the definition of $u^n_{j,n}(\xi)$ (for any $\xi \in [0, 1]$) commute with each other by Definition 3.5.

Lemma 3.13. *Suppose $N, n, k, i \in \mathbb{N}$ satisfy $N \le k \le i - N$ and $i \le n - N$, and that $\xi \in [0, 1]$. Then*

$$u^n_{k,n}(\xi) = U[\sigma^n_{i,n}] u^n_{k,i}(\xi) U[\sigma^n_{i,n}],$$

where $u^n_{k,n}(\xi)$, $u^n_{k,i}(\xi)$, and $\sigma^n_{i,n}$ are each products of N factors as defined in Definition 3.11.

Proof. By definition,

$$U[\sigma^n_{i,n}] u^n_{k,i}(\xi) = \left(\prod_{j=-(N-1)}^{0} U[(i + j \ \ n + j)]\right)\left(\prod_{j=-(N-1)}^{0} u_{(k+j \ i+j)}(\xi)\right). \tag{5}$$

Note that, for any $-(N - 1) \le j, j' \le 0$,

$$i + j \le i < n - (N - 1) \le n + j'$$

and

$$k + j \le i - N + j \le i - N < i - (N - 1) \le i + j'.$$

Thus, when $-(N-1) \le j, j' \le 0$ with $j \neq j'$, the permutations $(i+j' \;\; n+j')$ and $(k+j \;\; i+j)$ are disjoint, and hence, $U[(i+j' \;\; n+j')]$ and $u_{(k+j \;\; i+j)}(\xi)$ commute. Hence, equation (5) can be restated as

$$U[\sigma^n_{i,n}]u^n_{k,i}(\xi) = \prod_{j=-(N-1)}^{0} U[(i+j \;\; n+j)]u_{(k+j \;\; i+j)}(\xi).$$

By the same reasoning,

$$\begin{aligned} &U[\sigma^n_{i,n}]u^n_{k,i}(\xi)U[\sigma^n_{i,n}] \\ &\quad = \prod_{j=-(N-1)}^{0} U[(i+j \;\; n+j)]u_{(k+j \;\; i+j)}(\xi) \prod_{j=-(N-1)}^{0} U[(i+j \;\; n+j)] \\ &\quad = \prod_{j=-(N-1)}^{0} U[(i+j \;\; n+j)]u_{(k+j \;\; i+j)}(\xi)U[(i+j \;\; n+j)]. \end{aligned}$$

It is elementary to see that $U[(a \;\; b)]u_{(c \;\; b)}(\zeta)U[(a \;\; b)] = u_{(c \;\; a)}(\zeta)$ whenever $c \le b \le a$ and $\zeta \in [0,1]$. Hence,

$$U[\sigma^n_{i,n}]u^n_{k,i}(\xi)U[\sigma^n_{i,n}] = \prod_{j=-(N-1)}^{0} u_{(k+j \;\; n+j)}(\xi) = u^n_{k,n}(\xi),$$

which proves the lemma. □

Definition 3.14. For integers $1 \le j \le k \le n$, define

$$\gamma^n_{j,k} := (j \;\; j+1 \;\; \cdots \;\; k) \in S_n.$$

Lemma 3.15. *Assume that $N, n, t \in \mathbb{N}$ with $1 \le N < t \le n - N$, and suppose $(\xi_{N+1}, \ldots, \xi_{t-1})$ is a vector in $[0,1]^{t-1-N}$ whose final $N-1$ entries consist only of zeros. Then*

$$\begin{aligned} &U[\gamma^n_{1,n}]^N \Big(\prod_{k=N}^{t-2} u^n_{k,n}(\xi_{k+1})\Big) U[\sigma^n_{t-1,n}] \\ &\qquad = U[\gamma^n_{1,t-1}]^N \Big(\prod_{k=N}^{t-2} u^n_{k,t-1}(\xi_{k+1})\Big) U[\gamma^n_{t,n}]^N, \end{aligned}$$

where $u^n_{k,n}(\xi_{k+1})$, $u^n_{k,t-1}(\xi_{k+1})$, and $\sigma^n_{t-1,n}$ are each products of N factors as defined in Definition 3.11.

Proof. If $t-1 = N$, then the products on either side of the equality above are empty and the equation reduces to

$$U[\gamma^n_{1,n}]^N U[\sigma^n_{t-1,n}] = U[\gamma^n_{1,t-1}]^N U[\gamma^n_{t,n}]^N. \tag{6}$$

By Remark 3.12, it is elementary to see that equation (6) holds. Therefore, for the remainder of the proof, we may assume that $t-2 \ge N$.

For any $N \leq k \leq t-1-N$, we may apply Lemma 3.13 (recalling that $t-1 \leq n-N$) to conclude that

$$u^n_{k,n}(\xi_{k+1}) = U[\sigma^n_{t-1,n}]u^n_{k,t-1}(\xi_{k+1})U[\sigma^n_{t-1,n}]. \tag{7}$$

In fact, equation (7) holds for all $N \leq k \leq t-2$. Indeed, if $t-N \leq k \leq t-2$, then by the assumption of the lemma, it must be that $\xi_{k+1} = 0$. In this case, equation (7) reduces to $1_n = U[\sigma^n_{t-1,n}]^2$, which holds by Remark 3.12.

Therefore,

$$\begin{aligned}\prod_{k=N}^{t-2} u^n_{k,n}(\xi_{k+1}) &= \prod_{k=N}^{t-2} U[\sigma^n_{t-1,n}]u^n_{k,t-1}(\xi_{k+1})U[\sigma^n_{t-1,n}] \\ &= U[\sigma^n_{t-1,n}]\Big(\prod_{k=N}^{t-2} u^n_{k,t-1}(\xi_{k+1})\Big)U[\sigma^n_{t-1,n}],\end{aligned}$$

which, together with equation (6), yields that

$$\begin{aligned}&U[\gamma^n_{1,n}]^N \prod_{k=N}^{t-2} u^n_{k,n}(\xi_{k+1}) \\ &\quad = U[\gamma^n_{1,t-1}]^N U[\gamma^n_{t,n}]^N \Big(\prod_{k=N}^{t-2} u^n_{k,t-1}(\xi_{k+1})\Big)U[\sigma^n_{t-1,n}].\end{aligned} \tag{8}$$

Moreover, for each $k = N, \ldots, t-2$, the indices in each transposition-like unitary factor in $u^n_{k,t-1}(\xi_{k+1})$ are distinct from $t, \ldots, n$. Hence, $U[\gamma^n_{t,n}]^N$ and $\prod_{k=N}^{t-2} u^n_{k,t-1}(\xi_{k+1})$ commute, and so, using equation (8), it follows that

$$U[\gamma^n_{1,n}]^N \prod_{k=N}^{t-2} u^n_{k,n}(\xi_{k+1}) = U[\gamma^n_{1,t-1}]^N \Big(\prod_{k=N}^{t-2} u^n_{k,t-1}(\xi_{k+1})\Big)U[\gamma^n_{t,n}]^N U[\sigma^n_{t-1,n}].$$

Lemma 3.15 then follows by multiplying $U[\sigma^n_{t-1,n}]$ on the right of both sides in the above expression. □

Definition 3.16. Let $N \in \mathbb{N}$. Given $n \in \mathbb{N}$ $(n \geq N)$, define $W_n \in C([0,1]^n, M_n)$ to be the unitary

$$W_n(\xi_1, \ldots, \xi_n) := U[\gamma^n_{1,n}]^N \Big(\prod_{k=N}^{n-1} u^n_{k,n}(\xi_{k+1})\Big), \tag{9}$$

where $u^n_{k,n}$ is the product of N factors as in Definition 3.11. We adopt the convention that $W_n := 1_n$ if $n = N$.

Lemma 3.17. *Let $N \in \mathbb{N}$. Suppose n is an integer greater than N and $\vec{\xi} := (\xi_1, \ldots, \xi_n)$ is a vector in $[0,1]^n$ with the property that $\xi_1 = 1$, the final N entries are all zero, and for any consecutive N entries, at most one is nonzero. Suppose $K = \{1 = k_1 < k_2 < \cdots < k_m\}$ is any set of indices, containing 1, at which $\vec{\xi}$ is 1; put $k_{m+1} := n+1$. Then*

$$\begin{aligned}W_n(\vec{\xi}) = \operatorname{diag}\big(&W_{k_2-k_1}(\xi_{k_1}, \ldots, \xi_{k_2-1}), \ldots, \\ &W_{k_{m+1}-k_m}(\xi_{k_m}, \ldots, \xi_{k_{m+1}-1})\big),\end{aligned} \tag{10}$$

where N is the fixed positive integer used to define $W_n, W_{k_2-k_1}, \ldots, W_{k_{m+1}-k_m}$ in Definition 3.16.

Proof. Fix an integer $n > N$, a vector $\vec{\xi}$, and an associated set K, satisfying the hypotheses of the lemma. Let us proceed by induction on the size m of K. If $m = 1$, there is nothing to show. Fix $m \geq 2$, and suppose that Lemma 3.17 holds for every natural number $n' > N$, vector ζ, and associated set K' of size $m-1$, provided they satisfy the required hypotheses. Assume that $|K| = m$. Let us show equation (10) holds in this case.

Note that, by assumption, $\xi_{k_2-(N-1)}, \ldots, \xi_{k_2-1} = 0$ and

$$N < k_2 \leq n - N. \tag{11}$$

Therefore, we may apply Lemma 3.15 with N, n, k_2, and $(\xi_{N+1}, \ldots, \xi_{k_2-1})$ to conclude that

$$\begin{aligned} U[\gamma^n_{1,n}]^N \Bigl(\prod_{k=N}^{k_2-2} u^n_{k,n}(\xi_{k+1})\Bigr) U[\sigma^n_{k_2-1,n}] \\ = U[\gamma^n_{1,k_2-1}]^N \Bigl(\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1})\Bigr) U[\gamma^n_{k_2,n}]^N. \end{aligned} \tag{12}$$

By equation (9) and inequality (11),

$$\begin{aligned} W_n(\vec{\xi}) &= U[\gamma^n_{1,n}]^N \Bigl(\prod_{k=N}^{n-1} u^n_{k,n}(\xi_{k+1})\Bigr) \\ &= U[\gamma^n_{1,n}]^N \Bigl(\prod_{k=N}^{k_2-2} u^n_{k,n}(\xi_{k+1})\Bigr) u^n_{k_2-1,n}(\xi_{k_2}) \Bigl(\prod_{k=k_2}^{n-1} u^n_{k,n}(\xi_{k+1})\Bigr). \end{aligned}$$

Since $u^n_{k_2-1,n}(\xi_{k_2}) = u^n_{k_2-1,n}(1) = U[\sigma^n_{k_2-1,n}]$, we may apply equation (12) to obtain

$$W_n(\vec{\xi}) = U[\gamma^n_{1,k_2-1}]^N \Bigl(\prod_{k=N}^{k_2-2} u^n_{k,k_2-1}(\xi_{k+1})\Bigr) U[\gamma^n_{k_2,n}]^N \Bigl(\prod_{k=k_2}^{n-1} u^n_{k,n}(\xi_{k+1})\Bigr). \tag{13}$$

Let $\vec{\xi'} := (\xi_1, \ldots, \xi_{k_2-1})$ and $\vec{\xi''} := (\xi_{k_2}, \ldots, \xi_n)$. By inequality (11), $|\vec{\xi'}| \geq N$ so that

$$W_{k_2-k_1}(\vec{\xi'}) = U[\gamma^{k_2-1}_{1,k_2-1}]^N \Bigl(\prod_{k=N}^{(k_2-1)-1} u^{k_2-1}_{k,k_2-1}(\xi_{k+1})\Bigr) \in M_{k_2-k_1}. \tag{14}$$

Furthermore, $|\vec{\xi''}| > N$ so that

$$\begin{aligned} W_{n+1-k_2}(\vec{\xi''}) &= W_{n+1-k_2}(\xi_{k_2}, \ldots, \xi_n) \\ &= U[\gamma^{n+1-k_2}_{1,n+1-k_2}]^N \Bigl(\prod_{k=N}^{n+1-k_2-1} u^{n+1-k_2}_{k,n+1-k_2}(\xi_{k_2+k})\Bigr) \end{aligned} \tag{15}$$

$$= U[\gamma^{n+1-k_2}_{1,n+1-k_2}]^N \Bigl(\prod_{k=1}^{n-k_2} u^{n+1-k_2}_{k,n+1-k_2}(\xi_{k_2+k})\Bigr)$$
$$= U[\gamma^{n+1-k_2}_{1,n+1-k_2}]^N \Bigl(\prod_{k=k_2}^{n-1} u^{n+1-k_2}_{k+1-k_2,n+1-k_2}(\xi_{k+1})\Bigr) \in M_{n+1-k_2},$$

where the penultimate equality follows since $\xi_{k_2} = 1$ and at most one of any N consecutive entries of $\vec{\xi}$ is nonzero. Therefore,

$$\begin{aligned}\operatorname{diag}\bigl(&W_{k_2-k_1}(\vec{\xi'}), W_{n+1-k_2}(\vec{\xi''})\bigr)\\ &= \operatorname{diag}\bigl(W_{k_2-k_1}(\vec{\xi'}), 1_{n+1-k_2}\bigr)\operatorname{diag}\bigl(1_{k_2-k_1}, W_{n+1-k_2}(\vec{\xi''})\bigr)\\ &= U[\gamma^n_{1,k_2-1}]^N \Bigl(\prod_{k=N}^{(k_2-1)-1} u^n_{k,k_2-1}(\xi_{k+1})\Bigr) U[\gamma^n_{k_2,n}]^N \Bigl(\prod_{k=k_2}^{n-1} u^n_{k,n}(\xi_{k+1})\Bigr),\end{aligned}$$

where in the last equality the indices in the γ's and u's have been altered appropriately from the ones in equation (14) and equation (15) to accommodate for the identity factors in the diagonal. Combining this with equation (13) yields that

$$W_n(\vec{\xi}) = \operatorname{diag}\bigl(W_{k_2-k_1}(\vec{\xi'}), W_{n+1-k_2}(\vec{\xi''})\bigr). \tag{16}$$

We may apply the inductive hypothesis to $n' = |\vec{\xi''}| > N$, vector $\vec{\xi''}$, and associated set $K' = \{k_2, \dots, k_m\}$ of size $m-1$ to conclude that

$$\begin{aligned}W_{n+1-k_2}(\vec{\xi''}) = \operatorname{diag}\bigl(&W_{k_3-k_2}(\xi_{k_2}, \dots, \xi_{k_3-1}), \dots,\\ &W_{k_{m+1}-k_m}(\xi_{k_m}, \dots, \xi_{k_{m+1}-1})\bigr).\end{aligned}$$

Substituting this into equation (16) yields that

$$W_n(\vec{\xi}) = \operatorname{diag}\bigl(W_{k_2-k_1}(\xi_{k_1}, \dots, \xi_{k_2-1}), \dots, W_{k_{m+1}-k_m}(\xi_{k_m}, \dots, \xi_{k_{m+1}-1})\bigr),$$

which proves Lemma 3.17. □

3.18. **The main lemmas.** With the results of Section 3.2 in hand, we are now in position to prove the lemmas listed in Figure 1, which are needed to prove Theorem 3.30 in the sequel.

