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An Obata-type characterization of doubly-warped product Kähler manifolds

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Abstract. We give a characterization *à la Obata* for certain families of Kähler manifolds. These results are in the same line as other extensions of the well-known Obata rigidity theorem from [16], like for instance the generalizations in [17, 18]. Moreover, we give a complete description of the so-called Kähler doubly-warped product structures whose underlying metric is Einstein.

1. Introduction

This paper is the first of two papers devoted to the classification of complete Kähler manifolds carrying a real-valued function u whose Hessian is blue in particular black J-invariant and has pointwise at most two eigenvalues and one of them has as eigenvector the gradient of u. In this first paper, we consider the case where u has *no critical point*, and in [9], we treat the critical case, for which further constructions are needed. Before stating our main result, let us review some previous results that motivate our study.

In [16, Thm., p. 614], it is shown that the only complete Riemannian manifold (M^n, g) carrying a real-valued function u whose Hessian satisfies $\nabla^2 u = -u\,\mathrm{Id}$ is the round sphere. This result, known as the Obata theorem, has been generalized on Kähler manifolds in several papers such as [13, 17, 18]. Namely, in [17, Thm. 3], the authors proved that a complete Kähler manifold (M^{2n}, g, J) is biholomorphically isometric to $\mathbb{CP}^n$ with holomorphic sectional curvature 1 if and only if there exists a function u whose Hessian has at most two eigenvalues, namely $-\frac{u+1}{2}$ and $-u$ and where ∇u is an associated eigenvector (see also [13, Thm., p. 614] for a weaker version). In [18, Thm. 1], G. Santhanam proved that, given a function u on a complete Kähler manifold (M^{2n}, g, J) whose Hessian has the eigenvalues u and $\frac{u+1}{2}$ and where ∇u and $J\nabla u$ are both eigenvectors associated to u, then the manifold M is either isometric to the complex hyperbolic space $\mathbb{CH}^n$ of constant sectional curvature -1, or it is diffeomorphic to the normal bundle of some 2-codimensional totally geodesic submanifold M_0 of M such that the fiber over each point in M_0 is isometric to the hyperbolic plane $\mathbb{H}^2$ of constant curvature -1.

The main result of this paper is the following.

Theorem 1.1. *Let $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be a connected complete Kähler manifold of real dimension $2n \geq 4$ carrying a function $u \in C^\infty(\widetilde{M}, \mathbb{R})$ without critical points which satisfies the following three conditions:*

- *its Hessian $\widetilde{\nabla}^2 u$ is $\widetilde{J}$-invariant;*
- *its gradient $\widetilde{\nabla} u$ is a pointwise eigenvector of $\widetilde{\nabla}^2 u$ with some eigenvalue λ;*
- *the subbunble $\{\widetilde{\nabla} u, \widetilde{J}\widetilde{\nabla} u\}^\perp \to \widetilde{M}$ is a pointwise eigenspace of $\widetilde{\nabla}^2 u$ with some eigenvalue μ.*

Then the following claims hold true.

(i) *If μ vanishes at one point of $\widetilde{M}$, then μ vanishes identically on $\widetilde{M}$ and the triple $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is locally biholomorphically isometric to $(\mathbb{R}_t \times \mathbb{R}_s \times \Sigma, dt^2 \oplus \rho^2(t)\, ds^2 \oplus g_\Sigma)$ for some Kähler manifold $(\Sigma^{2n-2}, g_\Sigma)$ and $\rho(t) := |\widetilde{\nabla} u|(t, s, x)$, where $(\mathbb{R}_t \times \mathbb{R}_s \times \Sigma, dt^2 \oplus \rho(t)^2\, ds^2 \oplus g_\Sigma)$ is endowed with the complex structure that is naturally induced by the complex structure of $(\Sigma^{2n-2}, g_\Sigma)$.*

(ii) *If μ does not vanish at any point of $\widetilde{M}$, then we distinguish the following two cases.*

(a) *If $n > 2$, then up to changing u into $au + b$ with $a, b \in \mathbb{R}$, $a \neq 0$, the function u may be assumed to be positive and the Kähler manifold $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is biholomorphically isometric to a doubly-warped product*

$$\big(\mathbb{R} \times M^{2n-1}, dt^2 \oplus \rho(t)^2(\rho'(t)^2 \hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp})\big),$$

where M is a level hypersurface of u, the triple $(M^{2n-1}, \hat{g}, \hat{\xi})$ is Sasaki and $\rho(t) = \sqrt{u(t,x)}$ for any $(t,x) \in \mathbb{R} \times M$.

(b) *If $n = 2$, then up to changing u into $au + c$ with $a, b \in \mathbb{R}$, $a \neq 0$, the function u must be positive and the Kähler manifold $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is biholomorphically isometric to a doubly-warped product $(\mathbb{R} \times M^{2n-1}, dt^2 \oplus \rho(t)^2(\rho'(t)^2 \hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp}))$, where M is a level hypersurface of u, the triple $(M^{2n-1}, \hat{g}, \hat{\xi})$ is a minimal Riemannian flow that is basic conformally Sasaki and $\rho(t) = \sqrt{u(t,x)}$ for any $(t,x) \in \mathbb{R} \times M$.*

Moreover, in this case ($\mu \neq 0$), we have that

$$\lambda \circ F(t,x) = \frac{\partial^2 (u \circ F)}{\partial t^2}(t,x) \quad \textit{and} \quad \mu = \frac{|\widetilde{\nabla} u|^2}{2u}.$$

Note that there is one assumption missing in [9, Thm. 2], namely that the orthogonal complement of $\{\widetilde{\nabla} u, \widetilde{J}\widetilde{\nabla} u\}$ is a pointwise eigenspace of $\widetilde{\nabla}^2 u$.

The assumptions of Theorem 1.1 are related to various other well studied situations. First, it is easy to check that the condition of a $\widetilde{J}$-invariant Hessian $\widetilde{\nabla}^2 u$ is equivalent to the condition that $K := -\widetilde{J}\widetilde{\nabla} u$ is a Hamiltonian Killing vector field with moment map u, *i.e.* we have $L_K \widetilde{J} = 0 = L_K \widetilde{g}$ and $K \lrcorner\, \omega = du$, where ω denotes the Kähler form of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$.

Next, the condition that the gradient $\tilde{\nabla} u$ is a pointwise eigenvector of the Hessian of u, say for some eigenvalue λ, is equivalent to the equation $dx = 2\lambda\, du$, where x is the length function $x = |K|^2$. In particular, $dx \wedge du = 0$, and x has to be a function of u. Then the local S^1-action generated by K is *rigid* in the sense of V. Apostolov et al. (cp. [1]).

Our main result is also related to the work of A. Derdzinski and G. Maschler in [8], where they studied the question whether a given Kähler metric is conformal to an Einstein metric. As a necessary condition for the conformal factor u, they obtained that $\widetilde{J}\widetilde{\nabla} u$ has to be a Killing vector field and an eigenvector of the Hessian of u and of the Ricci tensor. They called such functions special Kähler–Ricci potentials.

Another equivalent way of formulating the assumptions of Theorem 1.1 is in terms of the distribution $\mathcal{D}_+$ spanned by K and JK. It turns out that equivalently this distribution has to be totally geodesic, holomorphic and conformal. The last condition follows from the assumption that the Hessian of u has at most two eigenvalues. Then our metric $\tilde{g}$ is locally of Calabi type, and there is a local classification due to S. Chiossi and P.-A. Nagy in [7]. From this point of view, it becomes clear that our manifolds are also *ambi-Kähler*, *i.e.* switching the sign of the complex structure $\widetilde{J}$ along the distribution $\mathcal{D}_+$ defines a new integrable complex structure I such that $(u^{-2}g, I)$ is again Kähler. We recall that there is a classification of compact ambi-Kähler manifolds in the work of F. Madani, A. Moroianu and M. Pilca in [12]. Moreover, $u\omega_+$ is a Hamiltonian 2-form of rank 1, where ω_+ is the restriction of the Kähler form to $\mathcal{D}_+$. Manifolds admitting Hamiltonian 2-forms are studied in a series of papers of V. Apostolov et al. including a global classification in the compact case (cp. [1, 2]). Independently, the assumptions of Theorem 1.1 can be reformulated in terms of so-called c-projectively equivalent Kähler metrics which have been locally described in [5, Thm. 1.6].

In contrast to the results mentioned so far, our main theorem gives a global description of the manifold without the additional compactness assumption. In fact, as a result, the underlying manifold in Theorem 1.1 cannot be compact.

The idea of the proof of Theorem 1.1 consists in identifying the manifold $\widetilde{M}$ with $I \times M$, where M denotes a level hypersurface of u, via the flow of the normalized gradient

$$\nu := \frac{\widetilde{\nabla} u}{|\widetilde{\nabla} u|} \in \Gamma(T\widetilde{M}),$$

which is geodesic. We show that the vector field $\xi := -\widetilde{J}\nu$ defines a minimal Riemannian flow on $(M, \widetilde{g}|_{T^*M \otimes T^*M})$, whose O'Neill tensor coincides with the complex structure $\widetilde{J}$, up to some factor depending on the eigenvalue μ.

Conversely, given any Kähler doubly-warped product (see Lemma 2.4 for the existence of such a structure) of the form $(I \times M^{2n-1}, dt^2 \oplus \rho^2((\rho')^2 \hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp}))$, where $I \subset \mathbb{R}$, $\rho, \rho' : I \to \mathbb{R}$ are positive functions and $(M, \hat{g} = \hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp}, \hat{\xi})$ is Sasaki, a direct computation shows that the function $u := \rho^2$ satisfies the second-order PDE

$$\widetilde{\nabla}^2 u = \widetilde{\nabla}^2 u(\nu,\nu)\cdot(\nu^\flat\otimes\nu+\xi^\flat\otimes\xi)+\frac{|\widetilde{\nabla}u|^2}{2u}\mathrm{Id}_{\{\xi,\nu\}^\perp},$$

where

$$\widetilde{\nabla}u := \mathrm{grad}_{\widetilde{g}}^{\widetilde{M}}(u),\quad \widetilde{\nabla}^2 u := \mathrm{Hess}_{\widetilde{g}}^{\widetilde{M}}(u),\quad \nu := \frac{\widetilde{\nabla}u}{|\widetilde{\nabla}u|}\quad\text{and}\quad \xi := -\widetilde{J}\nu.$$

Hence, the Hessian of u has two eigenvalues that coincide with λ and μ in Theorem 1.1. Note that the function ρ itself has no $\widetilde{J}$-invariant Hessian, whereas ρ^2 does, that is why we consider ρ^2.

The paper is organized as follows. In Section 2, we review some basic facts on doubly-warped products and characterize those which are Kähler. We recall that these structures were first introduced by Baier in his master thesis [3], in order to compute the Dirac spectrum of the complex hyperbolic space. In Section 3, we provide the proof of the main theorem. In the last section of the paper, we investigate when the metric of a Kähler doubly-warped product is Einstein and discuss the solutions of the differential equation that the warping function ρ has to satisfy. In the sequel, we will see that, when u has no critical points, this equation suffices to reconstruct the doubly warped product structure on $\widetilde{M}$.

2. Kähler doubly-warped products

In this section, we recall some basic facts on doubly-warped products. We characterize among these manifolds those which are Kähler and provide the necessary integrability conditions. We refer to [10] for more details.

Let M be a manifold and consider the product [10, Lem. 3.1]

$$(\widetilde{M} := I\times M, \widetilde{g} := \beta\, dt^2\oplus g_t),$$

where $I\subset\mathbb{R}$ is an open interval, g_t is a smooth 1-parameter family of Riemannian metrics on M and $\beta\in C^\infty(I\times M,\mathbb{R}_+^\times)$. We can easily see that the Koszul formula implies the following identities for all $X,Y\in\Gamma(\pi_2^*TM)$, where $\pi_2:\widetilde{M}\to M$ denotes the projection on the second factor:

$$\begin{aligned}
\widetilde{\nabla}_{\partial_t}\partial_t &= -\frac{1}{2}\,\mathrm{grad}_{g_t}(\beta(t,\cdot))+\frac{1}{2\beta}\frac{\partial\beta}{\partial t}\partial_t,\\
\widetilde{\nabla}_{\partial_t}X &= \frac{\partial X}{\partial t}+\frac{1}{2}g_t^{-1}\frac{\partial g_t}{\partial t}(X,\cdot)+\frac{1}{2\beta}\frac{\partial\beta}{\partial x}(X)\partial_t,\\
\widetilde{\nabla}_X\partial_t &= \frac{1}{2}g_t^{-1}\frac{\partial g_t}{\partial t}(X,\cdot)+\frac{1}{2\beta}\frac{\partial\beta}{\partial x}(X)\partial_t,\\
\widetilde{\nabla}_X Y &= \nabla_X^{M_t}Y-\frac{1}{2\beta}\frac{\partial g_t}{\partial t}(X,Y)\partial_t,
\end{aligned}\tag{1}$$

where $\frac{\partial X}{\partial t}=[\partial_t,X]$ and ∇^{M_t} is the Levi–Civita covariant derivative of (M,g_t). From now on, we assume $\hat{\xi}$ to be a unit Killing vector field; in other words,

$(M, \hat{g}, \hat{\xi})$ is a so-called minimal Riemannian flow. For more details on Riemannian flows, we refer to [6]. In this case, we have an orthogonal splitting $TM = \mathbb{R} \cdot \hat{\xi} \oplus \hat{\xi}^\perp$, and the normal bundle $Q := \hat{\xi}^\perp$ of the flow admits a so-called transversal Levi–Civita connection, denoted by $\hat{\nabla}$, which is defined for all $X \in \Gamma(TM)$ and $Z \in \Gamma(Q)$ as follows, cp. [19]:

$$\hat{\nabla}_X Z := \begin{cases} [\hat{\xi}, Z]^Q & \text{if } X = \hat{\xi}, \\ (\nabla^{\hat{M}}_X Z)^Q & \text{if } X \in \Gamma(Q), \end{cases}$$

where $(\,\cdot\,)^Q$ denotes the $\hat{g}$-orthogonal projection $TM \to Q$ and $\nabla^{\hat{M}}$ denotes the Levi–Civita covariant derivative of $(M, \hat{g})$. The connection $\hat{\nabla}$ is compatible with the induced metric $\hat{g}_{\hat{\xi}^\perp}$ on the bundle Q, and its curvature vanishes along $\hat{\xi}$. Recall also that a minimal Riemannian flow is characterized by the fact that the tensor $\hat{h} := \nabla^{\hat{M}} \hat{\xi}$, known as the O'Neill tensor [15], satisfies $\hat{h}(\hat{\xi}) = 0$ and is a skew-symmetric endomorphism field on Q equal to $\hat{g}(\hat{h}(Y), Z) = -\frac{1}{2}\hat{g}([Y, Z], \hat{\xi})$ for any $Y, Z \in \Gamma(Q)$.

