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Fredholm conditions and index for restrictions of invariant pseudodifferential operators to isotypical components

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Abstract. Let Γ be a compact group acting on a smooth, compact manifold M, let $P \in \psi^m(M; E_0, E_1)$ be a Γ-invariant, classical pseudodifferential operator acting between sections of two equivariant vector bundles $E_i \to M$, $i = 0, 1$, and let α be an irreducible representation of the group Γ. Then P induces a map $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$ between the α-isotypical components of the corresponding Sobolev spaces of sections. When Γ is finite, we explicitly characterize the operators P for which the map $\pi_\alpha(P)$ is Fredholm in terms of the principal symbol of P and the action of Γ on the vector bundles E_i. When $\Gamma = \{1\}$, that is, when there is no group, our result extends the classical characterization of Fredholm (pseudo)differential operators on compact manifolds. The proof is based on a careful study of the symbol C^*-algebra and of the topology of its primitive ideal spectrum. We also obtain several results on the structure of the norm closure of the algebra of invariant pseudodifferential operators and their relation to induced representations. As an illustration of the generality of our results, we provide some applications to Hodge theory and to index theory of singular quotient spaces.

Contents

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1. Introduction

Fredholm operators have been extensively studied and have been used in many applications in mathematical physics, in partial differential equations (linear and non-linear), in algebraic and differential geometry, in index theory, and in other areas. On a compact manifold, a classical pseudodifferential operator is Fredholm between suitable Sobolev spaces if, and only if, it is elliptic. In this paper, we obtain an analogous result for the restrictions to isotypical components of a classical pseudodifferential operator P invariant with respect to the action of a finite group Γ using C^*-algebra methods. Namely, *the restriction of P to the isotypical component corresponding to an irreducible representation α of Γ is Fredholm if, and only if, the operator is α-elliptic (Definition 1.3 and Theorem 1.5).*

Let us now formulate and explain this result in more detail.

1.1. **The setting and general notation.** We shall work essentially in the same setting as the one considered in [12], but for a general finite group Γ. Thus, *throughout this paper, Γ will be a finite group acting by diffeomorphisms on a smooth Riemannian manifold M.* As our main result is only valid for a compact manifold, we assume in the introduction that M is compact. For the main result (Theorem 1.5), we do need Γ to be discrete and finite. A related result for Γ a compact Lie group was announced in [13], but the statement, and especially the proof (although it is based on this paper), are significantly different. There is no loss of generality to assume that M is endowed with an invariant Riemannian metric, so we will assume that this is the case also throughout the paper.

As usual, $\widehat{\Gamma}$ denotes the finite set of equivalence classes of irreducible Γ-modules (or representations). Let $T : V_0 \to V_1$ be a Γ-equivariant linear map of Γ-modules and $\alpha \in \widehat{\Gamma}$. Then T induces by restriction a Γ-equivariant linear map

$$\pi_\alpha(T) : V_{0\alpha} \to V_{1\alpha} \tag{1}$$

between the α-isotypical components of the Γ-modules V_i, $i = 0, 1$.

We are mostly interested in this restriction morphism π_α in the following case. Let $P \in \psi^m(M; E_0, E_1)$ be a classical, Γ-invariant pseudodifferential operator acting between sections of two Γ-equivariant vector bundles $E_i \to M$, $i = 0, 1$. Then we obtain the operator

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha, \tag{2}$$

which acts between the α-isotypical components of the corresponding Sobolev spaces of sections. Our main result concerns this operator $\pi_\alpha(P)$. For simplicity, we will consider only classical pseudodifferential operators in this article [52, 53, 62, 97, 101]. The *main question* that we answer in this paper is to determine when the induced operator $\pi_\alpha(P)$ of equation (2) is Fredholm in terms of its "Γ-equivariant principal symbol" $\sigma_m^\Gamma(P)$ introduced next, see Theorem 1.5 below for the precise statement.

1.2. **The α-principal symbol and α-ellipticity.** To put our result into the right perspective, recall that a classical, order m, pseudodifferential operator P is called *elliptic* if its principal symbol

$$\sigma_m(P) \in \mathcal{C}^\infty(T^*M \setminus \{0\}; \mathrm{Hom}(E_0, E_1))$$

is invertible. Also, recall that a linear operator $T : X_0 \to X_1$ acting between Banach spaces is *Fredholm* if, and only if, the vector spaces

$$\ker(T) \coloneqq T^{-1}(0) \quad \text{and} \quad \mathrm{coker}(T) \coloneqq X_1/TX_0$$

are (both) finite-dimensional. Since M is compact, a very well-known and widely used result states that $P : H^s(M; E_0) \to H^{s-m}(M; E_1)$ is Fredholm if, and only if, P is elliptic; see, for instance, [52, 88, 92, 91, 98] and the references therein. Consequently, if P is elliptic, then $\pi_\alpha(P)$ is also Fredholm. The converse is not true, however, in general.

To state our main result characterizing the Fredholm property of $\pi_\alpha(P)$ in terms of the "α-principal symbol" $\sigma_m^\alpha(P)$ of P, Theorem 1.5, we shall need to introduce $\sigma_m^\Gamma(P)$, the "Γ-equivariant principal symbol" of P, which is a refinement of the principal symbol $\sigma_m(P)$ of P that takes into account the action of the group Γ. The *α-principal symbol* $\sigma_m^\alpha(P)$ of P is a suitable restriction of the Γ-equivariant principal symbol $\sigma_m^\Gamma(P)$. Let us formulate now the precise definition of these concepts.

The Γ-invariance of P implies that its principal symbol is also Γ invariant:

$$\sigma_m(P) \in \mathcal{C}^\infty(T^*M \setminus \{0\}; \mathrm{Hom}(E_0, E_1))^\Gamma.$$

Let $\Gamma_\xi \coloneqq \{\gamma \in \Gamma \mid \gamma\xi = \xi\}$ denote the isotropy of a $\xi \in T_x^*M$, $x \in M$, as usual. The isotropy Γ_x of $x \in M$ is defined similarly. Then $\Gamma_\xi \subset \Gamma_x$ acts on E_{0x} and on E_{1x}, the fibers of $E_0, E_1 \to M$ at x. If $Q \in \mathcal{C}^\infty(T^*M \setminus \{0\}; \mathrm{Hom}(E_0, E_1))^\Gamma$, then $Q(\xi) \in \mathrm{Hom}(E_{0x}, E_{1x})^{\Gamma_\xi}$. Let $\rho \in \widehat{\Gamma}_\xi$ be an irreducible representation of Γ_ξ. Then

$$\widehat{Q}(\xi, \rho) \coloneqq \pi_\rho[Q(\xi)] \in \mathrm{Hom}(E_{0x\rho}, E_{1x\rho})^{\Gamma_\xi}$$

denotes the restriction of Q to the isotypical component corresponding to ρ, with π_ρ defined in equation (1). Let

$$X_{M,\Gamma} \coloneqq \{(\xi, \rho) \mid \xi \in T^*M \setminus \{0\} \text{ and } \rho \in \widehat{\Gamma}_\xi\}. \tag{3}$$

Thus Q defines a function on $X_{M,\Gamma}$. Applying this construction to $\sigma_m(P) \in \mathcal{C}^\infty(T^*M \setminus \{0\}; \mathrm{Hom}(E_0, E_1))^\Gamma$, we obtain a function, the *Γ-principal symbol*

$$\begin{gathered} \sigma_m^\Gamma(P) : X_{M,\Gamma} \to \bigcup_{(x,\rho) \in X_{M,\Gamma}} \mathrm{Hom}(E_{0x\rho}, E_{1x\rho})^{\Gamma_\xi}, \\ \sigma_m^\Gamma(P)(\xi, \rho) \coloneqq \pi_\rho(\sigma_m(P)(\xi)) \in \mathrm{Hom}(E_{0x\rho}, E_{1x\rho})^{\Gamma_\xi}, \quad \xi \in T_x^*M. \end{gathered} \tag{4}$$

That is $\sigma_m^\Gamma(P) \coloneqq \widehat{\sigma_m(P)}$. The α-principal symbol $\sigma_m^\alpha(P)$ of P, $\alpha \in \widehat{\Gamma}$, is defined in terms of $\sigma_m^\Gamma(P)$, but we need a crucial additional ingredient that takes α into account.

Recall that $\Gamma_{g\xi} = g\Gamma_\xi g^{-1}$ and that this defines an action of Γ on the set of stabilizer subgroups $\mathrm{Stab}_\Gamma(T^*M) \coloneqq \{\Gamma_\xi \mid \xi \in T^*M\}$ given by $g \cdot \Gamma_\xi = \Gamma_{g\xi}$. For

$\rho \in \widehat{\Gamma}_\xi$, define $g \cdot \rho \in \widehat{\Gamma}_{g\xi}$ by $(g \cdot \rho)(h) = \rho(g^{-1}hg)$ for all $h \in \Gamma_{g\xi}$. Let $\Gamma_0 \subset \Gamma$ be a minimal element (for inclusion) among the isotropy groups Γ_x of elements $x \in M$. Such a minimal element exists trivially since Γ is finite. Moreover, if M/Γ is connected, then Γ_0 is unique up to conjugacy (see Subsection 2.13.3). If M/Γ is connected, then we let

$$X_{M,\Gamma}^{\alpha} \coloneqq \{(\zeta, \rho) \in X_{M,\Gamma} \mid \text{there exists } g \in \Gamma,\ \mathrm{Hom}_{\Gamma_0}(g \cdot \rho, \alpha) \neq 0\}. \tag{5}$$

(Note that it is implicit in the definition of $X_{M,\Gamma}^{\alpha}$ that $\Gamma_0 \subset \Gamma_{g\zeta} = g \cdot \Gamma_\zeta$.) In general (if M/Γ is not connected), we define $X_{M,\Gamma}^{\alpha}$ by taking the disjoint union of the corresponding spaces for each connected component of M/Γ, see Subsection 4.1. We are finally ready to introduce the α-principal symbol.

Definition 1.3. The *α-principal symbol* $\sigma_m^{\alpha}(P)$ of P is the restriction of the *Γ-principal symbol* $\sigma_m^{\Gamma}(P)$ to $X_{M,\Gamma}^{\alpha}$:

$$\sigma_m^{\alpha}(P) \coloneqq \sigma_m^{\Gamma}(P)|_{X_{M,\Gamma}^{\alpha}}.$$

We shall say that $P \in \psi^m(M; E_0, E_1)^{\Gamma}$ is *α-elliptic* if its α-principal symbol $\sigma_m^{\alpha}(P)$ is invertible everywhere on its domain of definition.

Note that, when $(\xi, \rho) \in X_{M,\Gamma}^{\alpha}$ is such that $E_{x\rho} = 0$, then $\sigma_m^{\Gamma}(P)(\xi, \rho) : 0 \to 0$ is invertible.

1.4. **Statement of the main result.** An alternative formulation of Definition 1.3 is that P is α-elliptic if, and only if, σ_m^{Γ} is invertible on $X_{M,\Gamma}^{\alpha}$ (this is, of course, a condition only for those ρ such that $E_{i\rho} \neq 0$ because, otherwise, we get an operator acting on the zero spaces, which we admit to be invertible). We then have the following result extending the classical result (i.e. $\Gamma = \{1\}$) and the one from [12] (i.e. Γ finite abelian) to a general finite group Γ.

Theorem 1.5. *Let Γ be a finite group acting on a smooth, compact manifold M, and let $P \in \psi^m(M; E_0, E_1)^{\Gamma}$ be a Γ-invariant classical pseudodifferential operator acting between sections of two Γ-equivariant bundles $E_i \to M$, $i = 0, 1$, $m \in \mathbb{R}$, and $\alpha \in \widehat{\Gamma}$. We have that*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$$

is Fredholm if, and only if, it is α-elliptic.

As in the abelian case, if Γ acts without fixed points on a *dense* open subset of M, then $X_{M,\Gamma} = X_{M,\Gamma}^{\alpha}$ for all $\alpha \in \widehat{\Gamma}$, by Corollary 5.16. Hence, in this case, P is α-elliptic if, and only if, it is elliptic. The ellipticity of P can thus be checked in this case simply by looking at the action of P on a single isotypical component. We stress, however, that if Γ is not discrete, this statement, as well as the statement of the above theorem, are no longer true. However, many intermediate results remain valid for compact Lie groups.

A motivation for our result comes from index theory. Let us assume that P is Γ-invariant and elliptic. Atiyah and Singer have determined, for any $\gamma \in \Gamma$, the value at γ of $\mathrm{ind}_\Gamma(P) \in R(G)$. More precisely, they have computed $\mathrm{ind}_\Gamma(P)(\gamma) \in \mathbb{C}$ in terms of data at the fixed points of γ on M. (Here

$R(G) := \mathbb{Z}^{\widehat{G}}$ is the representation ring of G and is identified with a subalgebra of $\mathcal{C}^\infty(G)^G$, the ring of conjugacy invariant functions on G via the characters of representations.) By contrast, the multiplicity of $\alpha \in \widehat{\Gamma}$ in $\operatorname{ind}_\Gamma(P)$ was much less studied. It did appear implicitly in the work of Brüning [18] who studied the "isotypical heat trace" $\operatorname{tr}(p_\alpha e^{-t\Delta})$ and its short time asymptotic expansion. Its heat trace is nothing but the heat trace of $\pi_\alpha(\Delta)$.

We obtain that the (Fredholm) index of $\pi_\alpha(P)$ depends only on the homotopy class of its α-principal symbol. (See Theorem 5.3 and in the remark following it.) In particular, this yields results on the index theory of singular quotient spaces. We therefore expect our results to have applications to the Hodge theory of algebraic varieties [2, 3, 16, 27, 48], see Remark 5.5. In the case of a non discrete compact Lie group, the computation of this index is related to the index class of G-transversally elliptic operators initiated in [7, 95]. Since then, this has been studied in K-theory [9, 11, 54, 56] and in equivariant cohomology [10, 15, 80].

Our proof relies in a significant way on C^*-algebras through the natural C^*-algebra completions of algebras of pseudodifferential operators. This point of view has been used and advocated of course by many people. Without attempting to provide a comprehensive list of references, let us mention that C^*-algebras were used very recently to obtain Fredholm conditions in [37, 60, 78], for example. Some of the algebras involved were groupoid algebras [6, 5, 26, 39, 75, 84]. The technique of "limit operators" [58, 66, 67, 82] is related to groupoids. Some of the most recent papers using related ideas include [4, 8, 25, 26, 28, 65, 76, 77, 103], to which we refer for further references. Fredholm conditions play an important role in the study of the essential spectrum of Quantum Hamiltonians [14, 44, 45, 50, 59]. Besides C^*-algebras, plain algebras of pseudodifferential operators (no norm completion) were also used to obtain Fredholm conditions; see, for example, [38, 49, 61, 70, 88, 89, 90] and the references therein. In addition to the works already mentioned, several general results on C^* and related algebras related to this work were obtained by Cordes and McOwen [33], Melo, Nest, and Schrohe [71], Melrose and Nistor [72], Rabinovich, Schulze, and Tarkhanov [83], Taylor [96], Voiculescu [104, 105], and many others. See [22, 23, 32, 46, 47, 51, 57, 73, 74] for some older, related results on singular integral operators. In [86, 87], Savin and Schrohe have obtained Fredholm conditions for G-operators.

1.6. **Contents of the paper.** We start in Section 2 with some preliminaries. We recall some facts about group actions, most notably the induction of representations and Frobenius reciprocity for finite groups. We also review some notions concerning the primitive spectrum of a C^*-algebra, as well as basic facts concerning (equivariant) pseudodifferential operators.

As in [12], we may assume that $E_0 = E_1 = E$ and that P is an order zero pseudodifferential operator. Let $A_M := \mathcal{C}_0(S^*M; \operatorname{End}(E))$. The most substantial technical results are in Section 3. There, we introduce the subset $\Omega_M := \{(\xi, \rho) \in S^*M \times \hat{\Gamma}_\xi,\ \rho \subset E_\xi\}$ of $X_{M,\Gamma}$ described above and identify the primitive spectrum of the C^*-algebra A_M^Γ of Γ-invariant symbols with the set Ω_M/Γ. Some care is taken to describe the corresponding topology on Ω_M/Γ. We then consider the canonical map from A_M^Γ to the Calkin algebra of $L^2(M;E)_\alpha$ and show that the closed subset of $\mathrm{Prim}(A_M^\Gamma)$ associated to its kernel is $X_{M,\Gamma}^\alpha/\Gamma$. These results are used in Section 4 to prove the main result of the paper, Theorem 1.5. The last section, Section 5, also addresses some particular cases of this theorem and gives a few examples. We also explain the relation with previously known results, namely, the particular formulation in the abelian case, which was established in [12], Fredholm conditions for transversally elliptic operators when the group Γ is not discrete, and Simonenko's localization principle for Fredholm operators.

2. Preliminaries

This section is devoted to background material. For the most part, it will consist of a brief review of [12, Sec. 2 and 3], where the reader will find more details, including some definitions and results not repeated here. Note, however, that we need certain preliminary results for the case Γ non-commutative that were not needed in the abelian case. Nevertheless, the reader familiar with [12] can skip this section at a first reading.

Except when otherwise stated, throughout this paper, M will be a smooth Riemannian manifold M, not necessarily compact, with a smooth isometric action of a finite group Γ such that M/Γ is connected. (In Subsection 4.1, we explain how the disconnected case reduces to the connected case.)

2.1. **Group representations.**

We follow the standard terminology and conventions. See, for instance, [12, 17, 93], where one can find further details. Most of the needed basic background material was recalled in greater detail in [12].

Throughout the paper, we let Γ be a *finite* group acting by isometries on a smooth, Riemannian manifold M (without boundary). We use the standard notation, see [12, 17, 93], to which we refer for further details. If $x \in M$, then Γx is the Γ orbit of x and

$$\Gamma_x := \{\gamma \in \Gamma \mid \gamma x = x\} \subset \Gamma$$

the isotropy group of the action at x.

We shall write $H \sim H'$ if the subgroups H and H' are conjugated in Γ. If $H \subset \Gamma$ is a subgroup, then $M_{(H)}$ will denote the set of elements of M whose isotropy Γ_x is conjugated to H (in Γ), that is, the set of elements $x \in M$ such that $\Gamma_x \sim H$.

Assuming that Γ acts on a space X, we let $\Gamma \times_H X$ denote the space

$$\Gamma \times_H X := (\Gamma \times X)/\sim, \tag{6}$$

where $(\gamma h, x) \sim (\gamma, hx)$ for all $\gamma \in \Gamma$, $h \in H$, and $x \in X$.

Let V and W be locally convex spaces and $\mathcal{L}(V;W)$ the set of continuous, linear maps $V \to W$. We let $\mathcal{L}(V) := \mathcal{L}(V;V)$. A representation of Γ on V is a group morphism $\Gamma \to \mathcal{L}(V)$; in that case, we also call V a Γ-module.