We start with a lemma that characterizes when a unital injective limit of subhomogeneous algebras is simple in terms of the corresponding maps between their spectra. This is essentially [5, Prop. 2.1], except that ours discusses the general unital subhomogeneous case. The proof is very similar.

Given any unital subhomogeneous C*-algebras A and B and a unital *-homomorphism $\psi : A \to B$, an irreducible representation π of B yields a representation $\pi \circ \psi$ of A. The finite-dimensional representation $\pi \circ \psi$ is unitarily equivalent to a direct sum $\tau_1 \oplus \cdots \oplus \tau_s$ of irreducible representations of A. In this way, we get a map $\hat{\psi} : \hat{B} \to \mathcal{P}(\hat{A})$ given by $\hat{\psi}([\pi]) := \{[\tau_1], \dots, [\tau_s]\}$, where multiplicities are ignored.

Lemma 3.19. *Suppose we have an inductive limit of the form*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i,$$

where A is unital and, for each $i \in \mathbb{N}$, A_i is subhomogeneous and ψ_i is injective. Let $\psi_{j,i} := \psi_{j-1} \circ \cdots \circ \psi_i$. Then the following statements are equivalent.

(i) *A is simple.*

(ii) *For all $i \in \mathbb{N}$ and all nonempty open $U \subset \hat{A}_i$, there is a $j > i$ such that $\hat{\psi}_{j,i}([\pi]) \cap U \neq \varnothing$ for all $[\pi] \in \hat{A}_j$.*

(iii) *For all $i \in \mathbb{N}$, if $f \in A_i$ is nonzero, there is a $j > i$ such that $\pi(\psi_{j,i}(f)) \neq 0$ for every nonzero irreducible representation π of A_j.*

Proof. For $n \in \mathbb{N}$, let $\mu_n : A_n \to A$ denote the map in the construction of the inductive limit. Since the ψ_j's are injective and A is unital, we may assume that the A_j's are all unital and that the μ_j's are injective and unit-preserving.

Let us start by showing that (i) implies (ii). Suppose that (ii) is false. To show (i) is false, let us construct a closed proper nonzero two-sided ideal of A. Choose $i \in \mathbb{N}$ and a nonempty open set $U \subset \hat{A}_i$ such that, for all $j > i$ there is a $[\pi] \in \hat{A}_j$ with $\hat{\psi}_{j,i}([\pi]) \cap U = \varnothing$. We may assume that $U \neq \hat{A}_i$. For $j > i$, set $F_j := \{[\pi] \in \hat{A}_j \mid \hat{\psi}_{j,i}([\pi]) \cap U = \varnothing\}$, and set $I_j := \{f \in A_j \mid f \in \bigcap_{[\pi]\in F_j} \ker \pi\}$. It is straight-forward to verify that, for all $j > i$, I_j is a closed proper nonzero two-sided ideal of A_j.

For $k > j > i$ and $[\pi] \in \hat{A}_k$, we have $\hat{\psi}_{k,i}([\pi]) = \hat{\psi}_{j,i}(\hat{\psi}_{k,j}([\pi]))$, from which it follows that $\hat{\psi}_{k,j}(F_k) \subset F_j$. Thus, $\psi_{k,j}(I_j) \subset I_k$ for all $k > j > i$. Hence, $\{\mu_j(I_j)\}_{j>i}$ is an increasing sequence of C*-algebras, and so $I := \overline{\bigcup_{j>i} \mu_j(I_j)}$ is a sub-C*-algebra of A. It is not hard to see that I is a closed two-sided ideal of A. Since the μ_j's are injective and the I_j's are nonzero, $I \neq \{0\}$. If $1_A \in I$, then for large enough j, I_j contains 1_{A_j}, contradicting that I_j is proper. Hence, I is the desired closed proper nonzero two-sided ideal of A. This proves that (i) implies (ii).

Let us now show that (ii) implies (iii). Fix $i \in \mathbb{N}$, and suppose $0 \neq f \in A_i$. Let $U := \{[\rho] \in \hat{A}_i \mid \rho(f) \neq 0\}$. Observe that U is a nonempty open subset of $\hat{A}_i$. By (ii), there is a $j > i$ such that $\hat{\psi}_{j,i}([\pi]) \cap U \neq \varnothing$ for all $[\pi] \in \hat{A}_j$. Thus, if π is any irreducible representation of A_j, $\pi(\psi_{j,i}(f)) \neq 0$, which proves (iii).

Finally, let us prove that (iii) implies (i). Suppose J is a nonzero closed two-sided ideal of A. For $j \in \mathbb{N}$, put $J_j := \mu_j^{-1}(J)$. Then, for all $j \in \mathbb{N}$, J_j is a closed two-sided ideal of A_j. It will be shown that $J_j = A_j$ for some $j \in \mathbb{N}$. Take $0 \neq a \in J$. It is well known that

$$J = \overline{\bigcup_{j=1}^{\infty} (\mu_j(A_j) \cap J)}.$$

Hence, there must be an i and an $a_i \in A_i$ such that $0 \neq \mu_i(a_i) \in J$. Thus, $a_i \neq 0$. By (iii), there is a $j > i$ such that, for all irreducible representations π of A_j, $\pi(\psi_{j,i}(a_i)) \neq 0$. Since $\mu_j(\psi_{j,i}(a_i)) = \mu_i(a_i) \in J$, it follows that $\psi_{j,i}(a_i) \in J_j$. The bijective correspondence between closed two-sided ideals of A_j and closed subsets of $\hat{A}_j$ thus forces J_j to be all of A_j. Hence, $1_A = \mu_j(1_{A_j}) \in J$, which shows that $J = A$. Therefore, A is simple, which proves (i). $\square$

Lemma 3.20. *Let A be a DSH algebra of length l. Let $\epsilon > 0$. Suppose that $f \in A$ is not invertible. Then there is an $f' \in A$ with $\|f - f'\| \leq \epsilon$ and there are unitaries $w, v \in A$ such that, for some $1 \leq i \leq l$, $(wf'v)_i$ has a zero cross at index 1 everywhere on some nonempty set $U \subset \hat{A} \cap (X_i \setminus Y_i)$, which is open with respect to the hull-kernel topology on $\hat{A}$. Moreover, there is a $\Delta \in A$ such that, for every $1 \leq j \leq l$ and $x \in X_j$, $\Delta_j(x)$ is a diagonal matrix with entries in $[0,1]$, where $\Delta_j(x)_{k,k} > 0$ implies $(wf'v)_j(x)$ has a zero cross at index k; moreover, $\Delta_i(z)_{1,1} = 1$ for all $z \in U$.*

Proof. Using Lemma 2.11, choose $1 \leq i \leq l$ and $x \in X_i \setminus Y_i$ such that $f_i(x)$ is a non-invertible matrix. We break the proof up into two cases.

Case one: x is not in the decomposition of any point in Y_j for any $j > i$. By Lemma 2.13, there is set $U_1 \subset X_i$ containing x, which is open in X_i and has the property that no point in it is in the decomposition of any point in Y_j for any $j > i$. Since Y_i is closed in X_i, the set $U_1 \cap (X_i \setminus Y_i)$ is open in X_i. By shrinking U_1, we may assume that $\|f_i(x) - f_i(z)\| \leq \epsilon$ for all $z \in U_1$. Choose a set U_2 that is open in X_i and satisfies $x \in U_2 \subset \overline{U_2}^{X_i} \subset U_1 \cap (X_i \setminus Y_i)$. Using Urysohn's Lemma, we can define a function $h \in C(X_i, M_{n_i})$ such that $h|_{\overline{U_2}^{X_i}} \equiv f_i(x)$, $h|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))} = f_i|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))}$, and $\|f_i - h\| \leq \epsilon$. Define f' coordinate-wise by $f' := (f_1, \ldots, f_{i-1}, h, f_{i+1}, \ldots, f_l)$. Since $h|_{Y_i} = f_i|_{Y_i}$, we have $(f_1, \ldots, f_{i-1}, h) \in A^{(i)}$. Since no point in U_1 is in the decomposition of any point in Y_j for any $j > i$, and because h may only differ from f_i on $U_1 \cap (X_i \setminus Y_i) \subset U_1$, this perturbation does not violate the diagonal decomposition at any point. Thus, $f' \in A$ since $f \in A$, and $\|f - f'\| \leq \epsilon$ because $\|f_i - h\| \leq \epsilon$. Since $f_i(x)$ is a non-invertible matrix, there are unitary matrices W and V in M_{n_i} with the property that $Wf_i(x)V$ has a zero cross at index 1. Since the unitary group in M_{n_i} is connected, we may, using the same reasoning as above, define unitaries $w, v \in A$ coordinate-wise with $w_j = v_j \equiv 1_{n_j}$ for all $j \neq i$ and $w_i, v_i \in C(X_i, M_{n_i})$ satisfying $w_i|_{\overline{U_2}^{X_i}} \equiv W$, $v_i|_{\overline{U_2}^{X_i}} \equiv V$, and $w_i|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))} = v_i|_{X_i \setminus (U_1 \cap (X_i \setminus Y_i))} \equiv 1_{n_i}$. Finally, choose a set U_3 that is open in X_i and satisfies $x \in U_3 \subset \overline{U_3}^{X_i} \subset U_2$. Define $\Delta \in A$ coordinate-wise as follows: $\Delta_j \equiv 0$ for $j \neq i$; let $g : X_i \to [0,1]$ be any continuous function such that $g|_{\overline{U_3}^{X_i}} \equiv 1$ and $g|_{X_i \setminus U_2} \equiv 0$, and put $\Delta_i := \mathrm{diag}(g, 0, \ldots, 0) \in C(X_i, M_{n_i})$. As argued above for f', we have $\Delta \in A$. Take $U := U_3$. Applying Lemma 2.12, we conclude U is open in $\hat{A}$. Since $(wf'v)_i$ has a zero cross at index 1 everywhere on U_2 and since Δ vanishes outside U_2, the lemma holds in this case.

Case two: There is a $j > i$ such that x is in the decomposition of some point in Y_j. In this case, we cannot define f' as above because we are not guaranteed a neighborhood around x in which we may freely perturb f while remaining in A. Let i' denote the largest integer for which x is in the decomposition of some point in $Y_{i'}$. Choose $y \in Y_{i'}$ such that x is in the decomposition of y. Then $f_{i'}(y)$ is a non-invertible matrix. Since x is not in the decomposition of any point in $Y_{j'}$ for any $j' > i'$, neither is y. Hence, by Lemma 2.13, there is a set $U_1 \subset X_{i'}$ containing y that is open in $X_{i'}$ with the property that no point in U_1 is in the decomposition of any point in $Y_{j'}$ for any $j' > i'$. Hence, as in case one, we are able to perturb f on $U_1 \cap (X_{i'} \setminus Y_{i'})$, while remaining in A. By shrinking U_1, we may assume that $\|f_{i'}(y) - f_{i'}(z)\| \leq \epsilon$ for all $z \in U_1$. By Lemma 2.10, we may assume that $Y_{i'}$ has empty interior and, thus, that there is a point $x' \in U_1 \cap (X_{i'} \setminus Y_{i'})$. Choose a set U_2 which is open in $X_{i'}$ and satisfies $x' \in U_2 \subset \overline{U_2}^{X_{i'}} \subset U_1 \cap (X_{i'} \setminus Y_{i'})$. As in case one, we may define $f' \in A$ with $\|f - f'\| \leq \epsilon$, $f'_{j'} = f_{j'}$ for $j' \neq i'$, $f'_{i'}|_{\overline{U_2}^{X_{i'}}} \equiv f_{i'}(y)$, and $f'_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))} = f_{i'}|_{X_{i'} \setminus (U_1 \cap (X_{i'} \setminus Y_{i'}))}$. Choose unitary matrices $W, V \in M_{n_{i'}}$ such that $W f_{i'}(y) V$ has a zero cross at index 1. Then the rest of the proof proceeds verbatim as the proof of case one with i' in place of i and x' in place of x. □

The following two lemmas guarantee the existence of certain indicator-function-like elements in DSH algebras. As outlined in Section 3.1, these unitaries are used, together with the results from Section 3.2, in the proofs of future lemmas to construct the unitaries needed to prove Theorem 3.30. It is for these two key lemma that we require the base spaces of a given DSH algebra to be metrizable.

Lemma 3.21. *Suppose A is a DSH algebra of length l. Suppose $M \in \mathbb{N}$ and $K := \{K_1 < K_2 < \cdots < K_m\}$ are such that $K_1 \geq 0$, $K_m < \mathfrak{s}(A) - M$, and $K_{t+1} - K_t \geq M$ for $1 \leq t < m$. Then there is a function $\Phi \in A$ such that*

(i) *for all $1 \leq i \leq l$ and $x \in X_i$, $\Phi_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final M diagonal entries are all 0, and such that at most one of every M consecutive diagonal entries is nonzero;*

(ii) *for all $1 \leq i \leq \ell$ and $1 \leq j \leq n_i$, $\Phi_i(x)_{j,j} = 1$ if and only if there is a $1 \leq t \leq m$ such that $x \in B_{i,j-K_t}$.*

Proof. We define Φ coordinate-wise inductively. Start off by putting $\Phi_1 \equiv \operatorname{diag}(\chi_K(0), \ldots, \chi_K(n_1 - 1))$, where χ_K is the indicator function corresponding to the set $K = \{K_1, \ldots, K_m\}$. By the assumption on the set K, condition (i) holds for Φ_1. To see that (ii) holds, suppose $\Phi_1(x)_{j,j} = 1$. Then $\chi_K(j-1) = 1$, so there is a $1 \leq t \leq m$ such that $j = K_t + 1$. By Lemma 2.9, $x \in X_1 = B_{1,1} = B_{1,j-K_t}$. Conversely, if there is a $1 \leq t \leq m$ such that $x \in B_{1,j-K_t}$, then by Lemma 2.9, $j - K_t = 1$ so that $\Phi_1(x)_{j,j} = \chi_K(j-1) = 1$, which proves (ii).

Now, suppose that we have a fixed $1 < i \leq l$ and assume that we have defined $(\Phi_1, \ldots, \Phi_{i-1}) \in A^{(i-1)}$ such that, for all $i' < i$ and $x \in X_{i'}$,

(I) the matrix $\Phi_{i'}(x)$ satisfies the properties of conditions (i) and (ii);

(II) $\Phi_{i'}(x)_{j,j} = \chi_K(j-1)$ for all $1 \leq j \leq \mathfrak{s}(A)$.