We consider in the following the general Ansatz

$$g_t := \rho(t)^2(\sigma(t)^2 \hat{g}_{\hat{\xi}} \oplus k(t,x)^2 \hat{g}_{\hat{\xi}^\perp}), \tag{2}$$

where $\rho, \sigma : I \to (0, \infty)$ and $k : I \times M \to (0, \infty)$ are *a priori* arbitrary smooth positive functions, and we define on the manifold $\widetilde{M} = I \times M$ the Riemannian metric $\widetilde{g} := dt^2 \oplus g_t$. In the next lemma, we make the Levi–Civita connection $\widetilde{\nabla}$ of $(\widetilde{M}, \widetilde{g})$ explicit and express it in terms of the transversal Levi–Civita connection $\hat{\nabla}$.

Lemma 2.1. *Let* $(\widetilde{M}, \widetilde{g}) := (I \times M, dt^2 \oplus \rho(t)^2(\sigma(t)^2 \hat{g}_{\hat{\xi}} \oplus k(t,x)^2 \hat{g}_{\hat{\xi}^\perp}))$, *where* $(M, \hat{g}, \hat{\xi})$ *is a minimal Riemannian flow. Then, for all* $Z, Z' \in \Gamma(\pi_2^* Q)$, *the following identities hold:*

$$\widetilde{\nabla}_{\partial_t} \partial_t = 0, \quad \widetilde{\nabla}_{\partial_t} \xi = 0, \quad \widetilde{\nabla}_{\partial t} Z = \frac{\partial Z}{\partial t} + \frac{1}{\rho k} \cdot \frac{\partial(\rho k)}{\partial t} Z,$$

$$\widetilde{\nabla}_\xi \partial_t = \frac{(\rho\sigma)'}{\rho\sigma}\xi, \quad \widetilde{\nabla}_\xi \xi = -\frac{(\rho\sigma)'}{\rho\sigma}\partial_t, \quad \widetilde{\nabla}_\xi Z = \hat{\nabla}_\xi Z + \frac{\xi(k)}{k} Z + \frac{\sigma}{\rho k^2}\hat{h} Z,$$

$$\widetilde{\nabla}_Z \partial_t = \frac{1}{\rho k} \cdot \frac{\partial(\rho k)}{\partial t} Z, \quad \widetilde{\nabla}_Z \xi = \frac{\xi(k)}{k} Z + \frac{\sigma}{\rho k^2}\hat{h} Z,$$

$$\begin{aligned} \widetilde{\nabla}_Z Z' = {} & \hat{\nabla}_Z Z' + Z(\ln(k))Z' + Z'(\ln(k))Z - \widetilde{g}(Z, Z')\,\mathrm{grad}^{\hat{\nabla}}(\ln(k)) \\ & - \xi(\ln(k))\widetilde{g}(Z, Z')\xi - \frac{\sigma}{\rho k^2}\widetilde{g}(\hat{h}Z, Z')\xi - \frac{\partial \ln(\rho k)}{\partial t}\widetilde{g}(Z, Z')\partial_t, \end{aligned}$$

where $\hat{h} := \nabla^{\hat{M}} \hat{\xi} \in \Gamma(\mathrm{End}(Q))$ *denotes the O'Neill tensor as above,* $\xi := \frac{1}{\rho\sigma}\hat{\xi}$, *and* $\mathrm{grad}^{\hat{\nabla}} f := (\nabla^{\hat{M}} f)^Q$ *for every function* f.

Proof. Since β is chosen to be equal to 1, the first identity in (1) becomes $\widetilde{\nabla}_{\partial_t} \partial_t = 0$, and the other three identities imply that, for all $X, Y \in \Gamma(\pi_2^* TM)$, the following relations hold:

$$\begin{aligned}\widetilde{\nabla}_{\partial_t} X &= \frac{\partial X}{\partial t} + \frac{1}{2} g_t^{-1} \frac{\partial g_t}{\partial t}(X, \cdot\,),\\ \widetilde{\nabla}_X \partial_t &= \frac{1}{2} g_t^{-1} \frac{\partial g_t}{\partial t}(X, \cdot\,),\\ \widetilde{\nabla}_X Y &= \nabla^{M_t}_X Y - \frac{1}{2}\frac{\partial g_t}{\partial t}(X, Y)\partial_t,\end{aligned}$$

with $\frac{\partial g_t}{\partial t} = 2(\rho\sigma)'(t)(\rho\sigma)(t)\hat{g}_{\hat{\xi}} + 2\frac{\partial(\rho k)}{\partial t}(t,x)(\rho(t)k(t,x))\hat{g}_{\hat{\xi}^\perp}$ and hence

$$\begin{aligned}g_t^{-1}\frac{\partial g_t}{\partial t}(X, \cdot\,) &= 2\frac{(\rho\sigma)'}{\rho\sigma}\hat{g}(X, \hat{\xi})\hat{\xi} \oplus \frac{2}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}X^\perp\\ &= 2\frac{(\rho\sigma)'}{\rho\sigma}\widetilde{g}(X, \xi)\xi \oplus \frac{2}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}X^\perp,\end{aligned}$$

where $\xi = \frac{1}{\rho\sigma}\hat{\xi}$ and $X = \hat{g}(X, \hat{\xi})\hat{\xi} + X^\perp$, with $X^\perp \in \Gamma(\pi_2^* Q)$. Note that the vector field ξ has unit length with respect to the metric $\widetilde{g}$. Thus, we obtain the following identities:

$$\begin{aligned}&\widetilde{\nabla}_{\partial_t}\partial_t = 0,\\ &\widetilde{\nabla}_{\partial_t} X = \frac{\partial X}{\partial t} + \frac{(\rho\sigma)'}{\rho\sigma}\widetilde{g}(X, \xi)\xi \oplus \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}X^\perp,\\ &\widetilde{\nabla}_X \partial_t = \frac{(\rho\sigma)'}{\rho\sigma}\widetilde{g}(X, \xi)\xi \oplus \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}X^\perp,\\ &\widetilde{\nabla}_X Y = \nabla^{M_t}_X Y - \Big(\frac{(\rho\sigma)'}{\rho\sigma}\widetilde{g}(X, \xi)\widetilde{g}(Y, \xi) + \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}\widetilde{g}(X^\perp, Y^\perp)\Big)\partial_t.\end{aligned}$$

Now, we need to compute $\nabla^{M_t}_X Y$ in a more precise way according to the components X and Y in the orthogonal splitting $TM = \mathbb{R}\cdot\xi \oplus Q$. Recall the Koszul formula, valid for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned}g_t(\nabla^{M_t}_X Y, Z) = \frac{1}{2}\{&X(g_t(Y, Z)) + Y(g_t(Z, X)) - Z(g_t(X, Y))\\ &+ g_t([X, Y], Z) - g_t([Y, Z], X) + g_t([Z, X], Y)\}.\end{aligned} \tag{3}$$

First, we consider the case when $Y = \xi$. For $X = \xi$, we have $g_t(\nabla^{M_t}_\xi \xi, \xi) = 0$ and, for every $Z \in \Gamma(Q)$,

$$g_t(\nabla^{M_t}_\xi \xi, Z) = -g_t([\xi, Z], \xi) = -\hat{g}([\hat{\xi}, Z], \hat{\xi}) = \hat{g}(Z, \underbrace{\nabla^{\hat{M}}_{\hat{\xi}}\hat{\xi}}_{0}) = 0$$

so that $\nabla^{M_t}_\xi \xi = 0$. For $X = Z \in \Gamma(Q)$, we have $g_t(\nabla^{M_t}_Z \xi, \xi) = 0$ and, for every $Z' \in \Gamma(Q)$,

$$\begin{aligned}g_t(\nabla^{M_t}_Z \xi, Z') = \frac{1}{2}\{&Z(g_t(\xi, Z')) + \xi(g_t(Z', Z)) - Z'(g_t(Z, \xi))\\ &+ g_t([Z, \xi], Z') - g_t([\xi, Z'], Z) + g_t([Z', Z], \xi)\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\xi((\rho k)^2\hat{g}(Z,Z')) - \frac{(\rho k)^2}{\rho\sigma}\hat{g}([\hat{\xi},Z],Z') \\
&\qquad - \frac{(\rho k)^2}{\rho\sigma}\hat{g}([\hat{\xi},Z'],Z) + (\rho\sigma)\hat{g}([Z',Z],\hat{\xi})\Big) \\
&= \frac{(\rho k)^2}{2\rho\sigma}\big(\hat{\xi}(\hat{g}(Z,Z')) - \hat{g}([\hat{\xi},Z],Z') - \hat{g}([\hat{\xi},Z'],Z)\big) \\
&\qquad + \frac{\xi(k)}{k}g_t(Z,Z') + \rho\sigma\hat{g}(\nabla^{\hat{M}}_Z\hat{\xi},Z') \\
&= \frac{\xi(k)}{k}g_t(Z,Z') + \frac{\sigma}{\rho k^2}g_t(\hat{h}Z,Z')
\end{aligned}
$$

so that $\nabla^{M_t}_Z\xi = \frac{\xi(k)}{k}Z + \frac{\sigma}{\rho k^2}\hat{h}Z$. In the last equality, we used the fact that $\hat{\xi}$ is a Killing vector field with respect to the metric $\hat{g}$. Let us now choose $Y = Z' \in \Gamma(Q)$ and compute as follows:

$$
\begin{aligned}
\nabla^{M_t}_\xi Z' &= \nabla^{M_t}_{Z'}\xi - [Z',\xi] \\
&= \frac{\xi(k)}{k}Z' + \frac{\sigma}{\rho k^2}\hat{h}Z' + \frac{1}{\rho\sigma}[\hat{\xi},Z'] \\
&= \frac{\xi(k)}{k}Z' + \frac{\sigma}{\rho k^2}\hat{h}Z' + \frac{1}{\rho\sigma}[\hat{\xi},Z']^Q + \frac{1}{\rho\sigma}\underbrace{\hat{g}([\hat{\xi},Z'],\hat{\xi})}_{0}\hat{\xi} \\
&= \frac{\xi(k)}{k}Z' + \frac{\sigma}{\rho k^2}\hat{h}Z' + \hat{\nabla}_\xi Z'.
\end{aligned}
$$

On the other hand, for every $Z \in \Gamma(Q)$, we have

$$
g_t(\nabla^{M_t}_Z Z',\xi) = -g_t(\nabla^{M_t}_Z\xi,Z') = -\frac{\xi(k)}{k}g_t(Z,Z') - \frac{\sigma}{\rho k^2}g_t(\hat{h}Z,Z'),
$$

and for any $Z'' \in \Gamma(Q)$, we compute

$$
\begin{aligned}
g_t(\nabla^{M_t}_Z Z',Z'') &= \frac{1}{2}\{Z(g_t(Z',Z'')) + Z'(g_t(Z'',Z)) - Z''(g_t(Z,Z')) \\
&\qquad + g_t([Z,Z'],Z'') - g_t([Z',Z''],Z) + g_t([Z'',Z],Z')\} \\
&= \frac{1}{2}\{Z((\rho k)^2\hat{g}(Z',Z'')) + Z'((\rho k)^2\hat{g}(Z'',Z)) \\
&\qquad - Z''((\rho k)^2\hat{g}(Z,Z')) + (\rho k)^2\hat{g}([Z,Z'],Z'') \\
&\qquad - (\rho k)^2\hat{g}([Z',Z''],Z) + (\rho k)^2\hat{g}([Z'',Z],Z')\} \\
&= \rho^2 Z(k)k\hat{g}(Z',Z'') + \rho^2 Z'(k)k\hat{g}(Z'',Z) \\
&\qquad - \rho^2 Z''(k)k\hat{g}(Z,Z') + (\rho k)^2\hat{g}(\nabla^{\hat{M}}_Z Z',Z'') \\
&= \frac{Z(k)}{k}g_t(Z',Z'') + \frac{Z'(k)}{k}g_t(Z'',Z) \\
&\qquad - \frac{Z''(k)}{k}g_t(Z,Z') + g_t(\hat{\nabla}_Z Z',Z'').
\end{aligned}
$$

This finishes the proof of the lemma. □

Let us now recall some standard definitions. On a Riemannian flow $(M, \hat{g}, \hat{\xi})$, a function f is said to be *basic* if $\hat{\xi}(f) = 0$, *i.e.* the function f depends only on the transversal variables. A *transversal* Kähler structure J on a Riemannian flow is defined as an almost-Hermitian structure $J : \Gamma(Q) \to \Gamma(Q)$, which is parallel with respect to the transversal Levi–Civita connection. The following lemma will be useful when considering basic transversal conformal changes of the metric on a Riemannian flow. Indeed, we will show that, for any conformal change of the transverse metric by a basic function, the flow will be still Riemannian.