For any two Γ-modules $\mathcal{H}$ and $\mathcal{H}_1$, we shall denote by

$$\mathrm{Hom}_\Gamma(\mathcal{H},\mathcal{H}_1) = \mathrm{Hom}(\mathcal{H},\mathcal{H}_1)^\Gamma = \mathcal{L}(\mathcal{H},\mathcal{H}_1)^\Gamma$$

the set of continuous linear maps $T : \mathcal{H} \to \mathcal{H}_1$ that commute with the action of Γ, that is, $T(\gamma\xi) = \gamma T(\xi)$ for all $\xi \in \mathcal{H}$ and $\gamma \in \Gamma$.

Let $\mathcal{H}$ be a Γ-module and α an irreducible Γ-module. Then p_α will denote the Γ-invariant projection onto the *α-isotypical component* $\mathcal{H}_\alpha$ of $\mathcal{H}$, defined as the largest (closed) Γ submodule of $\mathcal{H}$ that is isomorphic to a multiple of α. In other words, $\mathcal{H}_\alpha$ is the sum of all Γ-submodules of $\mathcal{H}$ that are isomorphic to α. Notice that $\mathcal{H}_\alpha \simeq \alpha \otimes \mathrm{Hom}_\Gamma(\alpha,\mathcal{H})$.

Since Γ is finite, it is, in particular, compact, and hence we have

$$\mathcal{H}_\alpha \neq 0 \iff \mathrm{Hom}_\Gamma(\alpha,\mathcal{H}) \neq 0 \iff \mathrm{Hom}_\Gamma(\mathcal{H},\alpha) \neq 0. \tag{7}$$

Let $p_\alpha \in \mathbb{C}[\Gamma]$ be the central projection associated to an irreducible representation α of Γ. If $T \in \mathcal{L}(\mathcal{H})^\Gamma$ (i.e. T is Γ-equivariant), then $T(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ and we let

$$\pi_\alpha : \mathcal{L}(\mathcal{H})^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha), \quad \pi_\alpha(T) := p_\alpha T|_{\mathcal{H}_\alpha}, \tag{8}$$

be the associated morphism, as in equation (1) of the introduction. The morphism π_α will play an essential role in what follows.

2.2. **Induction and Frobenius reciprocity.** We now recall some definitions and results for induced representations mainly to set-up notation and to obtain some intermediate results.

We let $V^{(I)} := \{f : I \to V\}$ for I finite. We let throughout this subsection $H \subset \Gamma$ be a subgroup and V an H-module. (So H is also finite.) We then define, as usual,

$$\mathrm{Ind}_H^\Gamma(V) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V \simeq \{f : \Gamma \to V \mid f(gh^{-1}) = hf(g)\} \simeq V^{(\Gamma/H)} \tag{9}$$

to be the *induced representation.* The last isomorphism is obtained using a set of representatives of the right cosets Γ/H. The action of the group Γ on $\mathrm{Ind}_H^\Gamma(V)$ is obtained from the left multiplication on $\mathbb{C}[\Gamma]$. The induction is a functor, that is, the Γ-module $\mathrm{Ind}_H^\Gamma(V)$ depends functorially on V.

Remark 2.3. Summarizing [12, Rem. 2.2], we have that

(i) if V is an H-algebra, then $\mathrm{Ind}_H^\Gamma(V)$ is an algebra for the pointwise product of functions $\Gamma \to V$,
(ii) if V is a left R-module (with compatible actions of Γ), then $\mathrm{Ind}_H^\Gamma(V)$ is a $\mathrm{Ind}_H^\Gamma(R)$ module, again with the pointwise multiplication,
(iii) the induction is compatible with morphisms of modules and algebras (change of scalars), which can again be seen from the functional picture in (9).

We shall use the *Frobenius reciprocity* in the form that states that we have an isomorphism

$$\begin{aligned}\Phi = \Phi_H^\Gamma &: \mathrm{Hom}_H(\mathcal{H}, V) \to \mathrm{Hom}_\Gamma(\mathcal{H}, \mathrm{Ind}_H^\Gamma(V)),\\ \Phi(f)(\xi) &:= \frac{1}{|H|}\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} f(g^{-1}\xi), \quad \xi \in \mathcal{H},\ f \in \mathrm{Hom}_H(\mathcal{H}, V).\end{aligned} \tag{10}$$

The version of the Frobenius reciprocity used in this paper is valid only for finite groups [17, 93] (although it can be suitably generalized to the compact case). We note that a more precise notation would be to write $\mathrm{Hom}_H(\mathrm{Res}^\Gamma_{\mathrm{H}}(\mathcal{H}), V)$ instead of our simplified notation $\mathrm{Hom}_H(\mathcal{H}, V)$.

Definition 2.4. Let A and B be finite groups, and let H be a subgroup of both A and B. Let $\alpha \in \widehat{A}$ and $\beta \in \widehat{B}$. We say that α and β are *H-disjoint* if $\mathrm{Hom}_H(\alpha, \beta) = 0$; otherwise, we say that they are *H-associated (to each other)*.

Let $\alpha \in \widehat{\Gamma}$, let $H \subset \Gamma$ be a subgroup, and $\beta \in \widehat{H}$. A useful consequence of the Frobenius reciprocity is that the multiplicity of α in $\mathrm{Ind}_H^\Gamma(\beta)$ is the same as the multiplicity of β in the restriction of α to H. In particular, α is contained in $\mathrm{Ind}_H^\Gamma(\beta)$ if, and only if, β is contained in the restriction of α to H, in which case, recall that we say that α and β are H-associated (Definition 2.4). On the other hand, recall that if β is *not* contained in the restriction of α to H, we say that α and β are *H-disjoint.*

Let V be an H-module and $\mathcal{H}$ be the trivial Γ-module $\mathbb{C}$. Then we obtain, in particular, an isomorphism

$$\begin{aligned}\Phi : V^H = \mathrm{Hom}_H(\mathbb{C}, V) &\simeq \mathrm{Hom}_\Gamma(\mathbb{C}, \mathrm{Ind}_H^\Gamma(V)) = \mathrm{Ind}_H^\Gamma(V)^\Gamma,\\ \Phi(\xi) &:= \frac{1}{|H|}\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} \xi = \sum_{x\in\Gamma/H} x \otimes \xi.\end{aligned} \tag{11}$$

Lemma 2.5. *If V is an algebra, then the map Φ is an isomorphism of algebras with $\Phi(1) = 1$ when the algebra is unital. Moreover, for any $T \in \mathcal{L}(V)^H$ and $\zeta \in V$, we have that $\Phi(T)\Phi(\zeta) = \Phi(T\zeta)$.*

Proof. See Remark 2.3. □

Let us also mention the following consequences for further use.

Remark 2.6. Let $H \subset \Gamma$ be a subgroup of Γ, as above, let β_j be nonisomorphic simple H-modules, $j = 1, \ldots, N$, and let

$$\beta := \bigoplus_{j=1}^N \beta_j^{k_j}.$$

We then have that $\mathrm{Ind}_H^\Gamma(\beta) \simeq \bigoplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$, and the Frobenius isomorphism gives

$$\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \simeq \mathrm{End}(\beta)^H \simeq \bigoplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H \simeq \bigoplus_{j=1}^N M_{k_j}(\mathbb{C}),$$

which is a semisimple algebra and where the first isomorphism is induced by Φ of equation (11).

We shall need the following refinement of the above remark.

Lemma 2.7. *Let* $\beta := \bigoplus_{j=1}^N \beta_j^{k_j}$ *be as in Remark 2.6, let*

$$T = (T_j) \in \mathrm{End}(\beta)^H \simeq \bigoplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H,$$

with $T_j \in \mathrm{End}(\beta_j^{k_j})^H$*, and let* $\xi_j \in \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$*. We let*

$$\xi := (\xi_j) \in \bigoplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\beta_j^{k_j}) \simeq \mathrm{Ind}_H^\Gamma(\beta).$$

Then $\Phi(T)(\xi) = (\Phi(T_j)\xi_j)_{j=1,\dots,N}$.

Proof. This follows from Lemma 2.5 and Remark 2.3. See [12, Lem. 2.4] for more details. □

For the abelian case, the following elementary result was proved in [12, Prop. 2.5]. That proof *does not* generalize to our case.

Proposition 2.8. *Let* $H \subset \Gamma$ *be a subgroup of* Γ*, as above. Let* $\beta := \bigoplus_{j=1}^N \beta_j^{k_j}$ *be as in Remark 2.6. Let* $J \subset \{1, 2, \dots, N\}$ *be the set of indices* j *such that* α *and* β_j *are* H*-disjoint (i.e.* β_j *is not contained in the restriction of* α *to* H*). Then the morphism*

$$\pi_\alpha : \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \to \mathrm{End}(p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta))$$

is such that

$$\ker(\pi_\alpha) = \bigoplus_{j\in J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma \quad \textit{and} \quad \mathrm{Im}(\pi_\alpha) \simeq \bigoplus_{j\notin J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma.$$

Proof. By Lemma 2.7, we can assume that $N = 1$. Therefore, the algebra $\mathrm{End}(\beta)^H$ is simple (more precisely, isomorphic to a matrix algebra $M_q(\mathbb{C})$, $q = k_1$). We shall use the isomorphism of Remark 2.6. The action of

$$\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \simeq \mathrm{End}(\beta)^H \simeq M_q(\mathbb{C})$$

on $\mathrm{Ind}_H^\Gamma(\beta)$ is unital (i.e. non-degenerate) by Lemma 2.5, so the morphism

$$M_q(\mathbb{C}) \simeq \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \to \mathrm{End}(p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta))$$

is injective if, and only if, $p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta) \neq 0$. Notice the following equivalences:

$$p_\alpha \,\mathrm{Ind}_H^\Gamma(\beta) \neq 0 \iff \mathrm{Hom}(\alpha, \mathrm{Ind}_H^\Gamma(\beta))^\Gamma \neq 0 \iff \mathrm{Hom}(\alpha, \beta)^H \neq 0.$$

The result then follows from equation (7). □

2.9. **The primitive ideal spectrum of a C^*-algebra.** We shall need a few basic concepts and facts about C^*-algebras. A general reference is [40]. Recall that a two-sided ideal $I \subset A$ of a C^*-algebra A is called *primitive* if it is the kernel of a nonzero, irreducible $*$-representation of A. Hence A is *not* a primitive ideal of itself. By $\mathrm{Prim}(A)$, we shall denote the set of primitive ideals of A, called the *primitive ideal spectrum* of A. The primitive ideal spectrum $\mathrm{Prim}(A)$ of A is endowed with the Jacobson topology. The open sets of the Jacobson topology are the sets of the form $V(I) := \mathrm{Prim}(I) = \{J \in \mathrm{Prim}(A) \mid I \not\subset J\}$, where I is a closed, two-sided ideal of A.

If X is a locally compact space, then $\mathcal{C}_0(X)$ denotes the space of continuous functions $X \to \mathbb{C}$ that vanish at infinity. The concept of primitive ideal spectrum is important for us since we have a natural homeomorphism

$$\mathrm{Prim}(\mathcal{C}_0(X)) \simeq X.$$

This identification lies at the heart of non-commutative geometry [29, 30]. See also [24, 68, 69].

If A is a type I C^*-algebra, then $\mathrm{Prim}(A)$ identifies with the set of isomorphism classes of irreducible representations of A. Any C^*-algebra with only finite-dimensional irreducible representations is a type I algebra [40]. Most of the algebras considered in this paper (a notable exception are the algebras of compact operators) have this property.

The following example from [12] will be used several times.

Example 2.10. Let H be a finite group and $\beta = \bigoplus_{j=1}^{N} \beta_j^{k_j}$ as in Remark 2.6. Then, as explained in that remark, we have $\mathcal{L}(\beta)^H \simeq \bigoplus_j M_{k_j}(\mathbb{C})$. The algebra $\mathcal{L}(\beta)^H = \mathrm{End}_H(\beta)$ is thus a C^*-algebra with only finite-dimensional representations, and we have natural homeomorphisms

$$\mathrm{Prim}(\mathrm{End}_H(\beta)) \leftrightarrow \{\beta_1, \beta_2, \dots, \beta_N\} \leftrightarrow \{1, 2, \dots, N\}.$$

The space $\mathrm{Prim}(A)$ is a topological space for the Jacobson topology; we refer to [40] for more details. We will recall some facts about this topology when needed, see Lemma 3.3 below.

We shall need the following "central character" map.

Remark 2.11. Let Z be a commutative C^*-algebra, and let $\phi : Z \to M(A)$ be a $*$-morphism to the multiplier algebra $M(A)$ of A (see [1, 21]). Assume that $\phi(Z)$ commutes with A and $\phi(Z)A = A$. Then Schur's lemma gives that every irreducible representation of A restricts to (a multiple of) a character of Z, and hence there exists a natural continuous map

$$\phi^* : \mathrm{Prim}(A) \to \mathrm{Prim}(Z), \tag{12}$$

which we shall call also the *central character map* (associated to ϕ).

We conclude our discussion with the following simple result.

Lemma 2.12. *We freely use the notation of Example 2.10. The inclusion of the unit $\mathbb{C} \to \mathrm{End}_H(\beta)$ induces a morphism*

$$j : \mathcal{C}_0(X) \to \mathcal{C}_0(X; \mathrm{End}_H(\beta)) \simeq \mathcal{C}_0(X) \otimes \mathrm{End}_H(\beta).$$

The resulting central character map is the first projection

$$j^* : \mathrm{Prim}(\mathcal{C}_0(X;\mathrm{End}_H(\beta))) \simeq X \times \{1,2,\dots,N\} \to X \simeq \mathrm{Prim}(\mathcal{C}_0(X)).$$

2.13. **Group actions on manifolds.** As before, we consider a finite group Γ acting by isometries on a smooth, Riemannian manifold M.

2.13.1. *Slices and tubes.* Given $x \in M$, the isotropy group Γ_x acts linearly and isometrically on T_xM. For $r > 0$, let $U_x := (T_xM)_r$ denote the set of vectors of length smaller than r in T_xM. It is known then that, for $r > 0$ small enough, the exponential map gives a Γ-equivariant, isometric diffeomorphism

$$W_x = \Gamma \times_{\Gamma_x} \exp(U_x) \simeq \Gamma \times_{\Gamma_x} U_x, \tag{13}$$

where W_x is a Γ-invariant neighborhood of x in M and $\Gamma \times_{\Gamma_x} U_x$ is defined in equation (6). More precisely, W_x is the set of $y \in M$ at distance smaller than r to the orbit Γx if $r > 0$ is small enough. The set W_x is called a *tube* around x (or Γx), and the set U_x is called the *slice* at x. When M is compact, the injectivity radius is bounded from below, so we may assume that the constant r does not depend on x.

2.13.2. *Equivariant vector bundles.* Consider now a Γ-equivariant smooth vector bundle $E \to M$. Let us fix $x \in M$ and consider as above the tube $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ around x, see equation (13). We use this diffeomorphism to identify U_x to a subset of M, in which case, we can also assume the restriction of E to the slice U_x to be trivial. Therefore, there exists a Γ_x-module β such that

$$E|_{U_x} \simeq U_x \times \beta \quad \text{and} \quad E|_{W_x} \simeq \Gamma \times_{\Gamma_x} (U_x \times \beta). \tag{14}$$

The second isomorphism is Γ-equivariant.

Assume E is endowed with a Γ-invariant hermitian metric. We then have isomorphisms of Γ-modules

$$\begin{aligned} L^2(W_x; E|_{W_x}) &\simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(L^2(U_x;\beta)),\\ \mathcal{C}_0(W_x; E|_{W_x}) &\simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(\mathcal{C}_0(U_x;\beta)). \end{aligned} \tag{15}$$

In view of the previous isomorphism, we will often identify W_x and $\Gamma \times_{\Gamma_x} U_x$, making no distinction between them to simplify the notation.

2.13.3. *The principal orbit bundle.* Recall that $M_{(H)}$ denotes the set of points of M whose stabilizer is conjugated in Γ to H. Recall that we have assumed that M/Γ is *connected*. It is known then [100] that there exists a *minimal isotropy* subgroup $\Gamma_0 \subset \Gamma$, in the sense that $M_{(\Gamma_0)}$ is a dense open subset of M whose complement has measure zero in M.

The principal orbit bundle $M_{(\Gamma_0)}$ has the following useful property. If we have $x \in M_{(\Gamma_0)}$, then Γ_x acts *trivially* on the slice U_x at x by the minimality of Γ_0. Hence Γ_0 acts trivially on T_x^*M as well, which implies that $\Gamma_0 \subset \Gamma_\xi$ for any $\xi \in T_x^*M$. If, on the other hand, $x \in M$ is arbitrary (not necessarily in the principal orbit bundle), then the isotropy of Γ_x will contain a subgroup conjugated to Γ_0.

2.14. **Pseudodifferential operators.** We continue to follow [12]. We also continue to assume that Γ is a finite group that acts smoothly and isometrically on a smooth Riemannian manifold M. Let $\psi^m(M;E)$ denote the space of order m, *classical* pseudodifferential operators on M with *compactly supported* distribution kernel.

Let $\overline{\psi^0}(M;E)$, $\overline{\psi^{-1}}(M;E)$ denote the respective norm closures of $\psi^0(M;E)$ and $\psi^{-1}(M;E)$. The action of Γ then extends to an action on $\psi^m(M;E)$, $\overline{\psi^0}(M;E)$, and $\overline{\psi^{-1}}(M;E)$. We shall denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators acting on a Hilbert space $\mathcal{H}$. Of course, we have $\overline{\psi^{-1}}(M;E) = \mathcal{K}(L^2(M;E))$ since we have considered only pseudodifferential operators with compactly supported distribution kernels.

Let S^*M denote the *unit cosphere bundle* of a smooth manifold M, that is, the set of unit vectors in T^*M, as usual. We shall denote, as usual, by $\mathcal{C}_0(S^*M;\mathrm{End}(E))$ the set of continuous sections of the *lift* of the vector bundle $\mathrm{End}(E) \to M$ to S^*M. We have the following well-known exact sequence:

$$0 \longrightarrow \mathcal{K}(L^2(M;E))^\Gamma \longrightarrow \overline{\psi^0}(M;E)^\Gamma \xrightarrow{\sigma_0} \mathcal{C}_0(S^*M;\mathrm{End}(E))^\Gamma \longrightarrow 0. \tag{16}$$

See, for instance, [12, Cor. 2.7], where references are given.

2.14.1. *The structure of regularizing operators.* From now on, all our vector bundles will be Γ-equivariant vector bundles. We want to understand the structure of the algebra $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$ for any fixed $\alpha \in \widehat{\Gamma}$ (see equations (1) and (8) for the definition of the restriction morphism π_α and of the projectors $p_\alpha \in C^*(\Gamma)$).