Let $\Phi'_i := \varphi_{i-1}((\Phi_1, \ldots, \Phi_{i-1})) \in C(Y_i, M_{n_i})$. Fix $y \in Y_i$, and suppose y decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \ldots, x_r \in X_{i_r} \setminus Y_{i_r}$. Let us first check that conditions (i) and (ii) hold for $\Phi'_i(y) = \operatorname{diag}(\Phi_{i_1}(x_1), \ldots, \Phi_{i_r}(x_r))$. By the inductive hypothesis, $\Phi_i(y)$ is a diagonal matrix with entries in $[0,1]$ and the last M diagonal entries of $\Phi'_i(y)$ are all 0. Given M consecutive entries down the diagonal of $\Phi'_i(y)$, if they are all contained in one of the diagonal blocks, then by the inductive hypothesis applied to that one block, at most one of these entries is nonzero. If instead the M consecutive entries span two blocks $\Phi_{i_q}(x_q)$ and $\Phi_{i_{q+1}}(x_{q+1})$, then by the inductive hypothesis, the last M diagonal entries of $\Phi_{i_q}(x_q)$ are 0 and at most 1 of the first M diagonal entries of $\Phi_{i_{q+1}}(x_{q+1})$ can be nonzero. This shows that (i) holds for $\Phi'_i(y)$. Let us now show that (ii) holds for $\Phi'_i(y)$. Fix $1 \leq j \leq n_i$. Let $1 \leq q \leq r$ and $1 \leq j' \leq n_{i_q}$ be such that $\Phi'_i(y)_{j,j} = \Phi_{i_q}(x_q)_{j',j'}$. Note that $j = n_{i_1} + \cdots + n_{i_{q-1}} + j'$. Given $1 \leq t \leq m$, we know by Lemma 2.9 that $y \in B_{i,j-K_t}$ if and only if there is a $1 \leq p \leq r$ such that

$$j' - K_t + n_{i_1} + \cdots + n_{i_{q-1}} = j - K_t = 1 + n_{i_1} + \cdots + n_{i_{p-1}} \tag{17}$$

(the right-hand side is 1 if $p = 1$). We claim that if equation (17) holds, then $p = q$. Indeed, using the upper and lower bounds on j' and K_t, we have

$$1 - \mathfrak{s}(A) < 1 - (\mathfrak{s}(A) - M - 1) \leq j' - K_t \leq n_{i_q},$$

whence

$$1 + n_{i_1} + \cdots + n_{i_{q-1}} - \mathfrak{s}(A) < 1 + n_{i_1} + \cdots + n_{i_{p-1}} \leq n_{i_1} + \cdots + n_{i_{q-1}} + n_{i_q}.$$

The first inequality and the definition of $\mathfrak{s}(A)$ imply that $q \leq p$, while the second inequality forces $q \geq p$ so that $p = q$. Therefore, since $x_q \in X_{i_q} \setminus Y_{i_q}$, the above and Lemma 2.9 show that

$$\begin{aligned} y \in B_{i,j-K_t} &\iff j - K_t = 1 + n_{i_1} + \cdots + n_{i_{q-1}} \\ &\iff j' - K_t = 1 \\ &\iff x_q \in B_{i_q, j'-K_t}. \end{aligned}$$

Since the matrix $\Phi_{i_q}(x_q)$ satisfies (ii) by the inductive hypothesis, it follows that there is a $1 \leq t \leq m$ with $y \in B_{i,j-K_t}$ if and only if $\Phi'_i(y)_{j,j} = \Phi_{i_q}(x_q)_{j',j'} = 1$, which proves that (ii) holds for $\Phi'_i(y)$.

Let us now define $\Phi_i \in C(X_i, M_{n_i})$ to be a suitable extension of Φ'_i. Write $\Phi'_i = \operatorname{diag}(h'_1, \ldots, h'_{n_i})$, where $h'_j \in C(Y_i, [0,1])$ for $1 \leq j \leq n_i$. We define $\Phi_i = \operatorname{diag}(h_1, \ldots, h_{n_i})$ by specifying each h_j to be a continuous function $h_j : X_i \to [0,1]$ that extends h'_j. For $1 \leq j \leq \mathfrak{s}(A)$, put $h_j \equiv \chi_K(j-1)$ to insure that (II) in the inductive hypothesis is verified, and set $h_j \equiv 0$ for $n_i - M + 1 \leq j \leq n_i$ (since (I) and (II) hold for $\Phi_1, \ldots, \Phi_{i-1}$, these h_j's do indeed extend the corresponding h'_j's). We define h_j for $\mathfrak{s}(A) + 1 \leq j \leq n_i - M$ inductively. Fix $\mathfrak{s}(A) + 1 \leq j \leq n_i - M$, and assume we have defined $h_1, \ldots, h_{j-1}$ so that the following property holds:

$$\bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{j-t})} \subset X_i \text{ is disjoint from } \operatorname{supp}(h'_j) \subset Y_i. \tag{$\clubsuit$}$$

Note that $\bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{\mathfrak{s}(A)+1-t})} = \varnothing$, and so ($\clubsuit$) holds for the base case $j = \mathfrak{s}(A) + 1$. Since X_i is a metric space and, hence, perfectly normal, we may use ($\clubsuit$) to extend h'_j to a function f_j in $C(X_i, [0,1])$ that vanishes on

$\bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{j-t})}$ and is strictly less than 1 on $X_i \setminus Y_i$. Define

$$g_j^0 := h_j' - \sum_{t=1}^{M-1} h_{j+t}' \in C(Y_i).$$

Then the range of g_j^0 is contained in $[-1,1]$ since by (I) at most one of $h_j', \dots, h_{j+M-1}'$ is nonzero at any given point in Y_i. Extend g_j^0 to a function g_j' in $C(X_i,[-1,1])$. Put $g_j := \max(g_j', 0)$, and note that $g_j|_{Y_i} = h_j'$. Since

$$h_j'(y) = 0 \quad \text{for each } y \in \bigcup_{t=1}^{M-1} \operatorname{supp}(h_{j+t}'),$$

we may choose an open subset $U \supset \bigcup_{t=1}^{M-1} \operatorname{supp}(h_{j+t}')$ of X_i on which g_j' is strictly negative so that $g_j|_U \equiv 0$. Define $h_j := \min(f_j, g_j) \in C(X_i,[0,1])$, and note that $h_j|_{Y_i} = h_j'$. Since $h_j|_U \equiv 0$, we have $\operatorname{supp}(h_j) \cap U = \varnothing$, from which it follows that $\overline{\operatorname{supp}(h_j)} \cap \big(\bigcup_{t=1}^{M-1} \operatorname{supp}(h_{j+t}')\big) = \varnothing$. This ensures that (♣) holds with $j+1$ in place of j and, hence, that $\Phi_i := \operatorname{diag}(h_1, \dots, h_{n_i})$ is well defined.

To conclude the proof, let us check that Φ_i satisfies (i) and (ii). In light of the analysis above, we may restrict ourselves to the diagonal entries $\mathfrak{s}(A)+1 \leq j \leq n_i - M$. By definition, the range of each h_j is contained in $[0,1]$. If $h_j(x) > 0$ for some $x \in X_i$, then $f_j(x) > 0$ and, hence, by the definition of f_j, $x \notin \bigcup_{t=1}^{M-1} \overline{\operatorname{supp}(h_{j-t})}$. This proves that at most one of any M consecutive entries down the diagonal of $\Phi_i(x)$ is nonzero. Hence, (i) is established. To prove (ii), suppose $x \in X_i$ satisfies $h_j(x) = 1$. Then $f_j(x) = 1$, which implies that $x \in Y_i$. Thus, $h_j'(x) = 1$, and we already established that $x \in B_{i,j-K_t}$ for some t in this case. Conversely, suppose $x \in B_{i,j-K_t}$ for some t. If $j - K_t \neq 1$, then by Lemma 2.9, $x \in Y_i$, and we already concluded in this case that $h_j(x) = h_j'(x) = 1$. If instead $j - K_t = 1$, then it must be that $j < \mathfrak{s}(A)$, and we previously defined $h_j \equiv 1$ in this case. Therefore, property (ii) holds.

We verified that both (I) and (II) hold for $\Phi_i = \operatorname{diag}(h_1, \dots, h_l)$, and since $\Phi_i|_{Y_i} = \Phi_i' = \varphi_{i-1}((\Phi_1, \dots, \Phi_{i-1}))$, it follows that $(\Phi_1, \dots, \Phi_i) \in A^{(i)}$. Thus, by induction, we obtain $\Phi := (\Phi_1, \dots, \Phi_l) \in A$, which satisfies the requirements of the lemma. ☐

Lemma 3.22. *Suppose A is a DSH algebra of length l. Suppose $M \in \mathbb{N}$ and $K := \{K_1 < K_2 < \cdots < K_m\}$ are such that $K_1 \geq 0$, $K_m < \mathfrak{s}(A) - M$, and $K_{t+1} - K_t \geq M$ for $1 \leq t < m$. Suppose that, for each $1 \leq i \leq l$ and $1 \leq j \leq n_i$, we have a set $F_{i,j} \subset X_i$ that is closed in X_i and disjoint from each set $B_{i,j-K_t}$ (see Definition 2.8) for $1 \leq t \leq m$. Then there is a function $\Theta \in A$ such that*

(i) *for all $1 \leq i \leq l$ and $x \in X_i$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final M diagonal entries are all 0, and such that at most one of every M consecutive diagonal entries is nonzero;*

(ii) *for all $1 \leq i \leq l$, $1 \leq j \leq n_i$, and $x \in F_{i,j}$, we have $\Theta_i(x)_{j,j} = 0$;*

(iii) *for all $1 \leq i \leq l$ and $1 \leq j \leq n_i$, there is a (possibly empty) open subset $U_{i,j} \subset X_i$ containing $B_{i,j}$ with the property that if $x \in U_{i,j}$, then*

$$\Theta_i(x)_{j+K_t, j+K_t} = 1 \quad \text{for all } 1 \leq t \leq m.$$

Proof. Using the hypotheses of this lemma, Lemma 3.21 furnishes a function $\Phi \in A$ such that

(a) for all $1 \leq i \leq l$ and $x \in X_i$, $\Phi_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final M diagonal entries are all 0, and such that at most one of every M consecutive diagonal entries is nonzero;

(b) for all $1 \leq i \leq \ell$ and $1 \leq j \leq n_i$, $\Phi_i(x)_{j,j} = 1$ if and only if there is a $1 \leq t \leq m$ such that $x \in B_{i,j-K_t}$.

Let us use Φ to construct a function $\Theta \in A$ satisfying conditions (i) to (iii).

Given $\delta \in [0,1)$, define $g : [0,1] \to [0,1]$ by

$$g(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq \delta, \\ \text{linear} & \text{if } \delta \leq x \leq \frac{1+\delta}{2}, \\ 1 & \text{if } \frac{1+\delta}{2} \leq x \leq 1. \end{cases}$$

For $1 \leq i \leq l$, define $\Theta_i : X_i \to M_{n_i}$ by

$$\Theta_i(x) := \operatorname{diag}\bigl(g(\Phi_i(x)_{1,1}), \ldots, g(\Phi_i(x)_{n_i,n_i})\bigr).$$

Then $\Theta := \bigoplus_{i=1}^{l} \Theta_i \in \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$. Since each diagonal entry of Φ is modified in the same way in the definition of Θ, it is straight-forward to check that Θ is compatible with the diagonal structure of A. Hence, $\Theta \in A$. Moreover, since $\Theta_i(x)_{j,j} = 0$ whenever $\Phi_i(x)_{j,j} = 0$, it is clear that Θ satisfies (i) since Φ satisfies (a).

To see that Θ satisfies (ii), fix $1 \leq i \leq l$ and $1 \leq j \leq n_i$. Since $F_{i,j}$ is disjoint from each $B_{i,j-K_t}$ (for $1 \leq t \leq m$), condition (b) guarantees that $\Phi_i(x)_{j,j} < 1$ for all $x \in F_{i,j}$. Since $F_{i,j}$ is compact, there is a $\delta_{i,j} \in [0,1)$ such that $\Phi_i(x)_{j,j} \leq \delta_{i,j}$ for all $x \in F_{i,j}$. On choosing $\delta := \max\{\delta_{i,j} \mid 1 \leq i \leq l,\, 1 \leq j \leq n_i\} \in [0,1)$ in our definition of g above, it follows that $\Theta_i(x)_{j,j} = 0$ whenever $1 \leq i \leq l$, $1 \leq j \leq n_i$, and $x \in F_{i,j}$, which proves (ii).

Finally, to see that Θ satisfies (iii), fix $1 \leq i \leq l$ and $1 \leq j \leq n_i$. If $j > n_i - (\mathfrak{s}(A) - 1)$, we may take $U_{i,j} = \varnothing$ since $B_{i,j} = \varnothing$ by Lemma 2.9 for such j. For $j \leq n_i - (\mathfrak{s}(A) - 1)$, note that if $x \in B_{i,j}$, then by (b), $\Phi_i(x)_{j+K_t,j+K_t} = 1$ for all $1 \leq t \leq m$. Since g is 1 in a neighborhood of 1, it follows that, for each t, there is an open set $U_t \supset B_{i,j}$ on which the function $\Theta_i(\,\cdot\,)_{j+K_t,j+K_t} : X_i \to [0,1]$ is equal to 1. Taking $U_{i,j} := \bigcap_{1 \leq t \leq m} U_t$ yields (iii) and proves the lemma. □

Given a sequence of DSH algebras $A_1, A_2, \ldots$, we denote by $l(j)$ the length of the DSH algebra A_j. We denote the base spaces of A_j by $X_1^j, \ldots, X_{l(j)}^j$ and the corresponding closed subspaces by $Y_1^j, \ldots, Y_{l(j)}^j$. We denote the size of the matrix algebras in the pullback definition of A_j by $n_1^j, \ldots, n_{l(j)}^j$. Finally, we denote the sets defined in Definition 2.8 corresponding to A_j by $B_{i,k}^j$.

Lemma 3.23. *Suppose*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

is a simple limit of infinite-dimensional DSH algebras with injective diagonal maps. Then, for all $j, m \in \mathbb{N}$, there is a $j' > j$ such that $\mathfrak{s}(A_{j'}) > m$ (where, recall, $\mathfrak{s}(A_j) = \min\{n_t^j \mid 1 \le t \le l(j)\}$).