Lemma 2.2. *Let $(M, \hat{g}, \hat{\xi})$ be a connected minimal Riemannian flow, and let $f \in C^\infty(M, \mathbb{R})$ be a basic function. Then $(M, g := \hat{g}_{\hat{\xi}} \oplus e^{2f} \cdot \hat{g}_{\hat{\xi}^\perp}, \xi := \hat{\xi})$ is a minimal Riemannian flow, whose O'Neill tensor is given by $h = e^{-2f}\hat{h}$ and whose Levi–Civita connection satisfies, for all $X \in \Gamma(TM)$ and $Z \in \Gamma(Q)$,*

$$\nabla_X Z = \hat{\nabla}_X Z + X^Q(f)Z + Z(f)X^Q - \hat{g}(X, Z)\hat{\nabla} f,$$

where $\hat{\nabla} f := \mathrm{grad}^{\hat{\nabla}} f$ is the pointwise projection of the $\hat{g}$-gradient of f onto Q. Moreover, if J is a transversal Kähler structure on $(M, \hat{g}, \hat{\xi})$, then J remains a transversal Kähler structure on (M, g, ξ) if and only if either $\mathrm{rk}(Q) = 2$ *or f is constant.*

Proof. First, we make use of the Koszul formula (3) to show that, for any $Z \in \Gamma(Q)$, we have

$$g(\nabla^M_\xi \xi, Z) = -g([\xi, Z], \xi) = -\hat{g}([\hat{\xi}, Z], \hat{\xi}) = \hat{g}(\nabla^{\hat{M}}_{\hat{\xi}} \hat{\xi}, Z) = 0.$$

Moreover, the Lie derivative of the transverse conformal metric in the direction vector field $\hat{\xi}$ is equal to

$$\mathcal{L}_\xi(e^{2f} \cdot \hat{g}_{\hat{\xi}^\perp}) = e^{2f}\xi(f)\hat{g}_{\hat{\xi}^\perp} + e^{2f}\mathcal{L}_\xi(\hat{g}_{\hat{\xi}^\perp}) = 0$$

since f is assumed to be a basic function. In particular, this shows that ξ is a unit Killing vector field with respect to the metric g, and therefore (M, g, ξ) is a minimal Riemannian flow. The relation between ∇ and $\hat{\nabla}$ is proven as in the usual case by the uniqueness of a compatible transversal torsion-free connection. To compare the corresponding O'Neill tensors, we just compute for any $Y, Z \in \Gamma(Q)$ as follows:

$$g(hY, Z) = -\frac{1}{2} g_\xi([Y, Z], \xi) = -\frac{1}{2}\hat{g}_{\hat{\xi}}([Y, Z], \hat{\xi}) = \hat{g}(\hat{h}Y, Z) = e^{-2f} g(\hat{h}Y, Z).$$

Let J be a transversal Kähler structure on $(M, \hat{g}, \hat{\xi})$. Then J remains an almost-Hermitian structure on Q and $\nabla J = 0$ if and only if, for all $Z, Z' \in \Gamma(Q)$, the following equality holds:

$$Z(f)JZ' + JZ'(f)Z - g(Z, JZ')\hat{\nabla} f = Z(f)JZ' + Z'(f)JZ - g(Z, Z')J\hat{\nabla} f,$$

which is equivalent to

$$Z'(f)JZ - JZ'(f)Z + g(Z, JZ')\hat{\nabla} f - g(Z, Z')J\hat{\nabla} f = 0.$$

In case $\mathrm{rk}(Q)=2$, this identity is trivially satisfied for all $Z,Z'\in\Gamma(Q)$, whereas for $\mathrm{rk}(Q)>2$, it is satisfied if and only if $\hat{\nabla}f=0$, that is, f is constant. This concludes the proof of Lemma 2.2. $\square$

Remark 2.3. Note that if we rescale the vector field $\hat{\xi}$ by some positive number, that is, we consider the metric $g:=\alpha^2\hat{g}_{\hat{\xi}}\oplus\hat{g}_{\hat{\xi}^\perp}$ for some positive α, then $(M,g,\xi:=\frac{1}{\alpha}\hat{\xi})$ is still a minimal Riemannian flow with O'Neill tensor $h=\alpha\hat{h}$.

Let $(M,\hat{g},\hat{\xi})$ be a minimal Riemannian flow, and assume the existence of a transversal Kähler structure J on $Q=\hat{\xi}^\perp$. We consider the almost-Hermitian structure $\widetilde{J}$ on $\widetilde{M}=I\times M$ defined by setting

$$\widetilde{J}(\partial_t):=-\frac{1}{\rho\sigma}\hat{\xi},\quad \widetilde{J}\Big(\frac{1}{\rho\sigma}\hat{\xi}\Big):=\partial_t\quad\text{and}\quad \widetilde{J}_{|_{\{\hat{\xi},\partial_t\}^\perp}}:=J, \tag{4}$$

where ρ,σ,k are the coefficients of the metric in (2). Similarly to [10, Lem. 3.4], we want to characterize in the next lemma those functions ρ,σ,k for which $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is *Kähler*, that is, for which $\widetilde{\nabla}\widetilde{J}=0$ holds on $\widetilde{M}$. Note in particular that M has odd dimension equal to $2n-1$.

Lemma 2.4. *Let*

$$(\widetilde{M},\widetilde{g}):=\big(I\times M,dt^2\oplus\rho(t)^2(\sigma(t)^2\hat{g}_{\hat{\xi}}\oplus k(t,x)^2\hat{g}_{\hat{\xi}^\perp})\big)$$

be a doubly-warped product, where $(M,\hat{g},\hat{\xi})$ is a minimal Riemannian flow and $\rho,\sigma:I\to\mathbb{R}_+^\times$, as well as $k:I\times M\to\mathbb{R}_+^\times$, are smooth positive functions. Let $\xi:=\frac{1}{\rho\sigma}\hat{\xi}$ and $\hat{h}:=\nabla^{\hat{M}}\hat{\xi}$. Furthermore, we assume $(M,\hat{g},\hat{\xi})$ carries a transversal Kähler structure J, and we define the almost-Hermitian structure $\widetilde{J}$ on $\widetilde{M}$ via (4). Then the following statements hold true.

- (i) *The structure $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is Kähler if and only if $\hat{\xi}(k)=0$, $\hat{h}=-\frac{k}{\sigma}\cdot\frac{\partial(\rho k)}{\partial t}J$ and, if $n>2$, $\mathrm{grad}^{\hat{\nabla}}(k)=0$ (and thus the function k only depends on t if M is connected). In this case, there exists a basic function C on M, which is constant if $n>2$, such that $\frac{\partial(\rho k)^2}{\partial t}=2\rho\sigma C$.*
- (ii) *If $\hat{h}$ vanishes at a point (for $n>2$) or vanishes identically (for $n=2$), then $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is Kähler if and only if it is locally isometric to a Kähler product $(\mathbb{R}_t\times\mathbb{R}_s\times\Sigma,dt^2\oplus\rho^2(t)\,ds^2\oplus g_\Sigma)$ for some Kähler manifold (Σ,g_Σ) and some positive function ρ on I (that plays the role of $\rho\sigma$).*
- (iii) *If $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is Kähler, $\hat{h}\neq0$ on M and k only depends on t, then, up to rescaling $\hat{\xi}$, turning t into $-t$ or setting $\overline{\rho}:=\rho k$, as well as $\overline{\sigma}:=\frac{\sigma}{k}$, we have $\rho'=\sigma$ on I and $k=1$, hence $\hat{h}=-J$ on M. In particular, $(M,\hat{g},\hat{\xi})$ is Sasaki.*
- (iv) *If $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is Kähler, $\hat{h}\neq0$ on M and k is of the form $k(t,x)=k_1(t)k_2(x)$ (hence $n=2$), then, up to turning t into $-t$, rescaling $\hat{\xi}$, setting $\overline{\rho}:=\rho k_1$, as well as $\overline{\sigma}:=\frac{\sigma}{k_1}$, we may assume that $k_1=1$ and there exists a basic positive function C on M such that $\rho'=\sigma$ on I and $k_2=\sqrt{C}$, hence $\hat{h}=-C\cdot J$ on M. In particular, $(M^3,\overline{g}:=\hat{g}_{\hat{\xi}}\oplus C\cdot\hat{g}_{\hat{\xi}^\perp},\hat{\xi})$ is Sasaki. In this case, we call $(M,\hat{g},\hat{\xi})$ basic conformally Sasaki.*

Proof. (i) We first compute $\widetilde{\nabla}\widetilde{J}$ using Lemma 2.1. We keep the same notation as in the proof of Lemma 2.1. Let $Z, Z' \in \Gamma(\pi_2^*Q)$ be arbitrary sections. First,

$$\widetilde{\nabla}_{\partial_t}(\widetilde{J}\partial_t) - \widetilde{J}(\widetilde{\nabla}_{\partial_t}\partial_t) = -\widetilde{\nabla}_{\partial_t}\xi + 0 = 0$$

as well as $\widetilde{\nabla}_{\partial_t}(\widetilde{J}\xi) - \widetilde{J}(\widetilde{\nabla}_{\partial_t}\xi) = 0$. Moreover,

$$\begin{aligned}\widetilde{\nabla}_{\partial_t}(\widetilde{J}Z) - \widetilde{J}(\widetilde{\nabla}_{\partial_t}Z) &= \frac{\partial JZ}{\partial t} + \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}JZ - \widetilde{J}\Big(\frac{\partial Z}{\partial t} + \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}Z\Big)\\ &= \Big[\frac{\partial}{\partial t}, J\Big]Z = 0,\end{aligned}$$

showing that $\widetilde{\nabla}_{\partial_t}\widetilde{J} = 0$. Now, differentiating in the direction of ξ, we first obtain

$$\widetilde{\nabla}_{\xi}(\widetilde{J}\partial_t) - \widetilde{J}(\widetilde{\nabla}_{\xi}\partial_t) = -\widetilde{\nabla}_{\xi}\xi - \frac{(\rho\sigma)'}{\rho\sigma}\widetilde{J}\xi = 0$$

as well as $\widetilde{\nabla}_{\xi}(\widetilde{J}\xi) - \widetilde{J}(\widetilde{\nabla}_{\xi}\xi) = 0$. Furthermore, using $\hat{\nabla}J = 0$,

$$\begin{aligned}\widetilde{\nabla}_{\xi}(\widetilde{J}Z) - \widetilde{J}(\widetilde{\nabla}_{\xi}Z) &= \hat{\nabla}_{\xi}(JZ) + \frac{\xi(k)}{k}JZ + \frac{\sigma}{\rho k^2}\hat{h}JZ\\ &\qquad - \widetilde{J}\Big(\hat{\nabla}_{\xi}Z + \frac{\xi(k)}{k}Z + \frac{\sigma}{\rho k^2}\hat{h}Z\Big)\\ &= \frac{\sigma}{\rho k^2}[\hat{h}, J]Z.\end{aligned}$$

Thus, $\widetilde{\nabla}_{\xi}\widetilde{J} = 0$ if and only if $[\hat{h}, J] = 0$ on M. It remains to look at differentiation in transversal directions,

$$\begin{aligned}\widetilde{\nabla}_{Z}(\widetilde{J}\partial_t) - \widetilde{J}(\widetilde{\nabla}_{Z}\partial_t) &= -\widetilde{\nabla}_{Z}\xi - \widetilde{J}\Big(\frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}Z\Big)\\ &= -\frac{\xi(k)}{k}Z - \frac{\sigma}{\rho k^2}\hat{h}Z - \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}JZ;\end{aligned}$$

in particular, we have $(\widetilde{\nabla}_{Z}\widetilde{J})\partial_t = 0$ for all $Z \in \Gamma(\pi_2^*Q)$ if and only if $\xi(k) = 0$ and $\hat{h} = -\frac{k}{\sigma}\cdot\frac{\partial\rho k}{\partial t}J$; note that the latter condition implies $[\hat{h}, J] = 0$. Similarly,

$$\begin{aligned}\widetilde{\nabla}_{Z}(\widetilde{J}\xi) - \widetilde{J}(\widetilde{\nabla}_{Z}\xi) &= \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}Z - \widetilde{J}\Big(\frac{\xi(k)}{k}Z + \frac{\sigma}{\rho k^2}\hat{h}Z\Big)\\ &= -\frac{\xi(k)}{k}JZ + \frac{1}{\rho k}\cdot\frac{\partial(\rho k)}{\partial t}Z - \frac{\sigma}{\rho k^2}J\hat{h}Z,\end{aligned}$$

so $(\widetilde{\nabla}_{Z}\widetilde{J})\xi = 0$ for all $Z \in \Gamma(\pi_2^*Q)$ if and only if $\xi(k) = 0$ and $\hat{h} = -\frac{k}{\sigma}\cdot\frac{\partial(\rho k)}{\partial t}J$, which is precisely what we had before. Last but not least, assuming the latter conditions are fulfilled, we obtain

$$\begin{aligned}
\widetilde{\nabla}_Z(\widetilde{J}Z') - \widetilde{J}(\widetilde{\nabla}_Z Z') &= \hat{\nabla}_Z(JZ') + Z(\ln(k))JZ' + JZ'(\ln(k))Z \\
&\quad - \widetilde{g}(Z,JZ')\,\mathrm{grad}^{\hat{\nabla}}(\ln(k)) - \xi(\ln(k))\widetilde{g}(Z,JZ')\xi \\
&\quad - \frac{\sigma}{\rho k^2}\widetilde{g}(\hat{h}Z,JZ')\xi - \frac{\partial\ln(\rho k)}{\partial t}\widetilde{g}(Z,JZ')\partial_t \\
&\quad - J(\hat{\nabla}_Z Z') - Z(\ln(k))JZ' - Z'(\ln(k))JZ \\
&\quad + \widetilde{g}(Z,Z')J(\mathrm{grad}^{\hat{\nabla}}(\ln(k))) + \xi(\ln(k))\widetilde{g}(Z,Z')\partial_t \\
&\quad + \frac{\sigma}{\rho k^2}\widetilde{g}(\hat{h}Z,Z')\partial_t - \frac{\partial\ln(\rho k)}{\partial t}\widetilde{g}(Z,Z')\xi \\
&= JZ'(\ln(k))Z - Z'(\ln(k))JZ + \widetilde{g}(JZ,Z')\,\mathrm{grad}^{\hat{\nabla}}(\ln(k)) \\
&\quad + \widetilde{g}(Z,Z')J(\mathrm{grad}^{\hat{\nabla}}(\ln(k))).
\end{aligned}$$