We shall need the following standard result about negative order operators. Recall that, for $\alpha \in \widehat{\Gamma}$, we let π_α be the representation of $\overline{\psi^0}(M;E)^\Gamma$ on $L^2(M;E)_\alpha$ defined by restriction as before, equation (1). Recall that p_α denotes the central projection in $\mathbb{C}[\Gamma]$ corresponding to the representation $\alpha \in \widehat{\Gamma}$.

Proposition 2.15. *We have natural isomorphisms*

$$p_\alpha\overline{\psi^{-1}}(M;E)^\Gamma \simeq \pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma) = \pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma) = \mathcal{K}(L^2(M;E)_\alpha)^\Gamma,$$

where the first isomorphism map is simply π_α and

$$\mathcal{K}(L^2(M;E))^\Gamma = \overline{\psi^{-1}}(M;E)^\Gamma \simeq \bigoplus_{\alpha\in\widehat{\Gamma}} \mathcal{K}(L^2(M;E)_\alpha)^\Gamma.$$

Proof. See, for example, [12, Sec. 3] for a proof. □

3. The principal symbol

Let us fix an irreducible representation α of Γ and consider the restriction morphism π_α to the α-isotypical component of $L^2(M;E)$. Recall that this morphism was first introduced in equation (1) and discussed in detail in Section 2.1. As in [12], we now turn to the identification of the quotient

$$\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma).$$

The methods used in this paper diverge, however, significantly from those of [12].

Since $\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$ was identified in the previous section, the promised identification of the quotient $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma)$ will give further insight into the structure of the algebra $\pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)$ and will provide us, eventually, with Fredholm conditions. Recall that, in this paper, we are assuming Γ to be finite.

3.1. **The primitive ideal spectrum of A_M^Γ.** As before, S^*M denotes the unit cosphere bundle of M. For the simplicity of the notation, we shall write

$$A_M := \mathcal{C}_0(S^*M;\mathrm{End}(E)),$$

as in the introduction. Recall from Corollary 16 that we have an algebra isomorphism

$$\overline{\psi^0}(M;E)^\Gamma/\overline{\psi^{-1}}(M;E)^\Gamma \simeq A_M^\Gamma. \tag{17}$$

In our case, the inclusion $j:\mathcal{C}_0(S^*M/\Gamma)\subset A_M^\Gamma$ as a central subalgebra induces, as in equation (12), a central character map

$$j^*:\mathrm{Prim}(A_M^\Gamma)\to S^*M/\Gamma$$

that underscores the local nature of the structure of the primitive ideal spectrum of A_M^Γ. We introduce the representation $\pi_{\xi,\rho}$ defined for any $f\in A_M^\Gamma$ by

$$\pi_{\xi,\rho}(f)=\pi_\rho(f(\xi)), \tag{18}$$

that is, $\pi_{\xi,\rho}(f)$ is the restriction of $f(\xi)\in\mathrm{End}(E_x)$ to the ρ-isotypical component of E_x. The central character map j^* was used in [12, Cor. 4.2] to obtain the following identification of $\mathrm{Prim}(A_M^\Gamma)$.

Proposition 3.2 ([12]). *Let Ω_M be the set of pairs (ξ,ρ), where $\xi\in S_x^*M$, $x\in M$, and $\rho\in\widehat{\Gamma}_\xi$ appears in E_x (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho,E_x)\neq 0$).*
(i) *The map $\Omega_M/\Gamma\ni\Gamma(\xi,\rho)\mapsto\ker(\pi_{\xi,\rho})\in\mathrm{Prim}(A_M^\Gamma)$ is bijective, where $\pi_{\xi,\rho}$ is as defined in equation* (18).
(ii) *The central character map $\Omega_M/\Gamma\simeq\mathrm{Prim}(A_M^\Gamma)\to S^*M/\Gamma$ maps $\Gamma(\xi,\rho)\in\Omega_M/\Gamma$ to $\Gamma\xi$ and is continuous and finite-to-one.*

We would like to point out that some results on the description of the primitive ideal spectrum of the C^*-algebra of invariant symbols

$$A_M^\Gamma=\mathcal{C}_0(S^*M,\mathrm{End}(E))^\Gamma$$

could be deduced from more general results contained in [42, 43]. More precisely, one can take $B=A_M$ and $X=S^*M$ in [43]. Furthermore, the description of the primitive ideal spectrum

$$\mathrm{Prim}(A^\Gamma_{M_{(\Gamma_0)}})=\Omega_{M_{(\Gamma_0)}}/\Gamma$$

above the principal orbit bundle is related with the study in [41] in the sense that the action on $S^*M_{(\Gamma_0)}$ is with continuously varying stabilizers.

The space $\mathrm{Prim}(A_M^\Gamma)$ is endowed with the Jacobson topology, which was recalled in Subsection 2.9; thus Proposition 3.2 allows us to use the central character map j^* to obtain a topology on Ω_M/Γ that will play a crucial role in what follows. We thus now turn to the study of this topology on Ω_M/Γ. We begin with the following standard lemma.

Lemma 3.3. *Let A be a C^*-algebra. The family $(V_a)_{a\in A}$ defined by*

$$V_a = \{J \in \operatorname{Prim} A \mid a \notin J\}$$

for any $a \in A$ is a basis of open sets for $\operatorname{Prim}(A)$.

Proof. Following [40], we know that the open, nonempty subsets of $\operatorname{Prim}(A)$ are exactly the sets

$$\{J \in \operatorname{Prim}(A) \mid I \not\subset J\} \simeq \operatorname{Prim}(I),$$

where I ranges through the closed, nonzero, two-sided ideals of A. If $a \in A$, let us denote by $I_a := \overline{AaA}$ the closed, two-sided ideal generated by a. Then $a \notin J \Leftrightarrow I_a \not\subset J$ since $a \in I_a$, and hence $V_a = \operatorname{Prim}(I_a)$. This shows that V_a is open.

Next, let $V \subset \operatorname{Prim}(A)$ be a nonempty open subset and $J_0 \in V$. We know then that there exists a closed, two-sided ideal I, where $0 \neq I \subset A$, such that $V = \operatorname{Prim}(I)$. We have $I \not\subset J_0$ by the definition of $\operatorname{Prim}(I)$, and hence we can choose $a \in I \setminus J_0$. If $J \subset A$ is a primitive ideal such that $J \in V_a$, then $a \notin J$ and, *a fortiori*, $I \not\subset J$. Therefore, $V_a \subset \operatorname{Prim}(I)$. This shows that $J_0 \in V_a \subset V$. Therefore, the family $(V_a)_{a\in A}$ is a basis for the topology on $\operatorname{Prim}(A)$. □

We shall use the bijection of Proposition 3.2 to conclude the following.

Corollary 3.4. *A basis for the induced topology on $\Omega_M/\Gamma \simeq \operatorname{Prim}(A_M^\Gamma)$ is given by the sets*

$$V_f := \{\Gamma(\xi,\rho) \in \Omega_M/\Gamma \mid \pi_{\xi,\rho}(f) \neq 0\},$$

where f ranges through the nonzero elements of A_M^Γ and where $\pi_{\xi,\rho}$ is as defined in equation (18).

3.5. **The restriction morphisms.** Let $\mathcal{O} \subset M$ be an open subset. Then $S^*\mathcal{O}$ is the restriction of S^*M to $\mathcal{O}$. We shall also consider the algebras

$$A_{\mathcal{O}} := \mathcal{C}_0(S^*\mathcal{O}; \operatorname{End}(E)) \quad \text{and} \quad B_{\mathcal{O}} := \overline{\psi^0}(\mathcal{O}; E). \tag{19}$$

Assume that $\mathcal{O} \subset M$ is Γ-invariant. The group Γ does not act, in general, as multipliers on the C^*-algebra $B_{\mathcal{O}} := \overline{\psi^0}(\mathcal{O}; E)$ (it does however act by conjugation), so the method used in [12] to identify $\overline{\psi^{-1}}(\mathcal{O}; E)^\Gamma \simeq \mathcal{K}(L^2(\mathcal{O}; E))^\Gamma$ does not extend to identify $B_{\mathcal{O}}^\Gamma$. We shall thus consider the natural, surjective map

$$\begin{aligned} \mathcal{R}_{\mathcal{O}} : A_{\mathcal{O}}^\Gamma := \mathcal{C}_0(S^*\mathcal{O}; \operatorname{End}(E))^\Gamma &\simeq B_{\mathcal{O}}^\Gamma / \overline{\psi^{-1}}(\mathcal{O}; E)^\Gamma \\ &\to \pi_\alpha(B_{\mathcal{O}}^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(\mathcal{O}; E)^\Gamma). \end{aligned}$$

Recall from Corollary 2.15 that $\pi_\alpha(\overline{\psi^{-1}}(M; E)^\Gamma) = \mathcal{K}(L^2(M; E)_\alpha)^\Gamma$. Therefore, for a given $P \in \overline{\psi^0}(M; E)^\Gamma$, we have that $\pi_\alpha(P)$ is Fredholm if, and only if, the principal symbol of P is invertible in $A_M^\Gamma/\ker(\mathcal{R}_M)$. This will be discussed in more detail in the next section.

We shall identify $\ker(\mathcal{R}_M) \subset A_M^\Gamma$ by determining the closed subset

$$\Xi := \mathrm{Prim}(A_M^\Gamma / \ker(\mathcal{R}_M)) \subset \mathrm{Prim}(A_M^\Gamma) \tag{20}$$

of the primitive ideal spectrum of A_M^Γ corresponding to $\ker(\mathcal{R}_M)$. Once we will have determined Ξ, we will also have determined $\ker(\mathcal{R}_M)$, in view of the definitions recalled in Subsection 2.9 that put in bijection the closed, two-sided ideals of a C^*-algebra with the closed subsets of its primitive ideal spectrum.

Since $\mathcal{C}_0(M/\Gamma) \subset B_M$, it follows from the definition of $\mathcal{R}_M$ that it is a $\mathcal{C}_0(M/\Gamma)$-module morphism, and hence that $\ker(\mathcal{R}_M)$ is a $\mathcal{C}_0(M/\Gamma)$-module. Let us also recall that

$$\mathcal{C}_0(M/\Gamma) = \mathcal{C}_0(M)^\Gamma \subset Z_M := \mathcal{C}_0(S^*M)^\Gamma \subset Z(A_M^\Gamma) \subset A_M^\Gamma \subset A_M.$$

The local nature of $\ker(\mathcal{R}_M)$ and of the space Ξ is explained in the following remark.

Remark 3.6. Let $M/\Gamma = \cup V_k$ be an open cover and

$$\ker(\mathcal{R}_M)_{V_k} := \mathcal{C}_0(V_k)\ker(\mathcal{R}_M) = \ker(\mathcal{R}_{V_k}).$$

If we determine each $\ker(\mathcal{R}_M)_{V_k}$, then we determine $\ker(\mathcal{R}_M)$ using a partition of unity through

$$\ker(\mathcal{R}_M) = \text{closure of } \sum_k{}' \phi_k \ker(\mathcal{R}_{V_k}),$$

where $\sum'$ refers to sums with only finitely many nonzero terms and (ϕ_k) is a partition of unity of M/Γ with continuous functions subordinated to the covering (V_k) (thus, in particular, $\mathrm{supp}(\phi_k) \subset V_k$). To determine $\mathcal{R}_M$, we can therefore replace M by any of the open sets V_k in the covering and study $\ker(\mathcal{R}_{V_k})$. We shall do that for the covering of M/Γ with the tubes $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ considered in Subsection 2.13.1, see equation (13). Thus our problem is local. This is related to Simonenko's localization principle, see Subsection 5.18.

3.7. **Local calculations.** In view of Remark 3.6, we shall concentrate now on the local structure of $\ker(\mathcal{R}_M)$, that is, on the structure of $\ker(\mathcal{R}_\mathcal{O})$ for suitable ("small") open, Γ-invariant subsets $\mathcal{O} \subset M$. Let us fix then $x \in M$, and let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ be the tube around x, equation (13). For simplicity, we shall write

$$A_x := A_{U_x} := \mathcal{C}_0(S^*U_x; \mathrm{End}(E)) \quad \text{and} \quad Z_x := Z(A_x^{\Gamma_x}).$$

For these algebras, the role of Γ will be played by Γ_x. For the statement of the following lemma, recall the definitions in Subsection 2.13, especially equation (13).

Lemma 3.8. *Let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$. Then $S^*W_x \simeq \Gamma \times_{\Gamma_x} S^*U_x$, and we have Γ-equivariant algebra isomorphisms*

$$A_{W_x} := \mathcal{C}_0(S^*W_x; \mathrm{End}(E)) \simeq \mathrm{Ind}_{\Gamma_x}^\Gamma(\mathcal{C}_0(S^*U_x; \mathrm{End}(E))) =: \mathrm{Ind}_{\Gamma_x}^\Gamma(A_x).$$

Consequently, the Frobenius isomorphism Φ *of equation* (11) *induces an isomorphism*

$$\Phi^{-1} : A_{W_x}^{\Gamma} \to A_x^{\Gamma_x}.$$

Proof. We have $E|_{W_x} \simeq \Gamma \times_{\Gamma_x} (E|_{U_x})$, hence $\mathrm{End}(E)|_{W_x} \simeq \Gamma \times_{\Gamma_x} (\mathrm{End}(E)|_{U_x})$. Equation (15) then gives that $\mathcal{C}_0(W_x, \mathrm{End}(E)) \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(\mathcal{C}_0(U_x, \mathrm{End}(E)))$. The rest follows right away from the Frobenius reciprocity (more precisely, from equation (11)) and from equation (15), with β replaced with $\mathrm{End}(E)$. $\square$

Remark 3.9. In view of equation (11), the isomorphism Φ of Lemma 3.8 can be written explicitly as follows. Let $f \in A_x^{\Gamma_x}$. Then, for any equivalence class $[\gamma, \xi] := \Gamma_x(\gamma, \xi) \in \Gamma \times_{\Gamma_x} S^*U_x \simeq S^*W_x$, we have

$$\Phi(f)([\gamma, \xi]) = [\gamma, f(\xi)],$$

where $[\gamma, f(\xi)] \in \Gamma \times_{\Gamma_x} (U_x \times \mathrm{End}(E_x))^{\Gamma_x} \simeq \Gamma \times_{\Gamma_x} \mathrm{End}(E|_{U_x})^{\Gamma_x} \simeq \mathrm{End}(E|_{W_x})^{\Gamma}$. This defines $\Phi(f) \in \mathcal{C}_0(S^*W_x; \mathrm{End}(E|_{W_x}))^{\Gamma} = A_{W_x}^{\Gamma}$.

Lemma 3.8 together with the following remark will allow us to reduce the study of the algebra A_M^{Γ} to that of its analogues defined for slices.

Remark 3.10. Let U be an open set of some Euclidean space, and let $W = U \times \{1, 2, \ldots, N\}$, where the space on the second factor is endowed with the discrete topology. For simplicity, we identify $L^2(W)$ with $L^2(U)^N$ using the map $f \mapsto (f(i))_{i=1,\ldots,N}$. Then

$$\psi^{-\infty}(W) = M_N(\psi^{-\infty}(U)) \simeq \psi^{-\infty}(U) \otimes M_N(\mathbb{C}),$$

and hence

$$\overline{\psi^{-\infty}}(W) = M_N(\overline{\psi^{-\infty}}(U)) \simeq \overline{\psi^{-\infty}}(U) \otimes M_N(\mathbb{C}).$$

(Note that, on the last line, considering operators of order -1 or of order $-\infty$ makes no difference.) On the other hand, if A^N denotes the direct sum of N-copies of the algebra A, then we have the following inclusions of algebras:

$$\psi^0(U)^N \subset \psi^0(W) \subset M_N(\psi^0(U)) \simeq \psi^0(U) \otimes M_N(\mathbb{C}),$$

and hence

$$\overline{\psi^0}(U)^N \subset \overline{\psi^0}(W) \subset M_N(\overline{\psi^0}(U)) \simeq \overline{\psi^0}(U) \otimes M_N(\mathbb{C}).$$

The following lemma makes explicit the group actions in the isomorphisms of the last remark. Thus, in analogy with the definitions of the algebras $A_{W_x} = \mathcal{C}_0(S^*W_x; \mathrm{End}(E))$ and $A_x = \mathcal{C}_0(S^*U_x; \mathrm{End}(E))$, we consider the algebras

$$B_{W_x} := \overline{\psi^0}(W_x; E) \quad \text{and} \quad B_x := \overline{\psi^0}(U_x; E). \tag{21}$$

We shall also use the standard notation $V^{(I)} := \{f : I \to V\}$ for I finite, as before.

Lemma 3.11. *We keep the notation of Lemma 3.8 and of equation* (21) *above. Then we have* Γ*-equivariant algebra isomorphisms*

$$B_{W_x} \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(B_x) + \overline{\psi^{-1}}(W_x; E).$$

Consequently, $B_{W_x}^{\Gamma} \simeq \Phi(B_x^{\Gamma_x}) + \overline{\psi^{-1}}(W_x; E)^{\Gamma}$.

Proof. Since $B_y = B_{U_y} \subset B_{W_x}$ for all $y \in \Gamma x$ and since U_x and U_y are diffeomorphic through any $\gamma \in \Gamma$ such that $\gamma x = y$, we obtain the inclusion $B_x^{(\Gamma/\Gamma_x)} \subset B_{W_x}$, as in Remark 3.10. Similarly, since $B_x \to A_x$ is surjective, we obtain the equality $B_{W_x} = B_x^{(\Gamma/\Gamma_x)} + \overline{\psi^{-1}}(W_x; E)$, as in the same remark. From equation (17) and Lemma 3.8, we know that

$$B_{W_x}/\overline{\psi^{-1}}(W_x; E) \simeq A_{W_x} \simeq A_{U_x}^{(\Gamma/\Gamma_x)} = \operatorname{Ind}_{\Gamma_x}^{\Gamma}(A_x),$$

and hence we obtain $B_{W_x} \simeq \operatorname{Ind}_{\Gamma_x}^{\Gamma}(B_x) + \overline{\psi^{-1}}(W_x; E)$. The last isomorphism follows from the Frobenius reciprocity (more precisely, from equation (15), with β replaced with B_x) and from the exactness of the functor $V \to V^{\Gamma}$. $\square$

To be able to make further progress, it will be convenient to look first at the case when $x \in M$ has minimal isotropy $\Gamma_x \sim \Gamma_0$, that is, when x belongs to the principal orbit bundle $M_{(\Gamma_0)}$. The notation Γ_0 will remain fixed from now on.

3.12. **Calculations for the principal orbit bundle.** We assume as before that M/Γ is connected. Let Γ_0 be a minimal isotropy group (which, we recall, is unique up to conjugation). Let $x \in M$ be our fixed point and Γ_x its isotropy, as before. The case when Γ_x is conjugated to Γ_0 is simpler since, as noticed already, then Γ_x acts trivially on U_x.

