Münster J. of Math. **14** (2021), 445–484
DOI 10.17879/06089644766
urn:nbn:de:hbz:6-06089645886

Münster Journal of Mathematics

Projective geometry for perfectoid spaces

Gabriel Dorfsman-Hopkins

(Communicated by Peter Schneider)

Abstract. We develop a perfectoid analog of projective geometry and explore how equipping a perfectoid space with a map to a certain analog of projective space can be a powerful tool to understand its geometric and arithmetic structure. In particular, we show that maps from a perfectoid space X to the perfectoid analog of projective space correspond to line bundles on X together with some extra data, reflecting the classical theory. We then use this notion to compare the Picard group of a perfectoid space and its tilt. Along the way, we give a complete classification of vector bundles on the perfectoid unit disk and compute the Picard group of the perfectoid analog of projective space.

1. Introduction

This paper is inspired by the goal of understanding vector bundles on perfectoid spaces and how they behave under the so-called *tilting correspondence* of Scholze [18]. To do so, we develop a perfectoid analog of projective geometry. We study the perfectoid analog of projective space defined in [18], which we call *projectivoid space* and denote by $\mathbb{P}^{n,\mathrm{perf}}$, and we show that maps from a perfectoid space X to $\mathbb{P}^{n,\mathrm{perf}}$ correspond to line bundles on X together with some extra data, giving an analog to the classical theory of maps to projective space. In particular, this gives us a direct and geometric way to compare the Picard groups of a perfectoid space and its tilt by applying the tilting equivalence to the corresponding maps to projectivoid space. We expect that the projectivoid geometry as well as the local results studied in this article will have applications in a variety of contexts. Theorem 1.3 below concerning Picard groups is one such application. It should be noted that one can also study line bundles on perfectoid spaces using cohomological methods; in particular, one can obtain a different proof of Theorem 1.3 using such methods. This will be discussed in another article.

To begin, we must first understand the theory of line bundles on projectivoid space itself. Das [5] worked toward computing the Picard group of the projectivoid line, $\mathbb{P}^{1,\mathrm{perf}}$, although Das' proof assumed the existence of certain local

This work was partially supported by the NSF.

trivializations of line bundles, requiring a perfectoid analog of the Quillen–Suslin theorem. Therefore, in order to begin developing the theory of so-called projectivoid geometry, we must first extend the Quillen–Suslin theorem to the perfectoid setting. Explicitly, we must show that there are no nontrivial projective modules on the *perfectoid Tate algebra* $K\langle T^{1/p^\infty}\rangle$, which is the ring of convergent power series over a perfectoid field K (of residue characteristic p), such that the indeterminates have all their p-power roots. We do this in Section 2, deducing it from the case of the Tate algebra $K\langle T\rangle$ with a limiting argument together with a theorem of Gabber and Romero [9] regarding the behavior of projective modules with respect to completion on generic fibers of henselian rings. The case of the Tate algebra, in turn, was deduced from the polynomial case by Lütkebohmert [16]. This completes Das' proof and lays the groundwork to begin studying vector bundles on more general perfectoid spaces

In Section 3, we develop the theory of line bundles on projectivoid space, extending Das' result for $n = 1$.

Theorem 1.1. $\operatorname{Pic}\mathbb{P}^{n,\mathrm{perf}} \cong \mathbb{Z}[1/p]$.

To prove this, we notice that $\mathbb{P}^{n,\mathrm{perf}}$ has a natural integral model whose special fiber is the scheme theoretic perfect closure of projective space over the residue field. As it is easily deduced that the Picard group of the perfect closure of projective space is $\mathbb{Z}[1/p]$, there remain two main steps: first that every line bundle over the residue fields deforms uniquely to the integral model, and second that every line bundle on $\mathbb{P}^{n,\mathrm{perf}}$ has a unique integral model. Strictly speaking, this is all done on the level of Čech cohomology with a standard cover by perfectoid closed disks (analogous to the usual affine cover of projective space). This cover also allows us to compute the cohomology of all line bundles on projectivoid space, extending a computation of Bedi [2] for $n = 2$.

In Section 4, we compute the functor of points of projectivoid space, showing that (much like in the classical theory) it is deeply connected to the theory of line bundles on perfectoid spaces.

Theorem 1.2. *Let X be a perfectoid space over a perfectoid field K. Morphisms $X \to \mathbb{P}^{n,\mathrm{perf}}$ over K correspond to tuples $(\mathscr{L}_i, s_j^{(i)}, \varphi_i)$, where the $\mathscr{L}_i$ are line bundles on X, $\{s_0^{(i)}, \dots, s_n^{(i)}\}$ are $n+1$ global sections of $\mathscr{L}_i$ which generate $\mathscr{L}_i$, and $\varphi_i : \mathscr{L}_{i+1}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_i$ are isomorphisms under which $(s_j^{(i+1)})^{\otimes p}$ maps to $s_j^{(i)}$.*

We also provide refinements of this theorem in characteristic p and see how it behaves under the tilting equivalence of Scholze.

In Section 5, we apply this formalism to compare the Picard groups of a perfectoid space X and its tilt $X^\flat$. In particular, since the tilting equivalence builds a correspondence between maps $X \to \mathbb{P}_K^{n,\mathrm{perf}}$ and maps $X^\flat \to \mathbb{P}_{K^\flat}^{n,\mathrm{perf}}$, we can chain this together with the correspondence of line bundles and maps to projectivoid space to compare line bundles on X and $X^\flat$, allowing us to prove the following result.

Theorem 1.3. *Suppose X is a perfectoid space over a perfectoid field K, let $X^\flat$ be the tilt of X, and let C be the completion of an algebraic closure of K. Suppose that $X^\flat$ has a weakly ample line bundle (cp. Definition 5.12) and that $\mathrm{H}^0(X_C, \mathscr{O}_{X_C}) = C$. Then there is a natural injection*

$$\theta : \operatorname{Pic} X^\flat \hookrightarrow \varprojlim_{\mathscr{L} \mapsto \mathscr{L}^p} \operatorname{Pic} X.$$

In particular, if $\operatorname{Pic} X$ has no p-torsion, then composing with projection onto the first coordinate gives an injection

$$\theta_0 : \operatorname{Pic} X^\flat \hookrightarrow \operatorname{Pic} X.$$

This paper relies heavily on Huber's theory of adic spaces developed in [12, 11]. We do not develop the theory here and instead refer the reader to Huber's original papers or Wedhorn's detailed notes [22]. A summary of the theory with an emphasis on perfectoid spaces can be found in the author's doctoral dissertation [6, Sec. 2–4] or in Kedlaya's detailed notes from the 2017 Arizona Winter School on the subject [14].

Notational conventions. Throughout the paper, we will fix a perfectoid field K, with valuation ring K°, and maximal ideal $K^{\circ\circ}$. The residue field will be denoted by k and has fixed prime characteristic p. If R is a complete topological K-algebra, we denote its subring of power bounded elements by R°, the ideal of topologically nilpotent elements by $R^{\circ\circ}$, and the residue ring by $\tilde{R}$. If R is perfectoid, then we denote its *tilt* by $R^\flat$. We will denote by $\sharp : R^\flat \to R$ the multiplicative map coming from composing projection onto the first coordinate with the isomorphism (of monoids) $R^\flat \cong \varprojlim_{x \mapsto x^p} R$.

We also fix a topologically nilpotent unit $\varpi \in K$ such that $\varpi^p | p$ in K°, called a *pseudouniformizer*. As in [18, Rem. 3.5], we choose ϖ to be in the image of $\sharp : K^\flat \to K$ so that it comes equipped with a complete set of p-power roots. In this way, the symbol ϖ^d makes sense for every $d \in \mathbb{Z}[1/p]$.

2. The perfectoid Tate algebra

Algebraic geometry studies the polynomial ring and its various quotients and localizations, allowing for commutative algebraic facts to be interpreted geometrically and vice-versa. In the world of perfectoid algebras, a natural analog is the perfectoid Tate algebra (defined below), whose commutative algebra controls many of the geometric structures we study. We therefore begin with a careful study of the algebraic object underlying most of this work.

Let K be a perfectoid field of residue characteristic p, with pseudouniformizer ϖ. In particular, K is a nonarchimedean field, and so we can define the *Tate algebra* $T_{n,K} = K\langle X_1, \dots, X_n \rangle$ of convergent power series over K. The module theory of the Tate algebra is well understood, in no small part due to Lütkebohmert, see [16]. Our main object of study in this section is the following.

Definition 2.1 (The perfectoid Tate algebra)**.** The *perfectoid Tate algebra* $T_{n,K}^{\mathrm{perf}} = K\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle$ is a ring of convergent power series over K, whose indeterminates have all of their p-power roots. More precisely, it is the ϖ-adic completion of the union of the Tate algebras over K in the variables X_i^{1/p^r} as r varies and is therefore a completed colimit of Tate algebras.

$$K\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle \coloneqq \Big(\bigcup_{r\geq 0} K\langle X_1^{1/p^r}, \dots, X_n^{1/p^r}\rangle\Big)^\wedge \cong \big(\varinjlim T_{n,K}\big)^\wedge.$$

Remark 2.2. The perfectoid Tate algebra consists of formal power series over K which converge on the perfectoid unit disk. Explicitly, letting $X = (X_1, \dots, X_n)$ be an n-tuple, we can write down the elements of this ring as follows:

$$T_{n,K}^{\mathrm{perf}} = \Big\{ \sum_{\alpha\in(\mathbb{Z}[1/p]_{\geq 0})^n} a_\alpha X^\alpha \;\Big|\; \text{for all } \lambda \in \mathbb{R}_{>0}, \text{ only finitely many} |a_\alpha| \geq \lambda \Big\}.$$

This ring inherits the Gauss norm, $\|\sum_\alpha a_\alpha X^\alpha\| = \sup\{|a_\alpha|\}$.

$(T_{n,K}^{\mathrm{perf}})^\circ$ is the subring $\{\|f\| \leq 1\}$ of power-bounded elements of $T_{n,K}^{\mathrm{perf}}$ and consists of power series with coefficients in K°. The ideal $(T_{n,K}^{\mathrm{perf}})^{\circ\circ}$ of topologically nilpotent elements consists of power series with coefficients in $K^{\circ\circ}$. The quotient is

$$\tilde{T}_{n,K}^{\mathrm{perf}} \coloneqq (T_{n,K}^{\mathrm{perf}})^\circ/(T_{n,K}^{\mathrm{perf}})^{\circ\circ} = k[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}],$$

where $k = K^\circ/K^{\circ\circ}$ is the residue field. Notice that every element in the quotient ring is a polynomial because a power series in $(T_{n,K}^{\mathrm{perf}})^\circ$ can only have finitely many coefficients of norm 1.

When there will be no confusion, we omit K from the notation.

Remark 2.3 (A note on convergence)**.** Suppose $f = \sum f_\alpha X^\alpha \in T_n^{\mathrm{perf}}$. Morally speaking, saying that the sum converges should mean that evaluating f at any point in the perfectoid disk should give an element of K. Since the sums are not taken over $\mathbb{Z}_{\geq 0}^n$, but rather $(\mathbb{Z}[1/p]_{\geq 0})^n$, we must be more careful in defining what convergence means. Let us begin by studying $f(1, 1, \dots, 1) = \sum f_\alpha$. This should converge, so we begin by defining partial sums

$$s_m = \sum_{\alpha\in(\frac{\mathbb{Z}}{p^m})^n | 0\leq\alpha_i\leq m} f_\alpha.$$

If the sequence (s_m) converges, we define the infinite sum to be the limit. Let us check that the convergence of the power series f implies convergence of $\sum f_\alpha$ in this sense. Fixing some $\varepsilon > 0$, there are only finitely many f_α with $|f_\alpha| \geq \varepsilon$. Therefore, there is some large N such that, for each such f_α, we have $\alpha = (\alpha_1, \dots, \alpha_n) \in \big(\frac{\mathbb{Z}}{p^N}\big)^n$ and $0 < \alpha_i < N$. Therefore, fixing $m \geq r > N$, the differences $s_m - s_r$ have no coefficients f_α with absolute value larger than ε so that, by the nonarchimedean property, $|s_m - s_r| < \varepsilon$. Thus the sum converges to an element $f(1) \in K$.

We remark now that if $|g_\alpha| \le 1$, the same argument would show that $\sum f_\alpha g_\alpha$ also converges. This should imply that $f(x)$ converges to a point in K whenever x is in the perfectoid unit disk, but we defer further discussion until after we have the relevant definitions (see Remark 2.10).

We record a useful normalization trick for further use down the line.

Lemma 2.4 (Normalization)**.** *Let $f \in T_n^{\mathrm{perf}}$ be nonzero. There is some $\lambda \in K$ such that $\|\lambda f\| = 1$.*

Proof. Since only finitely many coefficients in f have absolute value above $\|f\| - \varepsilon$, the supremum of that absolute values of the coefficients is achieved by some f_α. Taking $\lambda = f_\alpha^{-1}$ completes the proof. □

The group of units. As a first step towards understanding the perfectoid Tate algebra, we compute its group of units.

Proposition 2.5. *Let $f \in T_n^{\mathrm{perf}}$ with $\|f\| = 1$. The following are equivalent.*
(i) *f is a unit in (T_n^{perf}).*
(ii) *f is a unit in $(T_n^{\mathrm{perf}})^\circ$.*
(iii) *The image of $\overline{f}$ of f in $\tilde{T}_n^{\mathrm{perf}}$ is a nonzero constant $\lambda \in k^\times$.*
(iv) *$|f(0)| = 1$ and $\|f - f(0)\| < 1$.*

Proof. (i) ⇔ (ii). An inverse to f must have absolute value 1 and therefore would also lie in $(T_n^{\mathrm{perf}})^\circ$.

(ii) ⇒ (iii). The map $(T_n^{\mathrm{perf}})^\circ \to \tilde{T}_n^{\mathrm{perf}}$ must send units to units, and the group of units of $\tilde{T}_n^{\mathrm{perf}}$ is precisely the nonzero constant polynomials. Indeed, the inverse to any element of $\tilde{T}_n^{\mathrm{perf}}$ would also have to be a polynomial (in X^{1/p^m} for some m), implying that they both must be constants.

(iii) ⇔ (iv). This is immediate.

(iv) ⇒ (i). If $|f(0)| = 1$, then $f(0) \in K^\times \subseteq (T_n^{\mathrm{perf}})^\times$. Therefore, we have $1 - \frac{f}{f(0)} \in T_n^{\mathrm{perf}}$ and

$$\left\|1 - \frac{f}{f(0)}\right\| = \|f(0)\| \cdot \left\|1 - \frac{f}{f(0)}\right\| = \|f(0) - f\| < 1.$$

Therefore, $1 - \frac{f}{f(0)}$ is topologically nilpotent so that the following geometric series converges:

$$\frac{1}{f/f(0)} = \frac{1}{1 - (1 - f/f(0))} = \sum_{k=0}^{\infty} \left(1 - \frac{f}{f(0)}\right)^k.$$

Hence $\frac{f}{f(0)}$ is a unit. Since $f(0)$ is too, we can conclude that f is a unit. □

Corollary 2.6. *$f = \sum f_\alpha X^\alpha \in (T_n^{\mathrm{perf}})^\circ$ is a unit if and only if $|f_0| = 1$ and $|f_\alpha| < 1$ for all $\alpha \neq 0$.*

Corollary 2.7. *$f = \sum f_\alpha X^\alpha \in T_n^{\mathrm{perf}}$ is a unit if and only if $|f_\alpha| < |f_0|$ for all $\alpha \neq 0$.*

Proof. Using our normalization trick, we know $\|\lambda f\| = 1$ for some $\lambda \in K^\times$. Then f is a unit if and only if λf is, if and only if $|\lambda f_\alpha| < 1 = |\lambda f_0|$ for all $\alpha \neq 0$. Canceling shows this holds if and only if $|f_\alpha| < |f_0|$ for all $\alpha \neq 0$. □

Vector bundles on the perfectoid unit disk. In classical algebraic geometry, the polynomial ring (and its various quotients) form the local building blocks of most of the objects of study. Phrased geometrically, the prime spectrum of the polynomial ring of n variables is affine space $\mathbb{A}^n$ which covers or contains many of the spaces of interest. Like schemes which are locally prime spectra of rings, perfectoid spaces are build from perfectoid algebras using the adic spectrum functor of Huber (see [12, 11, 13]). The perfectoid Tate algebra plays the role of the polynomial ring, and the perfectoid unit disk (defined below) plays the role of affine space.