We now want to know when the right-hand side of the last identity vanishes. In case $2n-2=2$, *i.e.* $n=2$, it trivially vanishes pointwise for all $Z,Z'\in Q$. In case $n>2$, assuming $\mathrm{grad}^{\hat{\nabla}}(\ln(k))$ to be nonzero at a point, we may chose for instance $Z\in Q$ such that $Z, JZ, \mathrm{grad}^{\hat{\nabla}}(\ln(k)), J(\mathrm{grad}^{\hat{\nabla}}(\ln(k)))$ are linearly independent, in which case, it can be deduced from the vanishing of all terms in the right-hand side of the last identity that $\widetilde{g}(JZ,Z')=0$ for all $Z'\in Q$, which is a contradiction; thus $\mathrm{grad}^{\hat{\nabla}}(\ln(k))$ must vanish identically for $n>2$. To sum up, we have shown that $\widetilde{\nabla}\widetilde{J}=0$ on $\widetilde{M}$ if and only if $\xi(k)=0$, $\hat{h}=-\frac{k}{\sigma}\cdot\frac{\partial\rho k}{\partial t}J$ and, if $n>2$, $\mathrm{grad}^{\hat{\nabla}}(k)=0$—and hence k only depends on t if M is connected.

(ii) If $\hat{h}=0$ at a point if $n>2$ or identically if $n=2$ and if $(\tilde{M},\tilde{g},\tilde{J})$ is Kähler, then, by (i), $\hat{h}$ must vanish identically and the following identity $\frac{\partial(\rho k)}{\partial t}=0$ holds. Hence, ρk is a function depending only on x, $\rho(t)k(t,x)=D(x)$. If $n>2$, as k depends only on t as we have already shown, then D is constant, and in this case, we have $\tilde{g}=dt^2\oplus\rho^2(t)\sigma^2(t)\hat{g}_{\hat{\xi}}\oplus D^2\hat{g}_{\hat{\xi}^\perp}$. Rescaling $\hat{g}_{\hat{\xi}^\perp}$ and replacing $\rho\sigma$ by ρ, we obtain the desired product form. If $n=2$, then D may be nonconstant, but $\hat{\xi}(D)=0$, and in this case, (Σ,g_Σ) is a surface (hence any Hermitian metric on Σ is already Kähler).

(iii) If k only depends on t and $(\tilde{M},\tilde{g},\tilde{J})$ is Kähler, then we may assume, up to replacing ρ by ρk and σ by $\frac{\sigma}{k}$, that $k=1$. Since neither $\hat{h}$ nor J depend on t, there exists a constant C such that $\frac{\rho'}{\sigma}=C$. If $C=0$, then $\hat{h}=0$ and we are back in case (ii). If $C\neq 0$, then, up to rescaling $\hat{\xi}$, we may assume that $C=\pm 1$ and, up to turning t into $-t$, that $C=1$. Then $\rho'=\sigma$ on I, and $\hat{h}=-J$ on M.

(iv) In case k is not necessarily constant in x (and thus $n=2$), there exists a function C on M, which must be basic since both $\hat{h}$ and J are, such that $\frac{k}{\sigma}\cdot\frac{\partial\rho k}{\partial t}=C$, that is, $\frac{\partial(\rho k)^2}{\partial t}=2\rho\sigma\cdot C$ on $\widetilde{M}$. This is equivalent to

$$(\rho(t)k(t,x))^2-(\rho(0)k(0,x))^2=2C(x)\cdot\int_0^t\rho(s)\sigma(s)\,ds\quad\text{for all }(t,x)\in I\times M.$$

This shows that k^2 is the sum of two functions that are products of a function of t with a function of x; still k must not be itself in product form. In case k is of the form $k(t,x) = k_1(t)k_2(x)$, then we may assume as above that $k_1 = 1$ (up to changing ρ and σ by multiplying ρ by k_1 and σ by $\frac{1}{k_1}$). The identity $\frac{\partial(\rho k_2)^2}{\partial t} = 2\rho\sigma \cdot C$ yields $\rho' k_2^2 = \sigma \cdot C$. If ρ' vanishes at one point, then C must vanish identically, and then $\hat{h} = 0$, which is again case (ii). Otherwise, up to turning t into $-t$, we may assume $\rho' > 0$ on I, from which $k_2(x)^2 = \frac{\sigma}{\rho'}(t) \cdot C(x)$ follows; in particular, $\frac{\rho'}{\sigma}$ is constant. Up to rescaling $\hat{\xi}$, we may assume that $\rho' = \sigma$ on I, from which $k_2(x)^2 = C(x)$ follows for all $x \in M$ (showing on the way that C must be positive) and therefore $k_2 = \sqrt{C}$. The last claim follows from Lemma 2.2. This concludes the proof of Lemma 2.4. □

Remark 2.5. The product form assumed for the function k in Lemma 2.4 (iv), is fulfilled in the case where we apply this result; see Theorem 1.1.

We end this section by characterizing the completeness of doubly-warped products. We will consider the case when $\mathrm{grad}^{\hat{\nabla}} k = 0$ (or $k = 1$, up to rescaling the metric) in the expression (2) of the metric since these cases naturally arise in our classification results (see Theorems 4.1 and 1.1).

Lemma 2.6. *Let* $(\widetilde{M}, \widetilde{g}) := (I \times M, dt^2 \oplus \rho(t)^2(\sigma(t)^2 \hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp}))$ *be a doubly-warped product. Then* $(\widetilde{M}, \widetilde{g})$ *is complete if and only if* $(M, \hat{g})$ *is complete and* $I = \mathbb{R}$.

Proof. The proof follows that of the analog result for warped products [4, Lem. 7.2]. Assume $(\widetilde{M}, \widetilde{g})$ to be complete. Then $(M, \hat{g})$ must be complete because it is a closed subset of $\widetilde{M}$ and the metrics $g_t := \rho(t)^2(\sigma(t)^2 \hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp})$ and $\hat{g}$ are equivalent for any fixed t. Moreover, because the integral curves of ∂_t are geodesics, according to (1), then $I = \mathbb{R}$ must hold. Conversely, assume $(M, \hat{g})$ to be complete and $I = \mathbb{R}$. Let $((t_n, x_n))_n$ be any Cauchy sequence in $(\widetilde{M}, \widetilde{g})$; then, because the distance between projected $\mathbb{R}$-components is anyway smaller than the distance associated to $\widetilde{g}$, the sequence $(t_n)_n$ must be a Cauchy sequence in $\mathbb{R}$ and therefore must converge to some $T \in \mathbb{R}$. But since, furthermore, the sequence $(t_n)_n$ must be bounded, so must be the coefficients $\rho(t_n), \sigma(t_n)$ of the metric g_{t_n} independently of n; therefore all $(g_{t_n})_n$ are uniformly equivalent to $\hat{g}$. As a consequence, the sequence $(x_n)_n$ must be a Cauchy sequence on $(M, \hat{g})$ and therefore must converge to some $x \in M$. Because, again, $(g_{t_n})_n$ are uniformly equivalent to $\hat{g}$, the sequence $((t_n, x_n))_n$ must converge to $(T, x) \in \widetilde{M}$, which concludes the proof. □

3. Proof of the main result

In this section, we give the proof of Theorem 1.1.

Proof. Let $\nu := \frac{\widetilde{\nabla} u}{|\widetilde{\nabla} u|} \in \Gamma(T\widetilde{M})$ and $\xi := -\widetilde{J}\nu$. By assumption,

$$\widetilde{\nabla}^2 u = \lambda \cdot (\nu^\flat \otimes \nu \oplus \xi^\flat \otimes \xi) + \mu \cdot \mathrm{Id}_{\{\xi,\nu\}^\perp}$$

on $\widetilde{M}$ for smooth real-valued functions λ, μ on $\widetilde{M}$. Fix any value u_0 of u, and let $M := u^{-1}(\{u_0\})$, which is a real hypersurface of $\widetilde{M}$ with induced metric $g := \widetilde{g}_{|_M}$. Since by assumption $\widetilde{\nabla} u$ is a pointwise eigenvector for $\widetilde{\nabla}^2 u$, the vector field ν is actually geodesic on $(\widetilde{M}, \widetilde{g})$, and Proposition A.1 shows that the map $F : \mathbb{R} \times M \to \widetilde{M}$, given by the flow of ν, is a diffeomorphism pulling $\widetilde{g}$ back onto $dt^2 \oplus g_t$, where $g_t := (F_t)^* g_{|_{TM \times TM}}$ is a smooth one-parameter-family of Riemannian metrics on M that we next determine more precisely. In the proof of Proposition A.1, we show that $f := u \circ F$ only depends on t. Let t_0 be such that $f(t_0) = (u \circ F)(t_0) = u_0$. Moreover, we have $f'(t) = |\widetilde{\nabla} u|_{F(t,x)}$ and, since ν is geodesic, $f''(t) = \lambda \circ F(t, x)$. Therefore, as $[\xi, \nu] = \frac{\lambda}{|\widetilde{\nabla} u|} \xi$, which can be shown by a straight-forward computation, we have

$$\begin{aligned}
\frac{\partial}{\partial s}((F_s)_* \xi)_{F_{t_0}(x)}|_{s=t} &= \Big((F_t)_* \Big(\frac{\lambda}{|\widetilde{\nabla} u|} \xi\Big)\Big)_{F_{t_0}(x)} \\
&= \frac{\lambda}{|\widetilde{\nabla} u|} \circ F_{-t}(F_{t_0}(x)) \cdot ((F_t)_* \xi)_{F_{t_0}(x)} \\
&= \frac{\lambda}{|\widetilde{\nabla} u|} \circ F_{-t+t_0}(x) \cdot ((F_t)_* \xi)_{F_{t_0}(x)} \\
&= \frac{f''(-t+t_0)}{f'(-t+t_0)} \cdot ((F_t)_* \xi)_{F_{t_0}(x)}.
\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}
(5)\quad ((F_t)_* \xi)_{F_{t_0}(x)} &= \exp\Big(\int_0^t \frac{f''(-s+t_0)}{f'(-s+t_0)}\, ds\Big) \cdot \xi(F_{t_0}(x)) \\
&= \exp\Big(\int_{-t_0}^{t-t_0} \frac{f''(-s')}{f'(-s')}\, ds'\Big) \cdot \xi(F_{t_0}(x)), \text{ where } s' := s - t_0 \\
&= \frac{f'(t_0)}{f'(t_0 - t)} \xi(F_{t_0}(x)).
\end{aligned}$$

In particular, this leads to $(F_t^* \widetilde{g})(\xi, \xi) = \frac{f'(t)^2}{f'(0)^2}$. On the other hand, we have, for all $X, Y \in TM$,

$$\begin{aligned}
(\mathcal{L}_\nu \widetilde{g})(X, Y) = 2\widetilde{g}(\widetilde{\nabla}_X \nu, Y) &= \frac{2}{|\widetilde{\nabla} u|} \widetilde{\nabla}^2 u(X, Y) \\
&= \frac{2}{|\widetilde{\nabla} u|}((\lambda - \mu)\widetilde{g}(X, \xi)\widetilde{g}(Y, \xi) + \mu \widetilde{g}(X, Y)).
\end{aligned}$$

Hence, as in Proposition A.1, we get for all $X, Y \in \{\xi, \nu\}^\perp$ that

$$\frac{\partial}{\partial s} F_s^* \widetilde{g}(X, Y)_{|_{s=t}} = \frac{2}{f'(t)} \mu \circ F(t, x) \cdot (F_t^* \widetilde{g})(X, Y);$$

in particular, after integrating, we find

$$(F_t^* \widetilde{g})(X, Y) = \exp\Big(2 \int_0^t \frac{\mu \circ F(s, x)}{f'(s)}\, ds\Big) \cdot \widetilde{g}(X, Y).$$