Let us fix $x \in M$ with isotropy group $\Gamma_x = \Gamma_0$. As before, we let

$$W_x \simeq \Gamma \times_{\Gamma_0} U_x \quad \text{and} \quad E_{|W_x} \simeq \Gamma \times_{\Gamma_0} (U_x \times \beta),$$

where β is some Γ_0-module, as in equations (13) and (14). We decompose β into a direct sum of representations of the form $\beta_j^{k_j}$ for some nonisomorphic irreducible module (or representation) β_j of Γ_0,

$$E_x = \beta \simeq \bigoplus \beta_j^{k_j},$$

as in Remark 2.6.

Remark 3.13. We have noticed earlier that Γ_0 acts trivially on U_x, and hence also on T_x^*M. In particular, S^*M also has Γ_0 as minimal isotropy subgroup, and $S^*M_{(\Gamma_0)}$ is a dense subset of the principal orbit bundle of S^*M.

Corollary 3.14. *Let $x \in M$ be such that $\Gamma_x = \Gamma_0$ and $\beta = \bigoplus_{j=1}^N \beta_j^{k_j}$, for some nonisomorphic, irreducible Γ_0-modules β_j. Then*

$$A_{W_x}^{\Gamma} \simeq A_x^{\Gamma_x} \simeq \mathcal{C}_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta) \simeq \bigoplus_{j=1}^{N} M_{k_j}(\mathcal{C}_0(S^*U_x)).$$

In particular, the canonical central character map

$$\operatorname{Prim}(A_x^{\Gamma_0}) \to S^*U_x \simeq \operatorname{Prim}(\mathcal{C}_0(S^*U_x)^{\Gamma_0})$$

of Proposition 3.2 corresponds to the trivial finite covering

$$S^*U_x \times \operatorname{Prim}(\operatorname{End}_{\Gamma_0}(\beta)) \to S^*U_x.$$

Proof. The first isomorphism is repeated from Lemma 3.8. The second one is obtained from the following:
(i) from the definition of $A_x = A_{U_x}$,
(ii) from the assumption that $\Gamma_x = \Gamma_0$,
(iii) from the fact that Γ_0 acts trivially on U_x, and
(iv) from the identifications

$$A_x^{\Gamma_0} := \mathcal{C}_0(S^*U_x; \mathrm{End}(E))^{\Gamma_0} \simeq \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}(\beta)^{\Gamma_0}.$$

The last isomorphism follows from Example 2.10 and the isomorphism $M_n(\mathbb{C}) \otimes A \simeq M_n(A)$, valid for any algebra A.

The rest follows from Lemma 2.12. Indeed, as both $\mathcal{C}_0(S^*U_x)$ and $\mathrm{End}(\beta)^{\Gamma_0}$ have only finite-dimensional irreducible representations, we obtain

$$\mathrm{Prim}(A_x^{\Gamma_0}) = S^*U_x \times \mathrm{Prim}(\mathrm{End}_{\Gamma_0}(\beta)) \simeq S^*U_x \times \{1, 2, \dots, N\},$$

where we use the identification $\mathrm{Prim}(\mathcal{C}_0(S^*U_x)) \simeq S^*U_x$ and where the set $\{1, 2, \dots, N\}$ is in natural bijection with the primitive ideal spectrum of the algebra $\mathrm{End}_{\Gamma_0}(\beta) \simeq \bigoplus_{j=1}^N M_{k_j}(\mathbb{C})$. The inclusion $\mathcal{C}_0(S^*U_x) = \mathcal{C}_0(S^*U_x)^{\Gamma_0} \to A_x^{\Gamma_0}$ is given by the unital inclusion $\mathbb{C} \to \bigoplus_{j=1}^N M_{k_j}(\mathbb{C})$. Hence $\mathrm{Prim}(A_x^{\Gamma_0}) \to S^*U_x$ identifies with the first projection in $S^*U_x \times \{1, 2, \dots, N\} \to S^*U_x$. That is, it is a trivial covering, as claimed. □

The fibers of

$$\mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}}) \to M_{(\Gamma_0)}/\Gamma$$

are thus the simple factors of $\mathrm{End}(E_x)^{\Gamma_0}$, whose structure was determined in Example 2.10. We shall need the following remark similar to Remark 3.10, but simpler.

Remark 3.15. Let U be an open subset of a Euclidean space, let V be a finite-dimensional vector space, and let V denote, by abuse of notation, also the trivial vector bundle with fiber V. Then we have *natural* isomorphisms

$$\psi^{-1}(U; V) \simeq \psi^{-1}(U) \otimes \mathrm{End}(V) \quad \text{and} \quad \psi^0(U; V) \simeq \psi^0(U) \otimes \mathrm{End}(V).$$

Consequently, we also have the analogous isomorphisms for the completions

$$\overline{\psi^{-1}}(U; V) \simeq \overline{\psi^{-1}}(U) \otimes \mathrm{End}(V) \quad \text{and} \quad \overline{\psi^0}(U; V) \simeq \overline{\psi^0}(U) \otimes \mathrm{End}(V).$$

We are in position now to determine the kernel of $\mathcal{R}_{W_x}$, when x is in the principal orbit bundle. We will use the notation of Subsection 2.13 that was recalled at the beginning of this subsection as well as the notation of Subsection 2.2. In particular, recall that $\beta_j \in \widehat{\Gamma}_0$ and $\alpha \in \widehat{\Gamma}$ are said to be Γ_0-disjoint if β_j is *not* contained in the restriction of α to Γ_0. Also, Φ is the Frobenius isomorphism, equations (10) and (11) and Corollary 3.14.

Proposition 3.16. *Let $\Gamma_x = \Gamma_0$, let $E_x = \beta = \bigoplus_{j=1}^{N} \beta_j^{k_j}$, and let*

$$\Phi : \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta) \simeq A_x^{\Gamma_0} \to A_{W_x}^{\Gamma}$$

be the Frobenius isomorphism of Corollary 3.14. Then
(i) $\mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j}) \subset \Phi^{-1}(\ker(\mathcal{R}_{W_x}))$ *if β_j, α are Γ_0-disjoint, and*
(ii) $\mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j}) \cap \Phi^{-1}(\ker(\mathcal{R}_{W_x})) = 0$ *if β_j, α are Γ_0-associated.*
In particular, let $J \subset \{1, 2, \dots, N\}$ be the set of indices j such that β_j and α are Γ_0-disjoint. Then

$$\ker(\mathcal{R}_{W_x}) = \Phi\Big(\bigoplus_{j\in J} \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j})\Big),$$

$$\pi_\alpha(B_M^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(M;E)^\Gamma) \simeq \Phi\Big(\bigoplus_{j\notin J} \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j})\Big).$$

Proof. The proof is essentially a consequence of Proposition 2.8 by including U_x as a parameter, using also Lemma 3.11. To see how this is done, we will use the notation of that lemma, in particular, $W_x \simeq \Gamma \times_{\Gamma_0} U_x \simeq (\Gamma/\Gamma_0) \times U_x$ and $E \simeq \Gamma \times_{\Gamma_0} (U_x \times \beta)$. We identify W_x with $\Gamma \times_{\Gamma_x} U_x$, i.e. we work with $W_x = \Gamma \times_{\Gamma_x} U_x$.

Let π_α the fundamental morphism of restriction to the α-isotypical component, see equations (1) and (8). Recall that $B_x \coloneqq \overline{\psi^0}(U_x; E)$. Since Γ_x acts trivially on U_x, Remark 3.15 yields the Γ-equivariant isomorphisms

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}(B_x) \simeq \overline{\psi^0}(U_x) \otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta)) \subset B_{W_x}, \tag{22}$$

where the last inclusion is modulo the trivial identification given by

$$P \otimes f(s)(\gamma, x) = P(f(\gamma)s(\gamma))(x),$$

$P \in \overline{\psi^0}(U_x)$, $f \in \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))$ and $s \in C_c(W_x, \mathrm{End}(E))$. Combining further Remark 3.15 with Remark 3.10, we obtain the isomorphism

$$\overline{\psi^{-1}}(W_x; E) \simeq \overline{\psi^{-1}}(U_x) \otimes \mathrm{End}(\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)).$$

Lemma 3.11 and the exactness of the functor $V \to V^\Gamma$ give

$$\pi_\alpha(B_{W_x}^\Gamma) = \pi_\alpha \circ \Phi(B_x^{\Gamma_x}) + \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma).$$

Hence we obtain

$$\pi_\alpha(B_{W_x}^\Gamma)/\pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma) = \pi_\alpha \circ \Phi(B_x^{\Gamma_x})/\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) \cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma).$$

Let $\mathfrak{A}$ and $\mathfrak{J}$ be the image and, respectively, the kernel of

$$\pi_\alpha : \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))^\Gamma \to \mathrm{End}(p_\alpha \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)),$$

which have been identified in Proposition 2.8 in terms of the set J. Recall next from equation (15) that $L^2(W_x; E) = L^2(U_x) \otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)$, again Γ-equivariantly. Each time, the action is on the second component since $\Gamma_0 = \Gamma_x$ acts trivially on $\overline{\psi^0}(U_x)$. The action of $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(B_x) \subset B_{W_x}$ on $L^2(W_x; E) = L^2(U_x) \otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)$ is compatible with the tensor product decomposition of equation (22), in the sense that $\overline{\psi^0}(U_x)$ acts on $L^2(U_x)$ and $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))$ acts on $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)$. Moreover, we may identify

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}(B_x)^{\Gamma} \quad \text{with} \quad \overline{\psi^0}(U_x) \otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))^{\Gamma}$$

by the isomorphism in (22). We obtain that

$$\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) = \pi_\alpha(\mathrm{Ind}_{\Gamma_0}^{\Gamma}(B_x)^{\Gamma}) = \overline{\psi^0}(U_x) \otimes \mathfrak{A}.$$

On the other hand, Corollary 2.15 then gives that $\pi_\alpha(\overline{\psi^{-1}}(W_x; \mathrm{End}(E))^{\Gamma})$ is the algebra of Γ-invariant compact operators acting on $p_\alpha(L^2(W_x, \mathrm{End}(E)))$. Therefore, $\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} \subset \pi_\alpha(\overline{\psi^{-1}}(W_x; \mathrm{End}(E))^{\Gamma})$ since $\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A}$ consists of compact, Γ-invariant operators acting on $p_\alpha(L^2(W_x, E))$. Consequently,

$$\begin{aligned}\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} &\subset \pi_\alpha(\mathrm{Ind}_{\Gamma_0}^{\Gamma}(B_x)^{\Gamma}) \cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^{\Gamma}) \\ &\subset \overline{\psi^0}(U_x) \otimes \mathfrak{A} \cap \mathcal{K}(p_\alpha L^2(W_x; E))^{\Gamma} \subset \overline{\psi^{-1}}(U_x) \otimes \mathfrak{A},\end{aligned}$$

and hence we have equalities everywhere.

Recall from Corollary 3.14 that $A_{W_x}^{\Gamma} \simeq A_x^{\Gamma_x}$. We obtain that the map

$$\mathcal{R}_{W_x} : A_{W_x}^{\Gamma} \simeq B_{W_x}^{\Gamma}/\overline{\psi^{-1}}(W_x; E)^{\Gamma} \to \pi_\alpha(B_{W_x}^{\Gamma})/\pi_\alpha(\overline{\psi^{-1}}(W_x; E)^{\Gamma})$$

becomes, up to the canonical isomorphisms above, the map

$$\begin{aligned}A_x^{\Gamma_x} \simeq \mathcal{C}_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta) &\to \pi_\alpha(B_{W_x}^{\Gamma})/\pi_\alpha(\overline{\psi^{-1}}(W_x)^{\Gamma}) \\ &= \pi_\alpha \circ \Phi(B_x^{\Gamma_x})/\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) \cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^{\Gamma}) \\ &\simeq \overline{\psi^0}(U_x) \otimes \mathfrak{A}/\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} \simeq \mathcal{C}_0(S^*U_x) \otimes \mathfrak{A},\end{aligned}$$

with all maps being surjective and preserving the tensor product decompositions. This identifies the kernel of $\mathcal{R}_{W_x}$ with $\mathcal{C}_0(S^*U_x) \otimes \mathfrak{J}$ and the image of $\mathcal{R}_{W_x}$ with $\mathcal{C}_0(S^*U_x) \otimes \mathfrak{A}$. The rest of the statement follows from the identification of $\mathfrak{J}$ and $\mathfrak{A}$ in Proposition 2.8. □

Proposition 3.16 above and its proof give the following corollary.

Corollary 3.17. *We use the notation of Proposition 3.16, and we identify the space* $\mathrm{Prim}(\mathrm{End}(\beta))$ *with* $\{1, 2, \dots, N\}$ *as in Remark 2.10. Then the homeomorphism* $\mathrm{Prim}(A_{W_x}^{\Gamma}) \simeq S^*U_x \times \{1, 2, \dots, N\}$ *maps the set* $\Xi \cap \mathrm{Prim}(A_{W_x}^{\Gamma})$ *to* $S^*U_x \times J$. *In particular, the restriction* $\Xi \cap \mathrm{Prim}(A_{W_x}^{\Gamma}) \to S^*U_x$ *of the central character is a covering as well.*

Proof. Using the notation of the proof of Proposition 3.16, we have that $\ker(\mathcal{R}_{W_x})$ has primitive ideal spectrum $S^*U_x \times \mathrm{Prim}(\mathfrak{J})$. We have

$$\Xi \cap \mathrm{Prim}(A_{W_x}^{\Gamma}) = S^*U_x \times \mathrm{Prim}(\mathfrak{A}). \qquad \square$$

The same methods yield the following result (recall that $M_{(\Gamma_0)}$ is the principal orbit bundle).

Corollary 3.18. *The central character map* $\mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}}) \to S^*M_{(\Gamma_0)}/\Gamma$ *defined by the inclusion* $\mathcal{C}_0(S^*M_{(\Gamma_0)}/\Gamma) \subset Z(A^{\Gamma}_{M_{(\Gamma_0)}})$ *is a covering with typical fiber* $\mathrm{Prim}(\mathrm{End}(E_x)^{\Gamma_0})$ *such that*

$$\Xi \cap \mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}}) \to S^*M_{(\Gamma_0)}/\Gamma$$

is a subcovering, see (20) *for the definition of* Ξ*. In particular, we have that* $\Xi \cap \mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}})$ *is open and closed in* $\mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}})$.

Proof. We notice that $A^{\Gamma}_{M_{(\Gamma_0)}} = \mathcal{C}_0(S^*M_{(\Gamma_0)}/\Gamma)A^{\Gamma}_M$, so the central character map is defined. The first statement is true locally, by Corollary 3.14, and hence it is true globally. Indeed, let $x \in M_{(\Gamma_0)}$, let $\xi \in S^*_xM_{(\Gamma_0)}$, and let $\rho \in \widehat{\Gamma}_x$ that appears in E_x (so $(\xi, \rho) \in \Omega_M$). We let $W_x \subset M_{(\Gamma_0)} \subset M$ be the typical tube with minimal isotropy $\Gamma_x = \Gamma_0$, as before. Let $Z_x \coloneqq \mathcal{C}_0(S^*W_x)^{\Gamma} \subset Z_M = \mathcal{C}_0(S^*M)^{\Gamma}$. Then $\mathrm{Prim}(Z_xA^{\Gamma}_M)$ is an open neighborhood in $\mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}})$ of the primitive ideal $\ker(\pi_{\xi,\rho})$, see Proposition 3.2 and equation (18) for notation and details. We have that $Z_xA^{\Gamma}_M = A^{\Gamma}_{W_x}$, and hence, on $\mathrm{Prim}(Z_xA^{\Gamma}_M)$, the central character is a covering, by Corollary 3.14. Similarly, its restriction to $\Xi \cap \mathrm{Prim}(Z_xA^{\Gamma}_M)$ is a covering by Corollary 3.17. □

Putting Corollary 3.18 and Proposition 3.16 together, we obtain the following results.

Corollary 3.19. *Let* $M_{(\Gamma_0)}$ *be the principal orbit bundle of* M*. The ideal*

$$\ker(\mathcal{R}_{M_{(\Gamma_0)}}) = A^{\Gamma}_{M_{(\Gamma_0)}} \cap \ker(\mathcal{R}_M)$$

is defined by the closed subset

$$\Xi_0 \coloneqq \Xi \cap \mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}}) \quad \textit{of} \ \mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}})$$

consisting of the sheets of

$$\mathrm{Prim}(A^{\Gamma}_{M_{(\Gamma_0)}}) \to S^*M_{(\Gamma_0)}/\Gamma$$

that correspond to the simple factors $\mathrm{End}(E_{x\rho})^{\Gamma_0}$ *of* $\mathrm{End}(E_x)^{\Gamma_0}$ *with* ρ *and* α Γ_0*-associated.*

If Γ is abelian, then ρ and α are characters and saying that they are Γ_0-associated means, simply, that their restrictions to Γ_0 coincide: $\rho|_{\Gamma_0} = \alpha_{\Gamma_0}$. This is consistent with the definition given in [12].

3.20. **The non-principal orbit case.** As in the rest of the paper, we assume M/Γ to be connected. We will show in Theorem 3.22 that Ξ is the closure of Ξ_0 in $\mathrm{Prim}(A^{\Gamma}_M)$. To that end, we first construct a suitable basis of neighborhoods of $\mathrm{Prim}(A^{\Gamma}_M)$ using Lemma 3.3.

Remark 3.21. Let $\Gamma(\xi, \rho) \in \mathrm{Prim}(A^{\Gamma}_M)$, where we have used the description of $\mathrm{Prim}(A^{\Gamma}_M)$ provided in Proposition 3.2 as orbits of pairs $(\xi, \rho) \in \Omega_M$, that is, with $\xi \in S^*M$ and $\rho \in \widehat{\Gamma}_\xi$ such that $E_{\xi\rho} \neq 0$. We construct a basis of neighborhoods $(V_{\xi,\rho,n})_{n\in\mathbb{N}}$ of $\Gamma(\xi, \rho)$ in $\mathrm{Prim}(A^{\Gamma}_M)$ as follows. Let $\xi \in S^*_xM$ (that is, ξ sits above $x \in M$), and let us use the notation U_x and W_x of equation (13), as always. First, by choosing a different point ξ in its orbit, if necessary, we may assume that $\Gamma_0 \subset \Gamma_\xi$. Now let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a family of Γ_ξ-invariant neighborhoods of ξ in S^*U_x such that,

- for all n and $\gamma \in \Gamma \setminus \Gamma_\xi$, we have $\gamma\mathcal{O}_n \cap \mathcal{O}_n = \varnothing$,
- $\overline{\mathcal{O}_{n+1}} \subset \mathcal{O}_n \subset S^*U_x$ and $\bigcap_{n\in\mathbb{N}} \mathcal{O}_n = \{\xi\}$.