Definition 2.8. The *perfectoid unit disk* is the adic space associated to the perfectoid Tate algebra

$$\mathbb{D}^{n,\mathrm{perf}} = \mathrm{Spa}(T_n^{\mathrm{perf}}, (T_n^{\mathrm{perf}})^\circ).$$

Remark 2.9. The rigid unit disk is the adic spectrum associated to the Tate algebra. Since the perfectoid Tate algebra is the completed union of Tate algebras, using the *tilde limit* formalism of [20], we have

$$\mathbb{D}^{n,\mathrm{perf}} \sim \varprojlim_{\varphi} \mathbb{D}^{n,\mathrm{ad}},$$

where φ is the pth power map on coordinates. It is worth noting that the tilde limit is not the categorical inverse limit (since these are not in general unique in the category of adic spaces). Nevertheless, it should be thought of affinoid locally as corresponding to the completed directed limit, and if such a limit exists as a perfectoid space, it is unique among all perfectoid spaces and satisfies the usual universal property (among perfectoid spaces). See [20, Def. 2.4.1] and subsequent discussion.

Remark 2.10 (K-points of the perfectoid disk)**.** Let us compute the K-points of $\mathbb{D}_K^{n,\mathrm{perf}}$ and, in doing so, conclude the discussion of Remark 2.3. A K-point x is a map of adic spaces $x : \mathrm{Spa}(K, K^\circ) \to \mathbb{D}_K^{n,\mathrm{perf}}$, which is equivalent to a map of Huber pairs $\varepsilon_x : (T_K^{n,\mathrm{perf}}, (T_K^{n,\mathrm{perf}})^\circ) \to (K, K^\circ)$. This is a commutative diagram of continuous K-algebra homomorphisms

$$\begin{array}{ccc} K\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle & \longrightarrow & K \\ \uparrow & & \uparrow \\ K^\circ\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle & \longrightarrow & K^\circ. \end{array}$$

In particular, it is determined by the image of the X_i^{1/p^k} in K°. Let $\lambda = \varepsilon_x(X_i)$. Then $\lambda_1 = \varepsilon_x(X_i^{1/p})$ must be a pth root of λ, and $\lambda_2 = \varepsilon_x(X_i^{1/p^2})$ must be a pth root of λ_1. Continuing in this fashion, we see that choosing the image of the X_i^{1/p^k} as k varies is equivalent to fixing an element of $\varprojlim_{t\mapsto t^p} K^\circ = K^{\flat\circ}$. We have therefore computed the K-points of the perfectoid disk,

$$\mathbb{D}_K^{n,\mathrm{perf}}(K) = (K^{\flat\circ})^n.$$

The evaluation function $\varepsilon_x(f) = f(x)$ then amounts to plugging the coordinates of x as an element of $(K^{\flat\circ})^n$ into f, which we saw in Remark 2.3 converges to a point in K.

There is a well-known correspondence between finite projective modules over a ring, and finite-dimensional (algebraic) vector bundles over the associated affine scheme, and more generally, between vector bundles over a locally ringed space and locally free sheaves on that space (see, for example, [10, Ex. 2.5.18]). In [21], Serre conjectured that all finite projective modules over the polynomial ring $A = k[x_1, \dots, x_n]$ are free. This can be interpreted geometrically as saying there are no nontrivial algebraic vector bundles over affine space $\mathbb{A}^n = \operatorname{Spec} A$. In 1976, Quillen [17] and independently Suslin proved Serre's conjecture, which is now known as the Quillen–Suslin theorem. Lütkebohmert in [16] was shortly after able to extend the result to the Tate algebra $K\langle X_1, \dots, X_n\rangle$ of convergent power series over a complete nonarchimedean field.

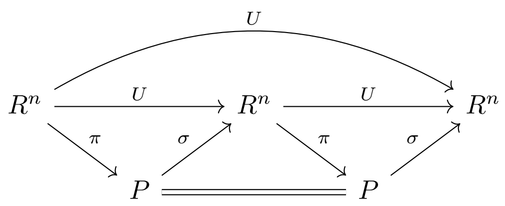
In what follows, we prove a perfectoid analog of the Quillen–Suslin theorem. Specifically, we prove that all finite projective modules on the perfectoid Tate algebra are trivial. This will imply that the perfectoid unit disk has no nontrivial finite vector bundles. Along the way, we will show that both the subring of integral elements $(T_n^{\mathrm{perf}})^\circ$, and the residue ring $\tilde{T}_n^{\mathrm{perf}}$ also have no nontrivial finite projective modules. Although these results are not necessary to establish the result for the perfectoid Tate algebra, they will be important in asserting the effectiveness of the Čech cohomology groups of certain sheaves in Section 3.

Finite projective modules on the residue ring. Let us begin by proving that finite projective modules are free over the residue ring

$$\tilde{T}_n^{\mathrm{perf}} = k[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}] = \bigcup_m k[X_1^{1/p^m}, \dots, X_n^{1/p^m}].$$

To see this, we first briefly review a (non-unique) correspondence between projective modules and idempotent matrices.

Let R be a commutative ring, and fix a finite projective R-module P. Consider a presentation $\pi : R^n \to P$, as well as a section of this projection σ. The composition $\sigma \circ \pi$ produces an idempotent matrix $U \in M_n(R)$.



Conversely, the image of an idempotent matrix U is always projective, with the section just given by the natural inclusion $\operatorname{im} U \subseteq R^n$. In this way, we get a (non-unique) correspondence between finite projective modules and idempotent matrices over R.

Lemma 2.11. *Suppose $R = \varinjlim_i R_i$ is a filtered colimit of commutative rings. Then every finite projective R-module is the base extension of a finite projective R_i-module.*

Proof. To a finite projective R-module M, we may associate a projector matrix U. Each entry in the matrix is defined over some R_i, and as the limit is filtered, U is defined over R_i for some (perhaps larger) i. Its image as a map from R_i^n to itself is therefore a projective R_i-module whose base extension is M. $\square$

Corollary 2.12. *Let $R = k[X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}]$. Every R-module is free. In particular (letting $r = n$), every $\tilde{T}_n^{\mathrm{perf}}$-module is free.*

Proof. As R is the filtered colimit of (Laurent) polynomial rings, Lemma 2.11 implies that a finite projective R-module M is the base extension of some N over a (Laurent) polynomial ring. By the Quillen–Suslin theorem, N is free, so M is too. $\square$

Finite projective modules on the subring of integral elements. We extend Corollary 2.12 to the subring of power-bounded elements of the perfectoid Tate algebra $(T_n^{\mathrm{perf}})^\circ$ using Nakayama's lemma. We first fix some notation.

Notation 2.13. For a commutative ring R and an ideal I contained in the Jacobson radical of R, we let $R_0 = R/I$. For an R-module M, we will denote by M_0 the R_0-module M/IM, and for a homomorphism ϕ of R-modules, we denote by ϕ_0 its reduction mod I. If $m \in M$, then we denote by $\overline{m}$ its image in M_0.

Lemma 2.14. *Let R be a commutative ring and I an ideal contained in the Jacobson radical of R. If M and N are two projective R-modules such that there exists an isomorphism $\phi : M_0 \xrightarrow{\sim} N_0$, then ϕ lifts to an isomorphism $\psi : M \xrightarrow{\sim} N$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} M & \overset{\psi}{\dashrightarrow} & N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ M_0 & \xrightarrow{\phi} & N_0. \end{array}$$

Indeed, a lift ψ exists because M is projective. Notice that $\psi_0 = \phi$. Indeed,

$$\psi_0(\overline{m}) = \overline{\psi(m)} = \rho\psi(m) = \phi\pi(m) = \phi(\overline{m}).$$

Since ϕ surjects, so does ψ by Nakayama's lemma. Since N is projective, ψ has a section σ which is necessarily injective. We claim that $\sigma_0 = \phi^{-1}$. We can check this after applying ϕ.

$$\phi\sigma_0(\overline{n}) = \phi\pi\sigma(n) = \rho\psi\sigma(n) = \rho(n) = \overline{n}.$$

Therefore, σ_0 surjects so that σ surjects by Nakayama's lemma. Thus σ is an isomorphism. □

Corollary 2.15. *With the same set-up as Lemma 2.14, we let P be a projective R-module. If P_0 is a free R_0-module, then P is free.*

Proof. Suppose $P_0 \cong R_0^m$. We also have $(R^m)_0 \cong R_0^m$ so that, by Lemma 2.14, $P \cong R^m$. □

We now apply this to the case at hand.

Lemma 2.16. *Let R be one of the following:*

$$R = K^\circ\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}\rangle,$$
$$R = K^\circ/\varpi^d[X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}].$$

The ideal of topologically nilpotent (resp. nilpotent) elements lies in the Jacobson radical of R.

Proof. If f is (topologically) nilpotent, then so is fg for all g. Thus the geometric series for $\frac{1}{1-fg}$ converges to an inverse of $1 - fg$ so that it is a unit. Since g was arbitrary, this shows that f is in the Jacobson radical. □

The desired result follows.

Corollary 2.17. *Let R be as in Lemma 2.16. Every finite projective R-module is free.*

Proof. Notice that R_0—the reduction of R modulo the ideal of (topologically) nilpotent elements—is the ring of Corollary 2.12. Let P be a finite projective R-module. Then P_0 is a finite projective R_0-module and therefore is free. Therefore, by Corollary 2.15, it suffices to show that the kernel of the reduction map is contained in the Jacobson radical, but this is Lemma 2.16. □

The Quillen–Suslin theorem for the perfectoid Tate algebra. We now prove the main result of this section.

Theorem 2.18. *Finite projective modules on the perfectoid Tate algebra are free. Equivalently, all finite vector bundles on the perfectoid unit disk are free.*

In fact, this will follow from something slightly more general.

Theorem 2.19. *Let R be a perfectoid Laurent series algebra,*

$$R = K\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}\rangle.$$

Then every finite projective R-module is free.

A result of Gabber and Romero will do most of the heavy lifting; we state the result here.

Proposition 2.20 ([9, Cor. 5.4.42]). *Let R be a commutative ring, $t \in R$ a nonzero divisor, and $I \subset R$ an ideal. Let $\hat{R}$ be the tI-adic completion of R, and suppose (R, tI) form a Henselian pair. Then the base extension functor $R[t^{-1}] - \mathbf{Mod} \to \hat{R}[t^{-1}] - \mathbf{Mod}$ induces a bijection between isomorphism classes of finite projective $R[t^{-1}]$-modules and finite projective $\hat{R}[t^{-1}]$-modules.*

Let us describe the objects we will feed into this result. Playing the role of R is

$$R = \bigcup_i K^\circ\langle X_1^{1/p^i}, \ldots, X_r^{1/p^i}, X_{r+1}^{\pm 1/p^i}, \ldots, X_n^{\pm 1/p^i}\rangle =: \bigcup_i R_i. \tag{1}$$

Playing the role of t, we have ϖ, and I will be the unit ideal. Therefore,

$$R[1/\varpi] = \bigcup_i K\langle X_1^{1/p^i}, \ldots, X_r^{1/p^i}, X_{r+1}^{\pm 1/p^i}, \ldots, X_n^{\pm 1/p^i}\rangle.$$

Then $\hat{R}$ is the ring of integral elements (cp. Lemma 2.16), and $\hat{R}[1/\varpi]$ is the perfectoid Tate algebra (or Laurent series algebra), for which we are trying to prove the Quillen–Suslin result. Theorem 2.19 will follow from the following two lemmas.

Lemma 2.21. *Finite projective modules on $R[1/\varpi]$ are all free.*

Proof. Applying Lemma 2.11, a finite projective $R[1/\varpi]$-module M is the base extension of a finite projective module N over a rigid analytic Tate algebra (or Laurent series algebra). But N is free due to the rigid analytic Quillen–Suslin theorem [16, Satz 1], so we are done. □

Lemma 2.22. *The pair $(R, (\varpi))$ is Henselian.*

Proof. Lemma 2.16 shows that ϖ is in the Jacobson radical of R. Suppose $f \in R[T]$ is monic and that $\overline{f} = g_0 h_0 \in (R/\varpi)[T]$ with g_0, h_0 monic. In fact, $f \in R_i[T]$ for some i (using the notation of equation (1) above). Perhaps increasing i, we can take the factorization of f to take place in $(R_i/\varpi)[T]$. As R_i is ϖ-complete, (R_i, ϖ) form a Henselian pair (see, for example, [24, Tag 0ALJ]). Therefore, the factorization lifts to $f = gh$ in $R_i[T] \subset R[T]$, with g and h monic. □

The main result now follows easily.

Proof of Theorem 2.19. Lemma 2.22 allows us to apply Proposition 2.20 and conclude that each finite projective module over $\hat{R}[1/\varpi]$ is the base extension of one on $R[1/\varpi]$, which by Lemma 2.21 must be free. □

3. Line bundles and cohomology on projectivoid space

In classical algebraic geometry, the notion of *projective geometry* is a very powerful tool to study properties of varieties and schemes. Indeed, one can learn a lot about a scheme by understanding its maps to various projective spaces, and this theory is intimately connected to the theory of line bundles on that space. In this and the following section, we develop an analogous theory for perfectoid spaces. Let us begin by defining a perfectoid analog of projective space.

Definition 3.1 (The projectivoid line)**.** Analogously to the construction of the Riemann sphere, we can build the perfectoid analog of the projective line by gluing two copies of the perfectoid unit disk along the perfectoid unit circle.

Explicitly, the inclusion $K\langle T^{1/p^\infty}\rangle \to K\langle T^{\pm 1/p^\infty}\rangle$ corresponds to the open immersion of the perfectoid unit circle into the perfectoid disk, $\mathbb{S}^{1,\mathrm{perf}} \hookrightarrow \mathbb{D}^{1,\mathrm{perf}}$. The map $K\langle T^{-1/p^\infty}\rangle \to K\langle T^{\pm 1/p^\infty}\rangle$ also corresponds to an open immersion of the circle into the disk, where now the disk has coordinate T^{-1}. Identifying the circles on each of these disks and gluing produces *the projectivoid line*, denoted $\mathbb{P}^{1,\mathrm{perf}}_K$.

Definition 3.2 (Projectivoid space)**.** As with the projectivoid line, we can define projectivoid n-space by gluing together $n+1$ perfectoid unit n-polydisks along their boundaries as in Definition 3.1. This equips projectivoid space with a cover by perfectoid unit disks which we will henceforth refer to as the *standard cover*.

In [19, Sec. 7], Scholze showed that we could also define projectivoid space in the following way. Let $\mathbb{P}^n_K$ be projective space over K, which can be viewed as an adic space by first being viewed as a rigid space using the rigid analytification functor, and then as an adic space as in [11]. Let $\varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be the morphism given by $(T_0 : \cdots : T_n) \mapsto (T_0^p : \cdots : T_n^p)$ in projective coordinates. Then

$$\mathbb{P}^{n,\mathrm{perf}}_K \sim \varprojlim_{\varphi} \mathbb{P}^n_K.$$

As with the perfectoid disk, "$\sim\varprojlim$" is the "tilde limit" of [20] (cp. Remark 2.9).

Remark 3.3. Notice that all the finite intersections of the standard cover will be the adic spectra associated to what we called perfectoid Laurent series algebras in the previous section. In particular, Theorem 2.19 shows that all their finite vector bundles are free.

Scholze showed in [18, Thm. 8.5] that the construction of projectivoid space is compatible with the tilting functor. In this section, we begin our exploration of so-called *projectivoid geometry* by developing the theory of line bundles on projectivoid space. In particular, we compute the Picard group of $\mathbb{P}^{n,\mathrm{perf}}$, as well as the sheaf cohomology of all line bundles. We continue developing the theory in the following section, where we will show how an arbitrary perfectoid space's maps to projectivoid space are intimately connected to its theory of line bundles, reflecting the situation in classical algebraic geometry, but with an extra arithmetic twist.

The Picard group of projectivoid space. The main result of this section follows.

Theorem 3.4. $\operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}} \cong \mathbb{Z}[1/p]$.

We outline the general strategy. We first need to define a few auxiliary sheaves related to the structure sheaf of an adic space. Let X be an adic space. We define the *sheaf of integral functions* $\mathscr{O}_X^+ \subseteq \mathscr{O}_X$ by the rule

$$U \mapsto \{f \in \mathscr{O}_X(U) \mid |f(x)| \le 1 \text{ for all } x \in U\}.$$

It is a sheaf of rings, and we denote its unit group by $\mathscr{O}_X^{+*}$. Moreover, $\mathscr{O}_X^+$ has an ideal sheaf $\mathscr{O}_X^{++}$ of *topologically nilpotent functions*, defined by the rule

$$U \mapsto \{f \in \mathscr{O}_X(U) \mid |f(x)| < 1 \text{ for all } x \in U\}.$$

We denote the quotient $\mathscr{O}_X^+/\mathscr{O}_X^{++}$ by $\tilde{\mathscr{O}}_X$. We now outline our proof. We use that, for any locally ringed space X, there is a natural isomorphism $\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathscr{O}_X^*)$ (see, for example, [10, Ex. III.4.5]). For the remainder of this section, we let $X = \mathbb{P}^{n,\mathrm{perf}}$. There are three main steps to the proof.