Proof. Since A_j is infinite-dimensional, at least one of the base spaces must be infinite. Let $1 \le i \le l(j)$ be the largest integer for which X_i^j is infinite. By Lemma 2.10, $X_i^j \setminus Y_i^j$ is also infinite and $Y_{i'}^j = \varnothing$ for $i < i' \le l(j)$. Choose pairwise-disjoint open in X_i^j sets $\mathcal{O}_1, \dots, \mathcal{O}_{m+1} \subset X_i^j \setminus Y_i^j$. Lemma 2.12 guarantees that $\mathcal{O}_1, \dots, \mathcal{O}_{m+1}$ are all open with respect to the hull-kernel topology on $\hat{A}_j$. By Lemma 3.19, there is a $j' > j$ such that, for all $x \in \hat{A}_{j'}$, $\hat{\psi}_{j',j}(x)$ contains a point from each of $\mathcal{O}_1, \dots, \mathcal{O}_{m+1}$. Hence, $n_i^{j'} \ge m+1$ for all $1 \le i \le l(j')$, which proves the lemma. □

Lemma 3.24. *Suppose*

$$A_1 \xrightarrow{\psi_1} A_2 \xrightarrow{\psi_2} A_3 \xrightarrow{\psi_3} \cdots \longrightarrow A := \varinjlim A_i$$

is a simple limit of infinite-dimensional DSH algebras with injective diagonal maps. Suppose that f is a non-invertible element belonging to some A_j and that $\epsilon > 0$. Then there exist $f' \in A_j$ with $\|f - f'\| \le \epsilon$ and $M \in \mathbb{N}$ such that, for all $N \in \mathbb{N}$, there exist $j' > j$ satisfying $\mathfrak{s}(A_{j'}) > NM$ and unitaries $V, V' \in A_{j'}$ with the following properties:

(i) *for any $1 \le i \le l(j')$ and $1 \le k \le n_i^{j'}$, there is a (possibly empty) open subset $U_{i,k}$ of $X_i^{j'}$ containing $B_{i,k}^{j'}$ such that, for all $x \in U_{i,k}$, $(V\psi_{j',j}(f')V')_i(x)$ has zero crosses at indices $k, k+M, k+2M, \dots, k+(N-1)M$;*

(ii) *for all $1 \le i \le l(j')$ and $x \in X_i^{j'}$, we have $\mathfrak{r}((V\psi_{j',j}(f')V')_i(x)) \le \mathfrak{S}(A_j) + M - 1$ (where, recall, $\mathfrak{S}(A_j) = \max\{n_t^j \mid 1 \le t \le l(j)\}$).*

Proof. Let $f', w, v, \Delta \in A_j$ and $U \subset \hat{A}_j$ be given as in Lemma 3.20 (when applied to f and ϵ), and set $g := wf'v$. Then, at every point in U, g has a zero cross at index 1 and the $(1,1)$-entry of Δ is 1. By Lemma 3.19 and Lemma 2.5, there is a $j'' > j$ such that $\hat{\psi}_{j'',j}([\mathrm{ev}_x])$ contains a point in U for all

$$x \in \bigsqcup_{i=1}^{l(j'')} (X_i^{j''} \setminus Y_i^{j''}).$$

Since $\psi_{j'',j}$ is diagonal, this means that, for $1 \le i \le l(j'')$ and $x \in X_i^{j''} \setminus Y_i^{j''}$, at least one of the points x decomposes into under $\psi_{j'',j}$ lies in U so that the matrices $\psi_{j'',j}(g)_i(x)$ and $\psi_{j'',j}(\Delta)_i(x)$ have a zero cross and a 1, respectively, at the same index along their respective diagonal. Owing to the decomposition structure of $A_{j''}$, these two results hold, in fact, for all $1 \le i \le l(j'')$ and $x \in X_i^{j''}$. Take $M := 2\mathfrak{S}(A_{j''})$, and let $N \in \mathbb{N}$ be arbitrary.

By Lemma 3.23, there is a $j' > j''$ such that

$$\mathfrak{s}(A_{j'}) > NM. \tag{18}$$

Let

$$\Delta' := \psi_{j',j}(\Delta) = \psi_{j',j''}(\psi_{j'',j}(\Delta)),$$
$$g' := \psi_{j',j}(g) = \psi_{j',j''}(\psi_{j'',j}(g)).$$

Given $1 \le i \le l(j')$ and $x \in X_i^{j'}$ and regarding Δ' as a diagonal image under $\psi_{j',j''}$, it follows from the definition of M that any M consecutive entries down the diagonal of $\Delta'_i(x)$ must contain a 1. Moreover, regarding g' and Δ' as diagonal images under $\psi_{j',j}$ shows that $g'_i(x)$ has a zero cross at index k whenever $\Delta'_i(x)_{k,k} > 0$ (as a consequence of the conclusion of Lemma 3.20) and that $\mathfrak{r}(g'_i(x)) \le \mathfrak{S}(A_j)$.

We now apply Lemma 3.22 with the natural number M, with $m = N$, $K_1 = 0$, $K_2 = M, \dots, K_N = (N-1)M$, and $F_{i,k} = \varnothing$ for $1 \le i \le l(j')$ and $1 \le k \le n_i^{j'}$ (note that $K_N < \mathfrak{s}(A_{j'}) - M$ by inequality (18)). This furnishes a function $\Theta \in A_{j'}$ with the following properties:

(I) for all $1 \le i \le l(j')$ and $x \in X_i^{j'}$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final M diagonal entries are all 0, and such that at most one of every M consecutive diagonal entries is nonzero;

(II) for all $1 \le i \le l(j')$ and $1 \le k \le n_i^{j'}$, there is a (possibly empty) open subset $U_{i,k} \subset X_i^{j'}$ containing $B_{i,k}^{j'}$ with the property that if $x \in U_{i,k}$, then $\Theta_i(x)_{k+aM,k+aM} = 1$ for all $0 \le a \le N-1$.

Fix $1 \le i \le l(j')$. Given $x \in X_i^{j'}$ and $1 \le k \le n_i^{j'} - (M-1)$, let

$$u_k^i(x) := \prod_{t=1}^{M-1} u^i_{(k\ k+t)}\big(\Theta_i(x)_{k,k}\Delta'_i(x)_{k+t,k+t}\big) \in M_{n_i^{j'}},$$

where each $u^i_{(k\ k+t)} : [0,1] \to M_{n_i^{j'}}$ is a connecting path of unitaries as described in Definition 3.5. Define $W_i \in C(X_i^{j'}, M_{n_i^{j'}})$ to be the unitary

$$W_i(x) := \prod_{k=1}^{n_i^{j'}-M} u_k^i(x).$$

Set $W := (W_1, \dots, W_{l(j')})$, and take $V := W\psi_{j',j}(w)$ and $V' := \psi_{j',j}(v)W^*$. Before showing that $W \in A_{j'}$, let us prove that statements (i) and (ii) of Lemma 3.24 hold.

Fix $x \in X_i^{j'}$. Note that if $\Theta_i(x)_{k',k'} = 0$, then $u^i_{k'}(x) = 1_{n_i^{j'}}$. Let $\{k_1 < \dots < k_s\}$ denote the set of indices r at which $\Theta_i(x)_{r,r} > 0$. Then

$$W_i(x) = u^i_{k_1}(x) \cdots u^i_{k_s}(x),$$

where, by (I) above, $k_{p+1} - k_p \ge M$ for $1 \le p < s$ and $k_s \le n_i^{j'} - M$. Note that conjugating any matrix by $u^i_{k_p}(x)$ only affects the $k_p, \dots, k_p + (M-1)$ rows and columns of that matrix. Thus, for $p \ne q$, the indices of the rows and columns affected when conjugating by $u^i_{k_p}(x)$ do not overlap with the indices of the rows and columns affected when conjugating by $u^i_{k_q}(x)$. This observation will be used to prove (i) and (ii) below.

To prove (i), fix $1 \leq k \leq n_i^{j'}$, and assume $x \in U_{i,k}$. For $p = s, s-1, \ldots, 1$, let

$$D_p := u^i_{k_p}(x) \cdots u^i_{k_s}(x) g'_i(x) u^i_{k_s}(x)^* \cdots u^i_{k_p}(x)^*$$

(setting $D_{s+1} := g'_i(x)$), and apply part (b) of Lemma 3.6 with M, $n = n_i^{j'}$, $l = k_p$, $\xi_{l+t} = \Theta_i(x)_{l,l} \Delta'_i(x)_{l+t,l+t}$ for $t = 0, 1, \ldots, M-1$, $D = D_{p+1}$, and $U = u^i_{k_p}(x)$ to conclude that D_p has a zero cross at any index among $\{1, \ldots, n_i^{j'}\} \setminus \{k_p, \ldots, k_p + M - 1\}$ whenever D_{p+1} does.

Now, fix an integer $0 \leq a \leq N-1$. Let us show that $W_i(x) g'_i(x) W_i(x)^*$ has a zero cross at index $k + aM$. By (II) above, $\Theta_i(x)_{k+aM,k+aM} = 1$. Let r denote the unique integer such that $k_r = k + aM$. Applying the result obtained just above inductively $s - r$ times, it follows that, for every $q \in \{k_r, \ldots, k_r + M - 1\}$, D_{r+1} has a zero cross at index q whenever $g'_i(x)$ does; in particular, for any such q, D_{r+1} has a zero cross at index q provided that $\Delta'_i(x)_{q,q} > 0$. Hence, since any M consecutive entries along the diagonal of $\Delta'_i(x)$ must contain a 1, the assumptions of part (c) of Lemma 3.6 are satisfied with M, $n = n_i^{j'}$, $l = k + aM$, $\xi_{l+t} = \Theta_i(x)_{l,l} \Delta'_i(x)_{l+t,l+t} = \Delta'_i(x)_{l+t,l+t}$ for $t = 0, 1, \ldots, M-1$, $D = D_{r+1}$, and $U = u^i_{k_r}(x)$. Thus, we may apply that part of the lemma to deduce that D_r has a zero cross at index $k + aM$. Appealing to the conclusion of the previous paragraph inductively $r - 1$ times, it follows that $D_1 = W_i(x) g'_i(x) W_i(x)^*$ has a zero cross at index $k + aM$ since $k + aM = k_r$ is not among the indices affected upon conjugation by $u^i_{k_1}(x) \cdots u^i_{k_{r-1}}(x)$. This proves (i).

Next, recall that g' is the diagonal image of g, which has bandwidth at most $\mathfrak{S}(A_j)$ at every point. To prove (ii), therefore, it suffices to show that, for any given matrix $D = (D_{q,t}) \in M_{n_i^{j'}}$, we have $\mathfrak{r}(W_i(x) D W_i(x)^*) \leq \mathfrak{r}(D) + M - 1$. This is most easily seen by drawing a picture and examining which rows and columns are potentially affected upon conjugation by the $u^i_{k_p}$'s:

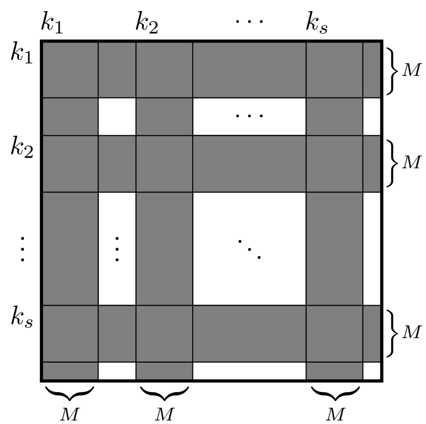

FIGURE 9. Affected rows/columns

Since the sets $\{k_p, \ldots, k_p + (M-1)\}$ for $1 \le p \le s$ are disjoint, the block rows and columns are disjoint. Suppose we are given an index (q,t) that lies in the shaded region of the diagram in Figure 9, and suppose that λ is the number at entry (q,t) of $W_i(x)DW_i(x)^*$. Upon partitioning this shaded region, it follows that the index (q,t) lies in one of the following three shaded subregions:

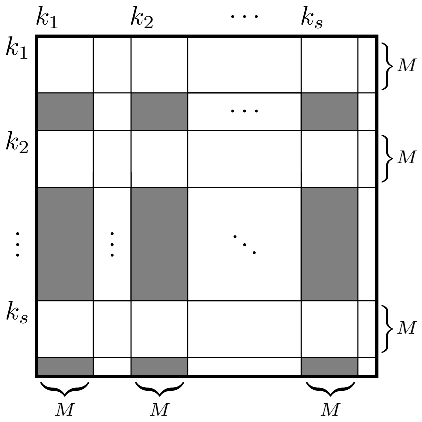

Figure 10. Region A

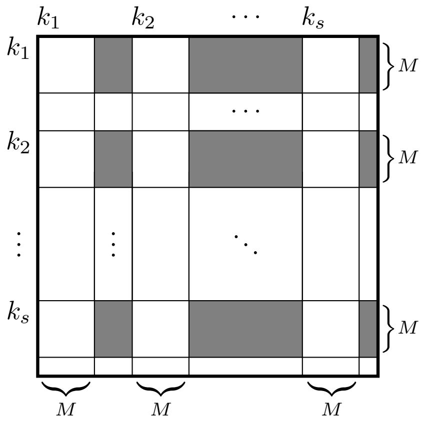

Figure 11. Region B

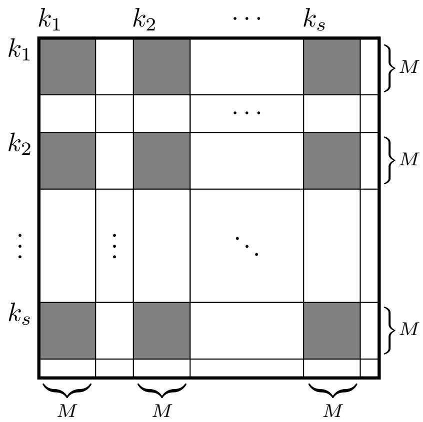

Figure 12. Region C

On Figure 10, the matrices $W_i(x)DW_i(x)^*$ and $DW_i(x)^*$ are equal. Hence, if (q,t) lies in Figure 10 and p is such that $k_p \le t \le k_p + M - 1$, then λ is a linear combination of $D_{q,k_p}, \ldots, D_{q,k_p+M-1}$. Thus, λ can be nonzero only if one of $D_{q,k_p}, \ldots, D_{q,k_p+M-1}$ is nonzero. Hence, no nonzero entry in this region is more than $M-1$ indices away from a nonzero entry in D. On Figure 11, the matrices $W_i(x)DW_i(x)^*$ and $W_i(x)D$ are equal, and so a symmetrical analysis shows that the same is true also for nonzero entries in this region. If (q,t) lies in one of the s^2 disjoint $M\times M$ blocks in Figure 12, then λ is a linear combination of the corresponding entries in D lying in that block. Hence, in this case, λ is 0 unless that $M\times M$ block in D contains a nonzero entry. Thus, no nonzero entry of $W_i(x)DW_i(x)^*$ in Figure 12 is more than $M-1$ units further away from the diagonal than a nonzero entry of D. This analysis proves that $\mathfrak{r}(W_i(x)DW_i(x)^*)\le\mathfrak{r}(D)+M-1$, yielding (ii).