For any $n \in \mathbb{N}$, we choose a function $\varphi_n \in \mathcal{C}_c(\mathcal{O}_n)^{\Gamma_\xi}$ such that $\varphi_n = 1$ on $\mathcal{O}_{n+1}$. Let $p_\rho \in \mathrm{End}(E_x)^{\Gamma_\xi}$ be the projection onto $E_{x\rho}$. We have $E_{x\rho} \neq 0$ since $(\xi,\rho) \in \Omega_M$, and hence $p_\rho \neq 0$. We can assume the bundle E to be trivial on U_x, and using that, we first extend p_ρ constantly on $\mathcal{O}_n$. We notice that this extension is Γ_ξ invariant since p_ρ is. Then we extend further p_ρ to an element $q_n \in \mathcal{C}_c(S^*U_x;\mathrm{End}(E_x))^{\Gamma_x}$ defined as

$$q_n := \begin{cases} \Phi_{\Gamma_\xi}^{\Gamma_x}(\varphi_n p_\rho) & \text{on } \Gamma_x\mathcal{O}_n, \\ 0 & \text{on } S^*U_x \setminus \Gamma_x\mathcal{O}_n, \end{cases}$$

with $\Phi_{\Gamma_\xi}^{\Gamma_x}$ the Frobenius isomorphism of equation (11). Set $\tilde{q}_n := \Phi_{\Gamma_x}^{\Gamma}(q_n) \in A_M^\Gamma$, where $\Phi_{\Gamma_x}^{\Gamma}$ is the Frobenius isomorphism of equation (11). Finally, we associate to $\tilde{q}_n$ the open set

$$V_{\xi,\rho,n} := \{J \in \mathrm{Prim}(A_M^\Gamma) \mid \tilde{q}_n \notin J\}.$$

Recall from 3.3 that $V_{\xi,\rho,n}$ is an open subset of $\mathrm{Prim}(A_M^\Gamma)$. Moreover, it follows from our definition that $V_{\xi,\rho,n+1} \subset V_{\xi,\rho,n}$ and that $\bigcap_{n\in\mathbb{N}} V_{\xi,\rho,n} = \{\Gamma(\xi,\rho)\}$.

Recall that we are assuming that M/Γ is connected.

Theorem 3.22. *Let* $\Xi := \mathrm{Prim}(A_M^\Gamma/\ker(\mathcal{R}_M)) \subset \mathrm{Prim}(A_M^\Gamma)$ *be the closed subset defined by the ideal* $\ker(\mathcal{R}_M)$*. Then* Ξ *is the closure in* $\mathrm{Prim}(A_M^\Gamma)$ *of the set* $\Xi_0 := \Xi \cap \mathrm{Prim}(A^\Gamma_{M_{(\Gamma_0)}})$*, where* $M_{(\Gamma_0)}$ *is the principal orbit bundle of* M*.*

Proof. We have that $\overline{\Xi}_0 \subset \Xi$ since $\Xi_0 \subset \Xi$ and the latter is a closed set.

Conversely, let $\mathfrak{P} \in \mathrm{Prim}(A_M^\Gamma) \setminus \overline{\Xi}_0$. We will show that $\mathfrak{P} \notin \Xi$. As in Proposition 3.2, let $\mathfrak{P} = \ker \pi_{(\xi,\rho)}$ correspond to $(\xi,\rho) \in \Omega_M$. We may assume that $\Gamma_0 \subset \Gamma_\xi$. Let x be projection of ξ onto M. Since the problem is local, we may also assume that $M = W_x := \Gamma \times_{\Gamma_x} U_x$ and that $E := \Gamma \times_{\Gamma_x} (U_x \times \beta)$ for some Γ_x-module β, where $U_x = (T_xM)_r \subset T_xM$.

We shall use freely the notation of Remark 3.21 throughout this proof. The assumption that $\mathfrak{P} \notin \overline{\Xi}_0$ and the fact that the family $V_{\xi,\rho,n}$ of Remark 3.21 is a basis for the system of neighborhoods of $\mathfrak{P} = \ker \pi_{(\xi,\rho)}$ implies that there exists $n > 0$ such that

$$V_{\xi,\rho,n} \cap \Xi_0 = \varnothing. \tag{23}$$

Let $\tilde{q}_n = \Phi_{\Gamma_x}^{\Gamma}(q_n)$ be the symbol defined in Remark 3.21. The description of Ξ_0 provided in Corollary 3.19, the definition of $V_{\xi,\rho,n}$, and the definition of $\tilde{q}_n$ imply that $\pi_{\zeta,\rho'}(\tilde{q}_n) = 0$ for any $\zeta \in S^*M_{(\Gamma_0)}$ and $\rho' \in \widehat{\Gamma}_0$ such that $\ker \pi_{(\zeta,\rho')} \in \Xi_0$, $\Gamma_\xi = \Gamma_0$, that is, such that ρ' and α are Γ_0-associated.

We next "quantize $\tilde{q}_n$" in an appropriate way, that is, we construct an operator $\widetilde{Q}_n \in B_{W_x}^\Gamma$ with symbol $\tilde{q}_n$ and with other convenient properties as follows. First, let $\chi \in \mathcal{C}_c^\infty(U_x)^{\Gamma_x}$ be such that $\chi\varphi_n = \varphi_n$, which is possible since φ_n has compact support. (See Remark 3.21.) Then let $\psi \in \mathcal{C}^\infty(T_x^*M)^{\Gamma_x}$ be such that $\psi(0) = 0$ if $|\eta| < 1/2$ and $\psi(\eta) = 1$ whenever $|\eta| \geq 1$. Recall that we have assumed in this proof that $M = \Gamma \times_{\Gamma_x} U_x$ through the exponential map and that $U_x \subset T_xM$ is hence identified with its image in M. Let then, for a suitable symbol a,

$$\mathrm{Op}(a)f(y) \coloneqq (2\pi)^{-n/2} \int_{T_x^*M} \int_{U_x} e^{i(y-z)\cdot\eta} a(y,z,\eta) f(z)\, dz\, d\eta.$$

We shall use this for $a_n(y,z,\eta) \coloneqq \chi(y)\psi(\eta)\tilde{q}_n\big(\frac{\eta}{|\eta|}\big)\chi(z) \in S^0_{\mathrm{cl}}(U_x \times U_x \times T_x^*M)$ so that $Q_n \coloneqq \mathrm{Op}(a_n)$ is defined. In particular,

$$Q_n f(y) \coloneqq (2\pi)^{-n/2} \int_{T_x^*M} \int_{U_x} e^{i(y-z)\cdot\eta} \chi(y)\psi(\eta)\tilde{q}_n\Big(\frac{\eta}{|\eta|}\Big)\chi(z) f(z)\, dz\, d\eta$$

is a classical, order zero pseudodifferential operator on U_x. The operator Q_n is Γ_x-invariant by construction. Using the Frobenius isomorphism of equation (11), we extend Q_n to the operator $\widetilde{Q}_n \coloneqq \Phi^\Gamma_{\Gamma_x}(Q_n)$, which acts on

$$M = W_x = \Gamma \times_{\Gamma_x} U_x$$

(see Lemma 3.11 as well as equation (15)). Then $\widetilde{Q}_n \in \Psi^0(M;E)^\Gamma$, that is, it is Γ-invariant, by construction, and has principal symbol $\sigma_0(\widetilde{Q}_n) = \tilde{q}_n$.

Now let $x_0 \in M_{(\Gamma_0)} \cap U_x$, where, we recall, $M_{(\Gamma_0)}$ denotes the principal orbit bundle. We have

$$L^2(W_{x_0};E) = \mathrm{Ind}^\Gamma_{\Gamma_0}(L^2(U_{x_0};\beta)) = L^2(U_{x_0};\mathrm{Ind}^\Gamma_{\Gamma_0}(\beta)),$$

where $\beta = E_{x_0} = E_x$ by the assumption that $E \coloneqq \Gamma \times_{\Gamma_x} (U_x \times \beta)$.

Let $\beta_j \in \widehat{\Gamma_0}$ be the isomorphism classes of the Γ_ξ-submodules of β, and $k_j \geq 0$ is the dimension of the corresponding β_j-isotypical component in β so that $\beta \simeq \bigoplus_{j=1}^N \beta_j^{k_j}$, as Γ_0-modules, as before. Thus

$$L^2(W_{x_0};E) \simeq \bigoplus_{j=1}^N L^2(U_{x_0};\mathrm{Ind}^\Gamma_{\Gamma_0}(\beta_j^{k_j})).$$

Recall that the α-isotypical component of $\mathrm{Ind}^\Gamma_{\Gamma_0}(\beta_j^{k_j})$ is

$$\alpha \otimes \mathrm{Hom}_\Gamma(\alpha, \mathrm{Ind}^\Gamma_{\Gamma_0}(\beta_j^{k_j})),$$

which is nonzero if, and only if, α and β_j are Γ_0-associated, by the Frobenius isomorphism. Hence, passing to the α-isotypical components, we have

$$L^2(W_{x_0};E)_\alpha \coloneqq p_\alpha L^2(W_{x_0};E) = \bigoplus_{j\in J^c} L^2(U_{x_0};\mathrm{Ind}^\Gamma_{\Gamma_0}(\beta_j^{k_j}))_\alpha, \tag{24}$$

where $J \subset \{1,\dots,N\}$ is the set of indices such that $\beta_j \in \widehat{\Gamma}_0$ and α are Γ_0-disjoint; J^c is its complement (i.e. $\beta_j \in \widehat{\Gamma}_0$ and α are Γ_0-associated).

Let $p_J \in \mathrm{End}(\beta)^{\Gamma_0}$ be the projection onto $\bigoplus_{j\in J^c} \beta_j^{k_j}$. Recall from equation (23) that $\pi_{\zeta,\beta_j}(\tilde{q}_n) = 0$ for any $(\zeta,\beta_j) \in S^*M_{(\Gamma_0)} \times \widehat{\Gamma}_0$ with $j \notin J$.

Therefore, $\tilde{q}_n(\zeta)p_J = 0$ for all $\zeta \in S^*M_{(\Gamma_0)}$. Since $S^*M_{(\Gamma_0)}$ is dense in S^*M, this implies that $\tilde{q}_n p_J = 0$ and hence that $\chi\psi\tilde{q}_n\chi p_J = \chi\psi\tilde{q}_n p_J\chi = 0$. Thus

$$\widetilde{Q}_n p_J = \mathrm{Op}(\chi\psi\tilde{q}_n\chi)p_J = \mathrm{Op}(\chi\psi\tilde{q}_n\chi p_J) = 0,$$

as operators on $L^2(M;E)$. Hence, for any $f \in L^2(W_{x_0};E)_\alpha$, we have that $\widetilde{Q}_n f = 0$ since $L^2(W_{x_0};E)_\alpha \subset p_J L^2(W_{x_0};E)$. This is true for any $x_0 \in M_{(\Gamma_0)}$, so we conclude that $\widetilde{Q}_n$ is zero on $L^2(M_{(\Gamma_0)};E)_\alpha$. Since $M_{(\Gamma_0)}$ has measure zero complement in M, we have $L^2(M_{(\Gamma_0)};E)_\alpha = L^2(M;E)_\alpha$; therefore, $\pi_\alpha(\widetilde{Q}_n) = 0$. This implies that $\mathcal{R}_M(\widetilde{q}_n) = 0$, while $\pi_{\xi,\rho}(\widetilde{q}_n) = 1$. Thus $\ker\pi_{(\xi,\rho)} \notin \Xi$, which concludes the proof. ☐

Our question now is to decide whether some given $\Gamma(\xi,\rho)$ is in Ξ or not. Recall that ρ and α are said to be Γ_0-associated if $\mathrm{Hom}_{\Gamma_0}(\rho,\alpha) \neq 0$. The set $X^\alpha_{M,\Gamma}$ was defined in the introduction as the set of pairs $(\xi,\rho) \in T^*M \setminus \{0\} \times \widehat{\Gamma}_\xi$ for which there is an element $g \in \Gamma$ such that $\Gamma_0 \subset g\cdot\Gamma_\xi = \Gamma_{g\xi}$ and such that $g\cdot\rho$ and α are Γ_0-associated.

Remark 3.23. Let us highlight the following interesting fact, implied by the proof of Theorem 3.22. We have $E_{x\rho} = 0$ for any $(\xi,\rho) \in X^\alpha_{M_{(\Gamma_0)},\Gamma}$ (with x the projection of ξ on $M_{(\Gamma_0)}$) if, and only if, $L^2(M;E)_\alpha = 0$. Indeed, for any $x \in M_{(\Gamma_0)}$ with $\Gamma_x = \Gamma_0$, we have noted in equation (24) that

$$L^2(W_x;E)_\alpha = \bigoplus_\rho L^2(U_x;\mathrm{Ind}^\Gamma_{\Gamma_0}(E_{x\rho}))_\alpha,$$

where the direct sum is indexed by the representations $\rho \in \widehat{\Gamma}_0$ that are Γ_0-associated to α. If $E_{x\rho} = 0$ for any such representation, then $L^2(W_x;E)_\alpha = 0$. Such open sets W_x cover $M_{(\Gamma_0)}$, so $L^2(M_{(\Gamma_0)};E)_\alpha = 0$. Since $M_{(\Gamma_0)}$ has measure zero complement, we conclude that $L^2(M;E)_\alpha = 0$.

Theorem 3.24. *We use the notation in the last two paragraphs. We have that $\Gamma(\xi,\rho) \in \Xi$ if, and only if, there is a $g \in \Gamma$ such that $\Gamma_0 \subset g\cdot\Gamma_\xi$ and such that $g\cdot\rho$ and α are Γ_0-associated.*

Proof. Let $\Gamma(\xi,\rho) \in \mathrm{Prim}(A^\Gamma_M)$, with $x \in M$ the base point of ξ. We can assume (by choosing a different element in the orbit if needed) that $\Gamma_0 \subset \Gamma_\xi$. Let $\tilde{q}_n \in A^\Gamma_M$ be the element defined in Remark 3.21 and $V_{\xi,\rho,n}$ the corresponding neighborhood of $\Gamma(\xi,\rho)$ in $\mathrm{Prim}(A^\Gamma_M)$.

There is a Γ_x-equivariant isomorphism $E|_{U_x} \simeq U_x \times \beta$, where $\beta = E_x$ is a Γ_x-module. Since $\Gamma_0 \subset \Gamma_x$, we may decompose β into Γ_0-isotypical components, i.e. $\beta = \bigoplus_{j=1}^N \beta_j^{k_j}$, with the usual notation. If $\eta \in \mathcal{O}_n \cap S^*M_{(\Gamma_0)}$, then $\pi_{\eta,\beta_j}(\tilde{q}_n) = \varphi_n(\eta)\pi_{\beta_j}(p_\rho)$. Therefore, for any $\eta \in S^*M$, we have

$$\pi_{\eta,\beta_j}(\tilde{q}_n) = 0 \iff \mathrm{Hom}_{\Gamma_0}(\beta_j,\rho) = 0 \text{ or } \varphi_n(\eta) = 0.$$

This implies that

$$V_{\xi,\rho,n} \cap \Xi_0 = \{\Gamma(\eta,\beta) \in \Xi_0 \mid \varphi_n(\eta) \neq 0 \text{ and } \mathrm{Hom}_{\Gamma_0}(\beta,\rho) \neq 0\}.$$

It follows from the determination of Ξ_0 in Corollary 3.19 that $V_{\xi,\rho,n} \cap \Xi_0 \neq \varnothing$ if, and only if, there exists $\beta \in \widehat{\Gamma}_0$ that is Γ_0-associated with α and Γ_0-associated with ρ. Thus $\beta \subset \alpha|_{\Gamma_0}$ and $\beta \subset \rho|_{\Gamma_0}$, and hence α and ρ are Γ_0-associated. Thus $V_{\xi,\rho,n} \cap \Xi_0 \neq \varnothing$ if, and only if, we have $\mathrm{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. Now $\Xi = \overline{\Xi}_0$ by Theorem 3.22. Since the open sets $(V_{\xi,\rho,n})_{n\in\mathbb{N}}$ form a basis of neighborhoods of $\Gamma(\xi, \rho)$, we conclude that $\Gamma(\xi, \rho) \in \Xi$ if, and only if, $\mathrm{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. □

Remark 3.25. Our definition of α-ellipticity for an operator $P \in \overline{\psi^0}(M; E)^\Gamma$ was stated in terms of the set $X^\alpha_{M,\Gamma}$, defined in equation (5). Theorem 3.24 establishes that $\Xi \simeq \tilde{X}^\alpha_{M,\Gamma}/\Gamma$, where $\tilde{X}^\alpha_{M,\Gamma}$ is the (possibly smaller) subset of pairs $(\xi, \rho) \in X^\alpha_{M,\Gamma}$ such that $E_{x\rho} \neq 0$ (with x the projection of ξ on M). Keeping in mind the fact that the null operator on a trivial vector space is invertible, we have that $\sigma^\Gamma_0(P)(\xi, \rho)$ is invertible for any $(\xi, \rho) \in X^\alpha_{M,\Gamma}$ if, and only if, it is invertible for any $(\xi, \rho) \in \tilde{X}^\alpha_{M,\Gamma}$. The pathological case $\Xi = \varnothing$, for which $E_{x\rho} = 0$ for any $(\xi, \rho) \in X^\alpha_{M,\Gamma}$, causes no problem: indeed, as noticed in Remark 3.23, we then have $L^2(M; E)_\alpha = 0$. In that case, $\pi_\alpha(P)$ is Fredholm for any $P \in \overline{\psi^0}(M; E)^\Gamma$, which is consistent with the invertibility of $\sigma^\alpha_0(P)(\xi, \rho) : 0 \to 0$ for any $(\xi, \rho) \in X^\alpha_{M,\Gamma}$.

We summarize part of the above discussions in the following proposition.

Proposition 3.26. *Let $\tilde{X}^\alpha_{M,\Gamma}$ be as in Remark 3.25. The primitive ideal spectrum $\Xi = \mathrm{Prim}(A^\Gamma_M / \ker(\mathcal{R}_M))$ is canonically homeomorphic to $\tilde{X}^\alpha_{M,\Gamma}/\Gamma$ via the restriction map from $A^\Gamma_M \coloneqq \mathcal{C}_0(S^*M; \mathrm{End}(E))^\Gamma$ to sections over $\tilde{X}^\alpha_{M,\Gamma}$.*

4. Proof of the Fredholm criterion

We now complete the proof of the main result of the paper, Theorem 1.5, on the characterization of Fredholm operators. We first explain how to reduce the proof to the case M/Γ connected. In this section, we shall assume that M is compact since, if M is not compact, none of our operators will be Fredholm.

4.1. Reduction to the connected case and α-ellipticity.

In this subsection, unlike most of the rest of the paper, we *do* not *assume that M/Γ is connected* in order to explain how to reduce the general case to the connected one.