(I) **(Extension)** The inclusion $\mathscr{O}_X^{+*} \hookrightarrow \mathscr{O}_X^*$ induces a map on cohomology $\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \to \mathrm{H}^1(X, \mathscr{O}_X^*)$, which we show is an isomorphism in Proposition 3.6. Morally, this can be thought of as saying that line bundles on X extend uniquely to an integral model.

(II) **(Deformation)** The projection $\mathscr{O}_X^{+*} \to \tilde{\mathscr{O}}_X^*$ induces a map on cohomology $\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \to \mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*)$. We show that it is an isomorphism in Proposition 3.9 by deforming invertible modules along the sheaves of algebras $\mathscr{O}_X^+/\varpi^d$ for $d \in \mathbb{Z}[1/p]_{>0}$ which interpolate between $\mathscr{O}_X^+$ and $\tilde{\mathscr{O}}_X$. Morally, this can be thought of as saying line bundles deform uniquely from the special fiber to the integral model. This is the most involved step.

(III) **(Comparison)** By the first two steps, it suffices to compute $\mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*)$. We do this in Corollary 3.16 by showing that it is isomorphic to the Picard group of the perfection of classical projective space over the residue field k.

We make frequent use of the fact that projectivoid space comes equipped with a standard cover $\mathfrak{U} = \{U_i \to X\}$, by $n+1$ open sets each isomorphic to a perfectoid unit disk, whose geometry we understand well due to the results of Section 2. In particular, any line bundle on $\mathbb{P}^{n,\mathrm{perf}}$ becomes trivial on the U_i and their various finite intersections due to Theorem 2.19. We use Čech cohomology with respect to the standard cover to accomplish the steps outlined above, noting that the passage between Čech and sheaf cohomology is safe due to the following lemma.

Lemma 3.5. *There are natural isomorphisms*

- $\mathrm{H}^1(X, \mathscr{O}_X^*) \cong \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X^*)$,
- $\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \cong \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X^{+*})$,
- $\mathrm{H}^1(X, (\mathscr{O}_X^+/\varpi^d)^*) \cong \check{\mathrm{H}}^1(\mathfrak{U}, (\mathscr{O}_X^+/\varpi^d)^*)$ *for all* $d \in \mathbb{Z}[1/p]_{>0}$,
- $\mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*) \cong \check{\mathrm{H}}^1(\mathfrak{U}, \tilde{\mathscr{O}}_X^*)$.

Proof. Let $\mathscr{R}$ be one of the following sheaves of rings: $\mathscr{O}_X, \mathscr{O}_X^+, \mathscr{O}_X^+/\varpi^d$ or $\tilde{\mathscr{O}}_X$, and let $\mathscr{R}^*$ be the associated sheaf of units. The Čech-to-derived spectral sequence [1, Exposé V Théorème 3.2] describes the following spectral sequence:

$$E_2^{p,q} : \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}^q(\mathscr{R}^*)) \implies \mathrm{H}^{p+q}(X, \mathscr{R}^*).$$

Let U be a finite intersection of elements in the standard cover. Then either Theorem 2.19, Corollary 2.17, or Corollary 2.12 implies that $\mathscr{R}(U)$ has no nontrivial invertible modules. In particular, $\mathscr{H}^1(\mathscr{R}^*)(U) = \mathrm{H}^1(U, \mathscr{R}^*) = 0$, as such a cohomology class would construct an invertible $\mathscr{R}(U)$-module. We may conclude, therefore, that the sequence of low degree terms for the spectral sequence degenerates to $\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{R}^*) \cong \mathrm{H}^1(X, \mathscr{R}^*)$. □

Extending line bundles to the integral model. We begin with the extension step, showing that it suffices to compute the cohomology of the sheaf of integral units.

Proposition 3.6. *The natural map $\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^{+*}) \to \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^*)$ is an isomorphism for all $i > 0$. If $i = 1$, the isomorphism also holds for derived functor cohomology.*

Proof. Denote the intersection $U_{i_1} \cap \cdots \cap U_{i_t}$ by $U_{i_1 \cdots i_t}$. We have a short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_i \mathscr{O}_X^{+*}(U_i) & \longrightarrow & \prod_i \mathscr{O}_X^{*}(U_i) & \longrightarrow & \prod_i |K^*| & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i,j} \mathscr{O}_X^{+*}(U_{ij}) & \longrightarrow & \prod_{i,j} \mathscr{O}_X^{*}(U_{ij}) & \longrightarrow & \prod_{i,j} |K^*| & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i,j,k} \mathscr{O}_X^{+*}(U_{ijk}) & \longrightarrow & \prod_{i,j,k} \mathscr{O}_X^{*}(U_{ijk}) & \longrightarrow & \prod_{i,j,k} |K^*| & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The left and middle complexes are the Čech complexes for $\mathscr{O}_X^{+*}$ and $\mathscr{O}_X^*$ respectively, and the map on the right is $|\cdot|$ which is plainly surjective. Also, the right-hand complex has kernel $|K^*|$ and is otherwise exact (arguing as in, for example, [24, Tag 02UW]) so that the long exact sequence on cohomology gives $\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^{+*}) \cong \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^*)$ for all $i > 0$. Lemma 3.5 extends the result to derived functor cohomology if $i = 1$. □

Deforming line bundles from the residue. We next take care of the deformation step. We will need the following lemma, whose proof we defer for now (see Proposition 3.25 and Remark 3.27).

Lemma 3.7. $\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^+) = 0$ *for all $i > 0$.*

We will also need the following lemma of commutative algebra.

Lemma 3.8. *Let $\pi : R \to S$ be a surjection of rings whose kernel I is contained in the Jacobson radical of R. Then the induced map on unit groups, $R^* \to S^*$, remains surjective.*

Proof. As I is contained in the Jacobson radical of R, it is contained in each maximal ideal of R. Therefore, the image in S of any maximal ideal of R is a proper (and even maximal) ideal of S. This implies that if $r \in R$ is not a unit, then $\pi(r)$ is contained in a proper ideal of S and is therefore not a unit either. It follows that if $s \in S^*$, any element mapping to s must be a unit. Such elements must exist since π was surjective to begin with. $\square$

We enumerate a few useful exact sequences.

$$0 \to \mathscr{O}_X^{++} \to \mathscr{O}_X^{+} \to \tilde{\mathscr{O}}_X \to 0. \tag{2}$$

Because $\mathscr{O}_X^{++}$ consists of topologically nilpotent functions, it is contained in the Jacobson radical of $\mathscr{O}_X^{+}$ so that, by Lemma 3.8, the right-hand map of the sequence remains surjective on unit groups.

$$1 \to 1 + \mathscr{O}_X^{++} \to \mathscr{O}_X^{+*} \to \tilde{\mathscr{O}}_X^{*} \to 1. \tag{3}$$

The goal for this section is to prove the following proposition.

Proposition 3.9. *The map* $\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \to \mathrm{H}^1(X, \tilde{\mathscr{O}}_X^{*})$ *induced by sequence* (3) *is an isomorphism.*

We prove this by deforming along intermediate sheaves. One can consider for every $d \in \mathbb{Z}[1/p]_{>0}$ the sheaf of principal ideals $\varpi^d \subseteq \mathscr{O}_X^{++} \subseteq \mathscr{O}_X^{+}$ and define a sheaf of algebras $\mathscr{A}_d$ via the following exact sequence:

$$0 \to \varpi^d \to \mathscr{O}_X^{+} \to \mathscr{A}_d \to 0. \tag{4}$$

As before, ϖ^d is contained in the Jacobson radical so that we also have

$$1 \to 1 + \varpi^d \to \mathscr{O}_X^{+*} \to \mathscr{A}_d^{*} \to 1. \tag{5}$$

For every $d' > d > 0$ in $\mathbb{Z}[1/p]$, the inclusion $\varpi^{d'} \subseteq \varpi^d$ gives a surjection $\mathscr{A}_{d'} \twoheadrightarrow \mathscr{A}_d$ whose kernel identifies with $\varpi^d/\varpi^{d'}$ by the snake lemma. Passing to unit groups, for all such $d' > d > 0$, we have the following morphisms of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1+\varpi^{d'} & \longrightarrow & \mathscr{O}_X^{+*} & \longrightarrow & \mathscr{A}_{d'}^{*} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & 1+\varpi^{d} & \longrightarrow & \mathscr{O}_X^{+*} & \longrightarrow & \mathscr{A}_{d}^{*} & \longrightarrow & 0. \end{array} \tag{6}$$

These transition maps make sequence (5) into both a directed and inverse system, indexed by positive $d \in \mathbb{Z}[1/p]_{>0}$. We examine the direct and inverse limits of this system (as d approaches 0 and ∞ respectively) to show that the sheaves $\mathscr{A}_d^{*}$ interpolate continuously between $\tilde{\mathscr{O}}_X^{*}$ and $\mathscr{O}_X^{+*}$.

Lemma 3.10. $\varinjlim \mathscr{A}_d^{*} \cong \tilde{\mathscr{O}}_X^{*}$ *and* $\varprojlim \mathscr{A}_d^{*} \cong \mathscr{O}_X^{+*}$.

Proof. To compute the direct limit, we first show that

$$\varinjlim_{d \in \mathbb{Z}[1/p]_{>0}} (1 + \varpi^d) = 1 + \mathscr{O}_X^{++}. \tag{7}$$

We interpret the colimit as a union, and notice equation (7) follows if

$$\mathscr{O}_X^{++} = \bigcup_{d \in \mathbb{Z}[1/p]_{>0}} (\varpi^d)$$

(or more precisely, if this equality holds evaluated on a basis of opens). To see this, we fix an affinoid open $U \subseteq X$ and a topologically nilpotent function $f \in \mathscr{O}_X^{++}(U)$. Since f is topologically nilpotent, we can take large r so that f^{p^r} lands in the ideal of $\mathscr{O}_X^{+}(U)$ generated by ϖ (as this ideal is an open neighborhood of 0). We write $f^{p^r} = g\varpi$ for $g \in \mathscr{O}_X^{+}(U)$. Rephrasing, we see

$$\left(\frac{f}{\varpi^{1/p^r}}\right)^{p^r} \in \mathscr{O}_X^{+}(U).$$

But $\mathscr{O}_X^{+}(U)$ is integrally closed in $\mathscr{O}_X(U)$, so this in turn implies that

$$f/\varpi^{1/p^r} \in \mathscr{O}_X^{+}(U)$$

or, equivalently, that f is contained in the ideal of $\mathscr{O}_X^{+}(U)$ generated by ϖ^{1/p^r}. In particular, f is contained in the union of the ideals generated by the ϖ^d for $d \in \mathbb{Z}[1/p]_{>0}$. This completes the verification of equation (7). Since colimits of abelian sheaves are exact, equation (7) implies that taking the colimit over all $d \in \mathbb{Z}[1/p]_{>0}$ of sequence (5) produces sequence (3), exhibiting the desired isomorphism.

For the inverse limit, notice that $\varprojlim(\mathscr{A}_d) \cong \mathscr{O}_X^{+}$ since $\mathscr{O}_X^{+}$ is ϖ-adically complete. Since the unit group functor commutes with inverse limits (indeed, it is right adjoint to the group ring functor), we are done. □

We next assert that sequences (2) through (5) induce long exact sequences on Čech cohomology.

Lemma 3.11. *Each of the sequences* (2) *through* (5) *remain exact when evaluated on any finite intersection of elements in the standard cover* $\mathfrak{U}$. *In particular, they each induce long exact sequences on Čech cohomology.*

Proof. We need only check exactness on the right. For sequences (2) and (4), these evaluate to

$$K^\circ\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}\rangle \\ \to k[X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}]$$

and

$$K^\circ\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}\rangle \\ \to K^\circ/\varpi^d[X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}, X_{r+1}^{\pm 1/p^\infty}, \dots, X_n^{\pm 1/p^\infty}]$$

which are plainly surjective. Sequences (3) and (5) come from applying the unit group functor to surjections above. Lemma 2.16 says that the kernel of each is contained in the Jacobson radical so that the maps remain surjective on unit groups by Lemma 3.8. □

Lemma 3.12. *For all $d \in \mathbb{Z}[1/p]_{>0}$ and $i > 0$, $\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{A}_d) = 0$.*

Proof. This follows from the long exact sequence on Čech cohomology associated to sequence (4) and Lemma 3.7, noticing that $\varpi^d \cong \mathscr{O}_X^+$ since it is a principal ideal. □

Lemma 3.13. *For all $d \in \mathbb{Z}[1/p]_{>0}$ and $i > 0$, the natural map*

$$\mathrm{H}^i(\mathfrak{U}, \mathscr{A}_{2d}^*) \to \mathrm{H}^i(\mathfrak{U}, \mathscr{A}_d^*)$$

is an isomorphism. If $i = 1$, the isomorphism holds for derived functor cohomology as well.

Proof. Consider the following case of diagram (6):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1+\varpi^{2d} & \longrightarrow & \mathscr{O}_X^{+*} & \longrightarrow & \mathscr{A}_{2d}^* & \longrightarrow & 0 \\ & & \big\downarrow & & \| & & \big\downarrow & & \\ 0 & \longrightarrow & 1+\varpi^{d} & \longrightarrow & \mathscr{O}_X^{+*} & \longrightarrow & \mathscr{A}_{d}^* & \longrightarrow & 0. \end{array}$$

The snake lemma exhibits the exact sequence

$$1 \to 1 + \varpi^d/\varpi^{2d} \to \mathscr{A}_{2d}^* \to \mathscr{A}_d^* \to 1. \tag{8}$$

Notice that $1 + \varpi^d/\varpi^{2d}$ has a natural $\mathscr{A}_d$-module structure making it isomorphic to $\mathscr{A}_d$, given locally by the map $a \mapsto 1 + a\varpi^d$. Indeed, the map is well-defined because ϖ^d/ϖ^{2d} is a square zero ideal, and the kernel is precisely ϖ^d (which is 0 in $\mathscr{A}_d$), while surjectivity is clear. In particular, by Lemma 3.12, $1 + \varpi^d/\varpi^{2d}$ has no higher Čech cohomology, so the conclusion follows if there is a long exact sequence on Čech cohomology associated sequence (8).

Let U be a finite intersection of elements of the standard cover. It remains to show that sequence (8) remains exact when evaluated on U. By Lemma 3.11,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(U, 1+\varpi^{2d}) & \longrightarrow & \Gamma(U, \mathscr{O}_X^{+*}) & \longrightarrow & \Gamma(U, \mathscr{A}_{2d}^*) & \longrightarrow & 0 \\ & & \big\downarrow & & \| & & \big\downarrow & & \\ 0 & \longrightarrow & \Gamma(U, 1+\varpi^{d}) & \longrightarrow & \Gamma(U, \mathscr{O}_X^{+*}) & \longrightarrow & \Gamma(U, \mathscr{A}_{d}^*) & \longrightarrow & 0 \end{array}$$

has exact rows. The ring map $\mathscr{A}_{2d}(U) \to \mathscr{A}_d(U)$ is surjective with nilpotent kernel so that, again applying Lemma 3.8, we see the vertical map on the right is surjective. Thus the snake lemma exhibits sequence (8) evaluated at U as an exact sequence, completing the proof. Lemma 3.5 extends the result to derived functor cohomology if $i = 1$. □

Lemma 3.14. *For all $d' > d > 0$ in $\mathbb{Z}[1/p]$ and $i > 0$, the natural map*

$$\mathrm{H}^i(\mathfrak{U}, \mathscr{A}_{d'}^*) \to \mathrm{H}^i(\mathfrak{U}, \mathscr{A}_d^*),$$

is an isomorphism. If $i = 1$, the isomorphism also holds for derived functor cohomology.

Proof. By Lemma 3.13, replacing d with $2^l d$ preserves the cohomology groups in question, so we may assume $d < d' < 2d$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} & \mathrm{H}^i(\mathfrak{U}, \mathscr{A}_{2d}^*) & \xrightarrow{\sim} & & \mathrm{H}^i(\mathfrak{U}, \mathscr{A}_d^*) \\ \nearrow & & \searrow^{\psi} & \nearrow & \\ \mathrm{H}^i(\mathfrak{U}, \mathscr{A}_{2d'}^*) & \xrightarrow{\sim} & & \mathrm{H}^i(\mathfrak{U}, \mathscr{A}_{d'}^*). & \end{array}$$

In particular, ψ is injective and surjective, hence an isomorphism. Lemma 3.5 extends the result to derived functor cohomology if $i = 1$. □

Now that we have this set-up, we can prove the main result of this section.