To conclude, let us show that $W\in A_{j'}$. Fix $1\le i\le l(j')$, and suppose that $y\in Y_i^{j'}$ decomposes into

$$x_1\in X_{i_1}^{j'}\setminus Y_{i_1}^{j'},\dots,x_s\in X_{i_s}^{j'}\setminus Y_{i_s}^{j'}.$$

For $1\le k\le s$, let $p_k:=1+n_{i_1}^{j'}+\cdots+n_{i_{k-1}}^{j'}$. Note that, by inequality (18), $p_s\le n_i^{j'}-\mathfrak{s}(A_{j'})+1\le n_i^{j'}-M$. Thus, we may write

$$W_i(y)=\prod_{k=1}^{n_i^{j'}-M}u_k^i(y)=\prod_{m=1}^{s-1}\prod_{k=p_m}^{p_{m+1}-1}u_k^i(y)\times\prod_{k=p_s}^{n_i^{j'}-M}u_k^i(y).\tag{19}$$

Fix $1\le m<s$. Then

$$\prod_{k=p_m}^{p_{m+1}-1}u_k^i(y)=\prod_{k=p_m}^{p_{m+1}-1}\prod_{t=1}^{M-1}u^i_{(k\ k+t)}\big(\Theta_i(y)_{k,k}\Delta_i'(y)_{k+t,k+t}\big).$$

By (I), the last M entries of $\Theta_{i_m}(x_m)$ are zero. Hence, on account of the diagonal decomposition of $\Theta_i(y)$, the quantity above is equal to

$$\prod_{k=p_m}^{p_{m+1}-1-M}\prod_{t=1}^{M-1}u^i_{(k\ k+t)}\big(\Theta_{i_m}(x_m)_{k-p_m+1,k-p_m+1}\Delta'_{i_m}(x_m)_{k-p_m+1+t,k-p_m+1+t}\big),$$

which, upon relabeling indices, becomes

$$\prod_{q=1}^{p_{m+1}-p_m-M}\prod_{t=1}^{M-1}u^i_{(q+p_m-1\ q+p_m-1+t)}\big(\Theta_{i_m}(x_m)_{q,q}\Delta'_{i_m}(x_m)_{q+t,q+t}\big).\tag{20}$$

For each $1\le q\le p_{m+1}-p_m-M$ and $1\le t\le M-1$, note that

$$u^i_{(q+p_m-1\ q+p_m-1+t)}=\operatorname{diag}(1_{p_m-1},u^{i_m}_{(q\ q+t)},1_{n_i^{j'}-p_{m+1}+1}).$$

Hence, we may rewrite (20) as

$$\operatorname{diag}\Big(1_{p_m-1},\prod_{q=1}^{p_{m+1}-p_m-M}u_q^{i_m}(x_m),1_{n_i^{j'}-p_{m+1}+1}\Big)$$
$$=\operatorname{diag}\big(1_{p_m-1},W_{i_m}(x_m),1_{n_i^{j'}-p_{m+1}+1}\big).$$

Therefore, for all $1\le m<s$,

$$\prod_{k=p_m}^{p_{m+1}-1}u_k^i(y)=\operatorname{diag}\big(1_{p_m-1},W_{i_m}(x_m),1_{n_i^{j'}-p_{m+1}+1}\big)$$

and, similarly,

$$\prod_{k=p_s}^{n_i^{j'}-M} u_k^i(y) = \operatorname{diag}\bigl(1_{p_s-1}, W_{i_s}(x_s)\bigr).$$

Plugging this into equation (19) yields $W_i(y) = \operatorname{diag}(W_{i_1}(x_1), \dots, W_{i_s}(x_s))$, which proves that $W \in A$. The proof of Lemma 3.24 is now complete. □

Definition 3.25 (Block point)**.** Given a matrix $D \in M_n$ and $1 \leq k \leq n$, we say that D has a *block point at index* k provided that $D_{i,j} = 0$ if either $i \geq k$ and $j < k$ or $i < k$ and $j \geq k$.

Lemma 3.26. *Suppose A is a DSH algebra of length l. Suppose $f \in A$ and $\epsilon > 0$. Then there is a $g \in A$ with $\|g - f\| \leq \epsilon$ and with the property that, for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, there are (possible empty) open sets $\mathcal{O}_{i,k} \supset B_{i,k}$ in X_i such that $g_i(x)$ has a block point at index k whenever $x \in \mathcal{O}_{i,k}$. Moreover, g can be chosen so that, for each $1 \leq i \leq l$ and $x \in X_i$, $g_i(x)$ has a zero cross at index k whenever $f_i(x)$ does, and $\mathfrak{r}(g_i(x)) \leq \mathfrak{r}(f_i(x))$.*

Proof. Given $1 \leq i \leq l$ and $1 \leq s, t \leq n_i$, let $f_i(\,\cdot\,)_{s,t} \in C(X_i)$ denote the function taking x into $f_i(x)_{s,t}$. Let $\delta = \epsilon/\mathfrak{S}(A)^2$. Define $h \in C(\mathbb{C})$ by

$$h(z) := \frac{z}{|z|} \cdot \max(0, |z| - \delta),$$

where it is understood that $h(0) = 0$. Note that, for any $z \in \mathbb{C}$, if $|z| \leq \delta$, then $|h(z) - z| = |z| \leq \delta$, and if $|z| > \delta$, then

$$|h(z) - z| = \left|\frac{z}{|z|}(|z| - \delta) - z\right| = \left|\frac{z}{|z|}\delta\right| = \delta.$$

Thus, for all $z \in \mathbb{C}$, $|h(z) - z| \leq \delta$ and, hence, $|f_i(x)_{s,t} - h(f_i(x)_{s,t})| \leq \delta$ given any $1 \leq i \leq l$, $1 \leq s, t \leq n_i$, and $x \in X_i$. Define $g_i(x)_{s,t} := h(f_i(x)_{s,t})$, and denote by g_i the matrix-valued function in $C(X_i, M_{n_i})$ given by $(g_i(\,\cdot\,)_{s,t})_{s,t}$. Set $g := (g_1, \dots, g_l) \in \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$. For $x \in X_i$,

$$\|f_i(x) - g_i(x)\| \leq \sum_{1 \leq s,t \leq n_i} \|f_i(x)_{s,t} - g_i(x)_{s,t}\| \leq n_i^2 \delta \leq \epsilon.$$

Hence, $\|f - g\| \leq \epsilon$.

To see that $g \in A$, observe that if $y \in Y_i$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \dots, x_t \in X_{i_t} \setminus Y_{i_t}$, then $f_i(y) = \operatorname{diag}(f_{i_1}(x_1), \dots, f_{i_t}(x_t))$. Applying h to each coordinate yields that $g_i(y) = \operatorname{diag}(g_{i_1}(x_1), \dots, g_{i_t}(x_t))$. Furthermore, since $h(0) = 0$, $g_i(x)$ must have a zero cross at any index that $f_i(x)$ does, and $\mathfrak{r}(g_i(x)) \leq \mathfrak{r}(f_i(x))$.

Lastly, fix $1 \leq i \leq l$ and $1 \leq k \leq n_i$. Let us show how to construct $\mathcal{O}_{i,k}$. If $B_{i,k} = \varnothing$, take $\mathcal{O}_{i,k} := \varnothing$. Otherwise, suppose $x \in B_{i,k}$. Then $f_i(x)$ has a block point at index k. Let $I \subset \{1, \dots, n_i\}^2$ denote the set of indices (s,t) such that $s \geq k$ and $t < k$ or such that $s < k$ and $t \geq k$. Given $(s,t) \in I$, it follows that $f_i(x)_{s,t} = 0$ and, hence, that $g_i(\,\cdot\,)_{s,t}$ is 0 on an open set $U_{s,t}(x) \subset X_i$ containing x. Then $U_{s,t} := \bigcup_{x \in B_{i,k}} U_{s,t}(x)$ is an open set containing $B_{i,k}$ on which $g_i(\,\cdot\,)_{s,t}$ vanishes. Take $\mathcal{O}_{i,k} := \bigcap_{(s,t)\in I} U_{s,t}$. By construction, then $g_i(x)_{s,t} = 0$ whenever $x \in \mathcal{O}_{i,k}$ and $(s,t) \in I$. Thus, $g_i(x)$ has a block point at index k provided that $x \in \mathcal{O}_{i,k}$, which completes the proof. □

Lemma 3.27. *Suppose A is a DSH algebra of length l and that $M, N \in \mathbb{N}$ with $NM < \mathfrak{s}(A)$. Suppose f is an element of A with the property that, for all $1 \le i \le l$ and $1 \le j \le n_i$, there is a (possibly empty) open set $U_{i,k} \supset B_{i,k}$ in X_i such that if $x \in U_{i,k}$, then $f_i(x)$ has zero crosses at indices $k, k+M, \dots, k+(N-1)M$ and a block point at index k. Then there exists a unitary $V \in A$ with the following properties:*

(i) *for all $1 \le i \le l$ and $1 \le k \le n_i$, there are open sets $\mathcal{O}_{i,k} \supset B_{i,k}$ in X_i such that $V_i(x)f_i(x)V_i(x)^*$ has zero crosses at indices $k, k+1, \dots, k+N-1$ whenever $x \in \mathcal{O}_{i,k}$;*

(ii) $\mathfrak{r}(V_i(x)f_i(x)V_i(x)^*) \le \mathfrak{r}(f_i(x)) + 2$ *for all $1 \le i \le l$ and $x \in X_i$.*

Proof. Apply Lemma 3.22 with the natural number NM, the index set $K = \{0\}$, and closed sets $F_{i,k} := X_i \setminus U_{i,k}$ for $1 \le i \le l$ and $1 \le k \le n_i$ to obtain a function $\Theta \in A$ possessing the following properties:

(I) for all $1 \le i \le l$ and $x \in X_i$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final NM diagonal entries are all 0 and such that at most one of any NM consecutive diagonal entries is nonzero;

(II) for all $1 \le i \le l$ and $1 \le k \le n_i$, if $x \notin U_{i,k}$, then $\Theta_i(x)_{k,k} = 0$;

(III) for all $1 \le i \le l$ and $1 \le k \le n_i$, there is a (possibly empty) open subset $\mathcal{O}_{i,k} \subset X_i$ containing $B_{i,k}$ with the property that $\Theta_i(x)_{k,k} = 1$ whenever $x \in \mathcal{O}_{i,k}$.

Now, fix $1 \le i \le l$. For $1 \le k \le n_i - NM$, let $u_k \in C(X_i, M_{n_i})$ be the unitary

$$u_k(x) := \operatorname{diag}\big(1_{k-1}, W(\Theta_i(x)_{k,k}), 1_{n_i-(NM+k-1)}\big),$$

where W is the unitary in $C([0,1], M_{NM})$ given by Lemma 3.10 with $z_1 := 1$, $z_2 := 1+M, \dots, z_N := 1+(N-1)M$. For $n_i - NM < k \le n_i$, set $u_k \equiv 1_{n_i}$. Define $V_i \in C(X_i, M_{n_i})$ to be the unitary

$$V_i := \prod_{k=1}^{n_i} u_k.$$

For $x \in X_i$, let $K(x) := \{1 \le k \le n_i \mid \Theta_i(x)_{k,k} > 0\}$, and write $K(x) = \{k_1, \dots, k_s\}$, where $k_1 < \dots < k_s$, and put $k_{s+1} := n_i + 1$. Note that $k_1 = 1$ by (III) above since $B_{i,1} = X_i$ by Lemma 2.9, and for $1 \le t \le s$, $k_{t+1} - k_t \ge NM$ by (I) above. If $k \notin K(x)$, then $u_k \equiv 1_{n_i}$. Hence, we may write

$$V_i(x) = \prod_{t=1}^{s} u_{k_t}(x) = \operatorname{diag}\big(W(\Theta_i(x)_{k_1,k_1}), 1_{d_1}, W(\Theta_i(x)_{k_2,k_2}), 1_{d_2}, \dots, W(\Theta_i(x)_{k_s,k_s}), 1_{d_s}\big), \tag{21}$$

where $d_t := k_{t+1} - (k_t + NM)$ for $1 \le t \le s$.

Let $V := (V_1, \dots, V_l)$. In order to prove Lemma 3.27, let us show that (i) holds, then that (ii) holds, and finally that $V \in A$.

To prove (i) and (ii), fix $1 \leq i \leq l$ and $x \in X_i$, and let $K(x) = \{k_1, \dots, k_s\}$ and k_{s+1} be defined as above. For $1 \leq t \leq s$, we have $\Theta_i(x)_{k_t,k_t} > 0$. Hence, by (II) above, it must be that $x \in U_{i,k_t}$ and, thus, $f_i(x)$ has a block point at index k_t and zero crosses at indices $k_t, k_t + M, \dots, k_t + (N-1)M$ by the assumption of the lemma. Thus, $f_i(x) = \operatorname{diag}(Q_1, Q_2, \dots, Q_s)$, where Q_t is a $k_{t+1} - k_t$ block for $1 \leq t \leq s$ and has zero crosses at $1, 1+M, \dots, 1+(N-1)M$. Therefore, in light of the decomposition of $V_i(x)$ in equation (21), we may view $V_i(x)f_i(x)V_i(x)^*$ as a block-diagonal matrix $\operatorname{diag}(B_1, \dots, B_s)$ with

$$B_t = \operatorname{diag}\big(W(\Theta_i(x)_{k_t,k_t}), 1_{d_t}\big) \cdot Q_t \cdot \operatorname{diag}\big(W(\Theta_i(x)_{k_t,k_t}), 1_{d_t}\big)^*.$$

Thus, to prove (ii), it suffices to show that $\mathfrak{r}(B_t) \leq \mathfrak{r}(Q_t) + 2$. Furthermore, if $x \in \mathcal{O}_{i,k}$ for some $1 \leq k \leq n_i$, then by (III), $\Theta_i(x)_{k,k} = 1 > 0$, and so $k = k_t$ for some $1 \leq t \leq s$. Since the block B_t begins at index k_t down the diagonal of $V_i(x)f_i(x)V_i(x)^*$, to prove (i), it suffices to show that B_t has zero crosses at indices $1, 2, \dots, N$ whenever $\Theta_i(x)_{k_t,k_t} = 1$.