Let $\pi_{M,\Gamma} : M \to M/\Gamma$ be the quotient map, and let us write $M/\Gamma = \bigcup_{i=1}^N C_i$ as the *disjoint* union of its connected components (we have a finite number of components since we have assumed M to be compact). We let $M_i \coloneqq \pi^{-1}_{M,\Gamma}(C_i)$ be the preimages of these connected components. Note that, in general, the submanifolds M_i are not connected, although, for each i, $M_i/\Gamma = C_i$ *is connected*. Therefore, the spaces $X^\alpha_{M_i,\Gamma}$ are defined as in (5), and we let

$$X^\alpha_{M,\Gamma} \coloneqq \bigsqcup_{i=1}^N X^\alpha_{M_i,\Gamma},$$

where the union is a disjoint union.

We shall decorate with the index i the restrictions of objects on M to M_i. Thus $E_i \coloneqq E|_{M_i}$, and so on and so forth. This almost works for an operator

$P \in \overline{\psi^0}(M;E)^\Gamma$. Indeed, we first notice that

$$L^2(M;E) \simeq \bigoplus_{i=1}^N L^2(M_i;E_i) \quad \text{and} \quad \bigoplus_{i=1}^N \overline{\psi^0}(M_i;E_i) \subset \overline{\psi^0}(M;E). \tag{25}$$

Recall that $\mathcal{K}(V)$ denotes the algebra of compact operators on a Hilbert space V. The following proposition provides the desired reduction to the case M/Γ connected.

Proposition 4.2. *Let $p_i : L^2(M;E) \to L^2(M_i;E_i)$ be the canonical orthogonal projection. For $P \in \overline{\psi^0}(M;E)$, we let $P_i := p_i P p_i \in \overline{\psi^0}(M_i;E_i)$. Then*

$$P - \sum_{i=1}^N P_i \in \mathcal{K}(L^2(M;E)).$$

If we regard $\sum_{i=1}^N P_i = \bigoplus_{i=1}^N P_i$ as an element of $\bigoplus_{i=1}^N \overline{\psi^0}(M_i;E_i)$, then we see that

$$\overline{\psi^0}(M;E) = \bigoplus_{i=1}^N \overline{\psi^0}(M_i;E_i) + \mathcal{K}(L^2(M;E)),$$

$$\overline{\psi^0}(M;E)/\mathcal{K}(L^2(M;E)) = \bigoplus_{i=1}^N \overline{\psi^0}(M_i;E_i)/\mathcal{K}(L^2(M_i;E_i)).$$

In particular, $\pi_\alpha(P) - \bigoplus_{i=1}^N \pi_\alpha(P_i)$ is compact, and hence $\pi_\alpha(P)$ is Fredholm if, and only if, each $\pi_\alpha(P_i)$ is Fredholm for $i = 1, \dots, N$.

Proof. If $i \neq j$, $p_i P p_j$ has zero principal symbol, and hence it is compact. Therefore, $P - \sum_{i=1}^N P_i = \sum_{i \neq j} p_i P p_j$ is compact. The rest follows from equation (25), its corollary $L^2(M;E)_\alpha \simeq \bigoplus_{i=1}^N L^2(M_i;E_i)_\alpha$, and the fact that π_α respects these direct sum decompositions. $\square$

Recall the algebras $A_{\mathcal{O}}$ of symbols of the previous section, see equation (19), where $\mathcal{O}$ is an open subset of M.

Remark 4.3. The Γ-principal symbol $\sigma_m^\Gamma(P)$ was defined in (4), and we stress that the definition of the space $X_{M,\Gamma}$ in equation (3) did not require that M/Γ be connected. The disjoint union definition of the space $X_{M,\Gamma} = \bigsqcup_{i=1}^N X_{M_i,\Gamma}$ means that $\sigma_m^\Gamma(P)|_{X_{M_i,\Gamma}} = \sigma_m^\Gamma(P_i)$ for each $i = 1, \dots, N$. The analogous disjoint union decomposition of $X_{M,\Gamma}^\alpha := \bigsqcup_{i=1}^N X_{M_i,\Gamma}^\alpha$ gives that P is α-elliptic if, and only if, for each i, P_i is α-elliptic.

This allows us to reduce the proof of our main theorem, Theorem 1.5, to the connected case since, assuming that the connected case has been proved, we have

$$\begin{aligned} \pi_\alpha(P) \text{ is Fredholm} &\iff \text{for all } i,\ \pi_\alpha(P_i) \text{is Fredholm} \\ &\iff \text{for all } i,\ P_i \text{ is } \alpha\text{-elliptic} \\ &\iff P \text{ is } \alpha\text{-elliptic}, \end{aligned}$$

where the first equivalence is by Proposition 4.2, the second equivalence is by the assumption that our main theorem has been proved in the connected case, and the last equivalence is by the first part of this remark.

4.4. **Proof of the main result.** We now complete the proof of Theorem 1.5. We continue to assume that M is a compact smooth manifold (otherwise, there will be no Fredholm operators in our main result).

We have the following Γ-equivariant version of Atkinson's theorem. (Recall that $\mathcal{K}(V)$ denotes the algebra of compact operators acting on the Hilbert space V.)

Proposition 4.5. *Let V be a Hilbert space with a unitary action of Γ, and let $P \in \mathcal{L}(V)^\Gamma$ be a Γ-equivariant bounded operator on V. We have that P is Fredholm if, and only if, it is invertible modulo $\mathcal{K}(V)^\Gamma$, in which case, we can choose the parametrix (i.e. the inverse modulo the compacts) to also be Γ-invariant.*

Proof. See, for example, [12, Prop. 5.1]. □

Since $\pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma) = \mathcal{K}(L^2(M;E)_\alpha)^\Gamma$, we obtain the following corollary.

Corollary 4.6. *Let $P \in \overline{\psi^0}(M;E)^\Gamma$ and $\alpha \in \widehat{\Gamma}$. We have that $\pi_\alpha(P)$ is Fredholm on $L^2(M;E)_\alpha$ if, and only if, $\pi_\alpha(P)$ is invertible modulo*

$$\pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma) \quad \textit{in } \pi_\alpha(\overline{\psi^0}(M;E)^\Gamma).$$

We are now in a position to prove the main result of this paper, Theorem 1.5.

Proof of Theorem 1.5. As in [12, Sec. 2.6], we may assume that $P \in \overline{\psi^0}(M;E)^\Gamma$. Corollary 4.6 then states that $\pi_\alpha(P)$ is Fredholm if, and only if, the image of its symbol $\sigma(P)$ is invertible in the quotient algebra

$$\mathcal{R}_M(A_M^\Gamma) = \pi_\alpha(\overline{\psi^0}(M;E)^\Gamma)/\pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma).$$

To complete the proof, we shall use the following general facts about C^*-algebras. Let A be a C^*-algebra with unit and $a \in A$. Then a is invertible (in A) if, and only if, $\pi(a)$ is invertible for all irreducible representations of A (see [40]).

We shall use this general fact about C^*-algebras for $A = A_M^G/\ker(\mathcal{R}_M)$. Recall that this algebra is a type I C^*-algebra, and hence there is a bijection between its primitive ideals and (unitary equivalence classes of) irreducible representations. The primitive ideal spectrum of A_M^G was identified with the set $\{\ker(\pi_{(\xi,\rho)}\}$, $(x,\rho) \in \Omega_M$, see Proposition 3.2. According to Theorem 3.24 and Remark 3.25 following it, the primitive spectrum Ξ of $\mathcal{R}_M(A_M^\Gamma)$ identifies with $\Omega_{M,\Gamma}^\alpha := \Omega_M \cap X_{M,\Gamma}^\alpha$. Therefore, $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, the endomorphism $\pi_{\xi,\rho}(\sigma(P))$ is invertible for all $(\xi,\rho) \in \Omega_{M,\Gamma}^\alpha$. Since, for $(\xi,\rho) \in X_{M,\Gamma}^\alpha \setminus \Omega_{M,\Gamma}^\alpha$, $\pi_{(\xi,\rho)}$ acts on the zero space (and hence $\pi_{(\xi,\rho)}(P)$ is automatically invertible, we obtain that $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, the endomorphism $\pi_{\xi,\rho}(\sigma(P))$ is invertible for all $(\xi,\rho) \in X_{M,\Gamma}^\alpha$. That is, $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, P is α-elliptic (see Definition 1.3). □

5. Applications and extensions

We now discuss some applications and extensions of our results. We continue to assume that M is a Riemannian manifold and that Γ acts by isometries on M. Other assumptions change in each subsection, so they will be reminded each time.

5.1. **Applications to Hodge and index theory.** We now point out the relevance of our results to Hodge and index theory. In this subsection, we do not need M/Γ to be connected.

Remark 5.2. Let $P : H^s(M; E_0) \to H^{s-m}(M; E_1)$ be an order m, classical pseudodifferential operator. Since the index of Fredholm operators is invariant under small perturbations and under compact perturbations, we obtain that the index of $\pi_\alpha(P)$ depends only on the homotopy class of its α-principal symbol $\sigma_m^\alpha(P)$.

An alternative approach to the Fredholm property (Theorem 1.5) can be obtained from the following theorem. Recall that $X_{M,\Gamma}^\alpha$ was defined in (5). Below, by ∂, we shall denote the connecting morphism in the six-term K-theory exact sequence associated to a short exact sequence of C^*-algebras. Recall that σ_0^α is the α-principal symbol map, see Definition 1.3.

Theorem 5.3. *Let $\Omega_{M,\Gamma}^\alpha \coloneqq X_{M,\Gamma}^\alpha \cap \Omega_M$, and let us denote by $\mathcal{C}(X_{M,\Gamma}^\alpha(E)/\Gamma)$ the algebra of restrictions of $A_M^\Gamma \coloneqq \mathcal{C}(S^*M; \mathrm{End}(E))^\Gamma$ to $X_{M,\Gamma}^\alpha(E)/\Gamma$. Using the notation of Corollary 4.6, we have an exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi_\alpha(\overline{\psi^0}(M; E)^\Gamma) \xrightarrow{\sigma_0^\alpha} \mathcal{C}(\Omega_{M,\Gamma}^\alpha/\Gamma) \longrightarrow 0.$$

Let $\partial : K_1(\mathcal{C}(X_{M,\Gamma}^\alpha(E)/\Gamma)) \to \mathbb{Z} \simeq K_0(\mathcal{K})$ be the associated connecting morphism, and let $P \in \overline{\psi^0}(M; E)^\Gamma$ be such that $\pi_\alpha(P)$ is Fredholm. Then

$$\mathrm{ind}(\pi_\alpha(P)) = \dim(\alpha)\partial[\sigma_0^\alpha(P)].$$

Proof. The exactness of the sequence follows from the proof of Corollary 4.6 and the fact that $\mathcal{K}(L^2(M; E)_\alpha)^\Gamma \simeq \mathcal{K}$, the algebra of compact operators on the Hilbert space $\mathcal{H} \coloneqq L^2(M; \alpha^* \otimes E)^G$. Under this isomorphism, the resulting representation of $\mathcal{K}$ on $L^2(M; E)_\alpha$ is isomorphic to $\dim(\alpha)$ times the standard representation of $\mathcal{K}$ on $\mathcal{H}$. This justifies the factor $\dim(\alpha)$. The rest follows from the fact that the index is the connecting morphism in K-theory for the Calkin exact sequence. See [79] for more details. □

Remark 5.4. As in [79], it follows that the index of $\pi_\alpha(P)$ with $P \in \psi^0(M; E)^\Gamma$ is the pairing between a cyclic cocycle ϕ on $\mathcal{C}^\infty(X_{M,\Gamma})$ (the algebra of principal symbols of operators in $\psi^0(M; E)^\Gamma$) and the K-theory class of the α-principal symbol of P (see [30, 29, 31]). Lemma 3.11 gives that the restriction of this cyclic cocycle to the principal orbit bundle is the usual Atiyah–Singer cocycle (i.e. the cocycle that yields the Atiyah–Singer index theorem in cyclic homology [31, 64, 79, 81], which thus corresponds, after suitable rescaling, to the Todd class). The full determination of the class of the index cyclic cocycle ϕ

requires, however, a nontrivial use of cyclic homology since the quotient algebra $\mathcal{C}^\infty(X_{M,\Gamma})$ is non-commutative, in general. See also [24, 34, 35, 36, 55, 63, 68, 69, 102] for more related results.

Remark 5.5. As for the case of compact complex varieties [48, 106], we can consider complexes of operators [19] and the corresponding notion of α-ellipticity. In particular, we obtain the finiteness of the corresponding cohomology groups if the complex is α-elliptic. This is related to the Hodge theory of singular spaces [2, 3, 16, 20, 27, 99].

5.6. **A closer look at the α and Γ equivariant principal symbols.** In this subsection, we *resume our assumption that M/Γ is connected,* for convenience. In particular, Γ_0 will be a minimal isotropy group, which is unique up to conjugacy (since we are again assuming that M/Γ is connected). We shall take a closer look next at the Γ- and α-principal symbols, so the following simple discussion will be useful. Recall that if $K \subset \Gamma$, $\rho \in \widehat{\Gamma}$, and $g \in \Gamma$, then $g \cdot K \coloneqq gKg^{-1}$ and $(g \cdot \rho)(\gamma) \coloneqq \rho(g^{-1}\gamma g)$ so that $g \cdot \rho$ is an irreducible representation of $g \cdot K$ (i.e. $g \cdot \rho \in \widehat{g \cdot K}$).

Remark 5.7. Let $\xi \in T^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$ (that is, $(\xi, \rho) \in X_{M,\Gamma}$). Then the following three statements are equivalent:
(i) the pair $(\xi, \rho) \in X^\alpha_{M,\Gamma}$;
(ii) there is $g \in \Gamma$ such that $\Gamma_0 \subset g \cdot \Gamma_\xi = \Gamma_{g\xi}$ and such that $g \cdot \rho$ and α are Γ_0-associated;
(iii) there is $\gamma \in \Gamma$ such that $\gamma \cdot \Gamma_0 \subset \Gamma_\xi$ and $\mathrm{Hom}_{\gamma\cdot\Gamma_0}(\rho, \alpha) \neq 0$.

Indeed, if (i) is satisfied, then the definition of $X^\alpha_{M,\Gamma}$ (see equation (5) and Definition 2.4) is equivalent to the existence of g, i.e. (i) $\Leftrightarrow$ (ii). Recalling that $g \cdot \rho \in \widehat{\Gamma_{g\xi}}$, we stress then that we need $\Gamma_0 \subset g \cdot \Gamma_\xi = \Gamma_{g\xi}$ for α and $g \cdot \rho$ to be associated.

To prove (ii) $\Leftrightarrow$ (iii), let $\gamma = g^{-1}$. We have that the relations $\Gamma_0 \subset g \cdot \Gamma_\xi$ and $\mathrm{Hom}_{\Gamma_0}(g \cdot \rho, \alpha) \neq 0$ are equivalent to $\gamma \cdot \Gamma_0 \subset \Gamma_\xi$ and $\mathrm{Hom}_{\gamma\cdot\Gamma_0}(\rho, \gamma \cdot \alpha) \neq 0$. The result follows from the fact that α and $\gamma \cdot \alpha$ are equivalent representations of $\gamma \cdot \Gamma_0$ (since $\gamma \in \Gamma$ and α is a representation of Γ).

We include in Proposition 5.9 below a reformulation of our α-ellipticity condition in terms of the fixed-point manifold $S^*M^{\Gamma_0}$, with Γ_0 a minimal isotropy subgroup as before. This result was suggested by some discussions with P.-E. Paradan, whom we thank for his useful input.

In the following, $\mathrm{Stab}_\Gamma(M)$ will denote the set of stabilizer subgroups K of Γ, that is, the set of subgroups $K \subset \Gamma$ such that there is $m \in M$ with $K = \Gamma_m$. It is a finite set since Γ is finite. Similarly, we let

$$\mathrm{Stab}^{\Gamma_0}_\Gamma(M) \coloneqq \{K \in \mathrm{Stab}_\Gamma(M) \mid \Gamma_0 \subset K\}.$$

Note that $\mathrm{Stab}_\Gamma(T^*M) = \mathrm{Stab}_\Gamma(M)$. Recall also that $(T^*M)^K = T^*(M^K)$, where M^K is the submanifold of fixed points of M by K, as usual. For a Γ-space X and $K \subset \Gamma$ a subgroup, we shall let $X_K \coloneqq \{x \in X \mid \Gamma_x = K\} \subset X^K$ denote the set of points of X with isotropy K. The set X_K should not be confused with the set $X_{(K)}$ of points of X whose isotropy is *conjugated* to K. Note that, in general, $T^*(M_K) \neq (T^*M)_{(K)}$.

Lemma 5.8. *The set $M_K := \{m \in M \mid \Gamma_m = K\}$ is a submanifold.*

Proof. Let $x \in M_K$, that is, $\Gamma_x = K$. The problem is local, so, using [100, Prop. 5.13], we see that it suffices to consider the case $M = \Gamma \times_K V$, where V is a K-representation. Then, if $z = (\gamma, y) \in \Gamma \times_K V$, we have $\Gamma_z = \gamma K_y \gamma^{-1}$, and hence, if $\Gamma_z = K$, we obtain $K = \gamma K_y \gamma^{-1}$, which, in turn, gives $K_y = K$ and $\gamma \in N(K) := \{g \in \Gamma \mid gKg^{-1} = K\}$. We thus obtain that

$$M_K = \{(\gamma, y) \in \Gamma \times_K M \mid K_y = \gamma^{-1} K \gamma\} = N(K) \times_K V^K,$$

which is a submanifold of M. □

Let $K \subset \Gamma$ be a subgroup and $\rho \in \widehat{K}$. Then $E_\rho := \bigsqcup_{x \in M^K} E_{x\rho}$ is a smooth vector bundle over M^K, the set of fixed points of M with respect to K. Similarly, $(E \otimes \rho)^K \to M^K$ is a smooth vector bundle (over M^K). Moreover, we have an isomorphism

$$\mathrm{End}(E_\rho)^K \simeq \mathrm{End}((E \otimes \rho)^K \otimes \rho)^K \simeq \mathrm{End}((E \otimes \rho)^K) \tag{26}$$

of vector bundles over M^K, where the last isomorphism comes from the fact that $\mathrm{End}(\rho)^K = \mathbb{C}$. In view of this discussion, we choose to state the following result in terms of the vector bundle $(E \otimes \rho)^K$ over M^K rather than in terms of E_ρ. This discussion shows also that it is enough in our proofs to assume that α is the trivial (one-dimensional) representation.