Proof of Proposition 3.9. We can factor the homomorphism in question into the following chain:

$$\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \cong \mathrm{H}^1(X, \varprojlim \mathscr{A}_d^*) \tag{9}$$
$$\to \varprojlim \mathrm{H}^1(X, \mathscr{A}_d^*) \tag{10}$$
$$\cong \mathrm{H}^1(X, \mathscr{A}_d^*) \tag{11}$$
$$\cong \varinjlim \mathrm{H}^1(X, \mathscr{A}_d^*) \tag{12}$$
$$\cong \mathrm{H}^1(X, \varinjlim \mathscr{A}_d^*) \tag{13}$$
$$\cong \mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*). \tag{14}$$

Let us justify the isomorphisms. (9) is Lemma 3.10, (11) follows because the transition maps of the inverse system are the isomorphisms from Lemma 3.14, and similarly for (12). (13) is an isomorphism because colimits of abelian sheaves are exact, and (14) is again Lemma 3.10. We therefore can make the necessary identifications to view the composition as a homomorphism between the following groups:

$$\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \to \varprojlim \mathrm{H}^1(X, \mathscr{A}_d^*).$$

We can view the first group as isomorphism classes of invertible $\mathscr{O}_X^+$-modules, and the second as isomorphisms classes of inverse systems of invertible $\mathscr{A}_d$-modules. In particular, elements of the target can be represented by inverse systems $\{\mathscr{M}_d\}_{d \in \mathbb{Z}[1/p]_{>0}}$ of invertible $\mathscr{A}_d$-modules such that, for every $d' > d$, the transition map $\mathscr{M}_{d'} \to \mathscr{M}_d$ induces an isomorphism

$$\mathscr{M}_{d'}/\varpi^d \mathscr{M}_{d'} \xrightarrow{\sim} \mathscr{M}_d.$$

The map in question can be represented by sending an invertible $\mathscr{O}_X^+$-module $\mathscr{L}$ to the inverse system

$$\{\mathscr{L}/\varpi^d \mathscr{L}\}_{d \in \mathbb{Z}[1/p]_{>0}}.$$

Then we observe that this has an inverse given by taking an inverse system $\{\mathscr{M}_d\}$ of invertible $\mathscr{A}_d$-modules to the isomorphism class of the inverse limit

$\varprojlim \mathscr{M}_d$. It is not hard to see that these constructions are inverses to each other. Indeed, there is a natural map

$$\mathscr{L} \to \varprojlim \mathscr{L}/\varpi^d \mathscr{L}.$$

Affinoid locally on $\mathrm{Spa}(R, R^+)$, we associate $\mathscr{L}$ to an invertible R-module M, and this map becomes

$$M \to \varprojlim M/\varpi^d M \cong \hat{M},$$

which is an isomorphism since M is already complete. Conversely, fix a system $\{\mathscr{M}_d\}$ of invertible $\mathscr{A}_d$-modules. For $e \in \mathbb{Z}[1/p]_{>0}$, we consider the natural map $\varprojlim \mathscr{M}_d \to \mathscr{M}_e$. For every $d' > d > e$ in $\mathbb{Z}[1/p]$, we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varpi^e \mathscr{M}_{d'} & \longrightarrow & \mathscr{M}_{d'} & \longrightarrow & \mathscr{M}_e & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \varpi^e \mathscr{M}_{d} & \longrightarrow & \mathscr{M}_{d} & \longrightarrow & \mathscr{M}_e & \longrightarrow & 0. \end{array}$$

Passing to the limit gives

$$0 \to \varprojlim \varpi^e \mathscr{M}_d \to \varprojlim \mathscr{M}_d \to \mathscr{M}_e \to \varprojlim{}^{(1)} \varpi^e \mathscr{M}_d.$$

Since the map $\mathscr{M}_{d'} \to \mathscr{M}_d$ is surjective, it remains so after scaling by ϖ^e. In particular, $\{\varpi^e \mathscr{M}_d\}_{d \in \mathbb{Z}[1/p]_{>e}}$ is an ML-system of abelian sheaves, and so its derived limits vanish. Therefore, $\varprojlim \mathscr{M}_d \to \mathscr{M}_e$ surjects. Since the kernel is $\varprojlim \varpi^e \mathscr{M}_d = \varpi^e \varprojlim \mathscr{M}_d$, we obtain an isomorphism of inverse systems

$$\{\varprojlim \mathscr{M}_d / \varpi^e \varprojlim \mathscr{M}_d\} \cong \{\mathscr{M}_d\}.$$

This completes the proof. ☐

Comparison to the perfection of the special fiber. By Propositions 3.6 and 3.9, there is an isomorphism $\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*)$. To complete the proof of Theorem 3.4, we compute the latter cohomology group explicitly. To do this, we will show that this group is closely related to the Picard group of the *special fiber* of X, which is the (scheme theoretic) perfection of projective space $\mathbb{P}_k^n$ over the residue field k (which is a perfect field of characteristic p). Denote the latter by $(\mathbb{P}_k^n)_{\mathrm{perf}}$, where the subscript perf denotes scheme theoretic perfection, given by taking the inverse limit along the Frobenius map.

Lemma 3.15. $\mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*) \cong \mathrm{Pic}(\mathbb{P}_k^n)_{\mathrm{perf}}$.

Proof. Let $V_i = \mathrm{Spec}\, k\big[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\big]$ so that $\mathfrak{V} = \{V_i \to \mathbb{P}_k^n\}$ is the standard cover of projective space over k. Taking (scheme theoretic) perfection and recalling that the Frobenius map is a homeomorphism [24, Tag 0CC8], we obtain an open cover $\mathfrak{V}_{\mathrm{perf}} = \{(V_i)_{\mathrm{perf}} \to (\mathbb{P}_k^n)_{\mathrm{perf}}\}$ by affines of the form

$$V_i = \mathrm{Spec}\, k\bigg[\bigg(\frac{T_0}{T_i}\bigg)^{1/p^\infty}, \ldots, \bigg(\frac{T_n}{T_i}\bigg)^{1/p^\infty}\bigg].$$

In particular, we can identify $\Gamma(U_i, \tilde{\mathscr{O}}_X) \cong \Gamma(V_i, \mathscr{O}_{(\mathbb{P}^n_k)_{\mathrm{perf}}})$. The same can be said for the various intersections, which correspond to inverting indeterminates, and the restriction morphisms, which are just inclusions. Passing to unit groups therefore gives identification of the Čech complexes for $\tilde{\mathscr{O}}_X^*$ and $\mathscr{O}^*_{(\mathbb{P}^n_k)_{\mathrm{perf}}}$ with respect to $\mathfrak{U}$ and $\mathfrak{V}_{\mathrm{perf}}$ respectively. After passing to cohomology, we obtain isomorphisms

$$\check{\mathrm{H}}^i(\mathfrak{U}, \tilde{\mathscr{O}}_X^*) \cong \check{\mathrm{H}}^i(\mathfrak{V}_{\mathrm{perf}}, \mathscr{O}^*_{(\mathbb{P}^n_k)_{\mathrm{perf}}}).$$

Letting $i = 1$ and appealing to Lemma 3.5 identifies the source with $\mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*)$. A similar argument identifies the target with $\mathrm{Pic}(\mathbb{P}^n_k)_{\mathrm{perf}}$. Indeed, due to Corollary 2.12, every line bundle on the perfection of projective space trivializes on the V_i and their various intersections, so the identification follows from the Čech-to-derived spectral sequence by the same logic as in Lemma 3.5. □

Corollary 3.16. $\mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*) \cong \mathbb{Z}[1/p]$.

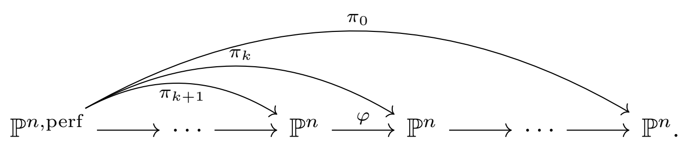
Proof. This follows from Lemma 3.15, noting that $\mathrm{Pic}(\mathbb{P}^n_k)_{\mathrm{perf}} \cong \mathbb{Z}[1/p]$, which follows, for example, from [4, Lem. 3.5]. This isomorphism is not hard to see directly, so we include a sketch for completeness. Since $(\mathbb{P}^n_k)_{\mathrm{perf}}$ is the inverse limit of $\mathbb{P}^n_k$ along affine transition morphisms, a globalization of Lemma 2.11 (cp. [24, Tag OB8W]) implies its Picard group is the colimit of $\mathrm{Pic}\,\mathbb{P}^n_k$ along Frobenius pullback, which is the pth power map. □

We now complete the computation of the Picard group of projectivoid space.

Proof of Theorem 3.4. By Propositions 3.6 and 3.9, and Corollary 3.16, we have

$$\mathrm{Pic}\, X \cong \mathrm{H}^1(X, \mathscr{O}_X^*) \cong \mathrm{H}^1(X, \mathscr{O}_X^{+*}) \cong \mathrm{H}^1(X, \tilde{\mathscr{O}}_X^*) \cong \mathbb{Z}[1/p]. \qquad \square$$

Twisting sheaves and integral twisting sheaves. Theorem 3.4 indexes isomorphism classes of line bundles on $\mathbb{P}^{n,\mathrm{perf}}$ by numbers $d \in \mathbb{Z}[1/p]$. Before moving on, we construct an explicit line bundle for each such d, which we will call the *twisting sheaf* $\mathscr{O}_X(d)$. Recall from Definition 3.2, $\mathbb{P}^{n,\mathrm{perf}} \sim \varprojlim_{\varphi} \mathbb{P}^n$. In particular, we have the following diagram:

(15)


Passing to Picard groups gives a map

$$\varinjlim_{\varphi^*} \mathrm{Pic}\,\mathbb{P}^n \to \mathrm{Pic}\,\mathbb{P}^{n,\mathrm{perf}}.$$

Since φ^* is the p-power map on $\mathrm{Pic}\,\mathbb{P}^n$, the source is isomorphic to $\mathbb{Z}[1/p]$ so that, by Theorem 3.4, the map is in fact an isomorphism. (Indeed, our technique of proof of Theorem 3.4 essentially consisted of deforming the analogous map for scheme theoretic perfections over the residue field k.) This allows us to make the following definition.

Definition 3.17. Fix $d = a/p^k \in \mathbb{Z}[1/p]$; taking the notation of diagram (15), we define the twisting sheaf

$$\mathscr{O}_X(d) = \mathscr{O}_X(a/p^k) \coloneqq \pi_k^* \mathscr{O}_{\mathbb{P}^n}(a).$$

We next evaluate $\mathscr{O}_X(d)$ on the elements of the standard cover of X and their various intersections, which sets us up for the cohomology computation of the following section.

Proposition 3.18. *Let $A = K[T_0^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]$, and fix $d \in \mathbb{Z}[1/p]$. Then*

$$\Gamma(U_{i_1} \cap \cdots \cap U_{i_r}, \mathscr{O}_X(d)) \cong (A_{T_{i_1} \cdots T_{i_r}})_d^\wedge,$$

where the right side is given by inverting T_{i_1} through T_{i_n} in A, taking the degree-d part, and then completing with respect to ϖ.

Proof. Let $B^{(k)} = K[T_0^{1/p^k}, \ldots, T_n^{1/p^k}]$ and $(B^{(k)})^\circ = K^\circ[T_0^{1/p^k}, \ldots, T_n^{1/p^k}]$. The standard (rigid analytic) cover of $\mathbb{P}^n$ by affinoid disks is $\mathfrak{W} = \{W_i\}$, where

$$W_i = \operatorname{Spa}((B^{(0)}_{T_i})_0^\wedge, ((B^{(0)})^\circ_{T_i})_0^\wedge).$$

Then $U_i \sim \varprojlim W_i$, and the restriction of diagram (15) to these affinoid opens corresponds to the following inclusions of rings:

$$(A_{T_i})_0^\wedge \leftarrow \cdots \leftarrow (B^{(k+1)}_{T_i})_0^\wedge \leftarrow (B^{(k)}_{T_i})_0^\wedge \leftarrow \cdots \leftarrow (B^{(0)}_{T_i})_0^\wedge. \tag{16}$$

Identical considerations hold for the intersections of the U_i and W_i by inverting more indeterminates.

Fix an integer $a \in \mathbb{Z}$, and consider $\mathscr{O}_{\mathbb{P}^n}(a) \in \operatorname{Pic}(\mathbb{P}^n)$. Let V_i be the complement of the coordinate hyperplane $V(T_i)$ in $\mathbb{P}^n$. Then it is classical that there is a natural identification

$$\Gamma(V_{i_1} \cap \cdots \cap V_{i_r}, \mathscr{O}(a)) = (B^{(0)}_{T_{i_1} \cdots T_{i_r}})_a$$

of the sections of $\mathscr{O}(a)$ at the intersection of the V_{i_j} with the degree-a part of the polynomial ring after inverting the corresponding T_{i_j}. Restriction to the affinoid open $W_{i_1} \cap \cdots \cap W_{i_r}$ corresponds to the algebraic operation of ϖ-adically completing so that

$$\Gamma(W_{i_1} \cap \cdots \cap W_{i_r}, \mathscr{O}(a)) = (B^{(0)}_{T_{i_1} \cdots T_{i_r}})_a^\wedge.$$

Now let $d = a/p^k$, and consider $\mathscr{O}_X(d) = \pi_k^* \mathscr{O}_{\mathbb{P}^n}(a)$. Identify $W_{i_1} \cap \cdots \cap W_{i_r}$ in the kth factor of the tower of diagram (15) with the adic space associated to $(B^{(k)}_{T_{i_1} \cdots T_{i_r}})_0$ as in diagram (16). Then we have

$$\Gamma(W_{i_1} \cap \cdots \cap W_{i_r}, \mathscr{O}(a)) \cong (B^{(k)}_{T_{i_1} T_{i_2} \cdots T_{i_r}})_d^\wedge.$$

Then (denoting the intersection $U_{i_1} \cap \cdots \cap U_{i_r}$ by $U_{i_1 \cdots i_r}$ and similarly for W)

$$\begin{aligned}
\Gamma(U_{i_1 \cdots i_r}, \mathscr{O}(a/p^k)) &\cong \Gamma(W_{i_1 \cdots i_r}, \mathscr{O}(a)) \otimes_{(B^{(k)}_{T_{i_1} \cdots T_{i_r}})_0^\wedge} (A_{T_{i_1} \cdots T_{i_r}})_0^\wedge \\
&\cong ((B^{(k)}_{T_{i_1} T_{i_2} \cdots T_{i_r}})_d \otimes_{(B^{(k)}_{T_{i_1} \cdots T_{i_r}})_0} (A_{T_{i_1} \cdots T_{i_r}})_0)^\wedge \\
&\cong (A_{T_{i_1} \cdots T_{i_r}})_d^\wedge.
\end{aligned}$$

□

Notice that Proposition 3.18 gives explicit descent data for $\mathscr{O}_X(d)$ with respect to the standard cover. Indeed, $\mathscr{O}_X(d)|_{U_i}$ is identified with the free rank-1 $\mathscr{O}_{U_i}$-module

$$(A_{T_i})^\wedge_d = T_i^d \cdot \left(K\left\langle \left(\frac{T_0}{T_i}\right)^{1/p^\infty}, \dots, \left(\frac{T_n}{T_i}\right)^{1/p^\infty} \right\rangle \right),$$

with gluing given by the identification

$$(A_{T_iT_j})^\wedge_d = (A_{T_jT_i})^\wedge_d.$$

This suggests a way of defining invertible $\mathscr{O}_X^+$-modules for each $d \in \mathbb{Z}[1/p]$.

Definition 3.19. The *integral twisting sheaf* $\mathscr{O}_X^+(d)$ is defined via the following descent data with respect to the natural cover. On U_i, we define

$$\mathscr{O}_{U_i}^+(d) := T_i^d \cdot \mathscr{O}_{U_i}^+$$

and obtain $\mathscr{O}_X^+(d)$ by gluing along the $\mathscr{O}_{U_i \cap U_j}$-module isomorphisms

$$\mathscr{O}_{U_i}^+(d)|_{U_j} \cong \mathscr{O}_{U_j}^+(d)|_{U_i},$$

given by multiplying by T_j^d/T_i^d.