To this end, fix $1 \leq t \leq s$, and write

$$Q_t = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where $D_{11} \in M_{NM}$, $D_{22} \in M_{d_t}$, and D_{12} and D_{21} are $NM \times d_t$ and $d_t \times NM$ matrices, respectively. Note that D_{11} has zero crosses at indices $1, 1+M, \dots,$ $1+(N-1)M$, while the rows of D_{12} and the columns of D_{21} at these same indices consist entirely of zeros. We may write

$$\begin{aligned} B_t &= \begin{pmatrix} W(\Theta_i(x)_{k_t,k_t}) & 0_{NM\times d_t} \\ 0_{d_t\times NM} & 1_{d_t} \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} W(\Theta_i(x)_{k_t,k_t})^* & 0_{NM\times d_t} \\ 0_{d_t\times NM} & 1_{d_t} \end{pmatrix} \\ &= \begin{pmatrix} W(\Theta_i(x)_{k_t,k_t}) D_{11} W(\Theta_i(x)_{k_t,k_t})^* & W(\Theta_i(x)_{k_t,k_t}) D_{12} \\ D_{21} W(\Theta_i(x)_{k_t,k_t})^* & D_{22} \end{pmatrix}. \end{aligned}$$

If $\Theta_i(x)_{k_t,k_t} = 1$, then

$$B_t = \begin{pmatrix} W(1) D_{11} W(1)^* & W(1) D_{12} \\ D_{21} W(1)^* & D_{22} \end{pmatrix}.$$

By our definition of W,it follows by Lemma 3.10 that $W(1)D_{11}W(1)^*$ has zero crosses at indices $1, 2, \dots, N$ and that the first $1, 2, \dots, N$ rows of $W(1)D_{12}$ and columns of $D_{21}W(1)^*$ consist only of zeros. It follows that B_t has zero crosses at indices $1, 2, \dots, N$, which, based on the aforementioned analysis, proves (i).

Let us now prove (ii) by showing that $\mathfrak{r}(B_t) \leq \mathfrak{r}(Q_t) + 2$. By our definition of W, we may apply Lemma 3.10 to obtain the following estimates:

$$\begin{gathered} \mathfrak{r}\big(W(\Theta_i(x)_{k_t,k_t}) D_{11} W(\Theta_i(x)_{k_t,k_t})^*\big) \leq \mathfrak{r}(D_{11}) + 2, \\ \mathfrak{r}\left(\begin{pmatrix} 0_{NM\times NM} & W(\Theta_i(x)_{k_t,k_t}) D_{12} \\ 0_{d_t\times NM} & 0_{d_t\times d_t} \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0_{NM\times NM} & D_{12} \\ 0_{d_t\times NM} & 0_{d_t\times d_t} \end{pmatrix}\right), \\ \mathfrak{r}\left(\begin{pmatrix} 0_{NM\times NM} & 0_{NM\times d_t} \\ D_{21} W(\Theta_i(x)_{k_t,k_t})^* & 0_{d_t\times d_t} \end{pmatrix}\right) \leq \mathfrak{r}\left(\begin{pmatrix} 0_{NM\times NM} & 0_{NM\times d_t} \\ D_{21} & 0_{d_t\times d_t} \end{pmatrix}\right). \end{gathered}$$

Combining these estimates gives

$$\begin{aligned}\mathfrak{r}(B_t) &= \mathfrak{r}\Bigg(\begin{pmatrix} W(\Theta_i(x)_{k_t,k_t})D_{11}W(\Theta_i(x)_{k_t,k_t})^* & W(\Theta_i(x)_{k_t,k_t})D_{12} \\ D_{21}W(\Theta_i(x)_{k_t,k_t})^* & D_{22}\end{pmatrix}\Bigg)\\ &\leq \mathfrak{r}(Q_t)+2,\end{aligned}$$

which proves (ii).

Finally, let us verify that $V \in A$. Suppose $y \in Y_i$ decomposes into $x_1 \in X_{i_1} \setminus Y_{i_1}, \dots, x_r \in X_{i_r} \setminus Y_{i_r}$. We have to show that $V_i(y) = \mathrm{diag}(V_{i_1}(x_1), \dots, V_{i_r}(x_r))$. Let $K(y) = \{1 \leq k \leq n_i \mid \Theta_i(y)_{k,k} > 0\}$, as defined above. Write $K(y) = \{k_1, \dots, k_s\}$, where $1 = k_1 < \cdots < k_s$, and put $k_{s+1} := n_i + 1$. As before, let $d_t := k_{t+1} - (k_t + NM)$ for $1 \leq t \leq s$. Define $B(y) := \{1 \leq k \leq n_i \mid y \in B_{i,k}\}$. By (III) above, $B(y) \subset K(y)$. Hence, by Lemma 2.9, for each $1 \leq j \leq r$, there is a $t_j \in \{1, \dots, s\}$ such that $1 + n_{i_1} + \cdots + n_{i_{j-1}} = k_{t_j}$ (where $k_{t_1} = 1 = k_1$ so that $t_1 = 1$); set $t_{r+1} := s + 1$ so that $k_{t_{r+1}} = k_{s+1} = n_i + 1$.

Now, fix $1 \leq j \leq r$, and observe that $\Theta_{i_j}(x_j)_{k,k} = \Theta_i(y)_{k_{t_j}+k-1,k_{t_j}+k-1}$. Therefore,

$$\begin{aligned}K(x_j) &= \{1 \leq k \leq n_{i_j} \mid \Theta_{i_j}(x_j)_{k,k} > 0\}\\ &= \{k - k_{t_j} + 1 \mid k \in K(y) \text{ and } k_{t_j} \leq k < k_{t_{j+1}}\}\\ &= \{k_t - k_{t_j} + 1 \mid t_j \leq t < t_{j+1}\}.\end{aligned}$$

Moreover, if $t_j \leq t < t_{j+1}$, then $(k_{t+1} - k_{t_j} + 1) - (k_t - k_{t_j} + 1 + NM) = d_t$. Given matrices $E_1, \dots, E_p$, let $\bigoplus_{q=1}^p E_q := \mathrm{diag}(E_1, \dots, E_p)$. Then, by the computation of $K(x_j)$ above and equation (21), it follows that

$$\begin{aligned}V_{i_j}(x_j) &= \bigoplus_{t_j \leq t < t_{j+1}} \mathrm{diag}\big(W(\Theta_{i_j}(x_j)_{k_t-k_{t_j}+1,k_t-k_{t_j}+1}),\\ &\qquad\qquad 1_{(k_{t+1}-k_{t_j}+1)-(k_t-k_{t_j}+1+NM)}\big)\\ &= \bigoplus_{t_j \leq t < t_{j+1}} \mathrm{diag}\big(W(\Theta_i(y)_{k_t,k_t}), 1_{d_t}\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{diag}\big(V_{i_1}(x_1), \dots, V_{i_r}(x_r)\big) &= \bigoplus_{1 \leq j \leq r} \bigoplus_{t_j \leq t < t_{j+1}} \mathrm{diag}\big(W(\Theta_i(y)_{k_t,k_t}), 1_{d_t}\big)\\ &= \bigoplus_{1 \leq t \leq s} \mathrm{diag}\big(W(\Theta_i(y)_{k_t,k_t}), 1_{d_t}\big) = V_i(y),\end{aligned}$$

where the last equality follows by equation (21). This shows that $V \in A$. The proof of Lemma 3.27 is now complete. □

Lemma 3.28. *Suppose A is a DSH algebra of length l and that $1 \leq N < \mathfrak{s}(A)$. Suppose $f \in A$ is such that, for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, there is an open subset $U_{i,k} \subset X_i$ containing $B_{i,k}$ with the property that if $x \in U_{i,k}$, then $f_i(x)$ has zero crosses at indices $k, k+1, \dots, k+N-1$, and such that $\mathfrak{r}(f_i(x)) \leq N$ for all $x \in X_i$. Then there is a unitary $V \in A$ such that, for all $1 \leq i \leq l$ and $x \in X_i$, the matrix $(fV)_i(x)$ is strictly lower triangular.*

Proof. Apply Lemma 3.22 with the natural number N, the index set $K = \{0\}$, and the closed sets $F_{i,k} := X_i \setminus U_{i,k}$ for $1 \leq i \leq l$ and $1 \leq k \leq n_i$. This yields a function $\Theta \in A$ with the following properties:

(I) for all $1 \leq i \leq l$ and $x \in X_i$, $\Theta_i(x)$ is a diagonal matrix with entries in $[0,1]$ whose final N entries are all 0 and such that at most one of every N consecutive diagonal entries is nonzero;

(II) for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, if $x \notin U_{i,k}$, then $\Theta_i(x)_{k,k} = 0$;

(III) for all $1 \leq i \leq l$ and $1 \leq k \leq n_i$, if $x \in B_{i,k}$, then $\Theta_i(x)_{k,k} = 1$.

Since $N < \mathfrak{s}(A)$, we may, for each $1 \leq i \leq l$, define $W_{n_i} \in C([0,1]^{n_i}, M_{n_i})$ as in Definition 3.16. For $1 \leq i \leq l$ and $x \in X_i$, define the unitary $V_i \in C(X_i, M_{n_i})$ by

$$V_i(x) := W_{n_i}(\Theta_i(x)_{1,1}, \ldots, \Theta_i(x)_{n_i,n_i}),$$

and set $V := (V_1, \ldots, V_\ell)$. Let us first argue that $(fV)_i(x)$ is strictly lower triangular for all $1 \leq i \leq l$ and $x \in X_i$, and then show that $V \in A$.

Fix $1 \leq i \leq l$ and $x \in X_i$. From equation (9), we have

$$\begin{aligned} (fV)_i(x) &= f_i(x) W_{n_i}(\Theta_i(x)_{1,1}, \ldots, \Theta_i(x)_{n_i,n_i}) \\ &= f_i(x) U[\gamma^{n_i}_{1,n_i}]^N \prod_{k=N}^{n_i-1} u^{n_i}_{k,n_i}(\Theta_i(x)_{k+1,k+1}). \end{aligned} \tag{22}$$

If we write $f_i(x) = [C_1 \mid \cdots \mid C_{n_i}]$, where C_j is the jth column of $f_i(x)$, then $f_i(x)U[\gamma^{n_i}_{1,n_i}]^N = [C_{N+1} \mid \cdots \mid C_{n_i} \mid C_1 \mid \cdots \mid C_N]$. By the assumption of the lemma, $\mathfrak{r}(f_i(x)) \leq N$. Hence, all nonzero entries in the first $n_i - N$ columns of the matrix $f_i(x)U[\gamma^{n_i}_{1,n_i}]^N$ must lie strictly below the diagonal. But by Lemma 2.9, $x \in B_{i,1}$ and, hence, by the assumptions of the lemma, $f_i(x)$ has zero crosses at indices $1, \ldots, N$. In particular, the columns $C_1, \ldots, C_N$ consist entirely of zeros. Therefore, $f_i(x)U[\gamma^{n_i}_{1,n_i}]^N$ is strictly lower triangular. To show that $(fV)_i(x)$ is strictly lower triangular, we thus only need to verify that $(fV)_i(x) = f_i(x)U[\gamma^{n_i}_{1,n_i}]^N$. To do this, it is enough, by equation (22), to check that, for each integer $N \leq k \leq n_i - 1$,

$$f_i(x)U[\gamma^{n_i}_{1,n_i}]^N u^{n_i}_{k,n_i}(\Theta_i(x)_{k+1,k+1}) = f_i(x)U[\gamma^{n_i}_{1,n_i}]^N. \tag{23}$$

To this end, fix $N \leq k \leq n_i - 1$. If $\Theta_i(x)_{k+1,k+1} = 0$, then there is nothing to show, since $u^{n_i}_{k,n_i}(\Theta_i(x)_{k+1,k+1}) = 1_{n_i}$ in this case. So we may assume $\Theta_i(x)_{k+1,k+1} > 0$. Then, by (II) above, necessarily, $x \in U_{i,k+1}$. Hence, by the assumption of the lemma, $f_i(x)$ has zero crosses at indices $k+1, \ldots, k+N$, from which it follows that the columns $C_{k+1}, \ldots, C_{k+N}$ consist entirely of zeros. As noted above, these correspond to the columns of $f_i(x)U[\gamma^{n_i}_{1,n_i}]^N$ at indices $k+1-N, \ldots, k$. By Definition 3.11, the columns of

$$f_i(x)U[\gamma^{n_i}_{1,n_i}]^N u^{n_i}_{k,n_i}(\Theta_i(x)_{k+1,k+1})$$

at indices $k-N+1, \ldots, k$ and $n_i - N + 1, \ldots, n_i$ are linear combinations of the same set of columns of $f_i(x)U[\gamma^{n_i}_{1,n_i}]^N$ (*i.e.*, of $C_{k+1}, \ldots, C_{k+N}, C_1, \ldots, C_N$). But every column in this latter set consists entirely of zeros. Hence, since multiplying by $u^{n_i}_{k,n_i}(\Theta_i(x)_{k+1,k+1})$ on the right only alters columns at indices $k-N+1,\dots,k$ and $n_i-N+1,\dots,n_i$, equation (23) holds.

Let us now ensure that $V\in A$. Suppose that $2\le i\le l$ and that $y\in Y_i$ decomposes into $x_1\in X_{i_1}\setminus Y_{i_1},\dots,x_s\in X_{i_s}\setminus Y_{i_s}$. Then

$$\begin{aligned}(\Theta_i(y)_{1,1},\dots,\Theta_i(y)_{n_i,n_i}) = (&\Theta_{i_1}(x_1)_{1,1},\dots,\Theta_{i_1}(x_1)_{n_{i_1},n_{i_1}},\dots,\\ &\Theta_{i_s}(x_s)_{1,1},\dots,\Theta_{i_s}(x_s)_{n_{i_s},n_{i_s}}).\end{aligned}$$

Define $B(y):=\{1\le k\le n_i \mid y\in B_{i,k}\}$. By Lemma 2.9, $B(y)=\{1=k_1<\cdots<k_s\}$, where $k_t=1+n_{i_1}+\cdots+n_{i_{t-1}}$ for $1\le t\le s$. Set $k_{s+1}:=n_i+1$, and note that $k_{t+1}-k_t=n_{i_t}$ for all $1\le t\le s$. By assumption, $n_i\ge\mathfrak{s}(A)>N$, and so, in light of (I) and (III) above, we may apply Lemma 3.17 with the vector $(\Theta_i(y)_{1,1},\dots,\Theta_i(y)_{n_i,n_i})$ and the set $B(y)$ to obtain

$$\begin{aligned}V_i(y) &= W_{n_i}(\Theta_i(y)_{1,1},\dots,\Theta_i(y)_{n_i,n_i})\\ &= \bigoplus_{j=1}^{s} W_{n_{i_j}}(\Theta_{i_j}(x_j)_{1,1},\dots,\Theta_{i_j}(x_j)_{n_{i_j},n_{i_j}})\\ &= \operatorname{diag}\big(V_{i_1}(x_1),\dots,V_{i_s}(x_s)\big),\end{aligned}$$

where $\bigoplus_{j=1}^{s}E_j:=\operatorname{diag}(E_1,\dots,E_s)$. Therefore, $V\in A$. This completes the proof of Lemma 3.28. □

3.29. **Proof of the main theorem.**

Theorem 3.30. *Suppose*

$$A_1\xrightarrow{\psi_1}A_2\xrightarrow{\psi_2}A_3\xrightarrow{\psi_3}\cdots\longrightarrow A:=\varinjlim A_i$$

is a simple inductive limit of DSH algebras with diagonal bonding maps. Then A has stable rank one.