Therefore,

$$F_t^*\widetilde{g} = dt^2 \oplus \frac{(f')^2(t)}{(f')^2(0)} g_\xi \oplus \exp\Big(2\int_0^t \frac{\mu\circ F(s,x)}{f'(s)}\,ds\Big) g_{\xi^\perp}, \tag{6}$$

where we recall that g is the induced metric $g = \widetilde{g}_{|M}$ and $g = g_\xi \oplus g_{\xi^\perp}$. We check now that (M, g, ξ) is a minimal Riemannian flow. Firstly, since, for all $X \in T\widetilde{M}$,

$$\widetilde{\nabla}_X \nu = \frac{1}{|\widetilde{\nabla} u|}(\widetilde{\nabla}^2_X u - g(\widetilde{\nabla}^2_X u, \nu)\nu) = \frac{1}{|\widetilde{\nabla} u|}(\widetilde{\nabla}^2_X u - \lambda g(X, \nu)\nu), \tag{7}$$

we have

$$\widetilde{\nabla}_\xi \xi = \widetilde{\nabla}_{\widetilde{J}\nu} \widetilde{J}\nu = \widetilde{J}(\widetilde{\nabla}_{\widetilde{J}\nu}\nu) = -\frac{1}{|\widetilde{\nabla} u|}\widetilde{\nabla}^2_\nu u = -\frac{\lambda}{|\widetilde{\nabla} u|}\nu$$

and hence $\nabla^M_\xi \xi = 0$. Secondly, for all $X \in TM \cap \xi^\perp$, we have

$$\nabla^M_X \xi = \widetilde{\nabla}_X \xi + \underbrace{g(\widetilde{\nabla}_X \nu, \xi)}_{0}\,\nu = -\widetilde{J}(\widetilde{\nabla}_X \nu) = -\frac{\mu}{|\widetilde{\nabla} u|}\widetilde{J}X. \tag{8}$$

Thus, $\nabla^M \xi$ is skew symmetric and vanishes on ξ; therefore (M, g, ξ) is a minimal Riemannian flow, and its O'Neill tensor is given by $h = -\frac{\mu}{|\widetilde{\nabla} u|}\widetilde{J}$. In the following, we denote by ∇ the transversal Levi–Civita connection of the flow. Hence, both connections are related by $\nabla_\xi Z = \nabla^M_\xi Z - \nabla^M_Z \xi$ and $\nabla_{Z'} Z = \nabla^M_{Z'} Z + g(\nabla^M_{Z'}\xi, Z)\xi$ for all sections Z, Z' of $Q := \{\xi, \nu\}^\perp \to M$. As for $J := \widetilde{J}$ on Q, we have, for every $Z \in \Gamma(Q)$,

$$\begin{aligned}
\nabla_\xi(JZ) &= \widetilde{\nabla}_\xi(\widetilde{J}Z) + \underbrace{g(\widetilde{\nabla}_\xi \nu, JZ)}_{0}\,\nu - \nabla^M_{JZ}\xi \\
&= \widetilde{J}(\widetilde{\nabla}_\xi Z) - \nabla^M_{JZ}\xi \\
&= \widetilde{J}(\widetilde{\nabla}_\xi Z - \nabla^M_Z \xi) \quad \text{since } [\nabla^M \xi, J] = 0 \\
&= \widetilde{J}(\nabla^M_\xi Z - \nabla^M_Z \xi) \quad \text{since } g(\widetilde{\nabla}_\xi Z, \nu) = 0 \\
&= J(\nabla_\xi Z)
\end{aligned}$$

and, for all $Z' \in Q$,

$$\begin{aligned}
\nabla_{Z'}(JZ) &= \widetilde{\nabla}_{Z'}(\widetilde{J}Z) + g(\widetilde{\nabla}_{Z'}\nu, JZ)\nu + g(\nabla^M_{Z'}\xi, JZ)\xi \\
&= \widetilde{J}(\widetilde{\nabla}_{Z'} Z) + \frac{\mu}{|\widetilde{\nabla} u|} g(Z', JZ)\nu - \frac{\mu}{|\widetilde{\nabla} u|} g(JZ', JZ)\xi \\
&= \widetilde{J}\Big(\widetilde{\nabla}_{Z'} Z + \frac{\mu}{|\widetilde{\nabla} u|} g(Z', Z)\nu - \frac{\mu}{|\widetilde{\nabla} u|} g(JZ', Z)\xi\Big) \\
&= J(\nabla_{Z'} Z).
\end{aligned}$$

Therefore, $\nabla J = 0$, and hence J defines a transversal Kähler structure on (M, g, ξ).

In the following, we show that the pullback of the almost complex structure $F^*\widetilde{J}$ on $\mathbb{R}\times M$ maps ξ onto ∂_t, ∂_t onto $-\xi$ and coincides with J on Q. For all $(t,x)\in\mathbb{R}\times M$ and $X\in\mathbb{R}\oplus T_xM$, we have

$$(F^*\widetilde{J})_{(t,x)}(X) = (d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}\circ(d_{(t,x)}F)(X).$$

For $X=\partial_t$, we have

$$\begin{aligned}(F^*\widetilde{J})_{(t,x)}(\partial_t) &= (d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}\circ(d_{(t,x)}F)(\partial_t)\\ &= (d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}(\partial_t)\\ &= -(d_{(t,x)}F)^{-1}(\xi_{F(t,x)})\\ &= -(F_{-t*}\xi)_x\\ &\overset{(5)}{=} -\frac{f'(0)}{f'(t)}\xi_x = -\xi_{(t,x)}\end{aligned}$$

as well as

$$\begin{aligned}(F^*\widetilde{J})_{(t,x)}(\xi_{(t,x)}) &= (d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}\circ(d_{(t,x)}F)(\xi_{(t,x)})\\ &= \frac{f'(0)}{f'(t)}(d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}((d_{(t,x)}F)(\xi_x))\\ &= \frac{f'(0)}{f'(t)}(d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}((F_{t*}\xi)_{F_t(x)})\\ &\overset{(5)}{=} \frac{f'(0)}{f'(t)}(d_{(t,x)}F)^{-1}\circ\widetilde{J}_{F(t,x)}\Big(\frac{f'(t)}{f'(0)}\cdot\xi_{F(t,x)}\Big)\\ &= (d_{(t,x)}F)^{-1}(\partial_t) = \partial_t.\end{aligned}$$

To show that $(F^*\widetilde{J})_{|Q} = J$, we compute the Lie derivative of $\widetilde{J}$ in ν-direction; for every $X\in T\widetilde{M}$,

$$\begin{aligned}(\mathcal{L}_\nu\widetilde{J})X &= [\nu,\widetilde{J}X]-\widetilde{J}[\nu,X] = \widetilde{J}(\widetilde{\nabla}_\nu X)-\widetilde{\nabla}_{\widetilde{J}X}\nu-\widetilde{J}[\nu,X] = \widetilde{J}\widetilde{\nabla}_X\nu-\widetilde{\nabla}_{\widetilde{J}X}\nu\\ &\overset{(7)}{=} \frac{1}{|\widetilde{\nabla}u|}\Big(\widetilde{J}\widetilde{\nabla}^2_X u-\lambda g(X,\nu)\widetilde{J}\nu\Big)-\frac{1}{|\widetilde{\nabla}u|}(\widetilde{\nabla}^2_{\widetilde{J}X}u-\lambda g(\widetilde{J}X,\nu)\nu)\\ &= \frac{\lambda}{|\widetilde{\nabla}u|}\cdot(g(X,\nu)\xi+g(X,\xi)\nu);\end{aligned}$$

therefore, $\mathcal{L}_\nu\widetilde{J} = \frac{\lambda}{|\widetilde{\nabla}u|}\cdot(\nu^\flat\otimes\xi+\xi^\flat\otimes\nu)$. Now, for any $(t,x)\in\mathbb{R}\times M$ and $Z\in T_xM\cap\hat{\xi}^\perp$,

$$\frac{\partial}{\partial s}(F_s^*\widetilde{J})(Z_x)_{|s=t} = \frac{\partial}{\partial s}(F_s^*\widetilde{J})_{|s=t}(Z_x) = (\mathcal{L}_\nu\widetilde{J})(d_xF_t(Z_x)) = 0$$

because F_t preserves Q. We deduce that $(F_t^*\widetilde{J})(Z_x) = \widetilde{J}_x(Z_x) = JZ_x$; therefore $(F^*\widetilde{J})_{|Q} = J$, as claimed.

Summing up, we have shown that, on the product manifold $I\times M$, the metric $F_t^*\widetilde{g}$ is determined by (6) and the complex structure $F_t^*\widetilde{J}$ has the form as in (4). Moreover, the manifold $(I\times M, F_t^*\widetilde{g}, F^*\widetilde{J})$ is Kähler, and the triple (M,g,ξ) is a minimal Riemannian flow equipped with a transversal Kähler structure $J = \widetilde{J}$. Recall that $g = g_\xi \oplus g_{\xi^\perp}$ is the induced metric $\widetilde{g}_{|M}$. In the following, we will apply Lemma 2.4 in order to obtain the classification result.

We begin with the case when $n > 2$. We write $F_t^*\widetilde{g}$ as a doubly-warped product in the following way:

$$F_t^*\widetilde{g} = dt^2 \oplus \rho^2(t)(\sigma^2(t)g_\xi \oplus k^2(t,x)g_{\xi^\perp}),$$

where ρ, σ, k are positive smooth functions that satisfy the system

$$\begin{cases} \rho^2(t)\sigma^2(t) = \dfrac{(f')^2(t)}{(f')^2(0)}, \\ \rho^2(t)k^2(t,x) = \exp\Big(2\displaystyle\int_0^t \frac{\mu\circ F(s,x)}{f'(s)}\,ds\Big) \end{cases}$$

for all $(t,x) \in \mathbb{R}\times M$. Therefore, as the hypotheses of Lemma 2.4 are fulfilled for the flow $(M, \hat{g}, \hat{\xi})$ with $\hat{g} = g$ and $\hat{\xi} = \xi$, we get the following cases.

To show (i), we assume that μ vanishes at one point of $\widetilde{M}$. Then, according to (8), also h vanishes at that point. Lemma 2.4 (ii) then implies that h, and thus also μ, vanishes identically. In this case, up to replacing ρk by 1 and $\rho\sigma$ by ρ, we obtain from the above system that $\rho = \frac{f'}{f'(0)}$. Furthermore, up to replacing u by $\frac{1}{f'(0)}u$ which does not change the statement of the theorem, we can assume that $f'(0) = 1$. Thus, we obtain $\rho = f'$, that is, $\rho(t) = |\widetilde{\nabla}u| \circ F(t,x)$ for all $(t,x) \in \mathbb{R}\times M$ and the triple $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is locally biholomorphically isometric to $(\mathbb{R}_t \times \mathbb{R}_s \times \Sigma, dt^2 \oplus \rho^2(t)\,ds^2 \oplus g_\Sigma)$ for some Kähler manifold $(\Sigma^{2n-2}, g_\Sigma)$ and some positive function ρ. Note that the eigenvalues λ and μ may be equal (to 0), which corresponds to the trivial case when $\widetilde{\nabla}u$ is a parallel vector field on $\widetilde{M}$.

To show (ii) (a), we assume that μ does not vanish at any point of $\widetilde{M}$. In this case, it follows from Lemma 2.4 (i) that the function k only depends on t so that $(t,x) \mapsto \mu\circ F(t,x)$ only depends on t as well. Setting $\overline{\rho} := \rho k$ as well as $\overline{\sigma} := \frac{\sigma}{k}$, we can assume that $k \equiv 1$. Furthermore, Lemma 2.4 (iii) implies after rescaling ξ and turning t into $-t$ (which amounts to changing u into $-u$) that $\rho' = \sigma$ and thus $h = -J$ on M. Hence, the first equation in the above system allows to get

$$(\rho\rho')^2 = \Big(\frac{f'}{f'(0)}\Big)^2, \quad \text{that is} \quad \rho\rho' = \frac{f'}{f'(0)},$$

or equivalently $\rho^2(t) - \rho^2(0) = \frac{2}{f'(0)}(f(t) - f(0))$ for all $t \in \mathbb{R}$. Thus, up to replacing u by $\frac{2}{f'(0)}(u - u_0) + \rho^2(0)$, which does not affect the assumptions on u, we may assume that $f'(0) = 2$ and $f(0) = \rho(0)^2 = 1$; in particular, we obtain $\rho^2 = f$ and thus $\rho = \sqrt{f} = \sqrt{u\circ F}$. We also deduce that

$$\lambda\circ F(t,x) = \widetilde{\nabla}^2u(\nu,\nu)\circ F(t,x) = f''(t),$$
$$\mu\circ F(t,x) = \frac{f'(t)\rho'(t)}{\rho(t)} = \frac{(f')^2(t)}{2f(t)} = \frac{|\widetilde{\nabla}u|^2}{2u}\circ F(t,x).$$