Remark 3.20. We record that (essentially by definition) an analog for Proposition 3.18 holds for $\mathscr{O}_X^+(d)$, replacing A with $A^\circ = K^\circ[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$, and that $\mathscr{O}_X^+(d) \otimes_{\mathscr{O}_X^+} \mathscr{O}_X \cong \mathscr{O}_X(d)$. Finally, these form a group under tensor product (over $\mathscr{O}_X^+$), which due to Proposition 3.9 and Corollary 3.16 is isomorphic to

$$\mathrm{H}^1(X, \mathscr{O}_X^{+*}) \cong \mathbb{Z}[1/p].$$

Cohomology of line bundles. We now compute the cohomology of the twisting sheaves and integral twisting sheaves defined above. There are partial results in this direction due to Bedi [2] in the case $n = 2$. Our general strategy is the following. Adopting the notation of the previous section, we let

$$A = K[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}] \quad \text{and} \quad A^\circ = K^\circ[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$$

and build the following complexes:

$$C^*(A) : 0 \to \prod_{i=0}^{n+1} A_{T_i} \to \prod_{i<j} A_{T_iT_j} \to \cdots \to A_{T_0\cdots T_n} \to 0,$$

$$C^*(A^\circ) : 0 \to \prod_{i=0}^{n+1} A^\circ_{T_i} \to \prod_{i<j} A^\circ_{T_iT_j} \to \cdots \to A^\circ_{T_0\cdots T_n} \to 0.$$

The differentials are given by the alternating sums of the obvious inclusions on the factors. These complexes are $\mathbb{Z}[1/p]$-graded, and by Proposition 3.18, the completion of the d-graded piece corresponds to the Čech complex of $\mathscr{O}_X(d)$ (respectively $\mathscr{O}_X^+(d)$) on the standard cover. Therefore (up to a completion), it will suffice to study the cohomology $C^*(A)$ and $C^*(A^\circ)$. To begin, we record some facts that allow us to safely take completions while preserving cohomological data.

Lemma 3.21. *Let M be a linearly topologized K°-module, and give $M \otimes K$ the induced topology. Then $\hat{M} \otimes K \cong (M \otimes K)^\wedge$.*

Proof. Since $M \otimes K$ is given the topology making M open, the basis of 0 given by the $\varpi^n M$ is also a basis for the topology $M \otimes K$. $\square$

We next record that completion of a complex of topological modules commutes with taking cohomology, adapting the following lemma from [7].

Lemma 3.22 ([7, Lem. 5.9])**.** *Let $M^* = M^0 \to M^1 \to \cdots \to M^n$ be a complex of linearly topologized K°-modules with countable systems of fundamental neighborhoods of 0. Then the following isomorphisms hold:*
(1) $\mathrm{H}^i(M^*)^\wedge \cong \mathrm{H}^i(\hat{M}^*)$,
(2) $\mathrm{H}^i(M^* \otimes K)^\wedge \cong \mathrm{H}^i((M^* \otimes K)^\wedge)$.

Proof. (1) is [24, Tag 0AS0]. To pass to the generic fiber, we use Lemma 3.21 and the fact that $\cdot \otimes K$ is an exact functor to establish the following chain of isomorphisms:

$$\begin{aligned}(\mathrm{H}^i(M^* \otimes K))^\wedge &\cong (\mathrm{H}^i(M^*) \otimes K)^\wedge \\ &\cong \mathrm{H}^i(M^*)^\wedge \otimes K \\ &\cong \mathrm{H}^i(\hat{M}^*) \otimes K \\ &\cong \mathrm{H}^i(\hat{M}^* \otimes K) \\ &\cong \mathrm{H}^i((M^* \otimes K)^\wedge).\end{aligned}$$ $\square$

Since $C^*(A)$ and $C^*(A^\circ)$ are complexes of $\mathbb{Z}[1/p]$-graded modules, we can extract their d-graded pieces,

$$C^*(A_d) : 0 \to \prod (A_{T_i})_d \to \prod (A_{T_iT_j})_d \to \cdots \to (A_{T_0 \cdots T_n})_d \to 0.$$

$$C^*(A_d^\circ) : 0 \to \prod (A^\circ_{T_i})_d \to \prod (A^\circ_{T_iT_j})_d \to \cdots \to (A^\circ_{T_0 \cdots T_n})_d \to 0.$$

Cohomology respects direct sums (for example, by [23, Ex. 1.2.1]), so the grading descends to a grading on cohomology

$$\mathrm{H}^i(C^*(A)) \cong \bigoplus_{d \in \mathbb{Z}[1/p]} \mathrm{H}^i(C^*(A_d)), \quad \mathrm{H}^i(C^*(A^\circ)) \cong \bigoplus_{d \in \mathbb{Z}[1/p]} \mathrm{H}^i(C^*(A_d^\circ)).$$

If $\mathfrak{U}$ is the standard cover of X, then Proposition 3.18 provides identifications

$$C^*(A_d)^\wedge \cong \check{\mathrm{C}}(\mathfrak{U}, \mathscr{O}_X(d)), \quad C^*(A_d^+)^\wedge \cong \check{\mathrm{C}}(\mathfrak{U}, \mathscr{O}_X^+(d)).$$

Putting this together, we arrive at our desired comparison.

Lemma 3.23. *For each $d \in \mathbb{Z}[1/p]$ and $i > 0$, we have isomorphisms*
- $\mathrm{H}^i(C^*(A_d))^\wedge \cong \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X(d))$,
- $\mathrm{H}^i(C^*(A_d^\circ))^\wedge \cong \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^+(d))$.

Proof. Applying Lemma 3.22, we observe

$$\mathrm{H}^i(C^*(A_d))^\wedge \cong \mathrm{H}^i(C^*(A_d)^\wedge) \cong \mathrm{H}^i(\check{\mathrm{C}}^*(\mathfrak{U}, \mathscr{O}_X(d))) = \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X(d)),$$

$$\mathrm{H}^i(C^*(A_d^\circ))^\wedge \cong \mathrm{H}^i(C^*(A_d^\circ)^\wedge) \cong \mathrm{H}^i(\check{\mathrm{C}}^*(\mathfrak{U}, \mathscr{O}_X^+(d))) = \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X^+(d)).$$ $\square$

Therefore, we have reduced to computing the cohomology of $C^*(A)$, $C^*(A^\circ)$ as $\mathbb{Z}[1/p]$-graded modules and considering the completions of the graded pieces. We will work slightly more generally.

Proposition 3.24. *Let R be a commutative domain, and let B be a "perfected" polynomial ring over R: $B = R[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$. Form the cochain complex*

$$C^*(B) : 0 \to \prod B_{T_i} \to \prod B_{T_iT_j} \to \cdots \to B_{T_0\cdots T_n} \to 0.$$

Then

$$\mathrm{H}^i(C^*(B)) = \begin{cases} B, & i = 0, \\ 0, & i \neq 0, n, \\ \langle T_1^{\alpha_1} \cdots T_n^{\alpha_n} \rangle_{\alpha_i \in \mathbb{Z}[1/p]_{<0}}, & i = n. \end{cases}$$

Explicitly, the third case is the free R-module generated by monomials in the T_i, all of whose exponents are negative elements of $\mathbb{Z}[1/p]$. The $\mathbb{Z}[1/p]$-grading is given by the sums of the exponents.

Proof. We first compute H^0, noting that an element $(f_i) \in \prod B_{T_i}$ is in the kernel of the first differential if and only if $f_i = f_j$ for all $i < j$. In particular, this allows us to identify the kernel with

$$\mathrm{H}^0(C^*(B)) = \bigcap_{i=0}^{n} B_{T_i} = B.$$

We next turn our attention to H^n. Notice that $B_{T_0\cdots T_n}$ is the free R-module generated by monomials $T_0^{\alpha_0} \cdots T_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}[1/p]$. The image of the $(n-1)$st differential is the free R-submodule generated by monomials where at least one of the $\alpha_i \geq 0$. Therefore, $\mathrm{H}^n(C^*(B))$, which is the cokernel of this differential, is the free R-module generated by monomials where each $\alpha_i < 0$.

To show that the remaining cohomology vanishes, we will exhibit $C^*(B)$ (in positive degrees) as a colimit of Koszul complexes. For all $s \in \mathbb{Z}[1/p]_{>0}$, let $T^s = (T_0^s, \dots, T_n^s)$. We claim that T^s is a regular sequence. Indeed, write $s = a/p^r$, and observe that T^{1/p^r} is a regular sequence for the polynomial ring $B_r \coloneqq R[T_0^{1/p^r}, \dots, T_n^{1/p^r}]$. Therefore, so is T^s (for example, by [24, Tag 07DV]). Returning to B, suppose T_i^s is a zero divisor in $B/(T_1^s, \dots, T_{i-1}^s)$. This would mean that there are $f, a_1, \dots, a_{i-1} \in B$ with

$$f \notin (T_1^s, \dots, T_{i-1}^s) \quad \text{and} \quad fT_i^s = a_1T_1^s + \cdots + a_{i-1}T_{i-1}^s.$$

But for r large enough, $f, T_1^s, \dots, T_i^s, a_1, \dots, a_{i-1} \in B_r$, contradicting that T^s forms a regular sequence for B_r, thereby establishing that T^s is a regular sequence for B. With this in hand, we consider the Koszul complex

$$K^*(T^s) : 0 \longrightarrow B \xrightarrow{T^s} B^{n+1} \xrightarrow{\wedge T^s} \Lambda^2 B^{n+1} \longrightarrow \cdots \longrightarrow \Lambda^{n+1} B^{n+1} \longrightarrow 0.$$

Since T^s is a B-regular sequence, we know by [8, Cor. 7.5] that $K^*(T^s)$ is exact, except at the very right (indeed, it is a free resolution of $B/(T_0^s, \dots, T_n^s)$). In particular, for all $i < n+1$,

$$\mathrm{H}^i(K^*(T^s)) = 0. \tag{17}$$

Before passing to the colimit as s goes to infinity, we record that, for any $f \in B$,

$$\varinjlim(B \xrightarrow{\cdot f} B \xrightarrow{\cdot f} \cdots) \cong B_f.$$

The isomorphism comes from mapping b in the rth coordinate of the directed system to b/f^r. This passes to a map from the colimit which is plainly surjective. On the other hand, suppose that an element b maps to zero in the localization. This says that $f^N b = 0$ in B for N large enough so that, in fact, b will map to 0 in the colimit as well, giving injectivity. With this in mind, we fit $K^*(T^s)$ into the following diagram, taking the colimit as s goes to infinity:

$$\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{T^s} & B^{n+1} & \xrightarrow{\cdot\wedge T^s} & \Lambda^2 B^{n+1} & \longrightarrow & \cdots & \longrightarrow & \Lambda^{n+1}B^{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \cdot T} & & \downarrow{\scriptstyle \cdot(T\wedge T)} & & & & \downarrow{\scriptstyle \cdot T^{\wedge(n+1)}} & & \\
0 & \longrightarrow & B & \xrightarrow{T^{s+1}} & B^{n+1} & \xrightarrow{\cdot\wedge T^{s+1}} & \Lambda^2 B^{n+1} & \longrightarrow & \cdots & \longrightarrow & \Lambda^{n+1}B^{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & \prod B_{T_i} & \longrightarrow & \prod B_{T_iT_j} & \longrightarrow & \cdots & \longrightarrow & B_{T_0\cdots T_n} & \longrightarrow & 0.
\end{array}$$

Since colimits of finite complexes commute with cohomology, equation (17) implies that $\varinjlim K^*(T^s)$ is exact, except at the very right. Observe that there is an obvious map

$$C^*(B)[1] \to \varinjlim K^*(T^s),$$

which is an isomorphism in degrees $\neq 0$. (Indeed, the source is just the "stupid" truncation $\sigma_{\geq 1} \varinjlim K^*(T^s)$.) Therefore, for all $0 < i < n$, we have

$$\mathrm{H}^i(C^*(B)) = \mathrm{H}^{i+1}(C^*(B)[1]) = \mathrm{H}^{i+1}(\varinjlim K^*(T^s)) = 0. \qquad \square$$

With Lemma 3.23 in mind, we may pass to the graded pieces and take completions to obtain the desired cohomology computations.

Proposition 3.25. *Let $X = \mathbb{P}^{n,\mathrm{perf}}$, and let $\mathfrak{U}$ be the standard cover. Fix $d \in \mathbb{Z}[1/p]$.*

(i) *If $d \geq 0$, then $\check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{O}_X^+(d)) = (A^\circ)^\wedge_d$.*

(ii) *If $d < 0$, then $\mathrm{H}^n(\mathfrak{U}, \mathscr{O}_X^+(d))$ is the completion of the free K°-module generated by monomials of degree d, where the degree of each indeterminate is strictly negative, that is,*

$$\check{\mathrm{H}}^n(\mathfrak{U}, \mathscr{O}_X^+(d)) = \Big\langle T_0^{\alpha_0} \cdots T_n^{\alpha_n} \;\Big|\; \alpha_i \in \mathbb{Z}[1/p]_{<0} \text{ and } \sum \alpha_i = d \Big\rangle^\wedge.$$

(iii) *In all other cases, $\check{\mathrm{H}}^r(\mathfrak{U}, \mathscr{O}_X^+(d)) = 0$.*

Theorem 3.26. *Let $X = \mathbb{P}^{n,\mathrm{perf}}$, and fix $d \in \mathbb{Z}[1/p]$.*
(i) *If $d \geq 0$, then $\mathrm{H}^0(X, \mathscr{O}_X(d)) = A_d^\wedge$.*
(ii) *If $d < 0$, then $\mathrm{H}^n(X, \mathscr{O}_X(d))$ is the completion of the K-vector space generated by monomials of degree d, where the degree of each indeterminate is strictly negative, that is,*

$$\mathrm{H}^n(X, \mathscr{O}_X(d)) = \Big\langle T_0^{\alpha_0} \cdots T_n^{\alpha_n} \;\Big|\; \alpha_i \in \mathbb{Z}[1/p]_{<0} \text{ and } \sum \alpha_i = d \Big\rangle^\wedge.$$

(iii) *In all other cases, $\mathrm{H}^r(X, \mathscr{O}_X(d)) = 0$.*

Proof. Proposition 3.25 follows immediately from Lemma 3.23 and the computation in Proposition 3.24, and for Theorem 3.26, one obtains the Čech cohomology the same way. Thus the only thing we have not already established is that

$$\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X(d)) \cong \mathrm{H}^i(X, \mathscr{O}_X(d)).$$

For this, we again use the Čech-to-derived functor spectral sequence as in the proof of Lemma 3.5,

$$E_2^{p,q} : \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{H}^q(\mathscr{O}_X(d))) \implies \mathrm{H}^{p+q}(X, \mathscr{O}_X(d)).$$

Let U be a finite intersection of elements of the standard cover. Then we have $\mathscr{O}_X|_U \cong \mathscr{O}_X(d)|_U$, and since U is affinoid perfectoid, $\mathrm{H}^i(U, \mathscr{O}_X) = 0$ for all $i > 0$. Therefore, the spectral sequence degenerates to our desired result. □

Remark 3.27. The last step of the proof does not immediately hold for integral line bundles because, a priori, $\mathscr{O}_X^+$ is only *almost acyclic* on affinoid perfectoids. This is why Proposition 3.25 is only stated on the level of Čech cohomology groups. Nevertheless, Čech cohomology is sufficient for the proof of Lemma 3.7, which is now complete.

4. Maps to projectivoid space

Suppose S is a scheme over K. There is a well-known correspondence between maps from $S \to \mathbb{P}^n$ over K and globally generated line bundles on S together with a choice of $n+1$ generating global sections (see, for example, [10, Thm. II.7.1]). In this section, we will prove an analog of this correspondence for perfectoid spaces.

Definition 4.1. To a perfectoid space X over K, we associate a groupoid $\mathfrak{L}_n(X)$ whose objects consist of tuples

$$(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \quad \text{for } i \geq 0 \text{ and } j = 0, \ldots, n,$$

where $\mathscr{L}_i$ are line bundles on X, $s_0^{(i)}, \ldots, s_n^{(i)} \in \Gamma(X, \mathscr{L}_i)$ are generating global sections, and $\alpha_i : \mathscr{L}_{i+1}^{\otimes p} \xrightarrow{\sim} \mathscr{L}_i$ are isomorphisms mapping $(s_j^{(i+1)})^{\otimes p} \mapsto s_j^{(i)}$. Morphisms are isomorphisms of line bundles which are compatible with the global sections and isomorphisms α_i.

If $f : X \to Y$ is a K-morphism, we get a pullback functor $f^* : \mathfrak{L}_n(Y) \to \mathfrak{L}_n(X)$ so that $\mathfrak{L}_n$ is a category fibered in groupoids.

Remark 4.2. Note that if some α_i exists, it is unique. Indeed, for each i, the global sections $s_j^{(i)}$ generate $\mathscr{L}_i$ so that an isomorphism $\mathscr{L}_{i+1}^{\otimes p} \to \mathscr{L}_i$ shows that the global sections $(\sigma_j^{(i+1)})^{\otimes p}$ generate $\mathscr{L}_{i+1}^{\otimes p}$. In particular, the isomorphism is completely determined by the images of these global sections.