Proof. For $n\in\mathbb{N}$, let $\mu_n:A_n\to A$ denote the map in the construction of the inductive limit, which is unital (since the bonding maps are) and, by Proposition 2.19, we may assume, injective. Furthermore, we may assume that the A_j's are infinite-dimensional.

Fix $\epsilon>0$ and $a\in A$. Our goal is to find an invertible element $a'\in A$ with $\|a-a'\|\le\epsilon$. To start, choose $j\in\mathbb{N}$ and $f\in A_j$ such that $\|a-\mu_j(f)\|\le\epsilon/4$. If f is invertible in A_j, then $\mu_j(f)$ is invertible in A, in which case we are finished. Thus, we may assume that f is not invertible in A_j.

Since A_j is infinite-dimensional, we may apply Lemma 3.24 with f, $\epsilon/4$, and $N=\mathfrak{S}(A_j)+M+1$, where M is the natural number depending on f and $\epsilon/4$, coming from the statement of Lemma 3.24. This yields a function $f'\in A_j$ with $\|f-f'\|\le\epsilon/4$, a $j'>j$ such that $\mathfrak{s}(A_{j'})>NM$, and unitaries $V,V'\in A_{j'}$ with the following two properties (we adopt the same notation for the decomposition of $A_{j'}$ introduced just above Lemma 3.24):

(i) for any $1\le i\le l$ and $1\le k\le n_i$, there is a (possibly empty) open subset $U_{i,k}$ of X_i containing $B_{i,k}$ such that $(V\psi_{j',j}(f')V')_i(x)$ has zero crosses at indices $k,k+M,k+2M,\dots,k+(N-1)M$;

(ii) for all $1 \le i \le l$ and $x \in X_i$, we have $\mathfrak{r}((V\psi_{j',j}(f')V')_i(x)) \le \mathfrak{S}(A_j) + M - 1$.

Let $f'' := V\psi_{j',j}(f')V' \in A_{j'}$.

Next, apply Lemma 3.26 with $A_{j'}$, f'', and $\epsilon/4$. This yields a function $g \in A_{j'}$ with $\|g - f''\| \le \epsilon/4$ and, for $1 \le i \le l$ and $1 \le k \le n_i$, open sets $\mathcal{O}_{i,k} \subset X_i$ containing $B_{i,k}$ on which g_i always has a block point at index k; moreover, for all $1 \le i \le l$ and $x \in X_i$, the matrix $g_i(x)$ has a zero cross at every index that $f''_i(x)$ does, and $\mathfrak{r}(g_i(x)) \le \mathfrak{r}(f''_i(x)) \le \mathfrak{S}(A_j) + M - 1$. Thus, intersecting the $\mathcal{O}_{i,k}$'s with the $U_{i,k}$'s, we may assume that $g_i(x)$ has zero crosses at indices $k, k+M, \dots, k+(N-1)M$ whenever $x \in \mathcal{O}_{i,k}$.

Since $\mathfrak{s}(A_{j'}) > NM$, we may now apply Lemma 3.27 on $A_{j'}$ with g and the $\mathcal{O}_{i,k}$'s above to obtain a unitary $W \in A_{j'}$ with the following properties:

(I) for all $1 \le i \le l$ and $1 \le k \le n_i$, there are open sets $\mathcal{O}'_{i,k} \supset B_{i,k}$ in X_i such that $W_i(x)g_i(x)W_i(x)^*$ has zero crosses at indices $k, k+1, \dots, k+N-1$ whenever $x \in \mathcal{O}'_{i,k}$;

(II) $\mathfrak{r}(W_i(x)g_i(x)W_i(x)^*) \le \mathfrak{r}(g_i(x)) + 2 \le \mathfrak{S}(A_j) + M + 1 = N$ for all $1 \le i \le l$ and $x \in X_i$.

Let $g' := WgW^* \in A_{j'}$.

Using these properties and the fact that $\mathfrak{s}(A_{j'}) > NM \ge N$, we may apply Lemma 3.28 on $A_{j'}$ with g' and the $\mathcal{O}'_{i,k}$'s to conclude that there is a unitary $W' \in A_{j'}$ such that, for all $1 \le i \le l$ and $x \in X_i$, the matrix $(g'W')_i(x)$ is strictly lower triangular. Thus, $g'W'$ is a nilpotent element. As observed in [21, Sec. 4], every nilpotent element of a unital C*-algebra is arbitrarily close to an invertible element. Thus, there is an invertible element $h \in A_{j'}$ such that $\|g'W' - h\| \le \epsilon/4$.

Take $a' := \mu_{j'}(V^*W^*h(W')^*W(V')^*)$, and observe that a' is invertible in A. Then, since the μ_n's are injective,

$$\begin{aligned}
\|\mu_j(f') - a'\| &= \|\psi_{j',j}(f') - V^*W^*h(W')^*W(V')^*\| \\
&= \|V^*W^*[WV\psi_{j',j}(f')V'W^*W' - h](W')^*W(V')^*\| \\
&\le \|V^*W^*\|\|Wf''W^*W' - h\|\|(W')^*W(V')^*\| \\
&= \|Wf''W^*W' - h\| \\
&\le \|Wf''W^*W' - WgW^*W'\| + \|WgW^*W' - h\| \\
&\le \|W\|\|f'' - g\|\|W^*W'\| + \|g'W' - h\| \\
&\le \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},
\end{aligned}$$

and

$$\|a - \mu_j(f')\| \le \|a - \mu_j(f)\| + \|\mu_j(f) - \mu_j(f')\| \le \frac{\epsilon}{4} + \|f - f'\| \le \frac{\epsilon}{2}.$$

Therefore,

$$\|a - a'\| \le \|a - \mu_j(f')\| + \|\mu_j(f') - a'\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. □

3.31. **Applications to dynamical crossed products.** Let T be an infinite compact metric space, and let $h : T \to T$ be a minimal homeomorphism. In the final portion of this paper, we present two applications of Theorem 3.30, both concerning the dynamical crossed product $A := \mathrm{C}^*(\mathbb{Z}, T, h)$. We show that A has stable rank one (see Corollary 3.36), thereby affirming a conjecture of Archey, Niu, and Phillips (see [3, Conj. 7.2]). We also apply a result of Thiel from [23] to conclude that classification for A is determined by strict comparison (see Corollary 3.37), which establishes the Toms–Winter Conjecture for minimal dynamical crossed products.

The Toms–Winter Conjecture dates back to 2008 and stipulates that, for separable, unital, simple, non-elementary, nuclear C*-algebras, three different notions of regularity are equivalent. Below is a precise statement of the conjecture.

Conjecture 3.32 (Toms–Winter [9, 27]). *Let A be a separable, unital, simple, non-elementary, nuclear* C*-*algebra. The following statements are equivalent.*
(i) *A has finite nuclear dimension.*
(ii) *A is $\mathcal{Z}$-stable; that is, $A \otimes \mathcal{Z} \cong A$.*
(iii) *A has strict comparison of positive elements.*

At the time, part of this conjecture had already been established by Rørdam, who, in [22, Thm. 4.5 and Cor. 4.6], proved that (ii) implies (iii). Winter showed in [26, Cor. 7.3] that (i) implies (ii). The work of various hands established that (ii) implies (i) in special cases, but very recently, in [4, Thm. A], this implication was shown to hold in full generality. Therefore, to establish the conjecture for a given C*-algebra, one needs only to check that strict comparison of positive elements yields $\mathcal{Z}$-stability.

Let $\sigma : C(T) \to C(T)$ denote the automorphism arising from h given by $\sigma(f) := f \circ h^{-1}$. Let u denote the unitary in the associated crossed product A implementing the σ action, *i.e.*, $ufu^* = \sigma(f)$ for all $f \in C(T)$. Then A is the C*-algebra generated by $C(T)$ and u. Given a closed set $S \subset T$ with nonempty interior, let A_S denote the orbit-breaking sub-C*-algebra of A associated to S, first introduced by Putnam in [19] for Cantor minimal systems and later by Q. Lin and Phillips for more general minimal systems (see [13, 14, 15]); that is, A_S is the C*-algebra generated by $\{f, ug \mid f \in C(T), g \in C_0(T \setminus S)\}$, where we adopt the shorthand $C_0(T \setminus S) := \{g \in C(T) \mid g|_S \equiv 0\}$. In [13, 14, 15], Q. Lin and Phillips showed that A_S is a recursive subhomogeneous algebra, and in fact a DSH algebra. We outline this below. For a more in-depth discussion, see [14, Thms. 3.1–3.3].

Given $s \in S$, let $\lambda_S(s) := \min\{n > 0 \mid h^n(s) \in S\}$ (the first return time of s to S). Since T is compact, it follows that $\sup_{s\in S} \lambda_S(s)$ is finite (see also [15, Lem. 2.2]). Thus, there exist $1 \le n_1^S < n_2^S < \cdots < n_{l(S)}^S$ such that $\{\lambda_S(s) \mid s \in S\} = \{n_i^S \mid 1 \le i \le l(S)\}$. For $1 \le i \le l(S)$, let

$$X_i^S := \overline{\lambda_S^{-1}(n_i^S)} \quad \text{and} \quad Y_i^S := X_i^S \setminus \lambda_S^{-1}(n_i^S).$$

Then, for given $1 \leq i \leq l(S)$ and $y \in Y_i^S$, there are indices $1 \leq i_1, \dots, i_p < i$ with $y \in X_{i_1}^S \setminus Y_{i_1}^S$ such that $n_{i_1}^S + \cdots + n_{i_p}^S = n_i^S$ and such that $h^k(y) \in S$ if and only if $k = n_{i_1}^S + \cdots + n_{i_j}^S$ for some $1 \leq j \leq p$. Note, too, that

$$h^{n_{i_1}^S + \cdots + n_{i_{j-1}}^S}(y) \in X_{i_j}^S \setminus Y_{i_j}^S \quad \text{for all } 2 \leq j \leq p.$$

Then A_S is isomorphic to a sub-C*-algebra of $\bigoplus_{i=1}^{l(S)} C(X_i^S, M_{n_i^S})$, where an element $(f_1, \dots, f_{l(S)})$ of $\bigoplus_{i=1}^{l(S)} C(X_i^S, M_{n_i^S})$ is in A_S if and only if, for given $1 \leq i \leq l(S)$ and $y \in Y_i^S$,

$$f_i(y) = \operatorname{diag}\big(f_{i_1}(y), f_{i_2}(h^{n_{i_1}^S}(y)), \dots, f_{i_p}(h^{n_{i_1}^S + \cdots + n_{i_{p-1}}^S}(y))\big),$$

where $i_1, \dots, i_p$ are as described above. It follows that A_S is a DSH algebra.

Lemma 3.33. *Suppose $R \subset S \subset T$. Let $1 \leq i \leq l(R)$ and $x \in X_i^R \setminus Y_i^R = \lambda_R^{-1}(n_i^R)$. Then there are indices $1 \leq i_1, \dots, i_q \leq l(S)$ such that*

(i) $n_{i_1}^S + \cdots + n_{i_q}^S = n_i^R$;

(ii) *for all $1 \leq k \leq n_i^R$, $h^k(x) \in S$ if and only if $k = n_{i_1}^S + \cdots + n_{i_j}^S$ for some $1 \leq j \leq q$;*

(iii) *$x \in X_{i_1}^S \setminus Y_{i_1}^S$ and, for all $2 \leq j \leq q$, $h^{n_{i_1}^S + \cdots + n_{i_{j-1}}^S}(x) \in X_{i_j}^S \setminus Y_{i_j}^S$.*

Proof. Since $R \subset S$ and the sets $X_j^S \setminus Y_j^S$ for $1 \leq j \leq l(S)$ partition S, there is a unique $1 \leq i_1 \leq l(S)$ such that $x \in X_{i_1}^S \setminus Y_{i_1}^S$. Moreover, $n_{i_1}^S = \lambda_S(x) \leq \lambda_R(x) = n_i^R$. If $n_{i_1}^S = n_i^R$, then there is nothing to show. Otherwise, there is an i_2 such that $h^{n_{i_1}^S}(x) \in X_{i_2}^S \setminus Y_{i_2}^S$. Note that $n_{i_2}^S = \lambda_S(h^{n_{i_1}^S}(x)) \leq n_i^R - n_{i_1}^S$. If $n_{i_2}^S = n_i^R - n_{i_1}^S$, the desired result follows. Otherwise, we let i_3 be such that $h^{n_{i_1}^S + n_{i_2}^S}(x) \in X_{i_3}^S \setminus Y_{i_3}^S$ and proceed as before. Eventually, this process terminates (when $n_{i_1}^S + \cdots + n_{i_q}^S = n_i^R$) and yields indices with the desired properties. This proves the lemma. $\square$

By [15, Prop. 2.4], there is a unique homomorphism

$$\gamma_S : A_S \to \bigoplus_{i=1}^{l(S)} C(X_i^S, M_{n_i^S})$$

with the property that, for $f \in C(T)$ and $g \in C_0(T \setminus S)$,

$$\gamma_S(f)_i = \operatorname{diag}\big(f \circ h|_{X_i^S}, f \circ h^2|_{X_i^S}, \dots, f \circ h^{n_i^S}|_{X_i^S}\big) \tag{24}$$

and

$$\gamma_S(ug)_i = \begin{pmatrix} 0 & & & & \\ g \circ h|_{X_i^S} & 0 & & & \\ & g \circ h^2|_{X_i^S} & \ddots & & \\ & & \ddots & 0 & \\ & & & g \circ h^{n_i^S - 1}|_{X_i^S} & 0 \end{pmatrix} \tag{25}$$

for $1 \leq i \leq l(S)$.

Now, fix $R \subset S \subset T$. By examining the generating sets, it follows that A_S is contained in A_R. Let $\psi : A_S \to A_R$ denote the inclusion map.