Now, we consider the case when $n = 2$. When μ vanishes identically, then, as before the triple $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is locally biholomorphically isometric to

$$(\mathbb{R}_t \times \mathbb{R}_s \times \Sigma, dt^2 \oplus \rho^2(t)\, ds^2 \oplus g_\Sigma)$$

for some surface (Σ^2, g_Σ) and some positive function ρ. In the following, we consider the case when μ does not vanish identically on $\widetilde{M}$. Let $x_0 \in \widetilde{M}$ be a point where $\mu(x_0) \neq 0$. Up to changing the regular value u_0 of u, we may assume that $x_0 \in M = u^{-1}(\{u_0\})$. This time, we write $\hat{g}_{\xi^\perp} := \frac{1}{\beta(x)^2} \cdot g_{\xi^\perp}$ for some positive basic function β that will be later determined. From Lemma 2.2, the triple $(M, \hat{g} := g_\xi \oplus \frac{1}{\beta(x)^2} \cdot g_{\xi^\perp}, \hat{\xi} := \xi)$ is still a minimal Riemannian flow and, as $n = 2$, is also endowed with the same transversal complex structure $J = \widetilde{J}$. Now we apply Lemma 2.4 to the flow $(M, \hat{g}, \hat{\xi})$ and put $F_t^* \widetilde{g}$ under the form $dt^2 \oplus \rho(t)^2(\sigma(t)^2 \hat{g}_\xi \oplus k(t,x)^2 \hat{g}_{\xi^\perp})$, where ρ, σ, k satisfy

$$\tag{9} \begin{cases} \rho^2(t)\sigma^2(t) = \dfrac{(f')^2(t)}{(f')^2(0)}, \\ \rho^2(t)k^2(t,x) = \beta^2(x) \cdot \exp\left(2 \displaystyle\int_0^t \frac{\mu \circ F(s,x)}{f'(s)}\, ds\right). \end{cases}$$

Lemma 2.4 (i) implies the existence of a basic function C on M such that $\frac{\partial(\rho k)^2}{\partial t} = 2\rho\sigma C$; in particular,

$$\rho^2(t)k^2(t,x) - \rho^2(0)k^2(0,x) = 2C(x) \cdot \int_0^t \rho(s)\sigma(s)\, ds \quad \text{for all } (t,x) \in \mathbb{R} \times M.$$

But $\int_0^t \rho(s)\sigma(s)\, ds = \frac{f(t)-f(0)}{f'(0)}$ by the first identity of (9) so that

$$\tag{10} \begin{aligned} &\rho^2(t)k^2(t,x) - \rho^2(0)k^2(0,x) \\ &\qquad = 2C(x) \cdot \frac{f(t)-f(0)}{f'(0)} \quad \text{for all } (t,x) \in \mathbb{R} \times M. \end{aligned}$$

Another consequence of (9) is that μ and C have the same sign everywhere; we have the identity $\int_0^t \frac{\mu \circ F(s,x)}{f'(s)}\, ds = \ln(\rho(t)k(t,x)) - \ln(\beta(x))$, from which we deduce that

$$\begin{aligned} \mu \circ F(t,x) &= f'(t) \cdot \frac{1}{2(\rho k)^2} \cdot \frac{\partial(\rho k)^2}{\partial t}(t,x) \\ &= \frac{f'(t)(\rho\sigma)(t)}{(\rho k)(t,x)^2} \cdot C(x) \quad \text{with } \frac{f'(t)(\rho\sigma)(t)}{(\rho k)(t,x)^2} > 0 \end{aligned}$$

as we recall that $f'(t) = |\widetilde{\nabla} u|_{F(t,x)}$. Therefore, we deduce that $C(x_0) \neq 0$ because of $\mu(x_0) \neq 0$. Note in particular that f (or, equivalently, u) is necessarily bounded below or above by identity (10); indeed, as $(\rho(t)k(t,x_0))^2 > 0$, we get

$$f(t) \geq f(0) - \frac{(\rho(0)k(0,x_0))^2 f'(0)}{2C(x_0)}$$

if $C(x_0) > 0$ and, if $C(x_0) < 0$, we have

$$f(t) \leq f(0) - \frac{(\rho(0)k(0,x_0))^2 f'(0)}{2C(x_0)} \quad \text{for all } t \in \mathbb{R}.$$

Hence, up to changing u into $-u \pm c$ for some $c \in \mathbb{R}$, we may assume that

$$u_0 = f(0) = \frac{(\rho(0)k(0,x_0))^2 f'(0)}{2C(x_0)} \quad \text{and} \quad f > 0.$$

As in this case $C(x_0) > 0$, hence $C > 0$ on some nonempty open neighborhood U of x_0 in M. Therefore, we may set

$$\beta(x) := \sqrt{\frac{2C(x)f(0)}{f'(0)}} \quad \text{for all } x \in U.$$

Since $(\rho(0)k(0,x))^2 = \beta(x)^2$ from the second identity in (9) evaluated at $t = 0$, then (10) implies that $(\rho(t)k(t,x))^2 = 2C(x)\frac{f(t)}{f'(0)}$; in particular, k is in product form. Again, Lemma 2.4 (iii) together with a possible further rescaling of ξ and a change of u into $\frac{2}{f'(0)}u$ yields the splitting result in (ii) (b) on U. Furthermore, as above, we have $\mu = \frac{|\widetilde{\nabla}u|^2}{2u}$ on $\mathbb{R} \times U$. Now we show that the closed and nonempty subset of $\widetilde{M}$ where $|\widetilde{\nabla}u|^2 = 2\mu u$ is also open. Indeed, if $|\widetilde{\nabla}u|^2 = 2\mu u$ is satisfied at some point $(t,z) \in \mathbb{R} \times M$, then, since we know that $u > 0$ on $\widetilde{M}$, we also know that $\mu(z) > 0$ and thus $C(z) > 0$ because $\mathrm{sgn}(C) = \mathrm{sgn}(\mu)$, and therefore $C > 0$ on some open connected neighborhood V of z in M. Repeating on V the argument performed on U, we obtain that $|\widetilde{\nabla}u|^2 = 2\mu u$ on $\mathbb{R} \times V$. This implies that $|\widetilde{\nabla}u|^2 = 2\mu u$ holds on $\widetilde{M}$ by connectedness. This concludes the proof of Theorem 1.1. □

Remark 3.1. Notice that, under the assumptions of Theorem 1.1 and in case $\mu \neq 0$, the Hessian of u satisfies

$$\widetilde{\nabla}^2 u = \widetilde{\nabla}^2 u(\nu,\nu) \cdot (\nu^\flat \otimes \nu + \xi^\flat \otimes \xi) + \frac{|\widetilde{\nabla}u|^2}{2u}\mathrm{Id}_{\{\xi,\nu\}^\perp}.$$

In particular, the two eigenvalues λ and μ cannot be equal in that case. Namely, assuming by contradiction that $\lambda = \mu$, then we know from Theorem 1.1 that without loss of generality u can be assumed to be positive on $\widetilde{M}$. Then it follows from the above established identities that $f''(t) = \frac{f'(t)^2}{2f(t)}$ for all $t \in \mathbb{R}$, where $f(t) := u \circ F(t,x)$ for all $(t,x) \in \mathbb{R} \times M$ as in Theorem 1.1. Thus, $2\frac{f''}{f'} = \frac{f'}{f}$, which further yields $2 \ln f' = \ln f + \text{const}$. We obtain $(f')^2 = cf$ for some positive constant c and therefore $f(t) = (at + b)^2$ for some real constants a, b. But, because of $u > 0$ everywhere, the coefficient a must vanish, and therefore f and hence u must be constant, contradicting the fact that $\mu \neq 0$. Note that the above computations show that $\lambda = \mu \neq 0$ is still possible locally on $\widetilde{M}$.

Corollary 3.2. *Let $(\widetilde{M}^{2n}, g, J)$ be a complete Kähler manifold admitting a function $u \in C^\infty(\widetilde{M}, \mathbb{R}_+^\times)$ with*

- $|\widetilde{\nabla}u| = 2u$,
- $\widetilde{\nabla}^2 u = 2u(\xi^\flat \otimes \xi + \nu^\flat \otimes \nu + \mathrm{Id}_{T\widetilde{M}})$.

Then $(\widetilde{M}^{2n}, g, J)$ *is biholomorphically isometric to a manifold of the form* $(\mathbb{R} \times M^{2n-1}, dt^2 \oplus e^{2t}(e^{2t}\hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp}))$*, where* $(M^{2n-1}, \hat{g}, \hat{\xi})$ *is Sasaki.*

Proof. Note that, by assumption, u has no critical point (because of the first condition and $u > 0$ everywhere), $\widetilde{\nabla}^2 u$ has pointwise two eigenvalues, $4u$ and $2u$ with $\ker(\widetilde{\nabla}^2 u - 4u\,\mathrm{Id}) = \mathrm{Span}(\widetilde{\nabla} u, J\widetilde{\nabla} u)$. By Theorem 1.1—that applies since by assumption $\frac{|\widetilde{\nabla} u|^2}{2u} = 2u$—it suffices to notice that $f(t) := u(F(t,x))$ satisfies $f(t) = f(0)e^{2t}$. But this obviously follows from the identities $f'(t) = |\widetilde{\nabla} u|(F(t,x)) = 2u(F(t,x)) = 2f(t)$. Choosing $M := u^{-1}(\{f(0)\})$, we see that we may choose $f(t) = e^{2t}$ and conclude with Theorem 1.1. □

Example 3.3. The generalized Taub-NUT metrics of Iwai–Katayama on $\mathbb{C}^2$ as described in [14, Ex. 2.2] are Ricci-flat doubly-warped product Kähler metrics and therefore are a particular case of our description in Section 4.

4. Kähler–Einstein doubly-warped products

The purpose of this section is to give a characterization of the Kähler doubly-warped products of the form $(\widetilde{M} = I \times M^{2n-1}, \widetilde{g} := dt^2 \oplus \rho^2((\rho')^2\hat{g}_{\hat{\xi}} \oplus \hat{g}_{\hat{\xi}^\perp}), \widetilde{J})$, whose underlying metric $\widetilde{g}$ is Einstein. Recall first that $(M, \hat{g}, \hat{\xi})$ is a minimal Riemannian flow endowed with a complex structure J and that the complex structure $\widetilde{J}$ on $\widetilde{M}$ is always the one given by (4). According to Lemma 2.4 (i), and since here $k = 1$ and $\sigma = \rho'$, the complex structure $\widetilde{J}$ is Kähler on $\widetilde{M}$, and we have $\hat{h} = -J$; hence $(M, \hat{g}, \hat{\xi})$ is a Sasakian manifold. We will show in the sequel that the Einstein condition on $(\widetilde{M}, \widetilde{g})$ is equivalent to $(M, \hat{g}, \hat{\xi})$ being η-Einstein and ρ satisfying an ODE of order 3. Depending on the sign of the Einstein constant, we will provide solutions of this ODE that in some cases might not be complete.

In the following, we will compute the Ricci curvature of $(\widetilde{M}, \widetilde{g})$ in terms of the transversal Ricci curvature which is associated to the transversal Levi–Civita connection $\hat{\nabla}$, by using the formulas in Lemma 2.1. For this, we denote by $(e_j)_{1 \le j \le 2n-1}$ a local orthonormal basis of TM with respect to the metric g_t with $e_{2n-1} = \xi = \frac{1}{\rho\rho'}\hat{\xi}$. Then, with our convention, $\widetilde{R}_{X,Y} = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$ for all X, Y, we compute

$$
\begin{aligned}
\widetilde{g}(\widetilde{R}_{\partial_t,\xi}\xi, \partial_t) &= -\widetilde{g}(\widetilde{R}_{\partial_t,\xi}\partial_t, \xi) \\
&= -\widetilde{g}(\widetilde{\nabla}_{\partial_t}\widetilde{\nabla}_\xi \partial_t, \xi) + \widetilde{g}(\widetilde{\nabla}_\xi \underbrace{\widetilde{\nabla}_{\partial_t}\partial_t}_{0}, \xi) + \widetilde{g}(\widetilde{\nabla}_{[\partial_t,\xi]}\partial_t, \xi) \\
&= -\widetilde{g}\Big(\widetilde{\nabla}_{\partial_t}\Big(\frac{(\rho\rho')'}{\rho\rho'}\xi\Big), \xi\Big) - \frac{(\rho\rho')'}{\rho\rho'}\widetilde{g}(\widetilde{\nabla}_\xi \partial_t, \xi) \\
&= -\Big(\frac{(\rho\rho')'}{\rho\rho'}\Big)' \underbrace{\widetilde{g}(\xi,\xi)}_{1} - \frac{(\rho\rho')'}{\rho\rho'} \underbrace{\widetilde{g}(\widetilde{\nabla}_{\partial_t}\xi, \xi)}_{0} - \Big(\frac{(\rho\rho')'}{\rho\rho'}\Big)^2 \widetilde{g}(\xi,\xi) \\
&= -\frac{(\rho\rho')''}{\rho\rho'},
\end{aligned}
$$

and for every $j \in \{1, \dots, 2n-2\}$, we similarly compute

$$\widetilde{g}(\widetilde{R}_{\partial_t, e_j} e_j, \partial_t) = -\frac{\rho''}{\rho} \widetilde{g}(e_j, e_j).$$

Therefore,

$$\widetilde{\mathrm{ric}}(\partial_t, \partial_t) = -(2n-2)\frac{\rho''}{\rho} - \frac{(\rho\rho')''}{\rho\rho'} = -(2n+1)\frac{\rho''}{\rho} - \frac{\rho'''}{\rho'}.$$