Remark 4.3. For each i, the data $(\mathscr{L}_i, s_j^{(i)})$ corresponds to a map to a projective space (as a rigid analytic variety) so that objects of the category $\mathfrak{L}_n(X)$ correspond to pth power root systems of maps to projective space.

The main result of this section is that the category $\mathfrak{L}_n(X)$ parametrizes K-morphisms $X \to \mathbb{P}^{n,\mathrm{perf}}$. In particular, viewing $\mathfrak{L}_n$ as a functor to sets, we construct a natural isomorphism $\mathrm{Hom}(\bullet, \mathbb{P}^{n,\mathrm{perf}}) \cong \mathfrak{L}_n$ of functors from perfectoid spaces over K to sets. First, we introduce a bit of notation.

Notation 4.4. We denote by $m_i : \mathscr{O}(1/p^{i+1})^{\otimes p} \xrightarrow{\sim} \mathscr{O}(1/p^i)$ the isomorphism of line bundles on $\mathbb{P}^{n,\mathrm{perf}}$ coming from multiplying factors together.

We now state the main theorem of this section (compare to [10, Thm. II.7.1]).

Theorem 4.5. *The natural transformation* $\mathrm{Hom}(\bullet, \mathbb{P}^{n,\mathrm{perf}}) \to \mathfrak{L}_n$, *which evaluated on* X *takes* $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}$ *to the tuple* $(\phi^*\mathscr{O}(1/p^i), \phi^* T_j^{1/p^i}, \phi^* m_i)$ *in* $\mathfrak{L}_n(X)$, *is an isomorphism of functors. In particular,* $\mathfrak{L}_n$ *is represented by projectivoid space.*

Since $\{T_j^{1/p^i}\}_{j=0}^n$ generates $\mathscr{O}(1/p^i)$, we have that $\{\phi^*(T_j^{1/p^i})\}_{j=0}^n$ generates $\phi^*(\mathscr{O}(1/p^i))$. Since multiplication $m_i : \mathscr{O}(1/p^{i+1})^{\otimes p} \xrightarrow{\sim} \mathscr{O}(1/p^i)$ sends $(T_j^{1/p^{i+1}})^{\otimes p}$ to T_j^{1/p^i}, pulling back these isomorphisms along ϕ gives us an element of $\mathfrak{L}_n(X)$, and so the natural transformation is well-defined. We construct an inverse to this transformation in Proposition 4.11 below, but first, we will need a bit of set-up.

$\mathscr{L}$-distinguished open sets. For this section, we let X be an adic space, $\mathscr{L}$ a line bundle on X, and $s_1, \dots, s_n$ global sections of $\mathscr{L}$ which generate it at every point. Let $D(s_i) = \{x \in X \mid s_i|_x \text{ generates } \mathscr{L}_x\}$ be the *does-not-vanish* set of the section s_i. Then the map $\mathscr{O}_X \to \mathscr{L}$ determined by s_i is an isomorphism on the stalks of every point of $D(s_i)$ and therefore restricts to an isomorphism on $D(s_i)$. We suggestively denote the inverse by $s \mapsto s/s_i$. Let us validate this notation with the following lemma.

Lemma 4.6. *On* $D(s_i) \cap D(s_j)$, *we have the following relation:*

$$\frac{s_i}{s_j} \cdot \frac{s_j}{s_i} = 1.$$

Proof. We have two isomorphisms

$$\Gamma(D(s_i) \cap D(s_j), \mathscr{O}_X) \underset{s_j}{\overset{s_i}{\rightrightarrows}} \Gamma(D(s_i) \cap D(s_j), \mathscr{L}).$$

Then we have

$$\frac{s_i}{s_j} = s_j^{-1} \circ s_i(1), \quad \frac{s_j}{s_i} = s_i^{-1} \circ s_j(1).$$

Since the maps $s_i^{-1} \circ s_j$ and $s_j^{-1} \circ s_i$ are inverses to each other, we win. $\square$

For every $x \in D(s_i)$, we can use the isomorphism s_i^{-1} to get a valuation on $\Gamma(X, \mathscr{L})$.

$$\begin{array}{ccccccc} \Gamma(X,\mathscr{L}) & \xrightarrow{\text{res}} & \Gamma(D(s_i),\mathscr{L}) & \xrightarrow{s_i^{-1}} & \Gamma(D(s_i),\mathscr{O}_X) & \xrightarrow{x} & \Gamma_x \cup \{0\}, \\ s & \longmapsto & & & & & |(s/s_i)(x)|. \end{array}$$

With this in hand, we can define the following open subsets of $D(s_i)$ for each i.

Definition 4.7. Let X be a perfectoid space, $\mathscr{L}$ a line bundle on X, and $s_1, \dots, s_n$ generating global sections of $\mathscr{L}$. An open set of X is called an *$\mathscr{L}$-distinguished open set* if it is of the form

$$X\left(\frac{s_1, \dots, s_n}{s_i}\right) = \{x \in D(s_i) \mid |(s_j/s_i)(x)| \le 1 \text{ for all } j\}.$$

For the case of classical projective space, we can build a map to projective space along the *does-not-vanish* sets of the given sections and glue them together. In the analytic topology, these are not affinoid, so we must use these smaller $\mathscr{L}$-distinguished open sets. Let us prove these smaller open sets cover X. Indeed, our notation suggests that one of $|(s_j/s_i)(x)|$ or $|(s_i/s_j)(x)|$ should be less than 1; let us check the details.

Lemma 4.8. *The $\mathscr{L}$-distinguished open sets $X_i = X\big(\frac{s_1,\dots,s_n}{s_i}\big)$ for $i = 1, \dots, n$ are open and cover X.*

Proof. The openness of X_i follows because it is in fact a *rational* open in the adic space $D(s_i)$, which is open in X. To show these cover X, fix some $x \in X$. We already know the $D(s_i)$ cover X because the s_i generate $\mathscr{L}$. Therefore, $I = \{i \in \{1, \dots, n\} \mid x \in D(s_i)\}$ is nonempty and finite. Order the elements of I via $i \le j$ if $|(s_i/s_j)(x)| \le 1$. Notice that, for any $i, j \in I$, we have $i \le j$ or $j \le i$. Indeed, applying Lemma 4.6 together with the multiplicativity of the valuation given by x, we have either $|(s_i/s_j)(x)| \le 1$ or $|(s_j/s_i)(x)| \le 1$. Also, if $i \le j$ and $j \le k$, then

$$|(s_i/s_k)(x)| = |(s_i/s_j)(x)| \cdot |(s_j/s_k)(x)| \le 1$$

so that $i \le k$. Finally, notice that if $i \le j$ and $j \le i$, then we have

$$|(s_i/s_j)(x)| = |(s_j/s_i)(x)| = 1.$$

Therefore, we can choose (not necessarily uniquely) some r which is maximal with respect to this ordering. Then $|(s_i/s_r)(x)| \le 1$ for all $i \in I$. For all other i, we have $x \notin D(s_i)$ so that $|(s_i/s_r)(x)| = 0 \le 1$. Therefore, $x \in X_r$, completing the proof. $\square$

Example 4.9. The standard cover of $\mathbb{P}^{n,\text{perf}}$ by perfectoid unit disks consists of the $\mathscr{O}(1)$-distinguished open sets $\mathbb{P}^{n,\text{perf}}\big(\frac{T_0,\dots,T_n}{T_i}\big)$.

The following lemma implies that if $\mathscr{L}^{\otimes p} \cong \mathscr{M}$ and s and t are global sections of $\mathscr{L}$ and $\mathscr{M}$ respectively, with $s^{\otimes p} = t$, then $D(s) = D(t)$. In particular, using the notation of Definition 4.1, this implies that if the $s_j^{(i)}$ generate $\mathscr{L}_i$, then the $s_j^{(i+1)}$ generate $\mathscr{L}_{i+1}$.

Lemma 4.10. *Let $(R, \mathfrak{m})$ be a local ring and M, N invertible R-modules such that $M^{\otimes r} \cong N$ for a positive integer r. Let $f \in M$ and $g \in N$ such that, under this identification, $f^{\otimes r} = g$. If g generates N, then f generates M.*

Proof. We show the contrapositive. If f does not generate M, Nakayama's lemma implies that $f \in \mathfrak{m}M$. Thus $f = a \cdot s$ for some $a \in \mathfrak{m}$ and $s \in M$. But then, under the appropriate identification,

$$g = f^{\otimes r} = (a \cdot s)^{\otimes r} = a^r \cdot s^{\otimes r} \in \mathfrak{m}^r N \subseteq \mathfrak{m}N.$$

Therefore, g cannot generate N. $\square$

Construction of the projectivoid morphism. We can now finish the proof of Theorem 4.5 by constructing an inverse to the natural transformation from the theorem. The result follows from the following proposition.

Proposition 4.11. *Let X be a perfectoid space over K and*

$$(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X).$$

There is a unique K-morphism $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}$ such that

$$\big(\phi^* \mathscr{O}(1/p^i), \phi^* T_j^{1/p^i}, \phi^* m_i\big) \cong (\mathscr{L}_i, s_j^{(i)}, \alpha_i).$$

Proof. Let

$$X_j = X\left(\frac{s_0^{(0)}, \dots, s_n^{(0)}}{s_j^{(0)}}\right)$$

be the cover of X by $\mathscr{L}_0$-distinguished opens. Let

$$U_j = \mathbb{P}^{n,\mathrm{perf}}\left(\frac{T_0, \dots, T_n}{T_j}\right) \subseteq \mathbb{P}^{n,\mathrm{perf}}$$

be the standard cover by affinoids. The U_i are isomorphic to the perfectoid unit polydisk and are naturally identified with

$$\mathrm{Spa}\left(K\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \dots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle, K^\circ\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \dots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle\right).$$

We build ϕ locally from maps $\phi_j : X_j \to U_j$. Since U_j is affinoid, it is equivalent to build a map of Huber pairs,

$$\left(K\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \dots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle, K^\circ\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \dots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle\right) \xrightarrow{\gamma_j} \big(\mathscr{O}_X(X_j), \mathscr{O}_X^+(X_j)\big).$$

That is, a ring map

$$K\left\langle \left(\frac{T_0}{T_j}\right)^{1/p^\infty}, \dots, \left(\frac{T_n}{T_j}\right)^{1/p^\infty} \right\rangle \xrightarrow{\gamma_j} \Gamma(X_j, \mathscr{O}_X),$$

satisfying

$$\gamma_j\left(K^\circ\left\langle\left(\frac{T_0}{T_j}\right)^{1/p^\infty},\dots,\left(\frac{T_n}{T_j}\right)^{1/p^\infty}\right\rangle\right)\subseteq\Gamma(X_j,\mathscr{O}_X^+).$$

We define γ_j on generators by the rule

$$\gamma_j\left(\left(\frac{T_r}{T_j}\right)^{1/p^i}\right)=\frac{s_r^{(i)}}{s_j^{(i)}}.$$

To make sure this is a ring homomorphism, we must check that

$$\left(\frac{s_r^{(i+1)}}{s_j^{(i+1)}}\right)^p=\frac{s_r^{(i)}}{s_j^{(i)}}.$$

First notice that, under the identification $\alpha_i:\mathscr{L}_{i+1}^{\otimes p}\cong\mathscr{L}_i$, the following diagram commutes (keeping in mind that the horizontal maps are not homomorphisms):

$$\begin{array}{ccc}\mathscr{O}_X & \xrightarrow{x\mapsto x^p} & \mathscr{O}_X\\ {\scriptstyle s_j^{(i+1)}}\downarrow & & \downarrow{\scriptstyle s_j^{(i)}}\\ \mathscr{L}_{i+1} & \xrightarrow{s\mapsto s^{\otimes p}} & \mathscr{L}_i.\end{array}$$

Indeed, the commutativity of this diagram follows directly from the multilinearity of tensor product together with the identification $(s_j^{(i+1)})^{\otimes p}=s_j^{(i)}$. Chasing this diagram, we see that

$$\left(\frac{s_r^{(i+1)}}{s_j^{(i+1)}}\right)^p=\big((s_j^{(i+1)})^{-1}(s_r^{(i+1)})\big)^p=(s_j^{(i)})^{-1}(s_r^{(i)})=\frac{s_r^{(i)}}{s_j^{(i)}},$$

as desired. Therefore, γ_j is a homomorphism. Finally, the definition of X_j implies that, for all $x\in X_j$,

$$\left|\gamma_j\left(\frac{T_i}{T_j}\right)(x)\right|=\left|\frac{s_i^{(0)}}{s_k^{(0)}}(x)\right|\leq 1$$

so that

$$\gamma_j\left(\frac{T_i}{T_j}\right)\in\Gamma(X_j,\mathscr{O}_X^+).$$

The multiplicativity of the valuation associated to x shows the same holds for all pth power roots so that

$$\gamma_j\left(K^\circ\left\langle\left(\frac{T_0}{T_j}\right)^{1/p^\infty},\dots,\left(\frac{T_n}{T_j}\right)^{1/p^\infty}\right\rangle\right)\subseteq\Gamma(X_j,\mathscr{O}_X^+).$$

Therefore, we get a morphism $\phi_j:X_j\to U_j\subseteq\mathbb{P}^{n,\mathrm{perf}}$ for each j. Notice also that $s_r^{(i)}/s_j^{(i)}$ is a p^ith root of $s_r^{(0)}/s_j^{(0)}$, as desired.

Finally, we check that these morphisms glue to a map $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}$. This amounts to showing that the restrictions of γ_j and γ_k are equal as maps from $\Gamma(U_j \cap U_k, \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}}) \to \Gamma(X_j \cap X_k, \mathscr{O}_X)$. That is, that

$$\gamma_j\left(\left(\frac{T_k}{T_j}\right)^{1/p^i}\right) = \gamma_k\left(\left(\frac{T_j}{T_k}\right)^{1/p^i}\right)^{-1}.$$

With our notation, this boils down to

$$\frac{s_k^{(i)}}{s_j^{(i)}} \cdot \frac{s_j^{(i)}}{s_k^{(i)}} = 1.$$

But this is just Lemma 4.6.

The rest is immediate from the construction. Since $\mathscr{O}_{\mathbb{P}^{n\mathrm{perf}}}(d)$ is generated by the monomials of degree d, the construction shows that $\phi^*\mathscr{O}(1/p^i) = \mathscr{L}_i$ and $\phi^*(T_j^{1/p^i}) = s_j^{(i)}$. Furthermore, any map $\psi : X \to \mathbb{P}^{n,\mathrm{perf}}$ with these properties is by definition given affinoid locally on the standard cover of the target by the ring map $(T_j/T_k)^{1/p^i} \mapsto s_j^{(i)}/s_k^{(i)}$. That is, $\psi|_{X_i} = \phi_i$ so that $\psi = \phi$. □

The positive characteristic case. If X is a perfectoid space of characteristic p, then the Frobenius morphism $\mathrm{Frob}: \mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x^p$ is an isomorphism. Therefore, the pth power map on $\mathrm{Pic}\, X$ is an isomorphism as well since it is $\mathrm{H}^1(X, \mathrm{Frob})$. This means that, given $(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X)$, the $\mathscr{L}_i$ for $i > 0$ are uniquely determined by $\mathscr{L}_0$. Similarly, since X is perfect, the map γ_i constructed in the proof of Proposition 4.11 is completely determined by where T_r/T_i goes for each $r \neq j$ because the pth roots of the image are unique. We summarize this in the following corollary.

Corollary 4.12. *If X is a perfectoid space over a perfectoid field K of characteristic p, a map $X \to \mathbb{P}^{n,\mathrm{perf}}$ is equivalent to a line bundle $\mathscr{L}$ on X and global sections $s_0, \dots, s_n$ that generate $\mathscr{L}$ or, equivalently, to a map to classical projective space $\mathbb{P}^n$ (as a rigid analytic variety).*

We can now leverage the tilting equivalence to say that maps to $X \to \mathbb{P}^{n,\mathrm{perf}}$ in any characteristic are governed by a single line bundle on $X^\flat$. Indeed, by the tilting equivalence, we have that $\mathrm{Hom}(X, \mathbb{P}_K^{n,\mathrm{perf}}) = \mathrm{Hom}(X^\flat, \mathbb{P}_{K^\flat}^{n,\mathrm{perf}})$. This implies the following corollary to Theorem 4.5.

Corollary 4.13. *If X is a perfectoid space over K of any characteristic, a map $X \to \mathbb{P}_K^{n,\mathrm{perf}}$ is equivalent to a single line bundle $\mathscr{L}$ on $X^\flat$ together with $n+1$ global sections generating $\mathscr{L}$.*

Using this corollary as an intermediary, we get a natural and geometric correspondence between certain inverse systems of line bundles on X and single line bundles on $X^\flat$.