Lemma 3.34. *ψ is a diagonal map (see Definition 2.4) between DSH algebras.*

Proof. Fix $1 \le i \le l(R)$ and $x \in X_i^R \setminus Y_i^R = \lambda_R^{-1}(n_i^R)$. By Lemma 3.33, there are indices $1 \le i_1, \dots, i_q \le l(S)$ such that
(i) $n_{i_1}^S + \dots + n_{i_q}^S = n_i^R$;
(ii) for all $1 \le k \le n_i^R$, $h^k(x) \in S$ if and only if $k = \beta_j$ for some $1 \le j \le q$, where $\beta_j := n_{i_1}^S + \dots + n_{i_j}^S$;
(iii) for all $1 \le j \le q$, $h^{\beta_{j-1}}(x) \in X_{i_j}^S \setminus Y_{i_j}^S$ (here $\beta_0 := 0$ so that $x \in X_{i_1}^S \setminus Y_{i_1}^S$).
Let us show that x decomposes into $h^{\beta_0}(x) = x, h^{\beta_1}(x), \dots, h^{\beta_{q-1}}(x)$ under ψ.

Suppose that $f \in C(T)$. Let us begin by verifying that

$$\gamma_R(\psi(f))_i(x) = \operatorname{diag}\big(\gamma_S(f)_{i_1}(x), \gamma_S(f)_{i_2}(h^{\beta_1}(x)), \dots, \gamma_S(f)_{i_q}(h^{\beta_{q-1}}(x))\big). \tag{26}$$

Fix $1 \le j \le q$. By equation (24),

$$\begin{aligned}\gamma_S(f)_{i_j}(h^{\beta_{j-1}}(x)) &= \operatorname{diag}\big(f(h(h^{\beta_{j-1}}(x))), \dots, f(h^{n_{i_j}^S}(h^{\beta_{j-1}}(x)))\big)\\ &= \operatorname{diag}\big(f(h^{\beta_{j-1}+1}(x)), \dots, f(h^{\beta_j}(x))\big).\end{aligned}$$

Hence,

$$\begin{aligned}&\operatorname{diag}\big(\gamma_S(f)_{i_1}(x), \gamma_S(f)_{i_2}(h^{\beta_1}(x)), \dots, \gamma_S(f)_{i_q}(h^{\beta_{q-1}}(x))\big)\\ &\quad = \operatorname{diag}\big(f(h^{\beta_0+1}(x)), \dots, f(h^{\beta_1}(x)), \dots, f(h^{\beta_{q-1}+1}(x)), \dots, f(h^{\beta_q}(x))\big)\\ &\quad = \operatorname{diag}\big(f(h(x)), \dots, f(h^{n_i^R}(x))\big)\\ &\quad = \gamma_R(f)_i(x),\end{aligned}$$

which yields equation (26).

Next, suppose $g \in C_0(T \setminus S)$. Let us show that

$$\gamma_R(\psi(ug))_i(x) = \operatorname{diag}\big(\gamma_S(ug)_{i_1}(x), \gamma_S(ug)_{i_2}(h^{\beta_1}(x)), \dots, \gamma_S(ug)_{i_q}(h^{\beta_{q-1}}(x))\big). \tag{27}$$

By equation (25),

$$\gamma_R(ug)_i(x) = \begin{pmatrix} 0 & & & & \\ g(h(x)) & 0 & & & \\ & g(h^2(x)) & \ddots & & \\ & & \ddots & 0 & \\ & & & g(h^{n_i^R-1}(x)) & 0 \end{pmatrix}.$$

For $1 \le j \le q$, we have $h^{\beta_j}(x) \in S$ by property (ii) above so that $g(h^{\beta_j}(x)) = 0$. Hence, partitioning $\{1, 2, \dots, n_i^R\}$ into the sets $\{\beta_{j-1}+1, \dots, \beta_j\}$ for $1 \le j \le q$, we may view $\gamma_R(ug)_i(x)$ as a block-diagonal matrix $\operatorname{diag}(B_1,\dots,B_q)$, where

$$\begin{aligned} B_j &:= \begin{pmatrix} 0 & & & & \\ g(h^{\beta_{j-1}+1}(x)) & 0 & & & \\ & g(h^{\beta_{j-1}+2}(x)) & \ddots & & \\ & & \ddots & 0 & \\ & & & g(h^{\beta_j-1}(x)) & 0 \end{pmatrix} \\ &= \gamma_S(ug)_{i_j}(h^{\beta_{j-1}}(x)), \end{aligned}$$

which yields equation (27).

We have shown that x decomposes into $h^{\beta_0}(x)=x, h^{\beta_1}(x),\dots,h^{\beta_{q-1}}(x)$ under ψ on the generators of A_S. Let us now use continuity to prove that this decomposition is maintained for all elements of A_S. Let $a\in A_S$ be arbitrary. By definition, we may write $a=\lim_{n\to\infty} w_n$, where for each $n\in\mathbb{N}$, w_n is a word in $C(T)\cup uC_0(T\setminus S)\cup C_0(T\setminus S)u^*$. By equations (26) and (27),

$$\begin{aligned} \gamma_R(\psi(w_n))_i(x) = \operatorname{diag}\big(&\gamma_S(w_n)_{i_1}(x), \gamma_S(w_n)_{i_2}(h^{\beta_1}(x)),\dots, \\ &\gamma_S(w_n)_{i_q}(h^{\beta_{q-1}}(x))\big) \end{aligned}$$

for all $n\in\mathbb{N}$. Hence, by continuity of *-homomorphisms,

$$\begin{aligned} \gamma_R(\psi(a))_i(x) &= \lim_{n\to\infty} \gamma_R(\psi(w_n))_i(x) \\ &= \lim_{n\to\infty} \operatorname{diag}\big(\gamma_S(w_n)_{i_1}(x), \gamma_S(w_n)_{i_2}(h^{\beta_1}(x)),\dots, \\ &\qquad\qquad \gamma_S(w_n)_{i_q}(h^{\beta_{q-1}}(x))\big) \\ &= \operatorname{diag}\big(\gamma_S(a)_{i_1}(x), \gamma_S(a)_{i_2}(h^{\beta_1}(x)),\dots,\gamma_S(a)_{i_q}(h^{\beta_{q-1}}(x))\big), \end{aligned}$$

which yields the desired diagonal decomposition and completes the proof of Lemma 3.34. □

Theorem 3.35. *Let T be an infinite compact metric space, and let $h:T\to T$ be a minimal homeomorphism. Given a non-isolated point $x\in T$, the orbit-breaking subalgebra $A_{\{x\}}$ of $A:=\mathrm{C}^*(\mathbb{Z},T,h)$ is a simple inductive limit of DSH algebras with diagonal maps. In particular, $A_{\{x\}}$ has stable rank one.*

Proof. Choose a sequence $S_1\supset S_2\supset\cdots$ of closed sets with nonempty interior such that $\bigcap_{n=1}^\infty S_n=\{x\}$. For each $n\in\mathbb{N}$, let $A_{S_n}\subset A$ denote the subalgebra as described above, and let $\psi_n: A_{S_n}\to A_{S_{n+1}}$ denote the canonical inclusion. Since $\overline{\bigcup_{n=1}^\infty A_{S_n}}=A_{\{x\}}$, it follows by Lemma 3.34 that $A_{\{x\}}$ is an inductive limit of DSH algebras with diagonal maps. Moreover, by [14, Thm. 1.2], $A_{\{x\}}$ is simple (see [16, Prop. 2.5] for a proof). Therefore, by Theorem 3.30, $A_{\{x\}}$ has stable rank one. □

Corollary 3.36 (cp. [3, Conj. 7.2])**.** *Let T be an infinite compact metric space, and let $h:T\to T$ be a minimal homeomorphism. The dynamical crossed product $A:=\mathrm{C}^*(\mathbb{Z},T,h)$ has stable rank one.*

Proof. Let x be any non-isolated point in T. By Theorem 3.35, $A_{\{x\}}$ has stable rank one. Since h is minimal and T is infinite, $h^n(x) \neq x$ for all $n \in \mathbb{N}$. Thus, on combining [18, Thm. 7.10] with [3, Thm. 4.6], it follows that $A_{\{x\}}$ is a centrally large subalgebra of A. But by [3, Thm. 6.3], any infinite-dimensional unital simple separable C*-algebra containing a centrally large subalgebra with stable rank one must itself have stable rank one. □

Corollary 3.37. *Let T be an infinite compact metric space, and let $h : T \to T$ be a minimal homeomorphism. The dynamical crossed product $A := \mathrm{C}^*(\mathbb{Z}, T, h)$ is $\mathcal{Z}$-stable if and only if it has strict comparison of positive elements.*

Proof. Let x be any non-isolated point in T. By Theorem 3.35, $A_{\{x\}}$ has stable rank one. Thus, by [23, Thm. 9.6], the Toms–Winter Conjecture (Conjecture 3.32) holds for $A_{\{x\}}$. In particular, $A_{\{x\}}$ is $\mathcal{Z}$-stable if and only if it has strict comparison of positive elements. But by [2, Thm. 3.3 and Cor. 3.5], A is $\mathcal{Z}$-stable if and only if $A_{\{x\}}$ is. Furthermore, by [18, Thm. 6.14], A has strict comparison if and only if $A_{\{x\}}$ does. Therefore, Corollary 3.37 follows. □

Remark 3.38. Using the same ideas as those in the proof of Theorem 3.35, one can show that the orbit-breaking simple subalgebras constructed by Deeley, Putnam, and Strung in [6] also have stable rank one, despite possibly not being $\mathcal{Z}$-stable. Although, the closed subset of the underlying infinite compact metric space used in their construction need not be a singleton set, it still has the property that it meets every orbit exactly once and, thus, is a simple inductive limit of DSH algebras with diagonal maps.

Acknowledgments. The authors would like to thank Zhuang Niu, Chris Phillips, and Andrew Toms for a lot of helpful encouragement, as well as George Elliott and Maria Grazia Viola for all of their support and for proof-reading an earlier draft of this paper. We are also very grateful to the referee for providing a lot of prudent comments concerning the structure of this paper and for bringing the observation in Remark 3.38 to our attention.

References

[1] M. Alboiu, *The stable rank of diagonal Ash algebras*, Ph.D. Thesis, University of Toronto, ProQuest LLC, Ann Arbor, MI, 2021. MR4297468

[2] D. Archey, J. Buck, and N. C. Phillips, Centrally large subalgebras and tracial $\mathcal{Z}$-absorption, Int. Math. Res. Not. IMRN **2018** (2018), no. 6, 1857–1877. MR3801476

[3] D. E. Archey and N. C. Phillips, Permanence of stable rank one for centrally large subalgebras and crossed products by minimal homeomorphisms, J. Operator Theory **83** (2020), no. 2, 353–389. MR4078704

[4] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, Nuclear dimension of simple C*-algebras, Invent. Math. **224** (2021), no. 1, 245–290. MR4228503

[5] M. Dădărlat, G. Nagy, A. Némethi, and C. Pasnicu, Reduction of topological stable rank in inductive limits of C^*-algebras, Pacific J. Math. **153** (1992), no. 2, 267–276. MR1151561

[6] R. J. Deeley, I. F. Putnam, and K. R. Strung, Classifiable C*-algebras from minimal $\mathbb{Z}$-actions and their orbit-breaking subalgebras, arXiv:2012.10947v1 [math.OA] (2020).

[7] J. Dixmier, *C^*-algebras*, translated from the French by Francis Jellett, North-Holland Math. Libr. 15, North-Holland Publishing, Amsterdam, 1977. MR0458185

[8] G. A. Elliott, T. M. Ho, and A. S. Toms, A class of simple C^*-algebras with stable rank one, J. Funct. Anal. **256** (2009), no. 2, 307–322. MR2476944

[9] G. A. Elliott and A. S. Toms, Regularity properties in the classification program for separable amenable C^*-algebras, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 2, 229–245. MR2383304

[10] R. Engelking, *General topology*, translated from the Polish by the author, second edition, Sigma Ser. Pure Math. 6, Heldermann Verlag, Berlin, 1989. MR1039321

[11] K. R. Goodearl, Notes on a class of simple C^*-algebras with real rank zero, Publ. Mat. **36** (1992), no. 2A, 637–654 (1993). MR1209829

[12] T. M. Ho, *On inductive limits of homogeneous C^*-algebras with diagonal morphisms between the building blocks*, Ph.D. Thesis, University of Toronto, ProQuest LLC, Ann Arbor, MI, 2006. MR2709772

[13] Q. Lin, Analytic structure of the transformation group C*-algebra associated with minimal dynamical systems, preprint.

[14] Q. Lin and N. C. Phillips, Ordered K-theory for C^*-algebras of minimal homeomorphisms, in *Operator algebras and operator theory (Shanghai, 1997)*, 289–314, Contemp. Math. 228, American Mathematical Society, Providence, RI, 1998. MR1667666

[15] Q. Lin and N. C. Phillips, Direct limit decomposition for C^*-algebras of minimal diffeomorphisms, in *Operator algebras and applications*, 107–133, Adv. Stud. Pure Math., 38, Math. Soc. Japan, Tokyo, 2004. MR2059804

[16] H. Lin and N. C. Phillips, Crossed products by minimal homeomorphisms, J. Reine Angew. Math. **641** (2010), 95–122. MR2643926

[17] N. C. Phillips, Recursive subhomogeneous algebras, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4595–4623. MR2320643

[18] N. C. Phillips, Large subalgebras, arXiv:1408.5546v1 [math.OA] (2014).

[19] I. F. Putnam, The C^*-algebras associated with minimal homeomorphisms of the Cantor set, Pacific J. Math. **136** (1989), no. 2, 329–353. MR0978619

[20] M. A. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras, Proc. Lond. Math. Soc. (3) **46** (1983), no. 2, 301–333. MR0693043

[21] M. Rørdam, On the structure of simple C^*-algebras tensored with a UHF-algebra, J. Funct. Anal. **100** (1991), no. 1, 1–17. MR1124289

[22] M. Rørdam, The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras, Internat. J. Math. **15** (2004), no. 10, 1065–1084. MR2106263

[23] H. Thiel, Ranks of operators in simple C^*-algebras with stable rank one, Comm. Math. Phys. **377** (2020), no. 1, 37–76. MR4107924

[24] J. Villadsen, Simple C^*-algebras with perforation, J. Funct. Anal. **154** (1998), no. 1, 110–116. MR1616504

[25] J. Villadsen, On the stable rank of simple C^*-algebras, J. Amer. Math. Soc. **12** (1999), no. 4, 1091–1102. MR1691013

[26] W. Winter, Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras, Invent. Math. **187** (2012), no. 2, 259–342. MR2885621

[27] W. Winter and J. Zacharias, The nuclear dimension of C^*-algebras, Adv. Math. **224** (2010), no. 2, 461–498. MR2609012

Received April 13, 2021; accepted January 14, 2022

Mihai Alboiu
Department of Mathematics, Yale University,
442 Dunham Lab, 10 Hillhouse Ave, New Haven, CT, 06511, USA
E-mail: mihai.alboiu@yale.edu
URL: https://campuspress.yale.edu/alboiu/

James Lutley
Department of Mathematics, University of Toronto,
40 St. George St., Room 6290 Toronto, Ontario, M5S 2E4 Canada
E-mail: jlutley@gmail.com