Note that, because $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is Kähler, we also have by $\widetilde{J}$-invariance of the Ricci-curvature that $\widetilde{\mathrm{ric}}(\xi, \xi) = \widetilde{\mathrm{ric}}(\partial_t, \partial_t)$, as well as $\widetilde{\mathrm{ric}}(\xi, \partial_t) = 0$. For every $Z \in \{\xi, \partial_t\}^\perp$, we now compute $\widetilde{\mathrm{ric}}(Z, \partial_t)$. Indeed,

$$\begin{aligned}
\widetilde{g}(\widetilde{R}_{Z, e_j} e_j, \partial_t) &= -\widetilde{g}(\widetilde{R}_{Z, e_j} \partial_t, e_j) \\
&= -\widetilde{g}(\widetilde{\nabla}_Z \widetilde{\nabla}_{e_j} \partial_t, e_j) + \widetilde{g}(\widetilde{\nabla}_{e_j} \widetilde{\nabla}_Z \partial_t, e_j) + \widetilde{g}(\widetilde{\nabla}_{[Z, e_j]} \partial_t, e_j) \\
&= -\frac{\rho'}{\rho} \underbrace{\widetilde{g}(\widetilde{\nabla}_Z e_j, e_j)}_{0} + \frac{\rho'}{\rho} \widetilde{g}(\widetilde{\nabla}_{e_j} Z, e_j) + \widetilde{g}(\widetilde{\nabla}_{[Z, e_j]^Q} \partial_t, e_j) \\
&\quad - 2\widetilde{g}([Z, e_j], \xi) \underbrace{\widetilde{g}(\widetilde{\nabla}_\xi \partial_t, e_j)}_{0} \\
&= \frac{\rho'}{\rho} \widetilde{g}([e_j, Z], e_j) + \frac{\rho'}{\rho} \widetilde{g}([Z, e_j], e_j) = 0,
\end{aligned}$$

and similarly, we obtain $\widetilde{g}(\widetilde{R}_{Z,\xi}\xi, \partial_t) = 0$ so that $\widetilde{\mathrm{ric}}(Z, \partial_t) = 0$. Consequently, by the $\widetilde{J}$-invariance of Ricci-curvature, we also have $\widetilde{\mathrm{ric}}(Z, \xi) = 0$. The last term to be computed is $\widetilde{\mathrm{ric}}(Z, Z)$ for any Z. Similarly to above, we find $\widetilde{g}(\widetilde{R}_{Z, \partial_t} \partial_t, Z) = \widetilde{g}(\widetilde{R}_{Z,\xi}\xi, Z) = -\frac{\rho''}{\rho} \widetilde{g}(Z, Z)$. In order to compute the remaining curvature term, we take for simplification e_j and Z to be parallel with respect to the connection $\hat{\nabla}$ at some point x (in this case $[Z, e_j]_x = -2\hat{g}(\hat{h}Z, e_j)\hat{\xi}_x$, so it is collinear to $\hat{\xi}_x$). Then, as $\hat{h} = -J$, we have, at the point x,

$$\begin{aligned}
\widetilde{g}(\widetilde{R}_{Z, e_j} e_j, Z) &= \widetilde{g}(\widetilde{\nabla}_Z \widetilde{\nabla}_{e_j} e_j, Z) - \widetilde{g}(\widetilde{\nabla}_{e_j} \widetilde{\nabla}_Z e_j, Z) - \widetilde{g}(\widetilde{\nabla}_{[Z, e_j]} e_j, Z) \\
&= \widetilde{g}\Big(\widetilde{\nabla}_Z \Big(\hat{\nabla}_{e_j} e_j - \frac{\rho'}{\rho} \widetilde{g}(e_j, e_j) \partial_t\Big), Z\Big) \\
&\quad - \widetilde{g}\Big(\widetilde{\nabla}_{e_j} \Big(\hat{\nabla}_Z e_j + \frac{\rho'}{\rho} \widetilde{g}(JZ, e_j)\xi - \frac{\rho'}{\rho} \widetilde{g}(Z, e_j)\partial_t\Big), Z\Big) \\
&\quad - \widetilde{g}([Z, e_j], \xi) \widetilde{g}(\widetilde{\nabla}_\xi e_j, Z) \\
&= \widetilde{g}(\hat{\nabla}_Z \hat{\nabla}_{e_j} e_j, Z) - \Big(\frac{\rho'}{\rho}\Big)^2 \widetilde{g}(Z, Z) - \widetilde{g}(\hat{\nabla}_{e_j} \hat{\nabla}_Z e_j, Z) \\
&\quad - \Big(\frac{\rho'}{\rho}\Big)^2 \widetilde{g}(JZ, e_j)^2 + \Big(\frac{\rho'}{\rho}\Big)^2 \widetilde{g}(Z, e_j)^2 - 2(\frac{\rho'}{\rho})^2 \widetilde{g}(Je_j, Z)^2 \\
&= \widetilde{g}(R^{\hat{\nabla}}_{Z, e_j} e_j, Z) + \Big(\frac{\rho'}{\rho}\Big)^2 (\widetilde{g}(Z, e_j)^2 - \widetilde{g}(Z, Z) - 3\widetilde{g}(JZ, e_j)^2).
\end{aligned}$$

For the last identity, recall that $R^{\hat\nabla}_{Z,Z'} = [\hat\nabla_Z, \hat\nabla_{Z'}] - \hat\nabla_{[Z,Z']}$ for all $Z, Z' \in Q = \{\xi, \partial_t\}^\perp$. Finally, we deduce that

$$\begin{aligned}\widetilde{\mathrm{ric}}(Z,Z) &= \sum_{j=1}^{2n-2}\Big(\widetilde{g}(R^{\hat\nabla}_{Z,e_j}e_j, Z) + \Big(\frac{\rho'}{\rho}\Big)^2(\widetilde{g}(Z,e_j)^2 - \widetilde{g}(Z,Z) - 3\widetilde{g}(JZ,e_j)^2)\Big)\\ &\qquad - 2\frac{\rho''}{\rho}\widetilde{g}(Z,Z)\\ &= \mathrm{ric}^{\hat\nabla}(Z,Z) - 2\Big(\frac{\rho\rho'' + n(\rho')^2}{\rho^2}\Big)\widetilde{g}(Z,Z).\end{aligned}$$

To sum up, the $(1,1)$-Ricci-tensor of $(\widetilde{M}, \widetilde{g})$ is given pointwise by

$$\begin{aligned}\widetilde{\mathrm{Ric}} = &-\Big(\frac{(2n+1)\rho''}{\rho} + \frac{\rho'''}{\rho'}\Big)\cdot(dt\otimes\partial_t \oplus \xi^\flat\otimes\xi)\\ &\oplus\Big(\frac{1}{\rho^2}\mathrm{Ric}^{\hat\nabla} - 2\Big(\frac{\rho\rho''+n(\rho')^2}{\rho^2}\Big)\cdot\mathrm{Id}_{\{\xi,\partial_t\}^\perp}\Big).\end{aligned}$$

In particular, the manifold $(I \times M^{2n-1}, dt^2 \oplus \rho^2((\rho')^2\hat g_{\hat\xi} \oplus \hat g_{\hat\xi^\perp}))$ is Kähler–Einstein if and only if there exists a constant $C \in \mathbb{R}$, which is equal to $\frac{\widetilde{\mathrm{Scal}}}{2n}$, such that

$$\begin{cases} -(2n+1)\dfrac{\rho''}{\rho} - \dfrac{\rho'''}{\rho'} = C,\\ \dfrac{1}{\rho^2}\mathrm{Ric}^{\hat\nabla} - 2\Big(\dfrac{\rho\rho''+n(\rho')^2}{\rho^2}\Big)\cdot\mathrm{Id}_{\{\xi,\partial_t\}^\perp} = C\cdot\mathrm{Id}_{\{\xi,\partial_t\}^\perp},\end{cases}$$

that is, such that

$$\begin{cases}\rho\rho''' = -(2n+1)\rho'\rho'' - C\rho\rho',\\ \mathrm{Ric}^{\hat\nabla} = \big(2(\rho\rho''+n(\rho')^2) + C\rho^2\big)\cdot\mathrm{Id}_{\{\xi,\partial_t\}^\perp}.\end{cases} \tag{11}$$

Note that, by the first equation, the factor $2(\rho\rho''+n(\rho')^2)+C\rho^2 = 2c$ is constant on I (its first derivative is twice the difference between left- and right-hand sides of the first equation). This shows, in particular, that the manifold $(M, \hat g, \hat\xi)$ is transversally Einstein, and therefore it is η-Einstein, as it is Sasakian. Notice also that c is an integration constant, which is equal to $\frac{\mathrm{Scal}^Q}{4(n-1)}$.

To solve this ODE, we consider the change of variables $z := (\rho')^2 + \varepsilon\rho^2$, where ε denotes the sign of $\widetilde{\mathrm{Scal}} = 2nC$, *i.e.* it is defined as follows:

$$\varepsilon := \begin{cases}-1 & \text{if } C<0,\\ 0 & \text{if } C=0,\\ 1 & \text{if } C>0.\end{cases}$$

After rescaling the metric in such a way that $C = 2(n+1)\varepsilon$, the derivative $z' = 2\rho'\rho'' + 2\varepsilon\rho\rho'$ can be computed as follows:

$$\begin{aligned} z' &= 2\rho'\Big(\frac{c}{\rho} - \frac{C}{2}\rho - n\frac{(\rho')^2}{\rho} + \varepsilon\rho\Big) \\ &= 2\rho'\Big(\frac{c}{\rho} - \varepsilon n\rho - n\frac{(\rho')^2}{\rho}\Big) = 2c\frac{\rho'}{\rho} - 2n\frac{\rho'}{\rho}z. \end{aligned}$$

Solving this linear first-order ODE in z, we obtain

$$z = \frac{c}{n} + D\rho^{-2n}$$

for some constant $D \in \mathbb{R}$. In turn, this leads to the following nonlinear first-order ODE in ρ since we recall that $\rho' > 0$:

$$\rho' = \sqrt{-\varepsilon\rho^2 + D\rho^{-2n} + \frac{c}{n}}. \tag{12}$$

In the following, we assume ρ to be defined at $t = 0$ with $\rho(0) > 0$. In order to study the solution of this ODE, we distinguish three cases, according to the sign of the constant D, as follows.

(1) If $D = 0$, then equation (12) can either be solved explicitly or it admits no solution, depending on the sign of ε.

If $\varepsilon = -1$, *i.e.* if $\widetilde{\mathrm{Scal}}$ is negative, then the ODE (12) becomes $\rho' = \sqrt{\rho^2 + \frac{c}{n}}$ and it always admits an explicit maximal solution, which is given as follows, according to the sign of c, *i.e.* the sign of transversal scalar curvature Scal^Q.

- If $c < 0$, then $\rho(t) = \sqrt{-\frac{c}{n}} \cdot \cosh(t + \arg\cosh(\rho(0)\sqrt{-\frac{n}{c}}))$ solves the ODE (12) on the maximal interval $I_{\max} =]-\arg\cosh(\rho(0)\sqrt{-\frac{n}{c}}), \infty[$. For establishing this maximal interval of definition, we use the fact that both functions ρ and ρ' must be positive.
- If $c = 0$, then $\rho(t) = \rho(0) \cdot e^t$ solves the ODE (12) on $\mathbb{R}$.
- If $c > 0$, then $\rho(t) = \sqrt{\frac{c}{n}}\sinh(t + \arg\sinh(\rho(0)\sqrt{\frac{n}{c}}))$ solves the ODE (12) on the interval $I_{\max} =]-\arg\sinh(\rho(0)\sqrt{\frac{n}{c}}), \infty[$.

If $\varepsilon = 0$, *i.e.* if $\widetilde{\mathrm{Scal}}$ vanishes, then the ODE (12) becomes $\rho' = \sqrt{\frac{c}{n}}$. Hence, it has no solution if $c \leq 0$. However, for $c > 0$, the function $\rho(t) = t\sqrt{\frac{c}{n}} + \rho(0)$ solves the ODE on $\mathbb{R}$, but ρ is positive only on $I_{\max} =]-\sqrt{\frac{n}{c}}\rho(0), \infty[$.

If $\varepsilon = 1$, *i.e.* if $\widetilde{\mathrm{Scal}}$ is positive, then the ODE (12) becomes $\rho' = \sqrt{\frac{c}{n} - \rho^2}$. Hence, it has no solution with positive derivative if $c \leq 0$. For $c > 0$, the function $\rho(t) = \sqrt{\frac{c}{n}}\sin(t + \arcsin(\rho(0)\sqrt{\frac{n}{c}}))$ solves the equation on the maximal interval $I_{\max} =]-\arcsin(\rho(0)\sqrt{\frac{n}{c}}), \frac{\pi}{2} - \arcsin(\rho(0)\sqrt{\frac{n}{c}})[$.

(2) If $D < 0$, then we consider the function

$$f :]0, \infty[\to \mathbb{R}, \quad f(x) := -\varepsilon x^2 + Dx^{-2n} + \frac{c}{n},$$

whose derivative is given by $f'(x) = -2x(\varepsilon + nDx^{-2n-2})$ for all $x > 0$. According again to the sign of ε, we distinguish the following three subcases.