Corollary 4.14. *An element of $\mathfrak{L}_n(X)$ is equivalent to a line bundle $\mathscr{L} \in \mathrm{Pic}\, X^\flat$ together with $n+1$ generating global sections.*

This will be a useful tool in understanding the relationship between $\mathrm{Pic}\, X$ and $\mathrm{Pic}\, X^\flat$.

5. Untilting line bundles

Recall that one of our motivations was to understand the behavior of vector bundles under the tilting equivalence. In this section, we use the tools of projectivoid geometry developed in Section 4 to compare the Picard groups of a perfectoid space X and its tilt $X^\flat$. Indeed, the theory of maps to projectivoid space allows us to pass between line bundles on X and $X^\flat$ by choosing (compatible) generating sections, constructing the associated map to projectivoid space, and then using the tilting equivalence to pass across characteristics. We remark that the theory of *pro-étale cohomology* on perfectoid spaces allows us to make this comparison cohomologically, but the theory we developed in the previous section gives us a firm geometric grasp.

Cohomological untilting. In [3], Bhatt and Scholze introduce the *pro-étale* site for schemes and perfectoid spaces. We review the definition here.

Definition 5.1. A map $f : Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$ of affinoid perfectoid spaces is called *affinoid pro-étale* if it can be written as a cofiltered limit of étale maps $Y_i = \mathrm{Spa}(S_i, S_i^+) \to X$ of affinoid perfectoid spaces. More generally, a map $f : Y \to X$ of perfectoid spaces is *pro-étale* if it is locally on the source and target affinoid pro-étale.

The (small) pro-étale site of X is the Grothendieck topology on the category of perfectoid spaces $f : Y \to X$ pro-étale over X on which a collection $\{f_i : Y_i \to X\}_{i\in I}$ is a covering if, for each quasicompact open $U \subseteq X$, there exist a finite subset $J \subseteq I$ and quasicompact open subsets $V_i \subseteq Y_i$ for $i \in J$ such that $U = \bigcup_{i\in J} f_i(V_i)$.

If $\mathscr{F}$ is a pro-étale sheaf on X (that is a sheaf on the pro-étale site of X), the pro-étale cohomology groups $\mathrm{H}^i(X_{\text{pro-ét}}, \mathscr{F})$ are the derived functor sheaf cohomology groups on the pro-étale site.

Let X be a perfectoid space over K. The pro-étale sheaf $\mathbb{G}_{m,X}$ maps U to $\Gamma(U, \mathscr{O}_U)^*$. We have the following theorem.

Proposition 5.2. $\mathrm{H}^1(X_{\text{pro-ét}}, \mathbb{G}_m) \cong \mathrm{Pic}\, X$.

Proof. For any site S, the cohomology group $\mathrm{H}^1(X_S, \mathbb{G}_m)$ parametrizes isomorphism classes of line bundles on X with respect to the topology of S. Due to [15, Thm. 3.5.8], vector bundles (of any finite rank) on a perfectoid space with respect to the pro-étale, étale, and analytic topologies coincide. □

We use the equivalence of the pro-étale topologies of X and $X^\flat$ to construct the tilt of $\mathbb{G}_m$ as a pro-étale sheaf on X,

$$\mathbb{G}_{m,X}^\flat : U \mapsto (\Gamma(U, \mathscr{O}_U)^\flat)^* = \Gamma(U^\flat, \mathscr{O}_{U^\flat})^* = \Gamma(U^\flat, \mathbb{G}_{m,X^\flat}).$$

The equivalence of the étale topologies on X and $X^\flat$ shows that $\mathbb{G}_{m,X}^\flat$ is indeed a sheaf. Better yet, the effectiveness of Čech cohomology on the pro-étale site shows that

$$\mathrm{H}^i(X_{\text{pro-ét}}, \mathbb{G}_{m,X}^\flat) \cong \mathrm{H}^i(X_{\text{pro-ét}}^\flat, \mathbb{G}_{m,X^\flat}).$$

In particular, $\mathrm{H}^1(X_{\text{pro-ét}}, \mathbb{G}_{m,X}^\flat) \cong \operatorname{Pic} X^\flat$. Now consider the Kummer sequence for various powers of p,

$$0 \to \mu_{p^n} \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,X} \to 0.$$

This is an exact sequence of sheaves on the pro-étale site of X. Indeed, this can be checked on the stalks, which on the pro-étale site are strictly Henselian local rings. Therefore, we can form an inverse system of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_p & \longrightarrow & \mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,X} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mu_{p^n} & \longrightarrow & \mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,X} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mu_{p^{n+1}} & \longrightarrow & \mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,X} & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \| & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where the vertical maps on the left and middle sides are $x \mapsto x^p$. Taking this limit gives the following sequence:

$$0 \to \mathbb{Z}_p(1) \to \mathbb{G}_{m,X}^\flat \xrightarrow{\sharp} \mathbb{G}_{m,X}.$$

The middle term is $\mathbb{G}_{m,X}^\flat$ essentially by definition. Indeed, the construction of the tilt of a perfectoid algebra R (cp. [18, Lem. 3.4]) induces a map of multiplicative monoids

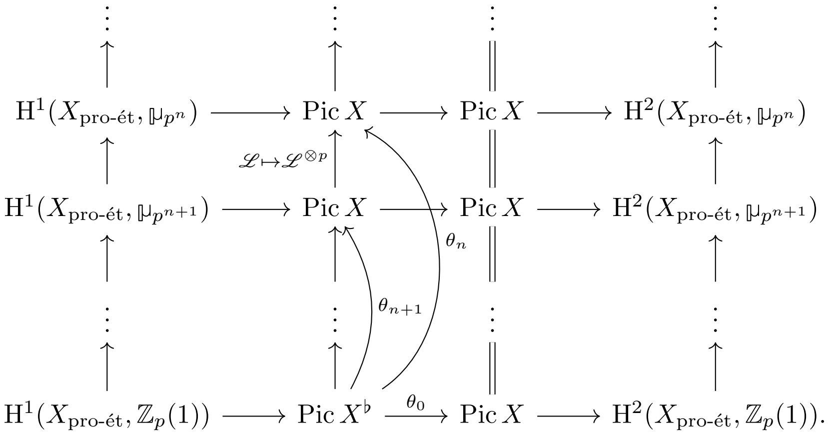
$$R^\flat \cong \varprojlim_{x \mapsto x^p} R,$$

which restricts to the desired isomorphism on unit groups. Finally, exactness on the right can be checked explicitly in the pro-étale topology. Indeed, adjoining a pth power root is an étale cover so that adjoining all the missing p-power roots gives a pro-étale cover on which $\sharp$ is surjective. Therefore, we have a short exact sequence of pro-étale sheaves

$$0 \to \mathbb{Z}_p(1) \to \mathbb{G}_{m,X}^\flat \xrightarrow{\sharp} \mathbb{G}_{m,X} \to 0.$$

Remark 5.3. If R is a perfectoid algebra, we always have a map of monoids $\sharp : R^\flat \to R$ given by projection onto the first coordinate. Although it is not a ring homomorphism unless R already had characteristic p, its restriction to unit groups $(R^\flat)^* \to R^*$ is a group homomorphism. This construction is another way of building the map $\sharp : \mathbb{G}_{m,X}^\flat \to \mathbb{G}_{m,X}$. The advantage of the above construction is that it explicitly exhibits the Tate module $\mathbb{Z}_p(1)$ as the kernel.

Taking long exact sequences in cohomology gives us the following diagram, where the rows are exact:



Taking the inverse limit of the θ_n gives a homomorphism of groups,

$$\theta : \operatorname{Pic} X^\flat \to \varprojlim_{\mathscr{L} \mapsto \mathscr{L}^{\otimes p}} \operatorname{Pic} X, \tag{18}$$

and θ_0 is this map composed with the projection onto the first coordinate.

Remark 5.4. In Corollary 4.14, we established that inverse systems of pth roots of line bundles (with generating sections) on X correspond to individual line bundles (with generating sections) on $X^\flat$. This seems to suggest that θ could be an isomorphism in cases where we have nice maps to projective space.

Untilting via maps to projectivoid space. We now give a geometric interpretation of θ and θ_0 in terms of maps to projectivoid space. Given a globally generated invertible sheaf $\mathscr{L} \in \operatorname{Pic} X^\flat$, one can choose n sections which generate $\mathscr{L}$. Corollary 4.12 associates to this data a unique morphism

$$\phi^\flat : X^\flat \to \mathbb{P}^{n,\text{perf}}_{K^\flat},$$

which is the tilt of a unique morphism $\phi : X \to \mathbb{P}^{n,\text{perf}}_K$. Let $\mathscr{L}_i = \phi^*(\mathscr{O}(1/p^i))$. This gives a system of $(\mathscr{L}_0, \mathscr{L}_1, \dots) \in \varprojlim \operatorname{Pic} X$. It turns out that the sheaves $\mathscr{L}_i$ do not depend on the choices of global sections of $\mathscr{L}$.

Proposition 5.5. *The construction in the previous paragraph is well-defined, and* $(\mathscr{L}_0, \mathscr{L}_1, \dots) = \theta(\mathscr{L})$*, where* θ *is the cohomological map defined in equation* (18) *above.*

Proof. Let $u : \mathbb{G}_{m,\mathbb{P}^{n,\text{perf}}_K} \to \phi_* \mathbb{G}_{m,X}$ be the unit of the adjunction of ϕ^* and ϕ_*. Then, passing to cohomology and composing with the natural map, we can exhibit $\phi^* : \operatorname{Pic} \mathbb{P}^{n,\text{perf}} \to \operatorname{Pic} X$ as the composition

$$H^1(\mathbb{P}^{n,\text{perf}}, \mathbb{G}_{m,\mathbb{P}^{n,\text{perf}}}) \to H^1(\mathbb{P}^{n,\text{perf}}, \phi_* \mathbb{G}_{m,X}) \to H^1(X, \mathbb{G}_{m,X}).$$

Pulling u back along the pth power map gives $\phi^{\flat *}$ the same way. Since the pth power map commutes with pullback, we get the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}}_{K^\flat} & \xrightarrow{\phi^{\flat *}} & \operatorname{Pic} X^\flat \\ \downarrow{\scriptstyle \theta_{\mathbb{P}^{n,\mathrm{perf}}}} & & \downarrow{\scriptstyle \theta_X} \\ \varprojlim \operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}}_K & \xrightarrow{\phi^*} & \varprojlim \operatorname{Pic} X. \end{array}$$

Since

$$\mathscr{L} = \phi^{\flat *} \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) \quad \text{and} \quad \mathscr{L}_i = \phi^* \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_K}(1/p^i),$$

we have reduced to proving the proposition for $\mathbb{P}^{n,\mathrm{perf}}_K$. Explicitly, we must show $\theta_{\mathbb{P}^{n,\mathrm{perf}}}(\mathscr{O}(1)) = (\mathscr{O}(1), \mathscr{O}(1/p), \mathscr{O}(1/p^2), \ldots)$. Since $\operatorname{Pic} \mathbb{P}^{n,\mathrm{perf}} = \mathbb{Z}[1/p]$, and is therefore uniquely p-divisible, it is enough to show that

$$\theta_{0,\mathbb{P}^{n,\mathrm{perf}}} \mathscr{O}(1) = \mathscr{O}(1).$$

θ_0 is obtained from $\sharp : \mathbb{G}_m^\flat \to \mathbb{G}_m$ by passing to cohomology. Scholze showed in [18, Prop. 5.20] that this map on the perfectoid Tate algebra takes $T_i \mapsto T_i$. View θ_0 as a map on Čech cohomology with respect to the standard affine covers, and view $\mathrm{H}^1(\mathbb{P}^{n,\mathrm{perf}}, \mathbb{G}_m)$ as descent data for building a line bundle (and similarly for the tilt). Then we see that $\sharp$ sends descent data for $\mathscr{O}(1)$ (which are monomials of degree one), to monomials of degree one, which build $\mathscr{O}(1)$ on $\mathbb{P}^{n,\mathrm{perf}}_K$.

Therefore, untilting line bundles via maps to projectivoid space is a well-defined process, as it agrees with the cohomological method which does not depend on the choice of sections. □

For the remainder of this paper, we use the techniques of Section 4 to study the injectivity of θ. We outline our general strategy. Let $\mathscr{L}, \mathscr{M} \in \operatorname{Pic}(X^\flat)$ be globally generated. Choose generating global sections of each, and untilt the associated maps to projectivoid space to obtain maps $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}_K$ and $\psi : X \to \mathbb{P}^{r,\mathrm{perf}}_K$. If $\phi^*(\mathscr{O}(1/p^i)) \cong \psi^*(\mathscr{O}(1/p^i)) =: \mathscr{L}_i$ for all i, we would like to conclude that $\mathscr{L} \cong \mathscr{M}$. We do so by considering the tuples

$$(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X) \quad \text{and} \quad (\mathscr{L}_i, t_j^{(i)}, \beta_i) \in \mathfrak{L}_r(X)$$

associated to ϕ and ψ respectively. We begin by settling the case where $\alpha_i = \beta_i$ by pulling back $\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1)$ along the tilt of the map associated to $(\mathscr{L}_i, \{s_j^{(i)}, t_k^{(i)}\}, \alpha_i) \in \mathscr{L}_{n+r+1}(X)$, and observing that it is isomorphic to both $\mathscr{L}$ and $\mathscr{M}$.

Proposition 5.6. *Let X be a perfectoid space over K. Suppose*

$$(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X) \quad \textit{and} \quad (\mathscr{L}_i, t_j^{(i)}, \alpha_i) \in \mathfrak{L}_r(X)$$

correspond to maps $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}_K$ and $\psi : X \to \mathbb{P}^{r,\mathrm{perf}}_K$ respectively. Then

$$\phi^{\flat *} \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) \cong \psi^{\flat *} \mathscr{O}_{\mathbb{P}^{r,\mathrm{perf}}_{K^\flat}}(1).$$

The rest of this section is devoted to the proof of Proposition 5.6. Fix $(\mathscr{L}_i, s_j^{(i)}, \alpha_i)$ corresponding to a map $\phi : X \to \mathbb{P}_K^{n,\mathrm{perf}}$. As a first step, we show that we can add one global section to each $\mathscr{L}_i$ that are compatible with the α_i without changing the line bundle we get over $X^\flat$. Suppose $t_i \in \Gamma(X, \mathscr{L}_i)$ is a global section such that $\alpha_i(t_{i+1}^{\otimes p}) = t_i$. For every

$$\lambda = (\lambda_0, \lambda_1, \ldots) \in \varprojlim K^* = K^{\flat *},$$

we let $\psi_\lambda : X \to \mathbb{P}_K^{n+1,\mathrm{perf}}$ be the projectivoid map corresponding to adding $\lambda_i t_i$, that is, corresponding to $(\mathscr{L}_i, \{s_j^{(i)}, \lambda_i t_i\}, \alpha_i)$. We hope to fit ϕ and ψ_λ in a commutative diagram. To do so, we must develop an analog of rational maps in this analytic context.