Proposition 5.9. *Let $\alpha \in \widehat{\Gamma}$ and $P \in \psi^m(M; E)$ for some $m \in \mathbb{R}$. Recall the vector bundle $(M \otimes \rho)^K \to M^K \supset M_K$. The following are equivalent.*

(i) *P is α-elliptic (Definition 1.3).*

(ii) *For all $K \in \mathrm{Stab}_\Gamma^{\Gamma_0}(M)$ and all $\rho \in \widehat{K}$ that are Γ_0-associated with α, we have that $(\sigma_m(P) \otimes \mathrm{id}_\rho)|_{(E\otimes\rho)^K}$ defines by restriction an invertible element of*

$$\mathcal{C}^\infty\big((T^*M \setminus \{0\})_K, \mathrm{End}((E \otimes \rho)^K)\big).$$

(iii) *The principal symbol $(\sigma_m(P) \otimes \mathrm{id}_\alpha)|_{(E\otimes\alpha)^{\Gamma_0}}$ defines by restriction an invertible element in*

$$\mathcal{C}^\infty\big(T^*M^{\Gamma_0} \setminus \{0\}; \mathrm{End}((E \otimes \alpha)^{\Gamma_0})\big).$$

Recall that, for representations α and β to be H-associated, they have to be defined, after restriction, on H. See Definition 2.4.

Proof. Recall that P is α-elliptic if the restriction of $\sigma_m^\Gamma(P)$ to $X^\alpha_{M,\Gamma}$ is invertible (see Remark 5.7 for a detailed definition and discussion of the space $X^\alpha_{M,\Gamma}$ appearing in the definition of α-ellipticity).

Let $K \in \mathrm{Stab}_\Gamma^{\Gamma_0}(M)$ (so $\Gamma_0 \subset K$), $\rho \in \widehat{K}$, and $\xi \in T^*_xM \setminus \{0\}$ with $\Gamma_\xi = K$. We have that $(\sigma_m(P) \otimes \mathrm{id}_\rho)|_{(E\otimes\rho)^K}$ is invertible at $\xi \in (T^*M)_K$ if, and only if, the restriction of $\sigma_m(P)(\xi)$ to $E_{x\rho}$ is invertible since they correspond to each other under the isomorphism of equation (26). Relation (ii) thus means that the restriction of the principal symbol $\sigma_m(P)$ is invertible on a *subset* of $X^{\alpha}_{M,\Gamma}$, so (i) implies (ii) right away.

Let us show next that (ii) implies (i), let $\xi \in T^*M \setminus \{0\}$, and let $K' \coloneqq \Gamma_\xi$. By definition, ξ belongs to $(T^*M)_{K'}$. Assume that $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$. This means that there exist $g \in \Gamma$ such that $\rho' \coloneqq g \cdot \rho$ and α are Γ_0 associated (see equation (5) and Definition 2.4; alternatively, this is also recalled in Remark 5.7). For this to make sense, it is implicit that

$$\Gamma_0 \subset \Gamma_{g\xi} = g \cdot \Gamma_\xi = g \cdot K' =: K$$

(again, see Remark 5.7). Then $g : (T^*M)_{K'} \to (T^*M)_K$ is a diffeomorphism. Condition (ii) for the group K gives that $\pi_{g\xi,\rho'}(\sigma_m(P))$ is invertible since the irreducible representation ρ' of $\Gamma_{g\xi}$ is Γ_0-associated to α (we have used here again the isomorphism (26)). Furthermore, $g : E_{\xi,\rho} \to E_{g\xi,\rho'}$ is an isomorphism. Now, by the Γ-invariance of $\sigma \coloneqq \sigma_m(P)$, we have

$$(g^{-1}\sigma)(\xi) = g^{-1}(\sigma(g\xi))g = \sigma(\xi);$$

therefore, $\pi_{\xi,\rho'}(\sigma)$ is invertible if, and only if, $\pi_{g\xi,\rho}(\sigma)$ is.

For the equivalence of (i) and (iii), we can assume that $m = 0$. Recall first that the density of Ξ_0 in Ξ established in Theorem 3.22 gives that the family of representations

$$\mathcal{F}_0 \coloneqq \{\pi_{\xi,\rho} \mid (\xi, \rho) \in X^{\alpha}_{M,\Gamma},\, \Gamma_\xi = \Gamma_0\}$$

is faithful for the C^*-algebra $A^{\Gamma}_M/\ker(\mathcal{R}_M)$ (see e.g. [85, Thm. 5.1]). In other words, the restriction morphism

$$A^{\Gamma}_M/\ker(\mathcal{R}_M) \to \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} \mathcal{C}\big((T^*M \setminus \{0\})_{\Gamma_0}, \mathrm{End}(E_\rho)^{\Gamma_0}\big)$$

is injective. Since $(T^*M)_{\Gamma_0}$ is dense in $T^*M^{\Gamma_0}$, it follows that the restriction morphism

$$R_M : A^{\Gamma}_M/\ker(\mathcal{R}_M) \to \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} \mathcal{C}\big(T^*M^{\Gamma_0} \setminus \{0\}, \mathrm{End}(E_\rho)^{\Gamma_0}\big)$$

is also injective.

Let us write $\alpha_{|\Gamma_0} = \bigoplus_{\rho\in\widehat{\Gamma}_0} m_\rho \rho$, with multiplicities $m_\rho \geq 0$. By considering the representations ρ with $m_\rho > 0$, we see that there is an injective vector bundle morphism over the manifold M^{Γ_0} defined by

$$\Psi : \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} \mathrm{End}(E_\rho)^{\Gamma_0} \simeq \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} \mathrm{End}((E \otimes \rho)^{\Gamma_0}) \hookrightarrow \mathrm{End}((E \otimes \alpha)^{\Gamma_0}),$$

where the last morphism maps any element $T \in \mathrm{End}((E \otimes \rho)^{\Gamma_0})$ to a direct sum of copies of T acting on the direct summand $[(E \otimes \rho)^{\Gamma_0}]^{m_\rho} \subset (E \otimes \alpha)^{\Gamma_0}$.

Condition (iii) amounts to the fact that

$$\Psi(R_M(\sigma^{\Gamma}_0(P))) \in \mathcal{C}^\infty(T^*M^{\Gamma_0} \setminus \{0\}; \mathrm{End}((E \otimes \alpha)^{\Gamma_0})$$

is invertible. To establish that (i) $\Leftrightarrow$ (iii), we thus need to prove that P is α-elliptic if, and only if, $\Psi(R_M(\sigma_0^\Gamma(P)))$ is invertible.

Recall the definition of the symbol algebras A_M from equation (19). We have that $P \in \overline{\psi^0}(M;E)$ is α-elliptic if, and only if, the image of $\sigma_0^\Gamma(P)$ in the quotient algebra $A_M^\Gamma/\ker(\mathcal{R}_M)$ is invertible (by the determination of $\ker(\mathcal{R}_M)$ in Remark 3.25 or Proposition 3.26). But since both Ψ and R_M are injective, $\Psi \circ R_M$ is injective on $A_M^\Gamma/\ker(\mathcal{R}_M)$. Thus $\sigma_0^\Gamma(P)$ is invertible in the quotient algebra $A_M^\Gamma/\ker(\mathcal{R}_M)$ if, and only if, $\Psi(R_M(\sigma_0^\Gamma(P)))$ is invertible. As we have seen above, this amounts to (i) $\Leftrightarrow$ (iii). $\square$

5.10. **Special cases.** We now specialize our main result to some particular cases. We no longer assume that $F = E$, and hence we consider *two* smooth vector bundles $E, F \to M$. Also, in this subsection, we continue to assume that M/Γ is connected and denote by Γ_0 a minimal isotropy group.

5.10.1. *The abelian group case* [12]. Let us begin by noticing that if Γ_i, $i = 1, 2$, are both abelian, then the irreducible representations $\alpha_i \in \widehat{\Gamma}_i$ are characters, that is, morphisms $\alpha_i : \Gamma_i \to \mathbb{C}^*$, and we have that they are H-associated for some subgroup H if, and only if, $\alpha_1|_H = \alpha_2|_H$.

Let us hence see how our statements simplify if Γ is abelian. So let us assume in this subsection that Γ is abelian. Let α be an irreducible representation of Γ. Since Γ is abelian, the conjugacy class of isotropy subgroups corresponding to the principal orbit bundle of the action of Γ on M has only one element, still denoted Γ_0. In that case, the set $X^\alpha_{M,\Gamma}$ defined in equation (5) of the introduction has the simpler expression

$$X^\alpha_{M,\Gamma} = \{(\xi, \rho) \mid \xi \in T^*M \setminus \{0\},\ \rho \in \widehat{\Gamma}_\xi,\ \rho|_{\Gamma_0} = \alpha|_{\Gamma_0}\}.$$

As a consequence, it is easier to check the α-ellipticity for an operator P in the abelian case. Let E, F be Γ-equivariant vector bundles over M, and set $\alpha_0 := \alpha|_{\Gamma_0}$. We then recover the main result of [12]. Indeed, Theorem 1.5 can then be stated as follows.

Theorem 5.11 ([12, Thm. 1.2])**.** *Let Γ be a finite,* abelian *group acting on a smooth, compact manifold M, and let $P \in \psi^m(M; E, F)^\Gamma$. Then, for any $s \in \mathbb{R}$, the following are equivalent:*

(i) *the operator $\pi_\alpha(P) : H^s(M;E)_\alpha \to H^{s-m}(M;F)_\alpha$ is Fredholm;*

(ii) *for all $(x,\xi) \in T^*M \setminus \{0\}$, $\rho \in \widehat{\Gamma}_\xi$ such that $\rho_{|\Gamma_0} = \alpha_0$, the restriction of $\sigma(P)(x,\xi)$ defines an isomorphism*

$$\pi_\rho(\sigma(P)(x,\xi)) : E_{x\rho} \to F_{x\rho}.$$

Let us notice, however, that, in the abelian case, the proof can be significantly simplified. Moreover, the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$ is quite important in applications since it can be used to deal with Dirichlet or Neumann boundary conditions.

5.11.1. *Scalar operators.* Our main theorem becomes quite explicit when we are dealing with scalar operators, i.e. when the vector bundles $E_i = M \times \mathbb{C}$, where $\mathbb{C}$ denotes the trivial representation of Γ.

Proposition 5.12. *Let $P : H^s(M) \to H^{s-m}(M)$ be a Γ-invariant pseudo-differential operator, and let $\alpha \in \widehat{\Gamma}$. Then P is α-elliptic if, and only if, $\sigma(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$ such that α is Γ_0-associated to the trivial (constant 1) representation of Γ_ξ.*

Proof. Let $\widehat{1}_{\Gamma_\xi}$ denote the trivial representation of Γ_ξ, and let $(\xi, \rho) \in X^\alpha_{M,\Gamma}$. If $\rho \neq \widehat{1}_{\Gamma_\xi}$, then $\mathbb{C}_\rho = 0$, and then $\pi_\rho(\sigma(P)(\xi)) : 0 \to 0$ is invertible. Now if $\rho = \widehat{1}_{\Gamma_\xi}$, then $(\xi, \rho) \in X^\alpha_{M,\Gamma}$ if, and only if, α is Γ_0-associated to $\widehat{1}_{\Gamma_\xi}$. □

Remark 5.13. Let us notice that, for all E (not necessarily trivial), we have

$$\Xi = \varnothing \iff \Xi_0 = \varnothing \iff L^2(M; E)_\alpha = 0.$$

See Remark 3.23.

This remark in the case $E = \mathbb{C}$ (one-dimensional, trivial) gives the following result.

Proposition 5.14. *Let $P : H^s(M) \to H^{s-m}(M)$ be a Γ-invariant pseudo-differential operator, let $\alpha \in \widehat{\Gamma}$, and assume that $L^2(M)_\alpha \neq 0$. We then have that P is α-elliptic if, and only if, P is elliptic.*

Proof. We have that $L^2(M)_\alpha \neq 0$ if, and only if, α contains a nonzero Γ_0-invariant vector. The result then follows from the previous proposition. □

5.14.1. *Trivial actions.* Assume that Γ acts trivially on M (in particular, M is then also connected). Our assumption implies that $\Gamma_0 = \Gamma_\xi = \Gamma$ for all $\xi \in T^*M \setminus \{0\}$. It follows that $\rho \in \widehat{\Gamma}_\xi$ is Γ_0-associated to $\alpha \in \widehat{\Gamma}$ if, and only if, $\alpha = \rho$.

Let $E \to M$ be a Γ-equivariant vector bundle. For any $x \in M$, recall that $E_{x\alpha}$ denotes the α-isotypical component of E_x. Assuming M to be connected, we have that $E_\alpha = \bigcup_{x \in M} E_{x\alpha}$ is a Γ-equivariant sub-vector bundle of E. Our main result then becomes the following statement.

Proposition 5.15. *Assume that Γ acts trivially on M, and let $\alpha \in \widehat{\Gamma}$. Let E and F be two Γ-equivariant vector bundles over M, and let $P \in \psi^m(M; E, F)^\Gamma$. Then, for any $s \in \mathbb{R}$, the following are equivalent:*

(i) $\pi_\alpha(P) : H^s(M; E_\alpha) \to H^{s-m}(M; F_\alpha)$ *is Fredholm;*

(ii) *for all $(x, \xi) \in T^*M \setminus \{0\}$, the morphism*

$$\pi_\alpha(\sigma(P)(x, \xi)) : E_{x\alpha} \to F_{x\alpha}$$

is invertible;

(iii) *for all $(x, \xi) \in T^*M \setminus \{0\}$, the morphism*

$$\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x, \xi) : \mathrm{Hom}_\Gamma(\alpha, E_x) \to \mathrm{Hom}_\Gamma(\alpha, F_x)$$

is invertible.

Of course, the above result is nothing but the classical condition that the elliptic operator $p_{F_\alpha} P p_{E_\alpha} \in \psi^m(M; E_\alpha, F_\alpha)$ be Fredholm.

Proof. The equivalence between (i) and (ii) is a direct consequence of Theorem 1.5. Let us check the equivalence of (i) and (iii). First note that

$$(H^s(M,E) \otimes \alpha)^\Gamma = H^s(M,(E \otimes \alpha)^\Gamma)$$

since the action of Γ on M is trivial. The operator $\pi_\alpha(P)$ is Fredholm if, and only if, the pseudodifferential operator

$$P_\alpha : H^s(M, \mathrm{Hom}(\alpha, E)^\Gamma) \to H^{s-m}(M, \mathrm{Hom}(\alpha, F)^\Gamma)$$

defined for any $v^* \in \alpha^*$ and $s \in \mathcal{C}^\infty(M,E)$ by $P_\alpha(v^* s) = v^* P s$ is Fredholm. Furthermore, the operator P_α is Fredholm if, and only if, it is elliptic, that is, if, and only if, $\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x,\xi) : \mathrm{Hom}_\Gamma(\alpha, E_x) \to \mathrm{Hom}_\Gamma(\alpha, F_x)$ is invertible for any $(x,\xi) \in T^*M \setminus \{0\}$. Note that the invertibility of $\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x,\xi)$ is equivalent to the invertibility of $\pi_\alpha(\sigma_m(P)(x,\xi))$ by definition, which is consistent with (ii). ☐

5.15.1. *Free action on a dense subset.* We have the following corollary of the last few results in Section 3.

Corollary 5.16. *Let us assume that* Γ *acts freely on a dense open subset of* M*. Then* $\Xi = \mathrm{Prim}(A_M^\Gamma)$.

Proof. The assumption on the action implies that $\Gamma_0 = \{1\}$. If $\xi \in T^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$, then ρ and α are always $\{1\}$-associated. The corollary then follows from Theorem 3.24. ☐

Similarly, we have the following result.

Proposition 5.17. *Assume that* Γ *acts freely on a dense subset in* M*, and let* $P \in \psi^m(M; E, F)^\Gamma$*. For any* $\alpha \in \widehat{\Gamma}$*, we have that* P *is* α*-elliptic if, and only if,* P *is elliptic.*

Proof. It follows from Corollary 5.16 that $X_{M,\Gamma}^\alpha = X_{M,\Gamma}$. Thus the operator P_α is α-elliptic if, and only if, the sum $\bigoplus_{\rho \in \widehat{\Gamma}_\xi} \pi_\rho(\sigma_m(P)(\xi)) = \sigma_m(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$, that is, if, and only if, P is elliptic. ☐

5.18. **Non-discrete compact groups and an equivariant version of Simonenko's localization principle.** Let us now say a few words about the case of nondiscrete compact groups in relation to Simonenko's localization principle [94]. A version of our main result, Theorem 1.5, for compact Lie groups was announced in [13]. Here we content to point out some connections with results of Atiyah, Singer, and Simonenko since they are relevant in explaining our approach by localization. In this subsection and in the rest of the paper, we consider the more general setting of a compact Lie group G instead of Γ. We assume that the compact Lie group G acts smoothly on M and that $\dim M > \dim G$, but we do not assume M/G to be connected.

Let $\mathcal{H} := L^2(M, E)$, and let $\mathcal{H}_\alpha$ be the α-isotypical component associated to $\alpha \in \widehat{G}$. We let $\Psi_M \subset \mathcal{L}(\mathcal{H}) = \mathcal{L}(L^2(M; E))$ denote the C^*-algebra consisting of all $P \in \mathcal{L}(\mathcal{H})$ such that $M_\phi P M_\psi \in \mathcal{K}(\mathcal{H})$ for all $\phi, \psi \in \mathcal{C}(M)$ with disjoint support. Then any $\phi \in \mathcal{C}(M)^G$ acts by multiplication on $\mathcal{H}_\alpha$, and we shall denote also by M_ϕ the induced multiplication operator.

Definition 5.19. We shall say that $P \in \mathcal{L}(\mathcal{H})^G$ is *locally α-invertible at $x \in M$* if there exist

(i) a G-invariant neighborhood V_x of Gx and

(ii) operators Q_1^x and $Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)$

such that, for any $\phi \in \mathcal{C}_c(V_x)^G \subset \mathcal{C}(M)^G$,

$$Q_1^x \pi_\alpha(P) M_\phi = M_\phi = M_\phi \pi_\alpha(P) Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha).$$

We have the following result, whose proof is a direct (but long) application of C^*-algebra techniques.

Proposition 5.20 (Simonenko's equivariant localization principle)**.** *Suppose that $P \in \Psi_M^G$ and $\dim M > \dim G$. Then P is locally α-invertible if, and only if, $\pi_\alpha(P)$ is Fredholm.*

Corollary 5.21. *Assume that M is a compact, smooth manifold and that Γ is a finite group acting smoothly on M. Let $P \in \psi(M; E, F)^\Gamma$ and $\alpha \in \widehat{\Gamma}$. Then the following are equivalent:*

(i) *$\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha$ is Fredholm for any $s \in \mathbb{R}$;*

(ii) *P is α-elliptic;*

(iii) *P is locally α-invertible.*

Proof. The first equivalence is given by Theorem 1.5. Now, since a finite group is compact, Proposition 5.20 implies that (i) is equivalent to (iii). ☐

Denote by $\mathfrak{g}$ the Lie algebra of G. Then any $X \in \mathfrak{g}$ defines as usual the vector field X_M^* given by $X_M^*(m) = \frac{d}{dt}_{|t=0} e^{tX} \cdot m$. Denote by $\pi : T^*M \to M$ the canonical projection, and let us introduce as in [7] the G-transversal space

$$T_G^*M := \{\alpha \in T^*M \mid \alpha(X_M^*(\pi(\alpha))) = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Recall that a G-invariant classical pseudodifferential operator P of order m is said to be *G-transversally elliptic* if its principal symbol is invertible on $T_G^*M \setminus \{0\}$ (see [7, 80]).