If $\varepsilon = -1$, then $f'(x) > 0$, for all $x > 0$, so the function f is increasing on $]0, \infty[$ with $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. Hence, there exists a unique $\rho_0 \in \,]0,\infty[$ with $f(x) < 0$ for $0 < x < \rho_0$, $f(\rho_0) = 0$ and $f(x) > 0$ for $x > \rho_0$. Necessarily, the solution ρ satisfies $\rho \geq \rho_0$, and actually $\rho > \rho_0$ unless ρ is constant, which is excluded because ρ' is positive everywhere. Integrating from 0 to some positive t, we obtain

$$\int_{\rho(0)}^{\rho(t)} \frac{d\rho}{\sqrt{\rho^2 + D\rho^{-2n} + \frac{c}{n}}} = t.$$

Since

$$\frac{1}{\sqrt{f(x)}} \sim_{\rho_0} \frac{1}{\sqrt{f'(\rho_0)\cdot(x-\rho_0)}},$$

the solution ρ runs backward to ρ_0 in finite time. On the other hand, because of $\frac{1}{\sqrt{f(x)}} \sim_\infty \frac{1}{x}$, the solution ρ becomes infinite only in infinite time. Summing up, we conclude that, in this case, the maximal solution ρ of the ODE (12) is defined on some time interval of the form $I_{\max} = \,]t_0,\infty[$, where $t_0 < 0$, and fulfils $\lim_{t\to t_0^+} \rho(t) = \rho_0$, $\lim_{t\to\infty} \rho(t) = \infty$ and $\lim_{t\to t_0^+} \rho'(t) = 0$ because $f(\rho_0) = 0$.

If $\varepsilon = 0$, then $f'(x) = -2nDx^{-2n-1} > 0$ for all $x > 0$, so the function f is increasing with $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \frac{c}{n}$. Hence, if $c \leq 0$, there is no solution of the ODE (12). If $c > 0$, then there exists a unique ρ_0 with $f(x) < 0$ for $0 < x < \rho_0$, $f(\rho_0) = 0$ and $f(x) > 0$ for $x > \rho_0$. The same argument as in the previous case shows the solution ρ necessarily satisfies $\rho > \rho_0$. Integrating again from 0 to some positive t, we obtain

$$\int_{\rho(0)}^{\rho(t)} \frac{d\rho}{\sqrt{D\rho^{-2n} + \frac{c}{n}}} = t.$$

Since

$$\frac{1}{\sqrt{f(x)}} \sim_{\rho_0} \frac{1}{\sqrt{f'(\rho_0)\cdot(x-\rho_0)}},$$

the solution ρ runs backward to ρ_0 in finite time. Summing up, we conclude that, in this case, the maximal solution ρ of the ODE (12) is defined on some time interval of the form $]t_0,\infty[$, where $t_0 < 0$, and fulfils $\lim_{t\to t_0^+} \rho(t) = \rho_0$, $\lim_{t\to\infty} \rho(t) = \infty$ and $\lim_{t\to t_0^+} \rho'(t) = 0$.

If $\varepsilon = 1$, then f defined above attains its maximum at $\rho_0 = (-nD)^{\frac{1}{2n+2}}$, where the function takes the value $f(\rho_0) = \frac{c}{n} - \frac{n+1}{n}(-nD)^{\frac{1}{n+1}}$. This leads us to consider the following subcases, according to the sign of $f(\rho_0)$, or equivalently, according to the value of c, as follows.

- If $c \leq (n+1)(-nD)^{\frac{1}{n+1}}$, then $f(x) \leq f(\rho_0) \leq 0$; hence, in this case, the ODE (12) has no solution with positive derivative.
- If $c > (n+1)(-nD)^{\frac{1}{n+1}}$, then there exist ρ_1, ρ_2 such that $0 < \rho_1 < \rho_0 < \rho_2$ and $f(\rho_1) = f(\rho_2) = 0$, which implies that, in this case, the solution ρ of the ODE (12) is bounded as follows: $\rho_1 < \rho(t) < \rho_2$. Integrating again from 0 to some positive t, we obtain

$$\int_{\rho(0)}^{\rho(t)} \frac{d\rho}{\sqrt{-x^2 + D\rho^{-2n} + \frac{c}{n}}} = t.$$

Since

$$\frac{1}{\sqrt{f(x)}} \sim_{\rho_1} \frac{1}{\sqrt{f'(\rho_1)\cdot(x-\rho_1)}},$$

the solution ρ runs backward to ρ_1 in finite time. Similarly, since

$$\frac{1}{\sqrt{f(x)}} \sim_{\rho_2} \frac{1}{\sqrt{f'(\rho_2)\cdot(x-\rho_2)}},$$

the solution ρ runs to ρ_2 in finite time. Summing up, we conclude that, in this case, the maximal solution ρ of the ODE (12) is defined on some time interval of the form $I_{\max} =]t_1, t_2[$, where $t_1 < 0 < t_2$, and fulfils $\lim_{t\to t_1^+}\rho(t) = \rho_1$ and $\lim_{t\to t_2^-}\rho(t) = \rho_2$.

(3) If $D > 0$, then one may proceed similarly to the analysis in case (2), also by remarking that the function $-f(x) = \varepsilon x^2 - Dx^{-2n} - \frac{c}{n}$ is the same as in case (2), as $-D < 0$.

Summing up, we have shown the following result.

Theorem 4.1. *Let $(\widetilde{M}^{2n}, g) = (I \times M^{2n-1}, dt^2 \oplus \rho^2((\rho')^2 g_{\hat\xi} \oplus g_{\hat\xi^\perp}))$ be a Kähler doubly-warped product, where I is an open interval containing 0, $\rho, \rho' : I \to \mathbb{R}$ are positive functions and $(M, \hat g, \hat\xi)$ is Sasaki. Then the following assertions hold.*

(i) *The manifold $(\widetilde{M}^{2n}, g)$ is an Einstein manifold with Einstein constant $2(n+1)\varepsilon$, where $\varepsilon \in \{-1, 0, 1\}$, if and only if there exist constants $c, D \in \mathbb{R}$ such that*

$$\begin{cases} \rho' = \sqrt{-\varepsilon\rho^2 + D\rho^{-2n} + \dfrac{c}{n}}, \\ \mathrm{Ric}^{\hat\nabla} = 2c\cdot \mathrm{Id}_Q, \end{cases}$$

where $Q := \{\xi, \nu\}^\perp$ and $\hat\nabla$ denotes the transverse connection on Q.

(ii) *In each of the two cases $D = 0$, $\varepsilon = -1$ and $c = 0$, or $D > 0$, $\varepsilon = -1$, $c = -(n+1)(nD)^{\frac{1}{n+1}}$ and $\rho(0) > (nD)^{\frac{1}{2n+2}}$, there exists a solution ρ of the ODE $\rho' = \sqrt{\rho^2 + D\rho^{-2n} + \frac{c}{n}}$ which is defined on $\mathbb{R}$. For any other values of c, D and ε, there exists a solution ρ of $\rho' = \sqrt{\rho^2 - D\rho^{-2n} + \frac{c}{n}}$ defined on a maximal interval $I_{\max} \subsetneq \mathbb{R}$ around 0.*

APPENDIX A

Let us recall here some general facts on warped product structures induced by smooth functions. The local version of the following can be found in the beautiful paper [11]; see in particular [11, Sec. D]. In the following proposition, the Levi–Civita connection of (M^n, g) is denoted by ∇ (and so differently from the Kähler setting, where ∇ denotes the natural connection on the transverse distribution on (M, g, ξ)).

Proposition A.1. *Let (M^n, g) be a connected complete Riemannian manifold. Assume that some $u \in C^\infty(M, \mathbb{R})$ has no critical point on M^n and satisfies $\nabla^2 u(\nabla u) = \lambda\nabla u$ for some $\lambda \in C^\infty(M, \mathbb{R})$. Then the manifold (M^n, g) is* *isometric to* $(\mathbb{R}\times\Sigma, dt^2\oplus g_t)$*, where* Σ *is a level hypersurface of* u *and* $(g_t)_{t\in\mathbb{R}}$ *is a one-parameter-family of Riemannian metrics on* Σ*.*

Proof. Fix $u_0\in u(M)$, and let $\Sigma := u^{-1}(\{u_0\})\subset M$. By assumption, Σ is a smooth hypersurface in M. Consider the map $F:\mathbb{R}\times\Sigma\to M$ given by the flow of ν, *i.e.* $f(t,x) := F_t^\nu(x)$ for all $(t,x)\in\mathbb{R}\times\Sigma$. Note that the flow $(F_t)_t = (F_t^\nu)_t$ is well-defined on $\mathbb{R}$ since ν is a bounded vector field on the complete Riemannian manifold (M^n, g). We show that F provides the desired isometry.

First, F is a local diffeomorphism; for any $(t,x)\in\mathbb{R}\times\Sigma$ and $(T,X)\in\mathbb{R}\times T_x\Sigma$, one has

$$\begin{aligned} d_{(t,x)}F(T,X) &= T\frac{\partial F}{\partial t}(t,x) + d_xF_t^\nu(X)\\ &= T\nu_{F(t,x)} + d_xF_t^\nu(X)\\ &= Td_xF_t^\nu(\nu_x) + d_xF_t^\nu(X) \quad \text{since } (F_t^\nu)_*\nu = \nu\\ &= d_xF_t^\nu(T\nu_x + X) \end{aligned}$$

so that $d_{(t,x)}F(T,X) = 0$ if and only if $T=0$ and $X=0$ (for $F_t^\nu : M\to M$ is a diffeomorphism). This shows the invertibility of $d_{(t,x)}F$ and hence that F is a local diffeomorphism.

In particular, $F(\mathbb{R}\times\Sigma)$ is open in M. But this also implies that $F(\mathbb{R}\times\Sigma)$ is closed in M, for one may define the equivalence relation $\sim$ on M via $x,y\in M$, $x\sim y$ if and only if there exists a $\hat{u}\in u(M)$ such that $x,y\in F(\mathbb{R}\times u^{-1}(\{\hat{u}\}))$, where F is defined by the flow of ν (starting this time from the hypersurface $u^{-1}(\{\hat{u}\})$ of M). By the preceding argument, each equivalence class is open in M and hence also closed in M. Since M is connected, this yields $F(\mathbb{R}\times\Sigma) = M$, *i.e.* F is surjective. The injectivity of F follows easily from the fact that, for any $x\in\Sigma$, the function $f_x := u\circ F(\cdot,x):\mathbb{R}\to\mathbb{R}$ is monotonously increasing, for it is smooth with $f_x'(t) = |\nabla u|_{F(t,x)} > 0$ for all $t\in\mathbb{R}$; if $F(t,x) = F(t',x')$ for some $(t,x),(t',x')\in\mathbb{R}\times\Sigma$, then the points $F(t,x)$ and $F(t',x')$ lie on the same integral curve of ν, but by the injectivity of f_x, that curve intersects Σ only in x; hence $x=x'$, and again by the injectivity of f_x, it follows $t=t'$. On the whole, F is a diffeomorphism. In particular, Σ itself must be connected.

We now look at the pullback metric F^*g on $\mathbb{R}\times\Sigma$. Obviously, we have $(F^*g)(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = 1$ since ν is a unit vector field. Moreover, as noticed in [18, Prop. 2], because ∇u is a pointwise eigenvector for $\nabla^2 u$, the vector field $\nu = \frac{\nabla u}{|\nabla u|}$ is geodesic. This has the important consequence for the splitting of the metric that, for any $(t,x)\in\mathbb{R}\times\Sigma$ and $X\in T_x\Sigma$, we have

$$\begin{aligned} (F^*g)_{(t,x)}(\frac{\partial}{\partial t},X) &= g_{F(t,x)}(\nu_{F(t,x)}, d_xF_t(X))\\ &= g_{F(t,x)}(d_xF_t^\nu(\nu_x), d_xF_t(X)) = (F_t^\nu)^*g(\nu_x,X), \end{aligned}$$

where

$$\frac{\partial}{\partial s}(F_s^*g)(\nu_x,X)_{|s=t} = (\mathcal{L}_\nu g)((F_t)_*\nu,(F_t)_*X)_{F(t,x)} = (\mathcal{L}_\nu g)(\nu,(F_t)_*X)_{F(t,x)}.$$

But, for all $X \in TM$,

$$(\mathcal{L}_\nu g)(\nu, X) = g(\nabla_\nu \nu, X) + g(\nabla_X \nu, \nu) = 0$$

by the fact that ν is geodesic of constant length. Therefore,

$$\frac{\partial}{\partial s}(F_s^* g)(\nu_x, X)_{|s=t} = 0 \quad \text{for all } t \in \mathbb{R},$$

and thus $(F_t^* g)(\nu_x, X) = (F_0^* g)(\nu_x, X) = g(\nu_x, X) = 0$ for all $(t, x) \in \mathbb{R} \times \Sigma$. This proves the splitting $F^* g = dt^2 \oplus g_t$, where $g_t := (F_t)^* g_{|T\Sigma \times T\Sigma}$. We note an important consequence of the splitting $F^* g = dt^2 \oplus g_t$, namely that the flow $(F_t^\nu)_t$ preserves the level hypersurfaces of u, or equivalently, that the function $f_x = u \circ F(\cdot, x)$ defined above actually does not depend on x. Given any further $y \in \Sigma$, consider any smooth curve $c : [0,1] \to \Sigma$ with $c(0) = x$ and $c(1) = y$. For a fixed $t \in \mathbb{R}$, look at the smooth function $h(s) := u \circ F(t, c(s))$, $s \in [0,1]$. Its first derivative is given by

$$\begin{aligned} h'(s) &= g_{F(t,c(s))}(\nabla u, d_{c(s)} F_t(\dot{c}(s))) \\ &= |\nabla u|_{F(t,c(s))} g_{F(t,c(s))}(\nu_{F(t,c(s))}, d_{c(s)} F_t(\dot{c}(s))) \\ &= |\nabla u|_{F(t,c(s))} (F_t^* g)(\nu_{c(s)}, \dot{c}(s)) = 0 \end{aligned}$$

so that h is constant, and hence $h(0) = u(F(t,x)) = h(1) = u(F(t,y))$, *i.e.* $f_x(t) = f_y(t)$. This concludes the proof. □

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