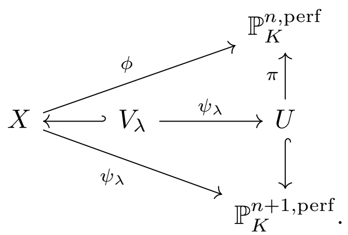
We would like the data $(\mathscr{O}(1/p^i), \{T_0^{1/p^i}, \ldots, T_n^{1/p^i}\}, m_i)$ to define a morphism $\mathbb{P}^{n+1,\mathrm{perf}} \to \mathbb{P}^{n,\mathrm{perf}}$, but this is not defined wherever $|T_{n+1}/T_i| > 1$ for all i since these points will not be contained in any of the $\mathscr{O}(1)$-distinguished opens for the given sections. In particular, it is only defined on the open set

$$U = \bigcup_{j \neq n+1} \mathbb{P}_K^{n+1,\mathrm{perf}}\left(\frac{T_0, \ldots, T_{n+1}}{T_j}\right).$$

This is the projectivoid analog of projecting away from the point at infinity where all of $T_0, \ldots, T_n$ vanish (here we are projecting away from a perfectoid disk at the "north pole"). Unfortunately, the image of ψ_λ does not a priori lie in U because there may be points x where $|(\lambda_0 t_0 / s_j^{(0)})(x)| > 1$ for all i so that $|(T_{n+1}/T_i)(\psi_\lambda(x))| > 1$. But by restricting to the open set

$$V_\lambda = \bigcup_j X\left(\frac{s_0^{(0)}, \ldots, s_n^{(0)}, \lambda_0 t_0}{s_j^{(0)}}\right),$$

we force the image of ψ_λ to lie in U. Thus we have the following commutative diagram for every λ:



Lemma 5.7. *The sets $V_{(\varpi^\flat)^r}$ form an open cover of X. As a consequence, the sets $V_{(\varpi^\flat)^r}^\flat$ cover $X^\flat$.*

Proof. Notice $(\varpi^\flat)^r = (\varpi^r, \varpi^{r/p}, \ldots)$. Fix $x \in X$. There is some j such that

$$x \in X\left(\frac{s_0^{(0)}, \ldots, s_n^{(0)}}{s_j^{(0)}}\right).$$

Furthermore, since ϖ is topologically nilpotent, there is some r such that

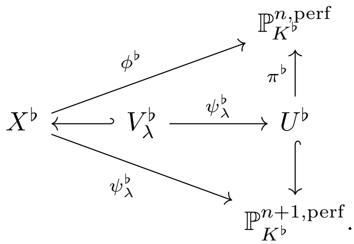
$$|(\varpi^r t_0/s_j^{(0)})(x)| = |\varpi^r| \cdot |(t_0/s_j^{(0)})(x)| < 1,$$

proving the first statement. The second is an immediate consequence of the tilting equivalence. ☐

Lemma 5.8. *For any $\lambda \in K^{\flat *}$,*

$$(\phi^{\flat *}\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1))|_{V^\flat_\lambda} \cong \psi^{\flat *}_\lambda(\mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1)|_{U^\flat}) \cong (\psi^{\flat *}_\lambda \mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1))|_{V^\flat_\lambda}.$$

Proof. This follows from the commutativity of the tilt of the diagram above, reproduced below, together with the fact that $\pi^\flat$ is given by the line bundle $\mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1)|_U$ together with the sections $T_0, \dots, T_n$:



☐

Lemma 5.9. *Fix any $\lambda, \xi \in \varprojlim K^* = K^{\flat *}$. Then*

$$\psi^{\flat *}_\lambda \mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1) \cong \psi^{\flat *}_\xi \mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1).$$

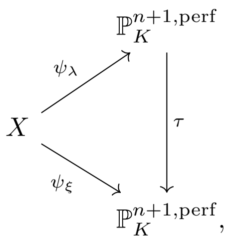
Proof. Let $\tau : \mathbb{P}^{n+1,\mathrm{perf}}_K \to \mathbb{P}^{n+1,\mathrm{perf}}_K$ be the map associated to the data

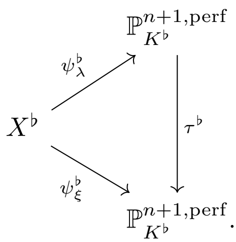
$$\left(\mathscr{O}(1/p^i), \left\{T_0^{1/p^i}, \dots, T_n^{1/p^i}, \frac{\lambda_i}{\xi_i} T_{n+1}^{1/p^i}\right\}, m_i\right).$$

Then τ is an isomorphism. Indeed, τ^{-1} corresponds to

$$\left(\mathscr{O}(1/p^i), \left\{T_0^{1/p^i}, \dots, T_n^{1/p^i}, \frac{\xi_i}{\lambda_i} T_{n+1}^{1/p^i}\right\}, m_i\right).$$

Observe also that $\tau^\flat$ is the map determined by $\mathscr{O}(1)$ and $T_0, \dots, T_n, \frac{\lambda}{\xi} T_{n+1}$. We have the following two commutative diagrams, the right-hand diagram being the tilt of the left:





Since $\tau^{\flat *}\mathscr{O}(1) = \mathscr{O}(1)$, we are done. ☐

Putting these three lemmas together, we conclude that

$$\phi^{\flat *}\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) \cong \psi_1^{\flat *}\mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1).$$

Indeed, the pullback of $\mathscr{O}(1)$ along $\psi_1^\flat$ agrees with the pullback along $\psi_{(\varpi^\flat)^r}$, for any r, but this agrees with the restriction of $\phi^{\flat *}\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1)$ to $V^\flat_{(\varpi^\flat)^r}$ for any r. Since these sets cover $X^\flat$, we are done.

In summary, we have proved the following proposition.

Proposition 5.10. *Let $(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X)$ define a map $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}_K$. Suppose that $t_i \in \Gamma(X, \mathscr{L}_i)$ is a global section such that $\alpha_i(t_{i+1}^{\otimes p}) = t_i$, and let $\psi : X \to \mathbb{P}^{n+1}_K$ be the map associated to $(\mathscr{L}_i, \{s_j^{(i)}, t_i\}, \alpha_i) \in \mathfrak{L}_{n+1}(X)$. Then*

$$\phi^{\flat *}\mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) \cong \psi^{\flat *}\mathscr{O}_{\mathbb{P}^{n+1,\mathrm{perf}}_{K^\flat}}(1).$$

Adding sections one at a time by induction completes the proof of Proposition 5.6.

Injectivity of $\boldsymbol{\theta}$. With these tools in hand, we can prove the injectivity of θ for certain perfectoid spaces X. We will first need one more lemma.

Lemma 5.11. *Let $(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X)$ define a map $\phi : X \to \mathbb{P}^{n,\mathrm{perf}}_K$. Fix*

$$\lambda = (\lambda_0, \lambda_1, \ldots) \in \Gamma(X, \mathscr{O}_X^{\flat *}),$$

that is, $\lambda_{i+1}^p = \lambda_i$, so that $(\mathscr{L}_i, \lambda_i s_0^{(i)}, \lambda_i\alpha_i) \in \mathfrak{L}_n(X)$ corresponds to a map $\psi : X \to \mathbb{P}^{n,\mathrm{perf}}_K$. Then $\phi = \psi$.

Proof. Multiplication by λ_i for each i defines an isomorphism

$$(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \xrightarrow{\sim} (\mathscr{L}_i, \lambda_i s_0^{(i)}, \lambda_i\alpha_i)$$

in $\mathfrak{L}_n(X)$. Then we are done by Theorem 4.5. □

Before we state the main theorem, we make the following definition.

Definition 5.12. A line bundle $\mathscr{L}$ on a perfectoid space X is said to be *weakly ample* if, for any other line bundle $\mathscr{M}$, there is some $N \gg 0$ such that, for all $r > N$, we have $\mathscr{M} \otimes \mathscr{L}^r$ globally generated.

Theorem 5.13. *Suppose X is a perfectoid space over a perfectoid field K, let $X^\flat$ be the tilt of X, and let C be the completion of an algebraic closure of K. Suppose that $X^\flat$ has a weakly ample line bundle and that $\mathrm{H}^0(X_C, \mathscr{O}_{X_C}) = C$. Then there is a natural injection*

$$\theta : \operatorname{Pic} X^\flat \hookrightarrow \varprojlim_{\mathscr{L} \mapsto \mathscr{L}^p} \operatorname{Pic} X.$$

In particular, if $\operatorname{Pic} X$ has no p-torsion, then composing with projection onto the first coordinate gives an injection

$$\theta_0 : \operatorname{Pic} X^\flat \hookrightarrow \operatorname{Pic} X.$$

Proof. Fix $\mathscr{L}, \mathscr{M} \in \operatorname{Pic} X^\flat$ with $\theta(\mathscr{L}) = \theta(\mathscr{M})$. We first reduce to the case that $\mathscr{L}, \mathscr{M}$ are globally generated. Indeed, letting $\mathscr{F}$ be a weakly ample line bundle, we have $\theta(\mathscr{L} \otimes \mathscr{F}^N) = \theta(\mathscr{M} \otimes \mathscr{F}^N)$. If the result holds for globally generated line bundles, for large enough N, we conclude that $\mathscr{L} \otimes \mathscr{F}^N \cong \mathscr{M} \otimes \mathscr{F}^N$ so that $\mathscr{L} \cong \mathscr{M}$.

Next we prove it for the case where K contains all pth power roots for all its elements. Choose generating sections $s_0, \ldots, s_n$ for $\mathscr{L}$ and $t_0, \ldots, t_r$ of $\mathscr{M}$, which give us maps

$$\phi^\flat : X^\flat \to \mathbb{P}^{n,\mathrm{perf}}_{K^\flat} \quad \text{and} \quad \psi^\flat : X^\flat \to \mathbb{P}^{r,\mathrm{perf}}_{K^\flat}$$

respectively. These untilt to

$$\phi : X \to \mathbb{P}^{n,\mathrm{perf}}_{K} \quad \text{and} \quad \psi : X \to \mathbb{P}^{r,\mathrm{perf}}_{K},$$

which in turn correspond to tuples

$$(\mathscr{L}_i, s_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X) \quad \text{and} \quad (\mathscr{L}_i, t_j^{(i)}, \beta_i) \in \mathfrak{L}_r(X).$$

Notice that α_i and β_i differ by an element

$$\lambda_i \in \operatorname{Isom}(\mathscr{L}_i, \mathscr{L}_i) = \Gamma(X, \mathscr{O}_X^*) = K^*.$$

That is, $\alpha_i = \lambda_i \beta_i$. Choose pth power roots λ_i^{1/p^j} for each i, j (these exist by assumption), and for all j, make the following definitions:

$$\begin{aligned}
\tilde{t}_j^{(0)} &= t_j^{(0)} \\
\tilde{t}_j^{(1)} &= \lambda_0^{-1/p} t_j^{(1)} \\
\tilde{t}_j^{(2)} &= \lambda_1^{-1/p} \lambda_0^{-1/p^2} t_j^{(2)} \\
&\vdots \\
\tilde{t}_j^{(i+1)} &= \lambda_i^{-1/p} \lambda_{i-1}^{-1/p^2} \cdots \lambda_0^{-1/p^{i+1}} t_j^{(i+1)} \\
&\vdots
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_i((\tilde{t}_j^{(i+1)})^{\otimes p}) &= \lambda_i \beta_i\big((\lambda_i^{-1/p} \lambda_{i-1}^{-1/p^2} \cdots \lambda_0^{-1/p^{i+1}} t_j^{(i+1)})^{\otimes p}\big) \\
&= \lambda_i \lambda_i^{-1} \lambda_{i-1}^{1/p} \cdots \lambda_0^{1/p^i} \beta\big((t_j^{(i+1)})^{\otimes p}\big) \\
&= \lambda_{i-1}^{1/p} \cdots \lambda_0^{1/p^i} t_j^{(i)} \\
&= \tilde{t}_j^{(i)}.
\end{aligned}$$

Therefore, the tuple $(\mathscr{L}_i, \tilde{t}_j^{(i)}, \alpha_i) \in \mathfrak{L}_n(X)$, and it also corresponds to ψ by Lemma 5.11. Furthermore, the isomorphisms corresponding to this data are now α_i in both cases so that, by Proposition 5.6,

$$\mathscr{L} = \phi^{\flat *} \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) \cong \psi^{\flat *} \mathscr{O}_{\mathbb{P}^{n,\mathrm{perf}}_{K^\flat}}(1) = \mathscr{M}.$$

For the general case, we let L/K be the extension given by adjoining all pth power roots of all elements of K. We have the following diagram:

$$\begin{array}{ccc} \operatorname{Pic} X_L^\flat & \overset{\theta_L}{\hookrightarrow} & \varprojlim \operatorname{Pic} X_L \\ \uparrow & & \uparrow \\ \operatorname{Pic} X^\flat & \xrightarrow{\theta} & \varprojlim \operatorname{Pic} X. \end{array}$$

θ_L injects by the argument we just made. Furthermore, since $X_L^\flat \to X^\flat$ is a pro-étale cover of pth power degree, the kernel of $\operatorname{Pic} X^\flat \to \operatorname{Pic} X_L^\flat$ is pth power torsion. Since $X^\flat$ is perfect, $\operatorname{Pic} X^\flat$ has no pth power torsion, so the map injects. Therefore, θ injects. □

Example 5.14. Let $X_0 \hookrightarrow \mathbb{P}^n_K$ be a geometrically connected projective variety over a perfectoid field of characteristic p. Then if we pull back along the map $\mathbb{P}^{n,\mathrm{perf}}_K \to \mathbb{P}^n$, we get $X_\infty := X_0 \times_{\mathbb{P}^n_K} \mathbb{P}^{n,\mathrm{perf}}_K$ which is Zariski closed in a perfectoid space and is therefore perfectoid. Furthermore, the pullback of $\mathscr{O}(n)$ to X_∞ will be weakly ample for any positive n. Therefore, if $K^\sharp$ is any untilt of K and $X^\sharp_\infty$ is the untilt of X_∞ over $K^\sharp$, we may conclude that $\operatorname{Pic}(X_\infty) \hookrightarrow \varprojlim \operatorname{Pic} X^\sharp_\infty$.

Remark 5.15. One can also study the map θ from Theorem 5.13 using homological methods by analyzing derived limits in the pro-étale site. Indeed, as part of forthcoming work, the author shows that θ is an isomorphism without appealing to a weakly ample line bundle to pass through maps to projectivoid space. We nevertheless include this proof here as it has a more geometric flavor and demonstrates an interesting application of the projectivoid theory.

Acknowledgments. The author thanks their doctoral advisor Max Lieblich for support while pursuing this work during graduate study, and Kiran Kedlaya for his generosity of ideas while serving as host for a term at the University of California in San Diego to work on this project. The author also thanks Peter Scholze for important comments and corrections to a previous draft, Bhargav Bhatt for many suggestions greatly simplifying the proofs and exposition, and the anonymous referee for extremely detailed and helpful comments. Thanks also to Daniel Bragg, Jacopo Brivio, Charles Godfrey, Tuomas Tajakka, Lucas Van Meter, Anwesh Ray, Manar Riman, Travis Scholl, Alex Voet, and Peter Wear for helpful conversations. Much of this work was completed while the author was supported under NSF grant DMS-1600813 during a semester visit to the University of California, San Diego, and under NSF grant DMS-1439786 while the author was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI.

References

[1] M. Artin, A. Grothendieck, and L. Verdier, *Théorie des topos et cohomologie étale des schémas. (SGA 4 Tome 2)*, Lecture Notes in Mathematics, Vol. 270. Springer-Verlag, Berlin-New York, 1972. MR0354653

[2] H. S. Bedi, *Line Bundles of Rational Degree Over Perfectoid Space*, PhD Thesis, George Washington University, 2018. MR3768607

[3] B. Bhatt and P. Scholze, The pro-étale topology for schemes, Astérisque No. 369 (2015), 99–201. MR3379634

[4] B. Bhatt and P. Scholze, Projectivity of the Witt vector affine Grassmannian, Invent. Math. **209** (2017), no. 2, 329–423. MR3674218

[5] S. Das, *Vector bundles on perfectoid spaces*, PhD thesis, University of California, San Diego, 2016. MR3553572

[6] G. Dorfsman-Hopkins, *Projective geometry for perfectoid spaces*, PhD thesis, University of Washington, 2018. MR4035417

[7] G. Dorfsman-Hopkins, A. Ray, and P. Wear, On Picard groups of perfectoid covers of toric varieties, arXiv:2011.03613v1 [math.AG] (2020).

[8] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995. MR1322960

[9] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Mathematics, 1800, Springer-Verlag, Berlin, 2003. MR2004652

[10] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. MR0463157

[11] R. Huber, A generalization of formal schemes and rigid analytic varieties, Math. Z. **217** (1994), no. 4, 513–551. MR1306024

[12] R. Huber, Continuous valuations, Math. Z. **212** (1993), no. 3, 455–477. MR1207303

[13] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, Vieweg, Braunschweig, 1996. MR1734903

[14] K. S. Kedlaya, *Sheaves, stacks, and shtukas*, Notes from the Arizona Winter School, March 2017.

[15] K. S. Kedlaya and R. Liu, Relative p-adic Hodge theory. II: Imperfect period rings, arXiv:1602.06899v1 [math.AG] (2016).

[16] W. Lütkebohmert, Vektorraumbündel über nichtarchimedischen holomorphen Räumen, Math. Z. **152** (1977), no. 2, 127–143. MR0430331

[17] D. Quillen, Projective modules over polynomial rings, Invent. Math. **36** (1976), 167–171. MR0427303

[18] P. Scholze, Perfectoid spaces, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR3090258

[19] P. Scholze, Perfectoid spaces and their applications, in *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, 461–486, Kyung Moon Sa, Seoul, 2014. MR3728623

[20] P. Scholze and J. Weinstein, Moduli of p-divisible groups, Camb. J. Math. **1** (2013), no. 2, 145–237. MR3272049

[21] J.-P. Serre, Faisceaux algébriques cohérents, Ann. of Math. (2) **61** (1955), 197–278. MR0068874

[22] T. Wedhorn, Adic spaces, arXiv:1910.05934v1 [math.AG] (2019).

[23] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994. MR1269324

[24] The Stacks Project Authors, *Stacks Project* (2018).

Received October 5, 2020; accepted May 5, 2021

Gabriel Dorfsman-Hopkins
University of California, Berkeley, 895 Evans Hall
Berkeley, CA 94720, USA
E-mail: `gabrieldh@berkeley.edu`
URL: `http://www.gabrieldorfsmanhopkins.com`