We may now state the classical result of Atiyah and Singer [7, Cor. 2.5].

Theorem 5.22 (Atiyah–Singer [7])**.** *Assume P is G-transversally elliptic. Then, for every irreducible representation $\alpha \in \widehat{G}$,*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$$

is Fredholm.

Note that this implies that Theorem 1.5 is not true anymore as stated if Γ is non-discrete, but see [13] for the announcement of the Fredholm characterization result in the case Γ non-discrete. In particular, we obtain the following consequence of the localization principle.

Corollary 5.23. *Assume that M is compact, that G is a compact Lie group, and that $\dim M > \dim G$. Let $P \in \psi^m(M,E)^G$ be a G-transversally elliptic operator. Then P is locally α-invertible for any $\alpha \in \widehat{G}$, as in Definition 5.19.*

Proof. Using Theorem 5.22, we obtain that $\pi_\alpha(P)$ is Fredholm. Therefore, by Proposition 5.20, P is α-invertible. □

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REFERENCES

[1] C. A. Akemann, G. K. Pedersen, and J. Tomiyama, Multipliers of C^*-algebras, J. Funct. Anal. **13** (1973), 277–301. MR0470685

[2] P. Albin, On the Hodge theory of stratified spaces, in *Hodge theory and L^2-analysis*, 1–78, Adv. Lect. Math. (ALM), 39, Int. Press, Somerville, MA, 2017. MR3751287

[3] P. Albin, E. Leichtnam, R. Mazzeo, and P. Piazza, Hodge theory on Cheeger spaces, J. Reine Angew. Math. **744** (2018), 29–102. MR3871440

[4] I. Androulidakis, O. Mohsen, and R. Yuncken, The convolution algebra of Schwarz kernels on a singular foliation, arXiv:1910.02623v3 [math.AP] (2020), to appear in J. Operator Theory.

[5] I. Androulidakis and G. Skandalis, The analytic index of elliptic pseudodifferential operators on a singular foliation, J. K-Theory **8** (2011), no. 3, 363–385. MR2863417

[6] I. Androulidakis and G. Skandalis, Pseudodifferential calculus on a singular foliation, J. Noncommut. Geom. **5** (2011), no. 1, 125–152. MR2746653

[7] M. F. Atiyah, *Elliptic operators and compact groups*, Lecture Notes in Mathematics, Vol. 401, Springer, Berlin, 1974. MR0482866

[8] K. Austin and J. Zhang, Limit operator theory for groupoids, Trans. Amer. Math. Soc. **373** (2020), no. 4, 2861–2911. MR4069235

[9] A. Baldare, The index of G-transversally elliptic families. I, J. Noncommut. Geom. **14** (2020), no. 3, 1129–1169. MR4170651

[10] A. Baldare, The index of G-transversally elliptic families. II, J. Noncommut. Geom. **14** (2020), no. 3, 1171–1207. MR4170652

[11] A. Baldare and M.-T. Benameur, The index of leafwise G-transversally elliptic operators on foliations, J. Geom. Phys. **163** (2021), 104128. MR4220749

[12] A. Baldare, R. Côme, M. Lesch, and V. Nistor, Fredholm conditions for invariant operators: finite abelian groups and boundary value problems, J. Operator Theory **85** (2021), no. 1, 229–256. MR4198971

[13] A. Baldare, R. Côme, and V. Nistor, Fredholm conditions for operators invariant with respect to compact Lie group actions, arXiv:2012.03944v1 [math.FA] (2020).

[14] S. Beckus, D. Lenz, M. Lindner, and Ch. Seifert, On the spectrum of operator families on discrete groups over minimal dynamical systems, Math. Z. **287** (2017), no. 3–4, 993–1007. MR3719524

[15] N. Berline and M. Vergne, L'indice équivariant des opérateurs transversalement elliptiques, Invent. Math. **124** (1996), no. 1-3, 51–101. MR1369411

[16] M. Brion and M. Vergne, An equivariant Riemann-Roch theorem for complete, simplicial toric varieties, J. Reine Angew. Math. **482** (1997), 67–92. MR1427657

[17] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, translated from the German manuscript, corrected reprint of the 1985 translation, Graduate Texts in Mathematics, 98, Springer, New York, 1995. MR1410059

[18] J. Brüning and E. Heintze, Representations of compact Lie groups and elliptic operators, Invent. Math. **50** (1978/79), no. 2, 169–203. MR0517776

[19] J. Brüning and M. Lesch, Hilbert complexes, J. Funct. Anal. **108** (1992), no. 1, 88–132. MR1174159

[20] J. Brüning and M. Lesch, Kähler-Hodge theory for conformal complex cones, Geom. Funct. Anal. **3** (1993), no. 5, 439–473. MR1233862

[21] R. C. Busby, Double centralizers and extensions of C^*-algebras, Trans. Amer. Math. Soc. **132** (1968), 79–99. MR0225175

[22] A. P. Calderon and A. Zygmund, On the existence of certain singular integrals, Acta Math. **88** (1952), 85–139. MR0052553

[23] A. P. Calderón and A. Zygmund, On a problem of Mihlin, Trans. Amer. Math. Soc. **78** (1955), 209–224. MR0068028

[24] A. L. Carey, V. Gayral, A. Rennie, and F. A. Sukochev, Index theory for locally compact noncommutative geometries, Mem. Amer. Math. Soc. **231** (2014), no. 1085, vi+130 pp. MR3221983

[25] C. Carvalho, R. Côme, and Y. Qiao, Gluing action groupoids: Fredholm conditions and layer potentials, Rev. Roumaine Math. Pures Appl. **64** (2019), no. 2–3, 113–156. MR4012600

[26] C. Carvalho, V. Nistor, and Y. Qiao, Fredholm conditions on non-compact manifolds: theory and examples, in *Operator theory, operator algebras, and matrix theory*, 79–122, Oper. Theory Adv. Appl., 267, Birkhäuser/Springer, Cham, 2018. MR3837632

[27] J. Cheeger, On the Hodge theory of Riemannian pseudomanifolds, in *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, 91–146, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, RI, 1980. MR0573430

[28] R. Côme, The Fredholm property for groupoids is a local property, Results Math. **74** (2019), no. 4, Paper No. 160, 33 pp. MR3998755

[29] A. Connes, Noncommutative differential geometry, Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360. MR0823176

[30] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994. MR1303779

[31] A. Connes and H. Moscovici, The local index formula in noncommutative geometry, Geom. Funct. Anal. **5** (1995), no. 2, 174–243. MR1334867

[32] H. O. Cordes, The algebra of singular integral operators in R^n, J. Math. Mech. **14** (1965), 1007–1032. MR0181911

[33] H. O. Cordes and R. C. McOwen, The C^*-algebra of a singular elliptic problem on a noncompact Riemannian manifold, Math. Z. **153** (1977), no. 2, 101–116. MR0431288

[34] J. Cuntz, Excision in periodic cyclic theory for topological algebras, in *Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995)*, 43–53, Fields Inst. Commun., 17, Amer. Math. Soc., Providence, RI, 1997. MR1478700

[35] J. Cuntz and D. Quillen, On excision in periodic cyclic cohomology. II. The general case, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), no. 1, 11–12. MR1260526

[36] J. Cuntz and D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289. MR1303029

[37] A. B. de Monvel-Berthier and V. Georgescu, Graded C^*-algebras in the N-body problem, J. Math. Phys. **32** (1991), no. 11, 3101–3110. MR1131694

[38] C. Debord, J.-M. Lescure, and F. Rochon, Pseudodifferential operators on manifolds with fibred corners, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 4, 1799–1880. MR3449197

[39] C. Debord and G. Skandalis, Pseudodifferential extensions and adiabatic deformation of smooth groupoid actions, Bull. Sci. Math. **139** (2015), no. 7, 750–776. MR3407514

[40] J. Dixmier, *Les C^*-algèbres et leurs représentations*, reprint of the second (1969) edition, Les Grands Classiques Gauthier-Villars., Éditions Jacques Gabay, Paris, 1996. MR1452364

[41] S. Echterhoff, The primitive ideal space of twisted covariant systems with continuously varying stabilizers, Math. Ann. **292** (1992), no. 1, 59–84. MR1141785

[42] S. Echterhoff and H. Emerson, Structure and K-theory of crossed products by proper actions, Expo. Math. **29** (2011), no. 3, 300–344. MR2820377

[43] S. Echterhoff and D. P. Williams, Structure of crossed products by strictly proper actions on continuous-trace algebras, Trans. Amer. Math. Soc. **366** (2014), no. 7, 3649–3673. MR3192611

[44] V. Georgescu, On the structure of the essential spectrum of elliptic operators on metric spaces, J. Funct. Anal. **260** (2011), no. 6, 1734–1765. MR2754891

[45] V. Georgescu and A. Iftimovici, Crossed products of C^*-algebras and spectral analysis of quantum Hamiltonians, Comm. Math. Phys. **228** (2002), no. 3, 519–560. MR1918787

[46] I. C. Gohberg, On an application of the theory of normed rings to singular integral equations, Uspehi Matem. Nauk (N.S.) **7** (1952), no. 2(48), 149–156. MR0048689

[47] I. C. Gohberg, On the theory of multidimensional singular integral equations, Soviet Math. Dokl. **1** (1960), 960–963. MR0124704

[48] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York, 1978. MR0507725

[49] L. Hartmann, M. Lesch, and B. Vertman, On the domain of Dirac and Laplace type operators on stratified spaces, J. Spectr. Theory **8** (2018), no. 4, 1295–1348. MR3870069

[50] B. Helffer and A. Mohamed, Caractérisation du spectre essentiel de l'opérateur de Schrödinger avec un champ magnétique, Ann. Inst. Fourier (Grenoble) **38** (1988), no. 2, 95–112. MR0949012

[51] L. Hörmander, Pseudo-differential operators, Comm. Pure Appl. Math. **18** (1965), 501–517. MR0180740

[52] L. Hörmander, *The analysis of linear partial differential operators. III*, reprint of the 1994 edition, Springer, Berlin, 2007.

[53] L. Jager, Pseudodifferential operators in infinite dimensional spaces: a survey of recent results, Rev. Roumaine Math. Pures Appl. **64** (2019), no. 2-3, 251–282. MR4012605

[54] P. Julg, Induction holomorphe pour le produit croisé d'une C^*-algèbre par un groupe de Lie compact, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 5, 193–196. MR0653572

[55] M. Karoubi, *K-theory*, Springer-Verlag, Berlin, 1978. MR0488029

[56] G. Kasparov, Elliptic and transversally elliptic index theory from the viewpoint of KK-theory, J. Noncommut. Geom. **10** (2016), no. 4, 1303–1378. MR3597146

[57] J. J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators, Comm. Pure Appl. Math. **18** (1965), 269–305. MR0176362

[58] B. V. Lange and V. S. Rabinovich, Math. USSR-Sb. **57** (1987), no. 1, 183–194; translated from Mat. Sb. (N.S.) **129(171)** (1986), no. 2, 175–185, 303. MR0832115

[59] Y. Last and B. Simon, The essential spectrum of Schrödinger, Jacobi, and CMV operators, J. Anal. Math. **98** (2006), 183–220. MR2254485

[60] R. Lauter, B. Monthubert, and V. Nistor, Pseudodifferential analysis on continuous family groupoids, Doc. Math. **5** (2000), 625–655. MR1800315

[61] R. Lauter and S. Moroianu, Fredholm theory for degenerate pseudodifferential operators on manifolds with fibered boundaries, Comm. Partial Differential Equations **26** (2001), no. 1–2, 233–283. MR1842432

[62] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators. Theory and Applications, 3, Birkhäuser Verlag, Basel, 2010. MR2599384

[63] M. Lesch, H. Moscovici, and M. J. Pflaum, Relative pairing in cyclic cohomology and divisor flows, J. K-Theory **3** (2009), no. 2, 359–407. MR2496452

[64] M. Lesch, H. Moscovici, and M. J. Pflaum, Connes–Chern character for manifolds with boundary and eta cochains, Mem. Amer. Math. Soc. **220** (2012), no. 1036, viii+92 pp. MR3025890

[65] J.-M. Lescure, D. Manchon, and S. Vassout, About the convolution of distributions on groupoids, J. Noncommut. Geom. **11** (2017), no. 2, 757–789. MR3669118

[66] M. Lindner, *Infinite matrices and their finite sections*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006. MR2253942

[67] M. Lindner and M. Seidel, An affirmative answer to a core issue on limit operators, J. Funct. Anal. **267** (2014), no. 3, 901–917. MR3212726

[68] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helv. **59** (1984), no. 4, 569–591. MR0780077

[69] Y. I. Manin, *Quantum groups and noncommutative geometry*, second edition, Springer, Cham, 2018.

[70] R. Mazzeo and R. B. Melrose, Pseudodifferential operators on manifolds with fibred boundaries, Asian J. Math. **2** (1998), no. 4, 833–866. MR1734130

[71] S. T. Melo, R. Nest, and E. Schrohe, C^*-structure and K-theory of Boutet de Monvel's algebra, J. Reine Angew. Math. **561** (2003), 145–175. MR1998610

[72] R. Melrose and V. Nistor, K-theory of C^*-algebras of b-pseudodifferential operators, Geom. Funct. Anal. **8** (1998), no. 1, 88–122. MR1601850

[73] S. G. Mihlin, Singular integral equations, Uspehi Matem. Nauk (N.S.) **3** (1948), no. 3(25), 29–112. MR0027429

[74] S. G. Mihlin, On the theory of multidimensional singular integral equations, Vestnik Leningrad. Univ. **11** (1956), no. 1, 3–24. MR0085453

[75] B. Monthubert, Pseudodifferential calculus on manifolds with corners and groupoids, Proc. Amer. Math. Soc. **127** (1999), no. 10, 2871–2881. MR1600121

[76] J. Mougel, Essential spectrum, quasi-orbits and compactifications: application to the Heisenberg group, Rev. Roumaine Math. Pures Appl. **64** (2019), no. 2-3, 325–343. MR4012608

[77] M. Măntoiu and V. Nistor, Spectral theory in a twisted groupoid setting: spectral decompositions, localization and Fredholmness, Münster J. Math. **13** (2020), no. 1, 145–196. MR4077101

[78] M. Măntoiu, R. Purice, and S. Richard, Spectral and propagation results for magnetic Schrödinger operators; a C^*-algebraic framework, J. Funct. Anal. **250** (2007), no. 1, 42–67. MR2345905

[79] V. Nistor, Higher index theorems and the boundary map in cyclic cohomology, Doc. Math. **2** (1997), 263–295. MR1480038

[80] P. Paradan and M. Vergne, Index of transversally elliptic operators, Astérisque **328** (2009), 297–338. MR2674881

[81] D. Perrot, Pseudodifferential extension and Todd class, Adv. Math. **246** (2013), 265–302. MR3091807

[82] V. Rabinovich, S. Roch, and B. Silbermann, *Limit operators and their applications in operator theory*, Operator Theory: Advances and Applications, 150, Birkhäuser Verlag, Basel, 2004. MR2075882

[83] V. Rabinovich, B.-W. Schulze, and N. Tarkhanov, C^*-algebras of singular integral operators in domains with oscillating conical singularities, Manuscripta Math. **108** (2002), no. 1, 69–90. MR1912948

[84] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, 793, Springer, Berlin, 1980. MR0584266

[85] S. Roch, Algebras of approximation sequences: structure of fractal algebras, in *Singular integral operators, factorization and applications*, 287–310, Oper. Theory Adv. Appl., 142, Birkhäuser, Basel, 2003. MR2167373

[86] A. Savin and E. Schrohe, Analytic and algebraic indices of elliptic operators associated with discrete groups of quantized canonical transformations, J. Funct. Anal. **278** (2020), no. 5, 108400, 45 pp. MR4046211

[87] A. Savin, E. Schrohe, and B. Sternin, Elliptic operators associated with groups of quantized canonical transformations, Bull. Sci. Math. **155** (2019), 141–167. MR3996327

[88] E. Schrohe, The symbols of an algebra of pseudodifferential operators, Pacific J. Math. **125** (1986), no. 1, 211–224. MR0860760

[89] E. Schrohe, Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance, Math. Nachr. **199** (1999), 145–185. MR1676318

[90] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, Pure and Applied Mathematics (New York), John Wiley & Sons, Ltd., Chichester, 1998. MR1631763

[91] R. T. Seeley, Complex powers of an elliptic operator, in *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, 288–307, Amer. Math. Soc., Providence, RI, 1967. MR0237943

[92] R. T. Seeley, The index of elliptic systems of singular integral operators, J. Math. Anal. Appl. **7** (1963), 289–309. MR0159247

[93] J.-P. Serre, *Linear representations of finite groups*, translated from the second French edition by Leonard L. Scott, Springer, New York, 1977. MR0450380

[94] I. B. Simonenko, A new general method of investigating linear operator equations of singular integral equation type. I, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 567–586. MR0179630

[95] I. M. Singer, Recent applications of index theory for elliptic operators, in *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, 11–31, Amer. Math. Soc., Providence, RI, 1973. MR0341538

[96] M. E. Taylor, Gelfand theory of pseudo differential operators and hypoelliptic operators, Trans. Amer. Math. Soc. **153** (1971), 495–510. MR0415430

[97] M. E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series, 34, Princeton University Press, Princeton, NJ, 1981. MR0618463

[98] M. E. Taylor, *Partial differential equations II. Qualitative studies of linear equations*, second edition, Applied Mathematical Sciences, 116, Springer, New York, 2011. MR2743652

[99] N. Teleman, Combinatorial Hodge theory and signature operator, Invent. Math. **61** (1980), no. 3, 227–249. MR0592692

[100] T. tom Dieck, *Transformation groups*, De Gruyter Studies in Mathematics, 8, Walter de Gruyter, Berlin, 1987. MR0889050

[101] F. Trèves, *Introduction to pseudodifferential and Fourier integral operators. Vol. 1*, Plenum Press, New York, 1980. MR0597144

[102] B. L. Tsygan, Homology of matrix Lie algebras over rings and the Hochschild homology, Uspekhi Mat. Nauk **38** (1983), no. 2(230), 217–218. MR0695483

[103] E. van Erp and R. Yuncken, A groupoid approach to pseudodifferential calculi, J. Reine Angew. Math. **756** (2019), 151–182. MR4026451

[104] D. Voiculescu, A non-commutative Weyl-von Neumann theorem, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113. MR0415338

[105] D. Voiculescu, Some results on norm-ideal perturbations of Hilbert space operators, J. Operator Theory **2** (1979), no. 1, 3–37. MR0553861

[106] R. O. Wells, Jr., *Differential analysis on complex manifolds*, third edition, Graduate Texts in Mathematics, 65, Springer, New York, 2008. MR2359489

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